

THE ICOSAHEDRAL LINE CONFIGURATION AND WALDSCHMIDT CONSTANTS

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ABSTRACT. There is a highly special point configuration in \mathbb{P}^2 of 31 points, naturally arising from the geometry of the icosahedron. The 15 planes of symmetry of the icosahedron projectivize to 15 lines in \mathbb{P}^2 , whose points of intersections yield the 31 points. Each point corresponds to an opposite pair of vertices, faces or edges of the icosahedron. The symmetry group of the icosahedron is $G = A_5 \times \mathbb{Z}_2$, one of finitely many exceptional complex reflection groups. The action of G on the icosahedron descends onto an action on the line configuration. We blow up \mathbb{P}^2 at the 31 points to study the line configuration. The Waldschmidt constant is a measure of how special a collection of points in \mathbb{P}^2 is. In this paper, we study negative G -invariant curves on this blow-up in order to compute the Waldschmidt constant of the ideal of the 31 singularities.

1. INTRODUCTION

In this paper, we study a line configuration \mathcal{A} of 15 lines and 31 points in $\mathbb{P}^2 = \mathbb{P}_{\mathbb{C}}^2$ that naturally arise from studying the symmetries of the icosahedron, the platonic solid studied heavily by Klein [Kle58]¹. The icosahedron has 15 mirror planes that projectivize to the 15 lines of \mathcal{A} . The 15 lines intersect at 6 quintuple points, 10 triple points and 15 double points. Each pair of opposite vertices, faces and edges correspond to a quintuple, triple and double point respectively.

We may consider $I \subseteq S = \mathbb{C}[x, y, z]$ to be the ideal of a reduced collection of points in \mathbb{P}^2 . Define the m -th symbolic power $I^{(m)}$ to be $\cap_i I_{p_i}^m$ [Har18, pg 6]. Geometrically, this ideal corresponds to curves having multiplicity at least m at each p_i . The *Waldschmidt constant* of an ideal I is defined to be

$$\hat{\alpha}(I) = \lim_{m \rightarrow \infty} \frac{\alpha(I^{(m)})}{m}$$

where $\alpha(I)$ is the least positive integer t such that the graded piece $I_t \neq 0$. In this paper, we compute the Waldschmidt constant for $I_{\mathcal{A}}$, the homogeneous ideal of singularities of \mathcal{A} . The singularities are highly non-general points of \mathbb{P}^2 . We demonstrate similar techniques to those in [BDRH19] in order to calculate $\hat{\alpha}(I_{\mathcal{A}})$. The following is the main result of the paper.

Theorem (Main Result). *Let $I_{\mathcal{A}}$ be the homogeneous ideal of singularities of \mathcal{A} . Then*

$$\hat{\alpha}(I_{\mathcal{A}}) = \frac{11}{2}.$$

Determining the Waldschmidt constant of a collection of points is difficult in general. The Nagata conjecture [Nag59] states that for $s \geq 10$ very general points in \mathbb{P}^2 , a curve C passing through each p_i with multiplicity at least m_i satisfies

$$\deg C \geq \frac{1}{\sqrt{s}} \sum_{i=1}^s m_i.$$

The formulation of Nagata's conjecture in terms of Waldschmidt constants states $\hat{\alpha}(I) = \sqrt{s}$. From this perspective, the Waldschmidt constant measures how special an arrangement of points in \mathbb{P}^2 is. The Waldschmidt constant can be regarded as the reciprocal of a multi-point Seshadri constant, a

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well-known measure of local positivity of a line bundle. It is always true that $1 \leq \hat{\alpha}(I) \leq \sqrt{s}$. For special configurations of points, such calculations of $\hat{\alpha}(I)$ are non-trivial but we expect $\hat{\alpha}(I)$ to be smaller than \sqrt{s} . We exploit a relationship between the 31 points and the group $A_5 \times \mathbb{Z}_2$ to alleviate the difficulty.

1.1. Emphasis of the group action on the point configuration. The 60 rotational symmetries of the icosahedron form a group isomorphic to A_5 . The full symmetry group of the icosahedron $A_5 \times \mathbb{Z}_2$ has order 120, obtained by introducing the reflection through the center of the icosahedron. This group is one of finitely many exceptional complex reflection groups. For a $(n+1)$ -dimensional complex vector space V , a *complex reflection group* $G \subseteq \mathrm{GL}(V)$ is a finite group that is generated by pseudoreflections: those elements that fix a hyperplane pointwise. These groups were studied and classified in [ST54]. Considering the collection of hyperplanes, given by pseudoreflections of G , and projectivizing V yields a hyperplane configuration in \mathbb{P}^n . In addition, the natural action of G on \mathbb{P}^n permutes the hyperplanes. Configurations admitted by complex reflection groups exhibit interesting characteristics [DS21]. For $n = 2$, we obtain a line configuration in \mathbb{P}^2 . Line configurations have been studied extensively [Har18, OT13] yet continue to be subject of recent research [BDRH19, AD09].

The group G then naturally acts on the homogeneous coordinate ring S of \mathbb{P}^2 . An important characteristic of complex reflection groups is that the invariant ring of such groups is finitely generated [DK15]. Let $G = A_5 \times \mathbb{Z}_2$. The invariant ring S^G is generated as a \mathbb{C} -algebra by homogeneous forms ϕ_d , where the subscript denotes the degree of the form. The polynomials ϕ_d turn out to be vital in considering curves vanishing to points of the point configuration. This will allow us to construct curves of high multiplicities at the singularities of \mathcal{A} in order to bound $\hat{\alpha}(I_{\mathcal{A}})$ from above.

