Given a vector feld F, we can integrate over a line C.

Sometimes C is a closed and simple curve

↳ start point and end pt are the same

↳ curve doesn't cross itself

not closed, not simple | closed ✓, not simple | not closed, simple ✓ | closed ✓, simple ✓

When C is closed and has an orientation, we denote the vector line integral as $\oint_C F \cdot d\vec{r}$. We say circulation of F around C.

(with this little circle.

By fundamental theorem of conservative vector fields,

$$\oint_C F \cdot d\vec{r} = 0$$

if C is a closed simple curve and F is conservative.

Ex 1. Compute $\oint_C F \cdot d\vec{r}$ where $F = \langle xy^2, x \rangle$ and C is unit circle oriented ccw.

$\oint_C F \cdot d\vec{r} = \oint_C xy^2\,dx + x\,dy$. Note that F is not conservative.

$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle \quad 0 \le t \le 2\pi.$

$r'(t) = \langle -\sin(t), \cos(t) \rangle$

$F(r(t)) = \langle \cos(t)\sin^2(t), \cos(t) \rangle$

$$\oint_C F \cdot d\vec{r} = \int_0^{2\pi} \cos^2(t) - \cos(t)\sin^3(t)\,dt = \text{bleh} \ldots = \pi$$

Attempt #2

Note $F_1 = xy^2$ and $F_2 = x$. Then consider $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 - 2xy.$

and instead of C, consider interior D bounded by C, $x^2+y^2 \le 1$. $(\partial D = C)$

$$\iint_D 1 - 2xy\,dA \Rightarrow \text{switch to polar coords.}$$

$0 \le r \le 1, \quad 0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \int_0^1 r - 2r^3 \sin\theta\cos\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} - \frac{1}{4}\sin(2\theta)\,d\theta$$

$$= \frac{1}{2}\theta + \frac{\cos(2\theta)}{8} \Bigg|_{\theta=0}^{\theta=2\pi} = \pi + \frac{1}{8} - \left(0 + \frac{1}{8}\right) = \pi.$$

So we have for $D: x^2+y^2 \le 1$

$$\oint_{\partial D} xy^2\,dx + x\,dy = \iint_D -2xy + 1\,dA = \pi.$$

Green's Theorem Let D be a domain whose boundary ∂D is a simple, closed curve oriented CCW. If F_1 and F_2 are differentiable in an open region containing D, then

$$\oint_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$$

Given a D, $\iint_D 1 \, dA = \text{area}(D)$. Can we choose F_1 and F_2 and apply Green's Thm? Sure!

$$F = \langle 0, x \rangle \Rightarrow \text{area}(D) = \oint_{\partial D} x \, dy$$

$$F = \langle -y, 0 \rangle \Rightarrow \text{area}(D) = \oint_{\partial D} -y \, dx$$

$$F = \left\langle \frac{-y}{2}, \frac{x}{2} \right\rangle \Rightarrow \text{area}(D) = \oint_{\partial D} \frac{-y}{2} dx + \frac{x}{2} dx$$

Ex 2. Compute area of an ellipse given by $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ using a line integral.

Get a weird polar eqn if we use $x = r\cos\theta$, $y = r\sin\theta$. Using a change of variables $u = \frac{x}{a}$ and $v = \frac{y}{b}$ suggests using a 2nd change of variable. Just use green's theorem!

Boundary is given by $\vec{r}(t) = \langle a\cos(t), b\sin(t) \rangle$ for $0 \le t \le 2\pi$.

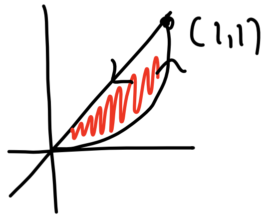
$$\text{area}(D) = \oint_C x \, dy = \int_0^{2\pi} F(\vec{r}(t)) \cdot r'(t) \, dt$$

$$= \int_0^{2\pi} \langle 0, a\cos(t) \rangle \cdot \langle a\sin(t), b\cos(t) \rangle \, dt$$

$$= ab\int_0^{2\pi} \cos^2(t)\,dt = \boxed{\pi ab}$$

Ex 3. Use Green's Theorem to compute $\oint_C F\cdot d\vec{r}$ where $F=\langle x^2, x^2\rangle$ and C consists of the arcs $y=x^2$ and $y=x$ for $0\le x\le 1$.

$$\frac{\partial F_2}{\partial} - \frac{\partial F_1}{\partial y} = 2x - 0 = 2x.$$



$$\oint_C F\cdot d\vec{r} = \int_0^1\int_{x^2}^{x} 2x\,dy\,dx$$

$$= \int_0^1 2xy\Big|_{x^2}^{x}dx = \int_0^1 2x^2 - 2x^3\,dx = \frac{2}{3}x^3 - \frac{x^4}{2}\Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{2} = \boxed{\frac{1}{6}}$$

Observe that a vector field $F=\langle F_1, F_2\rangle$ may be expressed as $F=\langle F_1, F_2, 0\rangle$. Then

$$\operatorname{curl}(F) = \left\langle 0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right\rangle$$

where the z-component is the term that shows up in Green's theorem. Denote the z-component by $\operatorname{curl}_z(F)$. We may reformulate Green's theorem to

$$\oint_C F\cdot d\vec{r} = \iint_D \operatorname{curl}_z(F)\,dA \approx \operatorname{curl}_z(F)\iint_D dA$$

in words, What is $curl_z(F)$? circulation!

So $\oint_{\partial D} F \cdot d\vec{r} \approx curl_z(F) \cdot area(D)$

$\Rightarrow curl_z(F) \approx \frac{1}{area(D)} \oint_{\partial D} F \cdot d\vec{r}$

curl is circulation (work) of around a small circle divided by area of circle

In other words,

$$\boxed{curl(F)(P) = \lim_{r \to 0} \frac{1}{|C_r|} \oint_{C_r} F \cdot d\vec{r}}$$

standard definition of curl

where C_r is a circle of radius r centered at P.

curl(F)(P) is a vector that tells us how much work is needed to go around a loop at P.

- direction of curl(F)(P) is axis of rotation at that point.
- length is magnitude of rotation.

Revisit the vector field $F = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

Observe that $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2}$. Is F conservative?

on $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \le t \le 2\pi$, so C is closed and simple.

$$\oint_C F \cdot d\vec{r} = \int_0^{2\pi} F(\vec{r}(t)) \cdot r'(t)\, dt = \int_0^{2\pi} 1\, dt = \boxed{2\pi}$$

But "by Green's Theorem",

$$\iint_{int(C)} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA = 0.$$

So whats going on?

① C is a simple closed curve. It's interior contains the origin but F_1 and F_2 are not defined there.

② in the limit, curl(F)(origin) DNE.

How to deal with $\oint_C F \cdot d\vec{r}$?

If F is conservative,

1) and C is closed simple loop, then $\oint_C F \cdot d\vec{r} = 0$

2) and C is a curve w/ distinct startpt and endpt, then use FTOCVF.

If F is not conservative,

3) and C is closed simple loop with no holes, then use Green's Thm.

4) and C is nasty, compute $\oint_C F \cdot d\vec{r}$ directly