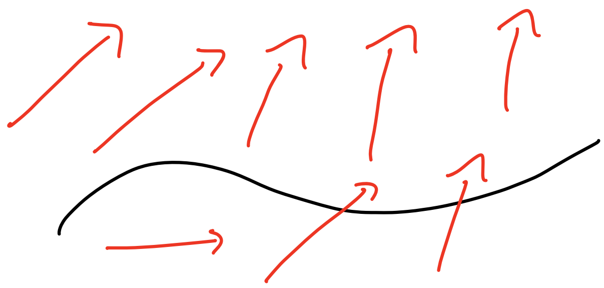
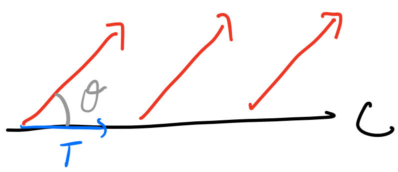
Suppose we have a vector field F and a curve C



We can consider the work done by the vector field F on the curve. If C were a straight path and F were constant, meaning F outputs the same vector at every pt, then



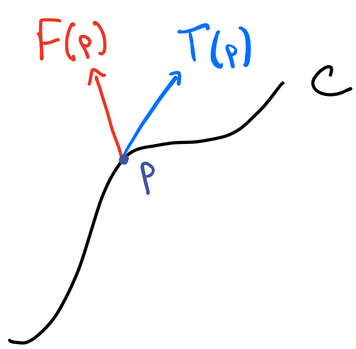
work $= \|F\| \cdot \cos\theta = F(p) \cdot T$, dot product w/ tangent direction.

Most curves not straight paths but we can consider the tangent lines to the curve.

Let $p \in C$ be a pt on the curve. Then denote

$F(p)$ to be the vector of vector field F at p.

$T(p)$ to be unit tangent vector of C at p.



So the work done at the point p is $\|F(p)\| \cdot \cos(\theta)$ where θ = angle between $F(p)$ and $T(p)$. Peform this for every point on C to obtain

dot product

$$\text{work} = W = \int_C (F \cdot T)\,ds$$

= scalar line integral of $F \cdot T$

Note that $T = \dfrac{\vec{r}'(t)}{\|r'(t)\|}$ and $ds = \|r'(t)\|\,dt$ so

$$W = \int_C (F \cdot T)\,ds = \int_a^b F(r(t)) \cdot \frac{r'(t)}{\|r'(t)\|}\,\|r'(t)\|\,dt$$

$$= \int_a^b F(\vec{r}(t)) \cdot r'(t)\,dt = \boxed{\int_C F \cdot d\vec{r}}$$

another notation you may see

ex 1. Calculate work done over the curve $C: \vec{r}(t) = \langle 3+5t^2, 3-t^2, t\rangle$ for $0 \le t \le 2$ by the force field $F(x,y,z) = \langle z^2, x, y\rangle$

① $F(\vec{r}(t)) = \langle t^2, 3+5t^2, 3-t^2\rangle$

② $r'(t) = \langle 10t, -2t, 1\rangle$

③ bounds were given as $0 \le t \le 2$

$$\Rightarrow \int_C F \cdot T\,ds = \int_0^2 \langle t^2, 3+5t^2, 3-t^2\rangle \cdot \langle 10t, -2t, 1\rangle\,dt$$

$$= \int_0^2 10t^3 - 6t - 10t^3 + 3 - t^2\,dt$$

$$= -\int_0^2 t^2 + 6t - 1\,dt$$

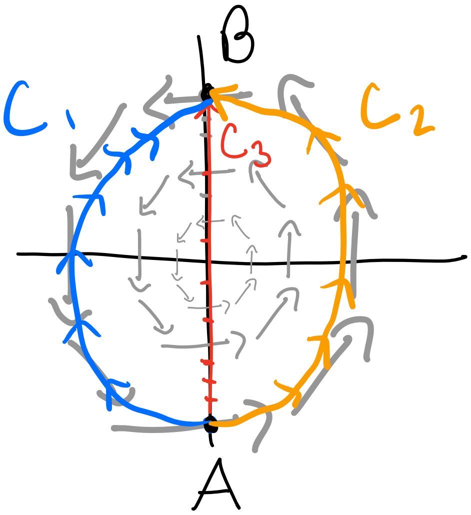
$\frac{30}{10} + \frac{8}{3}$

$$= -\left[\frac{t^3}{3} + 3t^2 - t\right]\Big|_0^2 = -\left[\frac{8}{3} + 12 - 2\right] = \frac{-38}{3}$$ units (joules)

Negative answers mean the work is done against the vector field, rather in the direction of it.

In general, the sign of work done by F over a path C can be determined by examining the graph.

$F = \langle -y, x \rangle$



① $\int_{C_1} F \cdot d\vec{r} < 0$ since curve C_1 is going against the flow

② $\int_{C_2} F \cdot d\vec{r} > 0$ since curve C_2 is going with the flow

③ $\int_{C_3} F \cdot d\vec{r} = 0$ since is perpendicular to the flow.

Another notation you may see is $\int_C F_1\,dx + F_2\,dy + F_3\,dz$

This equivalent to $\int_C F \cdot d\vec{r}$ where $F = \langle F_1, F_2, F_3 \rangle$

ex 2. Compute $\int_C y\,dx + x\,dz + z\,dy$ on $\vec{r}(t) = \langle 2+t^{-1}, t^3, t^2 \rangle$ on unit interval.

$\boxed{\frac{37}{10}}$

ex 3 $F(x,y,z) = \langle 2xy+z, x^2, x \rangle$ and $f(x,y,z) = x^2y + xz$ s.t. $F = \nabla f$

Evaluate $\int_C F \cdot d\vec{r}$ where C is a curve parameterized by

a) $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$ on $1 \le t \le 2$

both are 30

b) $\vec{r}_2(t) = (1-t)\langle 1,1,1 \rangle + t\langle 2,4,8 \rangle$ on $0 \le t \le 1$.

Notice that $f(2,4,8) - f(1,1,1) = (2)^2(4) + (2)(8) - 2 = 30$

Fundamental Theorem of Conservative Vector Fields

If $F = \nabla f$ and C is a curve is a path from P to Q, then

$$\int_C F \cdot d\vec{r} = f(Q) - f(P).$$