1.2. The blow-up of \mathbb{P}^2 at the 31 singularities. To give a lower bound of $\hat{\alpha}(I_{\mathcal{A}})$, we consider the Picard group of the blow-up $X_{\mathcal{A}}$ of \mathbb{P}^2 at the 31 singularities and we show certain divisor classes are not effective. This is equivalent to showing certain divisor classes are nef. In addition, the action of G on \mathbb{P}^2 extends to $\mathrm{Pic}(X_{\mathcal{A}})$. We will see that the divisor class $40H - 5E_5 - 7E_3 - 8E_2$ is G -invariant and nef, where H is the divisor class of a line and E_k is the sum of exceptional divisors over the points of multiplicity k of \mathcal{A} .

1.3. Outline. In section 3, we review group theoretic facts and some representation theory of G . We construct the corresponding line configuration \mathcal{A} while emphasizing its geometry with respect to the group action on \mathbb{P}^2 . In Section 4, we exhibit an upper bound for $\hat{\alpha}(I_{\mathcal{A}})$. We assume nefness of D to give a lower bound. In Section 5, we focus on studying G -invariant curves on $X_{\mathcal{A}}$ in order to prove D is nef. We follow the techniques and methods found in [BDRH19] to provide the new result 1. In addition, in Section 6 we compute the Waldschmidt constants of sub point-configurations naturally found in the configuration \mathcal{A} .

2. ACKNOWLEDGEMENTS

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3. PRELIMINARIES

3.1. The icosahedron. We return to the icosahedron. There are 15 mirror planes of the icosahedron in \mathbb{R}^3 . On the icosahedron, five planes meet at each vertex, three planes meet at the center of each face and two planes meet at the mid-point of each edge. Figure 1 displays the marked icosahedron alongside the line configuration \mathcal{A} in \mathbb{P}^2 . To obtain \mathcal{A} , fix a pair of opposite vertices $\{p, -p\}$ and identify the plane P wedged between the pair of vertices that slices the icosahedron in half. Project the mirror planes onto the plane containing p parallel to P . These are the lines of \mathcal{A}

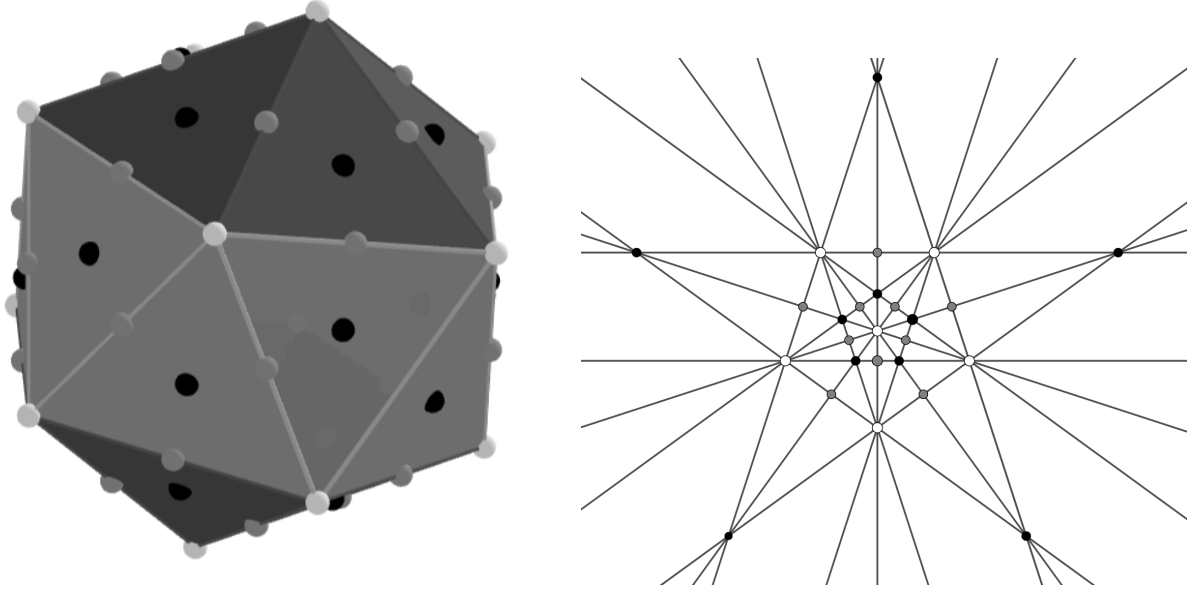


FIGURE 1. An icosahedron marked with quintuple (white), triple (black) and double (grey) points alongside the line configuration in the affine patch $y - \omega z \neq 0$. Note that 5 double points lie on the line at infinity.

and they meet at 6 quintuple points, 10 triple points and 15 double points. We say that \mathcal{A} is the line configuration corresponding to the group G .

3.2. The group $A_5 \times \mathbb{Z}_2$. The group G is generated by idempotent elements g, h, j . Let ω be the golden ratio. There is a 3-dimensional representation ρ given by

$$\rho(g) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(h) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(j) = \frac{-1}{2} \begin{pmatrix} \omega - 1 & \omega & 1 \\ \omega & -1 & \omega - 1 \\ 1 & \omega - 1 & -\omega \end{pmatrix}.$$

with the following character values

class	1	2A	2B	2C	3	5A	5B	6	10A	10B
size	1	1	15	15	20	12	12	20	12	12
χ_ρ	3	-3	1	-1	0	ω	$1 - \omega$	0	$\omega - 1$	$-\omega$

There are 15 pseudoreflections as observed from the character values of ρ . Indeed there 15 elements of conjugacy class $2B$ whose trace is 1. This implies for each such element, the dimension of the eigenspace E_1 is 2. This is an equivalent condition to fixing a hyperplane pointwise. Now considering the arrangement of these 15 hyperplanes, the projectivization of the arrangement in \mathbb{P}^2 admits \mathcal{A} . Let $\tilde{G} = A_5$, the image of G in $\text{PGL}(3, \mathbb{C})$.

Remark 3.1. The action of \tilde{G} on \mathbb{P}^2 with respect to the line configuration is the following. Notice that the lines and points may be defined over $\mathbb{Q}(\omega)$.

