A triple integral of a continuous fctn $f(x,y,z)$ over a box $B=[a,b]\times[c,d]\times[e,f]$ is given by

$$\iiint_B f(x,y,z)\,dV = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx$$

hypervolume

Ex 1. Compute $\iiint_B x^2 e^{y+3z}\,dV$ where $B=[1,4]\times[0,2]\times[2,6]$.

$$= \int_1^4 \int_0^2 \int_2^6 x^2 e^{y+3z}\,dz\,dy\,dx = \int_1^4 \int_0^2 \left(\frac{1}{3}x^2 e^{y+3z}\right)\Bigg|_{z=2}^{z=6} dy\,dx$$

$$= \frac{1}{3}\int_1^4 \int_0^2 x^2 e^{18+y} - x^2 e^{6+y}\,dy\,dx$$

$$= \frac{1}{3}\int_1^4 \left(x^2 e^{18+y} - x^2 e^{6+y}\right)\Bigg|_{y=0}^{y=2} dx$$

$$= \frac{1}{3}\int_1^4 (x^2 \cdot e^{20} - x^2 e^8) - (x^2 e^{18} - x^2 e^6)\,dx$$

$$= \frac{1}{3}\int_1^4 x^2(e^{20} - e^8 - e^{18} + e^6)\,dx$$

$$= \frac{1}{3}(e^{20} - e^8 - e^{18} + e^6)\left(\frac{x^3}{3}\right)\Bigg|_1^4$$

$$= 7\left(e^{20} - e^{8} - e^{18} + e^{6}\right)$$

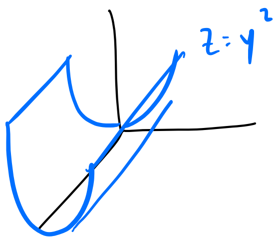
//

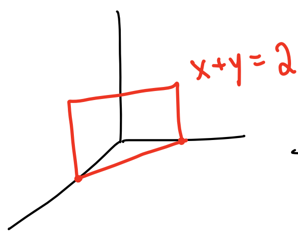
We still have $\iiint_W 1\, dV$ gives the volume of a region W.

ex 2 Find volume of region bounded by xy-, xz-, yz- planes and the planes $x+y=2$ and the parabolic cylinder $z=y^2$.

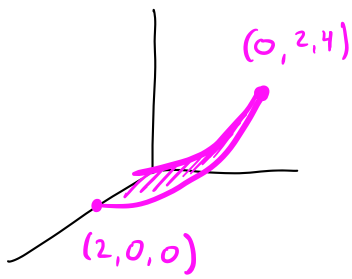
1) in first octant so $0 \le x$, $0 \le y$, $0 \le z$

2) draw a picture.





$\longrightarrow$



$dz\,dy\,dx$

$0 \le x \le 2$

$0 \le y \le 2-x$

$0 \le z \le y^2$

$dz\,dx\,dy$

$0 \le y \le 2$

$0 \le x \le 2-y$

$0 \le z \le y^2$

$dx\,dy\,dz$

$0 \le z \le 4$

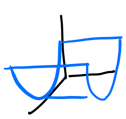
$\sqrt{z} \le y \le 2$

$0 \le x \le 2-y$

$$\boxed{\frac{4}{3}}$$

Q: why not $0 \le z \le (2-x)^2$?

Compare $z=(2-x)^2$ and $z=y^2$

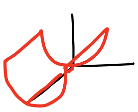


$0 \le x \le 2$

$0 \le y \le 2-x$

$0 \le z \le (2-x)^2$

On $z=4$, there is a line segment of form $(0,t,4)$, $0 \le t \le 2$.



$0 \le x \le 2$

$0 \le y \le 2-x$

$0 \le z \le y^2$

On $z=4$, there is only one point $(0,2,4)$

In addition, the region is bounded between the surfaces $z=y^2$ and $z=0$ and we are integrating over the surface $D = \left\{(x,y,0) \;\middle|\; \begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le 2x \end{array}\right\}$.

We call this z-simple.

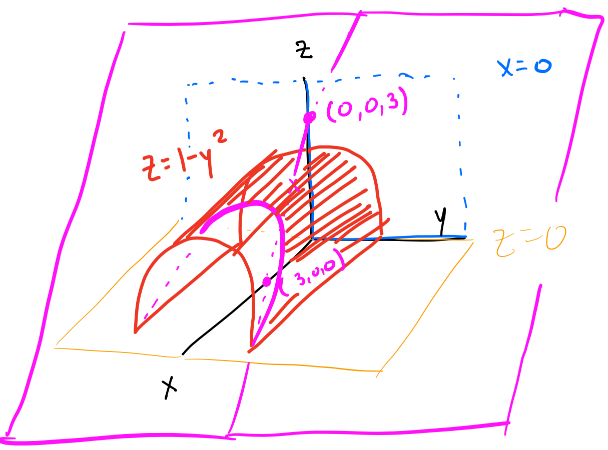
ex 3. What is average distance of pts lying in the region bounded by $z=1-y^2$, $x=0$, $z=0$ and $z+x=3$? Only set up the integral

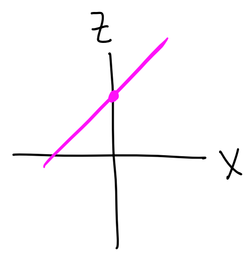
Recall that the average value of a fctn f on an interval $[a,b]$ is given by $\text{avg}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

We have natural generalizations in 2-, 3-dimensions

$$f(x,y) \rightsquigarrow \text{avg}(f) \text{ on } R \text{ is } \frac{1}{\text{area}(R)}\iint_R f(x,y)\,dA$$

$$f(x,y,z) \rightsquigarrow \text{avg}(f) \text{ on } R \text{ is } \frac{1}{\text{vol}(R)}\iiint_R f(x,y,z)\,dV.$$

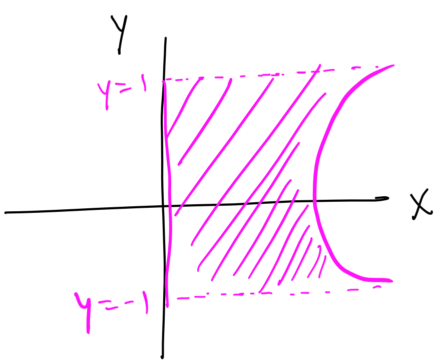




on $y=0$

$$3-x = 0 = 1-y^2$$

$$x = 2+y^2$$



$$-1 \le y \le 1$$

$$0 \le x \le 2+y^2$$

$$\text{vol}(R) = \int_{-1}^{1}\int_0^{2+y^2}\int_0^{1-y^2} dz\,dx\,dy = \int_{-1}^{1}\int_0^{2+y^2} 1-y^2\,dx\,dy = \int_{-1}^{1}(1-y^2)(2+y^2)\,dy$$

$$\frac{-3}{15}-\frac{5}{15}+\frac{30}{15}$$

$$= \int_{-1}^{1} -y^4-y^2+2\,dy = \frac{-y^5}{5}-\frac{y^3}{3}+2y\Big|_{-1}^{1} = \frac{-1}{5}-\frac{1}{3}+2-\left(\frac{1}{5}+\frac{1}{3}-2\right) = 2\left(\frac{-1}{5}-\frac{1}{3}+2\right) = \boxed{\frac{44}{15}}$$

$$\text{avg}(f) = \frac{15}{44} \int_{-1}^{1} \int_{0}^{2+y^2} \int_{0}^{1-y^2} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy.$$