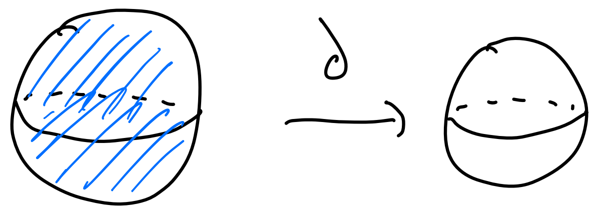
Let S be a closed surface and it encloses a region W. Then for a vector field F, the Divergence Thm says

$$\underbrace{\iint_S}_{\text{the regions boundary}} \underbrace{F}_{F} \cdot d\vec{S} = \underbrace{\iiint_W}_{\text{a region in } \mathbb{R}^3} \underbrace{\operatorname{div}(F)}_{\text{Kinda like a derivative of } F}\, dV$$

So $\operatorname{div}(F)$ measures flux per unit volume.

A region enclosed by a surface.

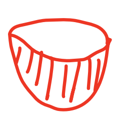


W = Solid ball $\qquad$ S = sphere

Ex 1. Let S be closed surface made of the part of $z = x^2 + y^2$ where $z \le 9$ and the disk on top, oriented outwards.

Let $F = \langle x^3 - 5yz^2,\ 3\sin(x) + e^z,\ x + xy^2 - z^2 \rangle$. Set up $\oiint_S F$ using divergence

The way we would normally do this is decompose S into 2 parts,

S_1 =  the body

S_2 =  the lid

Parameterize each surface, find normal vector of both, compute each surface integral separately.

Use divergence instead: $\operatorname{div}(F) = 3x^2 - 2z$. Then

$$\oiint_S F \cdot d\vec{S} = \iiint_W 3x^2 - 2z\, dV$$

convert to polar coords

$0 \le r \le 3$

$0 \le \theta \le 2\pi$

$r^2 \le z \le 9$

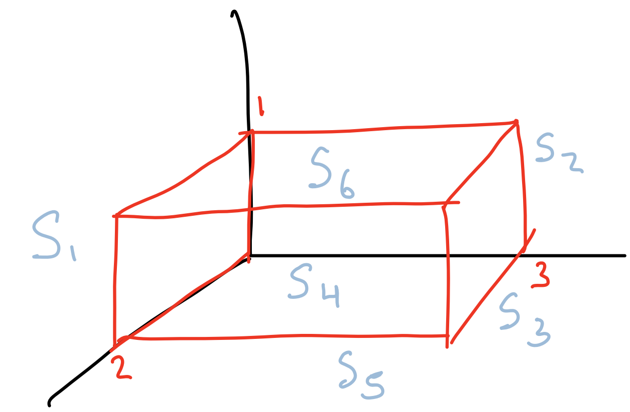
$$= \int_0^{2\pi} \int_0^3 \int_{r^2}^9 \left(3(r^2\cos^2\theta) - 2z\right) r \, dz\, dr\, d\theta = \frac{1215\pi}{4}.$$

b) Can you compute $\oiint_{S_1} F \cdot d\vec{S}_1$ using part a)?

The idea here is that

$$\oiint_{S_1} F \cdot d\vec{S}_1 + \oiint_{S_2} F \cdot d\vec{S}_2 \underset{\substack{\text{additivity} \\ \text{of integration}}}{=} \oiint_S F \cdot d\vec{S} \underset{\text{divergence thm.}}{=} \iiint_W \operatorname{div}(F)\, dV$$

Ex 2. Verify Divergence Thm works for $\iint_S \langle x^2, z^4, e^z \rangle \cdot dS$ where S is the rectangle $[0,2] \times [0,3] \times [0,1]$.



S_2: $G(y,z) = (0,y,z)$ $\quad \int_0^3 \int_0^1 0 \, dA = 0$

$N = \langle -1, 0, 0 \rangle$

S_6: $0 \le x \le 2$, $0 \le y \le 3$, $z = 1$ $\quad G(x,y) = (x,y,1)$, $N = \langle 0,0,1 \rangle$

$$\int_0^2 \int_0^3 \langle x^2, 1, e \rangle \cdot \langle 0,0,1 \rangle \, dx\, dy = \boxed{6e}$$

More quick, divergence theorem gives

$\operatorname{div}(\langle x^2, z^4, e^z \rangle) = 2x + e^z$. Then

$$\int_0^2 \int_0^3 \int_0^1 2x + e^z \, dz\, dy\, dx = 6e + 6.$$

//

Ex 3. Compute $\underset{S}{\iint}\langle z^2+xy^2, \cos(x+z), e^{-y}-zy^2\rangle$ over unit sphere.

flux

$= 0$ since $\operatorname{div}(F) = y^2 + 0 + (-y^2) = 0$

Ex 4. Compute $\underset{S}{\iint} \operatorname{curl}(\nabla f)\, d\vec{S}$ where f is a function anyone wants.

$= 0$

$$\underset{\text{fctn}}{f} \xrightarrow{\ \nabla\ } \underset{\text{vector field}}{F = \nabla f} \xrightarrow{\text{curl}} \underset{\text{vector field}}{A = \operatorname{curl}(F)} \xrightarrow{\text{div}} \underset{\text{fctn}}{g}$$

A vector field F is solenoidal if $F = \operatorname{curl}(A)$ for some v.f. A.
A is called a vector potential of F.

Note that $\operatorname{curl}(\nabla f) = 0$ and $\operatorname{div}(\operatorname{curl}(A)) = 0$.

Recall that curl measures circulation in a nbhd about a pt and divergence measures the expansion/contraction of F in a nbhd about a pt.

<u>Stoke's Thm</u>: if $F = \langle F_1, F_2, F_3 \rangle$ and C is a curve in $\mathbb{R}^3$ bounding a surface $S \subseteq \mathbb{R}^3$, then

$$\oint_C F \cdot d\vec{r} = \oiint_S \operatorname{curl}(F) \cdot d\vec{S} = \iint_S \operatorname{curl}(F)(G(u,v)) \cdot N(u,v)\, du\, dv$$

where $G(u,v)$ is a parameterization of S and $N(u,v)$ is the positively-oriented normal vector of S.

<u>Divergence Thm</u>: if $F = \langle F_1, F_2, F_3 \rangle$ and S is a surface in $\mathbb{R}^3$ bounding a region $W \subseteq \mathbb{R}^3$, then

$$\iint_S F \cdot d\vec{S} = \iiint_W \operatorname{div}(F)\, dV = \text{regular triple integral}$$

How can we deal with vector surface integrals $\iint_S F \cdot d\vec{S}$ given a v.f. $F = \langle F_1, F_2, F_3 \rangle$ and a surface S?

	F is not solenoidal	F is solenoidal.
S is not closed (has boundary C)	$\iint_S \vec{F} \cdot d\vec{S} = \iint_S F(G(u,v)) \cdot N(u,v)\, du\, dv$ direct computation	*"Can"* use stokes $\oiint_S F \cdot d\vec{S} = \oint_C A \cdot d\vec{r}$ Stokes or direct computation
S is closed (no boundary)	$\oiint_S F \cdot d\vec{S} = \iiint_W \mathrm{div}(\vec{F})\, dV = 0$ Divergence Thm	$\oiint_S F \cdot d\vec{S} = 0$ by divergence thm.