- (1) The line $x = 0$ is a line of \mathcal{A} . The 15 lines of \mathcal{A} form a single orbit of size 15.
- (2) The point $p_2 = [0 : 0 : 1]$ is a double point and has stabilizer $\tilde{G}_{p_2} = D_4$, the Klein four-group. The double points of \mathcal{A} form a single orbit of size 15 and each double point has stabilizer isomorphic to D_4 .

- (3) The point $p_3 = [1 : 1 : 1]$ is a triple point and has stabilizer $\tilde{G}_{p_3} = D_6$. The triple points of \mathcal{A} form a single orbit of size 10 and each triple point has stabilizer isomorphic to D_6 .
- (4) The point $p_5 = [\omega : 0 : 1]$ is a quintuple point and has stabilizer $\tilde{G}_{p_5} = D_{10}$. The quintuple points of \mathcal{A} form a single orbit of size 6 and each quintuple point has stabilizer isomorphic to D_{10} .

The group G acts naturally on the homogeneous coordinate ring S of \mathbb{P}^2 . We may then consider the ring of invariants S^G , the polynomials that are invariant under the action of G . The curves given by these polynomials are G -invariant. The ring S^G is generated by particular polynomials ϕ_2, ϕ_6 , and ϕ_{10} of degrees 2, 6, and 10 [Kle58]. The invariant ring $S^{\tilde{G}}$ is generated by these polynomials in addition to a fourth invariant ϕ_{15} . The polynomials ϕ_2, ϕ_6 and ϕ_{10} are algebraically independent, while ϕ_{15}^2 can be expressed in terms of the other generators. To obtain the generating polynomials ϕ_d , note that G is an orthogonal group since it is the symmetry group of the regular icosahedron. Since each element of G preserves distance, we have $\phi_2 = x^2 + y^2 + z^2$ to be a degree 2 invariant. Recall from Section 3.1 the pair of opposite vertices and the unique plane P corresponding to this pair. There are 6 such pairs of opposite vertices, thus 6 unique planes. We take ϕ_6 to be the product of the defining equations of these planes

$$\phi_6 = x^4 y^2 + y^4 z^2 + x^2 z^4 + 4\omega x^2 y^2 z^2 - (\omega + 1)(x^2 y^4 + y^2 z^4 + x^4 z^2).$$

Note that each plane P contains 5 double points and so for each double point p_2 , we have $\phi_6 \equiv 0$ modulo $\mathfrak{m}_{p_2}^2$, the maximal ideal corresponding to p_2 . Similarly, there are ten pairs of opposite faces with opposite centers. There is a unique plane sandwiched between a pair of opposite faces that cuts the icosahedron in half. We take ϕ_{10} to be the product of the defining equations of the planes

$$\begin{aligned} \phi_{10} = & x^8 y^2 + x^2 z^8 + y^8 z^2 + (3\omega - 5)(x^2 y^8 + x^8 z^2 + y^2 z^8) + (3\omega - 7)(x^6 y^4 + x^4 z^6 + y^6 z^4) \\ & - (6\omega - 11)(x^4 y^6 + x^6 z^4 + y^4 z^6) - (30\omega - 40)(x^6 y^2 z^2 + x^2 y^6 z^2 + x^2 y^2 z^6) \\ & + (45\omega - 60)(x^2 y^4 z^4 + x^4 y^2 z^4 + x^4 y^4 z^2) \end{aligned}$$

Similarly, we have $\phi_{10} \equiv 0 \pmod{\mathfrak{m}_{p_2}^2}$ for each double point. The polynomial ϕ_{15} is the defining polynomial of the line configuration \mathcal{A} . It may be obtained as the Jacobian determinant

$$\phi_{15} = \begin{vmatrix} \partial\phi_2/\partial x & \partial\phi_6/\partial x & \partial\phi_{10}/\partial x \\ \partial\phi_2/\partial y & \partial\phi_6/\partial y & \partial\phi_{10}/\partial y \\ \partial\phi_2/\partial z & \partial\phi_6/\partial z & \partial\phi_{10}/\partial z \end{vmatrix}.$$

The polynomial ϕ_{15} satisfies the following relation

$$\begin{aligned} c\phi_{15}^2 = & 125\phi_{10}^3 + (1300\omega - 2275)\phi_2^2\phi_6\phi_{10}^2 + (-12\omega + 16)\phi_2^5\phi_{10}^2 + (46800\omega - 75600)\phi_2\phi_6^3\phi_{10} \\ & - (6360\omega - 10335)\phi_2^4\phi_6^2\phi_{10} + (200\omega - 320)\phi_2^7\phi_6\phi_{10} + (343872\omega - 556416)\phi_2^5 \\ & - (84624\omega - 136912)\phi_2^3\phi_6^4 + (6916\omega - 11193)\phi_2^6\phi_6^3 - (188\omega - 304)\phi_2^9\phi_6^2 \end{aligned}$$

for an appropriate constant $c \in \mathbb{C}^*$.

Remark 3.2. Continuing the theme of Remark 3.1, we record how \tilde{G} acts on orbits of \mathbb{P}^2 . The representation ρ is irreducible and so G fixes no point of \mathbb{P}^2 .

- (1) The double points of \mathcal{A} form an orbit of 15 points.
- (2) The triple points of \mathcal{A} form an orbit of 10 points.
- (3) The quintuple points of \mathcal{A} form an orbit of 6 points.

- (4) The curves $\phi_2 = 0$ and $\phi_6 = 0$ intersect at and form an orbit of 12 points. These points do not lie on \mathcal{A} . One of the points is

$$\left[\sqrt{\frac{1}{2}(5 + \sqrt{5})} : 1 : 1 - \omega \right]$$

and this point is not defined over $\mathbb{Q}(\omega)$.

- (5) The curves $\phi_2 = 0$ and $\phi_{10} = 0$ intersect at and form an orbit of 20 points. These points do not lie on \mathcal{A} . One of these points is

$$[\zeta : \overline{-\zeta} : 1]$$

where $\zeta = e^{2\pi i/3}$. Similarly, one would need to extend to the field $\mathbb{Q}(\omega, \zeta)$ to define these intersections.

- (6) The curves $\phi_6 = 0$ and $\phi_{10} = 0$ intersect at the double points of \mathcal{A} . Each curve passes through a double point twice, therefore by Bezout's theorem, the curves only meet at the double points.
- (7) A non-singular point of the line configuration has an orbit of 30 points.
- (8) Otherwise a point has an orbit of 60 points.

4. WALDSCHMIDT COMPUTATIONS ON THE BLOW-UP OF \mathbb{P}^2

In this section, we compute an upper bound for $\widehat{\alpha}(I_{\mathcal{A}})$ and discuss our approach to proving a lower bound. To give such bounds, we consider the blow-up of \mathbb{P}^2 at the singularities of \mathcal{A} . In particular, we study certain divisor classes on this blow-up. These divisor classes correspond to curves on \mathbb{P}^2 with incidence relations to \mathcal{A} . To obtain a lower bound, we rely on the nefness of a specific divisor D . We prove the nefness of D in Section 5. In Section 6, we consider the 3 point configurations corresponding to the three types of k -uple points and directly compute the Waldschmidt constant of these configurations.

4.1. Divisor classes on the blow-up. Denote the blow-up of \mathbb{P}^2 at the singularities of \mathcal{A} by $X_{\mathcal{A}}$. Then $\text{Pic}(X_{\mathcal{A}})$ is generated by the pullback of a line H and 31 exceptional divisors. The divisor class of the proper transform of the line configuration on the blow-up is

$$A = 15H - 5E_5 - 3E_3 - 2E_2$$

where E_m is the sum of the exceptional divisors lying over the points of multiplicity m for $m \geq 2$. These divisor classes on the blow-up are subject to the following intersections

$$H^2 = 1, \quad E_5^2 = -6, \quad E_3^2 = -10, \quad E_2^2 = -15, \quad \text{and} \quad H \cdot E_i = E_i \cdot E_j = 0$$

for $i \neq j$. For example, $A^2 = -75$. With this, we now are able to bound $\widehat{\alpha}(I_{\mathcal{A}})$ from above.

Lemma 4.1. *Let $I_{\mathcal{A}}$ be the homogeneous ideal corresponding to the singularities of \mathcal{A} . Then*

$$\widehat{\alpha}(I_{\mathcal{A}}) \leq \frac{11}{2}.$$

Proof. Consider the divisor class

$$D = 40H - 5E_5 - 7E_3 - 8E_2.$$

For $k \geq 1$, we calculate the expected dimension of the linear series $|kD + 2H|$ to be

$$\chi(kD + 2H) = \binom{40k + 4}{2} - 6\binom{5k + 1}{2} - 10\binom{7k + 1}{2} - 15\binom{8k + 1}{2} = 6 + 30k > 0.$$

Therefore $kD + 2H + kA = (55k + 2)H - 10kE_5 - 10kE_3 - 10kE_2$ is an effective divisor class, and there is some element in $I_{\mathcal{A}}^{(10k)}$ of degree $55k + 2$. This gives

$$\widehat{\alpha}(I_{\mathcal{A}}) = \lim_{m \rightarrow \infty} \frac{\alpha(I^{(m)})}{m} \leq \lim_{k \rightarrow \infty} \frac{55k + 2}{10k} = \frac{11}{2}. \quad \square$$

To give a lower bound for $\widehat{\alpha}(I_{\mathcal{A}})$, we assume the nefness of D .

Lemma 4.2. *If $D = 40H - 5E_5 - 7E_3 - 8E_2$ is a nef divisor, then $\widehat{\alpha}(I_{\mathcal{A}}) = \frac{11}{2}$.*

Proof. Assume D is nef and suppose for contradiction there exists a $\beta \in \mathbb{Q}$ such that

$$\widehat{\alpha}(I_{\mathcal{A}}) < \beta < \frac{11}{2},$$

so that $F = \beta H - E_5 - E_3 - E_2$ is effective. Then

$$F \cdot D = 40\beta - 30 - 70 - 120 = 40 \left(\beta - \frac{11}{2} \right) < 0.$$

This contradicts the nefness of D . Therefore $\widehat{\alpha}(I) \geq \frac{11}{2}$. By the preceding lemma, we have equality. \square

5. G -IRREDUCIBLE AND G -INVARIANT CURVES

In this section, we prove the divisor D is nef. Together with Lemma 4.2, Theorem 5.4 proves the main result of the paper.

5.1. A decomposition of the divisor class D . A curve \mathcal{C} is *G -irreducible* if its irreducible components are in a single orbit under the action of G . For example, G acts transitively on the lines of \mathcal{A} and so the curve $\phi_{15} = 0$ is G -irreducible. A curve \mathcal{C} is *G -invariant* if for every $g \in G$, $g(\mathcal{C}) = \mathcal{C}$. Observe that G -irreducibility of a curve implies the curve is G -invariant. We first consider the point-configuration of quintuple points.

Remark 5.1. The group G acts on \mathbb{P}^2 . This action extends to a G -action on $\text{Pic}(X_{\mathcal{A}})$. If a divisor in $\text{Pic}(X_{\mathcal{A}})$ is effective, then we ask if there is a G -invariant curve in the corresponding linear series. This suggests to we pay attention to curves invariant under the action of G .

Observation 5.2. Consider the following divisor classes

$$\begin{aligned} A &= 15H - 5E_5 - 3E_3 - 2E_2, \\ B &= 6H - 2E_2, \\ C &= 30H - 2E_5 - 6E_3 - 6E_2, \\ D &= 40H - 5E_5 - 7E_3 - 8E_2. \end{aligned}$$

- (1) The line configuration \mathcal{A} is G -irreducible.
- (2) A curve \mathcal{B} of divisor class B can be given by the vanishing of ϕ_6 . The curve \mathcal{B} is a union of 6 lines (Figure 3), all in a single orbit under the action of G . It follows that \mathcal{B} is G -invariant.
- (3) The divisor classes A, B, C have negative self-intersection and $D^2 = 0$.
- (4) The divisor class D may be written as the linear combination $6D = 4A + 5B + 5C$.

Proposition 5.3. *The divisor class C is effective and there is a G -irreducible curve \mathcal{C} of class C .*

We dedicate Section 5.3 to give an equation for a G -irreducible curve \mathcal{C} of divisor class C . This proposition allows us to prove the main result.

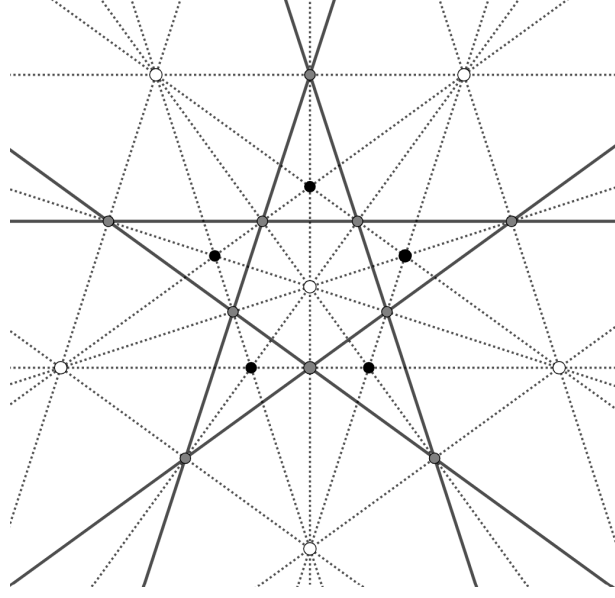


FIGURE 2. The curve $\phi_6 = 0$ is a union of the five solid lines and the line at infinity.

Theorem 5.4. *The divisor class $D = 40H - 5E_5 - 7E_3 - 8E_2$ is nef. Consequently*

$$\hat{\alpha}(I_A) = \frac{11}{2}.$$

Proof. By Observation 5.2 and Proposition 5.3, the divisor classes A, B , and C are effective and G -irreducible. The decomposition $6D = 4A + 5B + 5C$ shows that D is effective. Since an effective divisor will only intersect negatively on irreducible components, it suffices to use the G -action on $\text{Pic}(X_A)$ to show that D intersects non-negatively on its G -irreducible components. Since $D \cdot A = D \cdot B = D \cdot C = 0$, D is nef. The Waldschmidt constant $\hat{\alpha}(I_A) = 11/2$ follows from Lemma 4.2. \square

5.2. On G -invariant curves. To prove Proposition 5.3, we produce an equation for the curve \mathcal{C} and prove it is G -irreducible. Since G -irreducible curves are also G -invariant, the following lemma says such defining equations are contained in the subalgebra $T = \mathbb{C}[\phi_2, \phi_6, \phi_{10}] \subseteq S$. Consider the map

$$\begin{aligned} \varphi : \mathbb{P}^2 &\rightarrow \mathbb{P}(2, 6, 10) \\ p &\mapsto [\phi_2(p) : \phi_6(p) : \phi_{10}(p)] \end{aligned}$$

where $\mathbb{P}(2, 6, 10)$ is a weighted projective space. This space is isomorphic to the quotient space \mathbb{P}^2/G and φ is the quotient map.

Lemma 5.5. *Let $\mathcal{C} \subseteq \mathbb{P}^2$ be a G -invariant curve which does not contain the line configuration \mathcal{A} . Then the defining equation $f \in S$ of \mathcal{C} is G -invariant and is contained in T .*

Proof. We recall the invariants ϕ_2, ϕ_6 and ϕ_{10} from the preliminaries. The map φ is a 60-uple covering away from the line configuration and the finitely many points in (1) – (6) of Remark 3.2. In particular, the map φ is a local isomorphism at a general point $p \in \mathcal{C}$. We have then that $\varphi(\mathcal{C})$ is defined by a weighted homogeneous equation $g(w_0, w_1, w_2) = 0$ where w_0, w_1, w_2 are the coordinates of $\mathbb{P}(2, 6, 10)$. Therefore the pullback $\varphi^*g = f$ defines \mathcal{C} and is contained in T . \square

This prompts the following definition: for $m_5, m_3, m_2 \geq 0$, let

$$T_d(-m_5E_5 - m_3E_3 - m_2E_2)$$

be the finite-dimensional subspace of T of homogeneous degree d forms that are m_5 -uple at the 6 quintuple points, m_3 -uple at the 10 triple points and m_2 -uple at the 15 double points of \mathcal{A} . Elements of this space yield G -invariant curves in the linear series $|dH - m_5E_5 - m_3E_3 - m_2E_2|$. Consider the following curves. There is a unique curve given by the vanishing of $\psi_2 \in T_2(0)$. Let $\psi_6 \in T_6(0)$ be the unique (up to scale) degree 6 invariant such that the curve $\psi_6 = 0$ passes through a double point and $\psi_{10} \in T_{10}(0)$ be the unique (up to scale) invariant such that the curve $\psi_{10} = 0$ passes through a double point and a triple point. Additionally, there is a unique invariant $\psi'_6 \in T_6(0)$ that passes through a triple point. The invariants ψ_6, ψ'_6 and ψ_{10} enjoy incidence relations with respect to \mathcal{A} , as the following lemma shows.

Lemma 5.6. *Let $p, p' \in \mathcal{A} \subseteq \mathbb{P}^2$ be k -uple points. If C is a G -invariant curve passing through p , then C also passes through p' .*

Proof. By Remark 3.2, there is an element $g \in G$ such that $g(p) = p'$. Since p vanishes at C , then $p' = g(p)$ vanishes at $C = g(C)$. \square

We take advantage of the vanishings at k -uple points of $\tilde{\psi}_d = 0$ to give an equation for \mathcal{C} . An additional exploit is to observe the action of the stabilizer $\tilde{G}_p \subseteq \tilde{G} \subseteq \text{PGL}(3, \mathbb{C})$ on the local ring $(\mathcal{O}_p, \mathfrak{m}_p)$ at a singularity $p \in \mathcal{A}$.

Lemma 5.7. *Let $p \in \mathcal{A} \subseteq \mathbb{P}^2$ be a singularity and $\tilde{G}_p \subseteq G$ the stabilizer of p . Let w be a linear form not passing through p . If $\psi \in T$ is a \tilde{G} -invariant homogeneous form and vanishes at p , then ψ vanishes to order at least 2 at p and $\tilde{\psi} = \psi/w^d$ lies in the 1-dimensional trivial \tilde{G}_p -submodule of $\mathfrak{m}_p^2/\mathfrak{m}_p^3$.*

Proof. The proof is analogous to [BDRH19, Lemma 4.8] but we describe the upshot in the current setting. From Remark 3.1, the stabilizer \tilde{G}_p for a singularity $p \in \mathcal{A}$ is D_{2n} for $n = 2, 3$ or 5 . The key observation used in the proof is that the linear characters of D_{2n} have order either 1 or 2, and therefore must act trivially on a tangent line at p . \square

Therefore, the previous two lemmas combine to state that if a G -invariant curve passes through a k -uple point, it passes through every k -uple point at least twice. This will allow us to restrict the terms of invariant form vanishing at p .

5.3. Equation of a curve of divisor class C . For constants $\lambda_i \in \mathbb{C}$, define the invariant

$$\psi_{30} := \lambda_1 \psi_{10}^3 + \lambda_2 \psi_2^2 \psi_6 \psi_{10}^2 + \lambda_3 \psi_2 \psi_6^2 \psi'_6 \psi_{10} + \lambda_4 \psi_6^3 \psi'^2_6.$$

For the invariants $\psi_{10}^3, \psi_2^2 \psi_6 \psi_{10}^2, \psi_2 \psi_6^2 \psi'_6 \psi_{10}$ and $\psi_6^3 \psi'^2_6$, Lemma 5.7 says each invariant vanishes at order 6 and 4 at the double and triple points. To ensure that the curve \mathcal{C} defined by ψ_{30} is of divisor class C , we intend to select appropriate scalars λ_i so that ψ_{30} vanishes to two more orders at the triple points and twice at the quintuple points.

We now apply Lemma 5.7 to a triple point p_3 . Choose local affine coordinates \tilde{x}, \tilde{y} centered at p_3 so that $\mathfrak{m}_{p_3}^k = (\tilde{x}, \tilde{y})^k$. Let w be a linear polynomial not vanishing at p_3 and denote $\tilde{\psi}_d := \psi_d/w^d \in \mathcal{O}_{p_3}$. We have the following decompositions

$$\begin{aligned} \tilde{\psi}_{10} &\equiv A_2 + A_3 \pmod{\mathfrak{m}_{p_3}^4} \\ \tilde{\psi}'_6 &\equiv B_2 + B_3 \pmod{\mathfrak{m}_{p_3}^4} \\ \tilde{\psi}_6 &\equiv C_0 + C_1 \pmod{\mathfrak{m}_{p_3}^2} \\ \tilde{\psi}_2 &\equiv D_0 + D_1 \pmod{\mathfrak{m}_{p_3}^2}. \end{aligned}$$

where A_i, B_i, C_i, D_i are homogeneous polynomials of degree i in $\mathbb{C}[\tilde{x}, \tilde{y}]$. We therefore have the following relations

$$\begin{aligned} A_2 &= \mu B_2 \\ C_0 &= \nu D_0 \\ C_1 &= 3\nu D_1 \end{aligned}$$

for some $\mu, \nu \in \mathbb{C}^*$, where the third equation is obtained by computing the image of $C_0^3 \tilde{\psi}_2^3 - D_0 \tilde{\psi}_6$ in \mathfrak{m}_{p_3} . Therefore we have

$$\begin{aligned} 0 \equiv C_0^3 \tilde{\psi}_2^3 - D_0 \tilde{\psi}_6 &\equiv D_0^3(C_0 + C_1) - C_0(D_0 + D_1)^3 \pmod{\mathfrak{m}_{p_3}^2} \\ &\equiv D_0^3(C_1 - 3\nu D_1) \pmod{\mathfrak{m}_{p_3}^2}. \end{aligned}$$

Note that C_0 and D_0 are non-zero since ψ_2 and ψ_6 do not vanish at a triple point. The constants μ, ν and D_0 are dependent on the choice of triple point p_3 and linear form w . Despite this, we may observe that $\alpha := \nu/\mu \in \mathbb{C}^*$ satisfies

$$\alpha \psi_2 \psi_{10} \equiv \psi_6 \psi'_6 \pmod{I_{p_3}^3}.$$

This equation is G -invariant and thus does not depend on p_3 or w . The constant of proportionality α is determined by the invariants ψ_d .

Lemma 5.8. *The curve ψ_{30} is 5-uple at p_3 if*

$$\lambda_2 + \alpha \lambda_3 + \alpha^2 \lambda_4 = 0.$$

Proof. We consider $\tilde{\psi}_{30}$ modulo $\mathfrak{m}_{p_3}^5$

$$\tilde{\psi}_{30} \equiv \mu^2 \nu D_0^3 B_2^2 \lambda_2 + \mu \nu^2 D_0^3 B_2^2 \lambda_3 + \nu^3 D_0^3 B_2^2 \lambda_4 \pmod{\mathfrak{m}_{p_3}^5} \equiv 0.$$

The linear condition follows from factoring out by $D_0^3 B_2^2$ and dividing by $\mu^2 \nu$. □

Lemma 5.9. *The curve ψ_{30} is 6-uple at p_3 if it is 5-uple and*

$$\begin{aligned} 5\lambda_2 + 7\alpha\lambda_3 + 9\alpha^2\lambda_4 &= 0 \\ \lambda_3 + 2\alpha\lambda_4 &= 0 \\ 2\lambda_2 + \alpha\lambda_3 &= 0. \end{aligned}$$

Proof. Suppose the linear condition of Lemma 5.8 holds. Consider only the degree 5 terms of $\tilde{\psi}_{30}$ modulo $\mathfrak{m}_{p_3}^6$.

$$\begin{aligned} \tilde{\psi}_{30} &\equiv \lambda_2(5\mu^2 \nu D_0^2 D_1 B_2^2 + 2\mu \nu D_0^3 B_2 A_3) + \lambda_3(7\mu \nu^2 D_0^2 D_1 B_2^2 + \mu \nu^2 D_0^3 B_2 B_3 + \nu^2 D_0^3 B_2 A_3) \\ &\quad + \lambda_4(9\nu^3 D_0^2 D_1 B_2^2 + 2\nu^3 D_0^3 B_2 B_3) \pmod{\mathfrak{m}_{p_3}^5}. \end{aligned}$$

Observe that we may group these according to the terms $D_0^2 D_1 B_2^2, D_0^3 B_2 A_3$ and $D_0^3 B_2 B_3$.

$$\begin{aligned} D_0^2 D_1 B_2^2(5\mu \nu \lambda_2 + 7\mu \nu^2 \lambda_3 + 9\nu^3 \lambda_4) &= 0 \\ D_0^3 B_2 A_3(2\mu \nu \lambda_2 + \nu^2 \lambda_3) &= 0 \\ D_0^3 B_2 B_3(\mu \nu^2 \lambda_3 + 2\nu^3 \lambda_4) &= 0. \end{aligned}$$

We now fix multiples for the invariants ψ_d . The choice of multiples is due to experimentation that simplifies the polynomial ψ_{30} as much as possible.

$$\begin{aligned}\psi_2 &= 3\phi_2 \\ \psi_6 &= -3(\omega - 1)\phi_6 \\ \psi'_6 &= -25(\phi_2^3 - 27(\omega - 1)\phi_6) = -25\left(\frac{1}{27}\psi_2^3 + 9\psi_6\right) \\ \psi_{10} &= \frac{15}{4}(25(\omega - 1)\phi_2^2\phi_6 - (9\omega + 3)\phi_{10}).\end{aligned}$$

Note that for the double and triple points $p_2 = [0 : 0 : 1]$ and $p_3 = [1 : 1 : 1]$, we have

$$\begin{aligned}\varphi(p_2) &= [1 : 0 : 0] \\ \varphi(p_3) &= [3 : \omega : 45\omega - 60]\end{aligned}$$

from which we see that the ψ_d 's have the required vanishing conditions.

Lemma 5.10. *The curve ψ_{30} is double at a quintuple point p_5 if and only if*

$$32\lambda_1 = 9\lambda_2 + 15\lambda_3 + 25\lambda_4$$

Proof. Select the quintuple point $p_5 = [\omega : 0 : 1]$. It is straightforward to evaluate the terms ψ_{10}^3 , $\psi_2^2\psi_6\psi_{10}^2$, $\psi_2\psi_6^2\psi'_6\psi_{10}$ and $\psi_6^3\psi_6'^2$ at p_5 to obtain that $\psi_{30}(p_5) = 0$ if and only if

$$32\lambda_1 - 9\lambda_2 - 15\lambda_3 - 25\lambda_4 = 0. \quad \square$$

Observing that the selection of multiples gives $\alpha = -1$, then the linear system governing these vanishing conditions is the following

$$\begin{pmatrix} -32 & 9 & 15 & 25 \\ 0 & 1 & -1 & 1 \\ 0 & 5 & -7 & 9 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Theorem 5.11. *The unique solution to the system above is $(2, 1, 2, 1)$. The invariant ψ_{30} then has the form*

$$\psi_{30} = 2\psi_{10}^3 + \psi_2^2\psi_6\psi_{10}^2 + 2\psi_2\psi_6^2\psi'_6\psi_{10} + \psi_6^3\psi_6'^2.$$

Additionally, the curve \mathcal{C} defined by ψ_{30} is G -irreducible.

Proof. We verify the G -irreducibility of the curve \mathcal{C} defined by ψ_{30} . It is sufficient to check that the curve \mathcal{C}' in $\mathbb{P}(2, 6, 10)$ defined by

$$g : w_{10}^3 + w_2^2w_6w_{10}^2 + 2w_{10}w_6^2w_2w'_6 + w_6^3w_6'^2$$

is irreducible, where w_2, w_6, w_{10} are the coordinates of $\mathbb{P}(2, 6, 10)$ and $w'_6 = \frac{1}{27}\psi_2^3 + 9\psi_6$. Consider the quotient map giving an isomorphic quotient space to that in Lemma 5.5

$$\begin{aligned}\tilde{\varphi} : \mathbb{P}^2 &\rightarrow \mathbb{P}(2, 6, 10) \\ p &\mapsto [\psi_2(p) : \psi_6(p) : \psi_{10}(p)].\end{aligned}$$

The pullback $\tilde{\varphi}^*\mathcal{C}'$ is \mathcal{C} and so we prove irreducibility in terms of w_i . Firstly, assume g has the factorization

$$(F_2w_{10}^2 + F_1w_{10} + F_0)(G_1w_{10} + G_0)$$

where $F_i, G_i \in \mathbb{C}[w_2, w_6]$ are homogeneous polynomials of appropriate degree to make the factors homogeneous. Since $F_2G_1 = 1$, then G_1 is constant and G_0 must be also be degree 10. Since G_0

divides $w_6^3 w_6'^2$, G_0 must necessarily divide w_6' or $w_6'^2$. This contradicts the degree polynomial G_0 . On the other hand, if the factorization of g were of the form

$$(F_1 w_{10} + F_0)(G_1 w_{10} + G_0)(H_1 w_{10} + H_0).$$

Then the F_1, G_1, H_1 must be constant, forcing F_0, G_0, H_0 to be of degree 10. A similar reasoning as above concludes that the curve C is G -irreducible. \square

6. EXPLORING THE SUB-POINT CONFIGURATIONS OF \mathcal{A}_{23}

In this section, we consider the sub point-configurations corresponding to the quintuple, triple and double points of \mathcal{A}_{23} . For $k = 2, 3$ and 5, denote I_k to be the intersection of homogeneous ideals of k -uple points. In contrast to the Waldschmidt constant of $I_{\mathcal{A}}$, the Waldschmidt constant of I_k is easier to obtain due to the k -uple points being in a single orbit.

Proposition 6.1. *Let I_2 be the homogeneous ideal of double points. Then $\hat{\alpha}(I_2) = 3$.*

Proof. Consider the effective divisor $B = 6H - 2E_2$. Note that this divisor has negative self-intersection. We know there is a union of 6 lines that pass through each double point twice (Figure 3), therefore $\hat{\alpha}(I_2) \leq 3$.

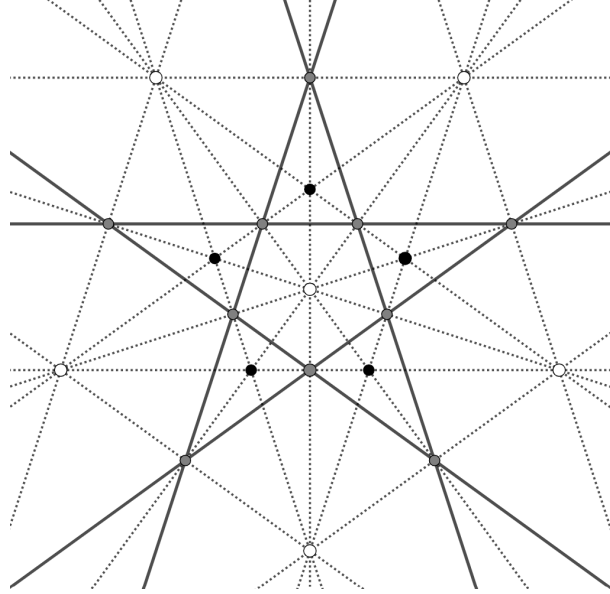


FIGURE 3. A curve of divisor class $6H - 2E_2$ drawn solid.

Suppose there is an effective \mathbb{Z} -divisor class $F = dH - mE_2$ such that $d/m < 3$. This divisor class is G -invariant. Observe that

$$B \cdot F = (6H - 2E_2) \cdot (dH - mE_2) = 6d - 30m < 0.$$

Considering the curves \mathcal{B} and \mathcal{F} of classes B and F respectively, the curve \mathcal{B} is G -irreducible and $\mathcal{B} \cdot \mathcal{F} < 0$. Suppose \mathcal{B} has irreducible components $\mathcal{B}_1, \dots, \mathcal{B}_k$. Since \mathcal{B} is G -irreducible, for any i we have $g(\mathcal{B}_i \cdot \mathcal{F}) = \mathcal{B}_j \cdot \mathcal{F}$ for some j . Then \mathcal{F} intersects each \mathcal{B}_i negatively and therefore contains \mathcal{B}_i . Thus the divisor class $F - B$ is effective.

Notice that $F - B$ also intersects B negatively. Performing a similar analysis, we may conclude by induction that $F - kB$ is effective for $k \geq 1$. This leads to a contradiction as $F - kB$ will eventually have negative degree. Therefore $\hat{\alpha}(I_2) \geq 3$. \square

We now state the Waldschmidt constants for the remaining point configurations.

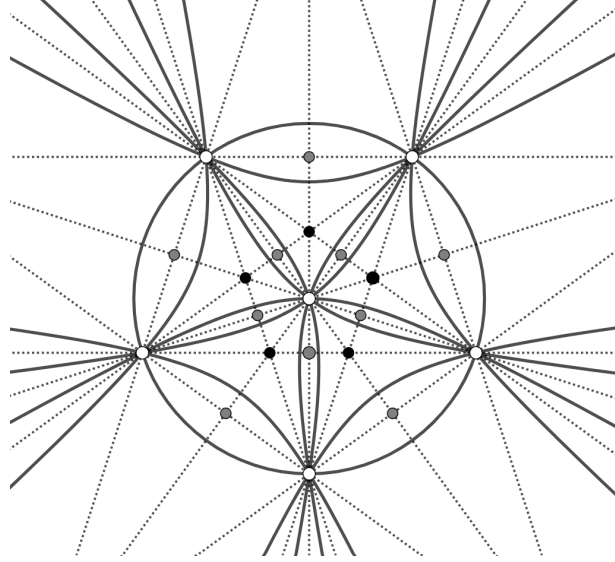


FIGURE 4. A curve of divisor class $12H - 5E_5$ drawn solid.

Proposition 6.2. *For the ideals I_3 and I_5 , we have $\hat{\alpha}(I_3) = 3$ and $\hat{\alpha}(I_5) = \frac{12}{5}$.*

The values $\hat{\alpha}(I_3)$ and $\hat{\alpha}(I_5)$ are obtained similarly. We see in Section 5.2, the divisor class $6H - 2E_3$ is effective. For I_5 , there is a degree 12 invariant vanishing to order 5 at the quintuple points thus the divisor $12H - 5E_5$. Indeed for each choice of 5 quintuple points, there is a unique conic passing through each point (Figure 4). The important property these divisors share with B is that they have negative self-intersection.

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