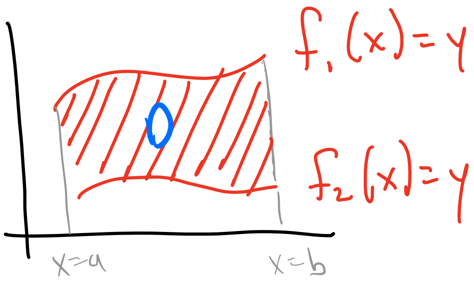
We would like to broaden the kinds of domains we would like to integrate over. For example,

For any domain D with an interior & boundary, we have

$$\text{area}(D) = \iint_D 1 \, dA.$$

Recall from Calc 1 we were able to calculate the area of region wedged between 2 curves.



$$\text{area}(D) = \int_a^b f_1(x) - f_2(x) \, dx$$

$$= \int_a^b y \Big|_{y=f_2(x)}^{y=f_1(x)} dx$$

$$= \int_a^b \int_{f_2(x)}^{f_1(x)} 1 \, dy\,dx$$

$$= \iint_D 1 \, dA.$$

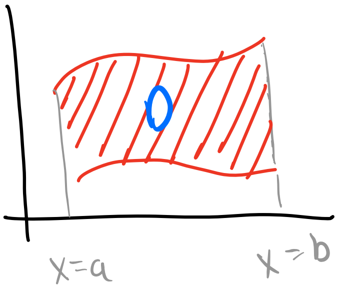
$a \leq x \leq b$

$f_2(x) \leq y \leq f_1(x)$

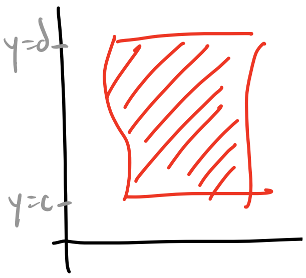
Ex1.

$$\int_0^2 \int_0^{2x} (2x+3y) \, dy\,dx$$

notice that the bound is no longer a curve.



is an example where the region is vertically simple:
on a fixed interval $[a,b]$ on the x-axis
the region is wedged between two curves
$y=f_1(x)$ and $y=f_2(x)$



Alternatively, a region may be horizontally simple
on a fixed interval $[c,d]$ on the y-axis
the region is wedged between two curves
$x=g_1(y)$ and $x=g_2(y)$

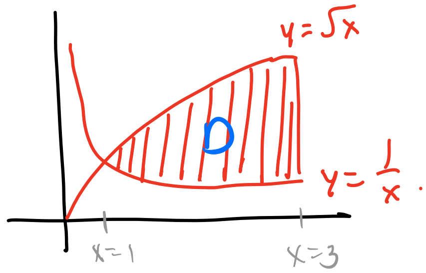
Given an integrable fctn $f(x,y)$ on a

1) vertically simple domain, we have $\displaystyle\int_a^b \int_{f_2(x)}^{f_1(x)} f(x,y)\,dy\,dx$

2) horizontally simple domain, we have $\displaystyle\int_c^d \int_{g_2(y)}^{g_1(y)} f(x,y)\,dx\,dy$

<u>Ex 1.</u> Consider the region D.

a) Compute $\displaystyle\iint_D x^2y\,dA$



$1 \le x \le 3$

$\frac{1}{x} \le y \le \sqrt{x}$

$$\iint_D x^2y\,dA = \int_1^3 \int_{\frac{1}{x}}^{\sqrt{x}} x^2y\,dy\,dx = \int_1^3 \left(\frac{x^2y^2}{2}\right)\Bigg|_{y=\frac{1}{x}}^{y=\sqrt{x}} dx$$

$$= \int_1^3 \left(\frac{x^3}{2} - \frac{1}{2}\right) dx = \left(\frac{x^4}{8} - \frac{1}{2}x\right)\Bigg|_{y=1}^{x=3} = \left(\frac{81}{8} - \frac{12}{8} - \left(\frac{1}{8} - \frac{4}{8}\right)\right)$$

$$= \left(\frac{69}{8} - \frac{-3}{8}\right) = 9$$

b) Does fubini's give us same answer?

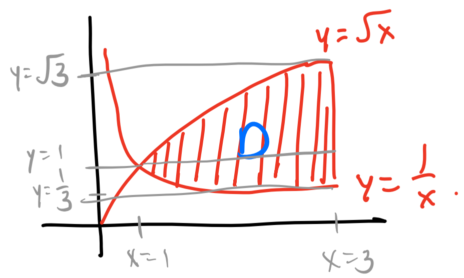
$$\int_{\frac{1}{x}}^{\sqrt{x}} \int_{1}^{3} x^2 y \, dx \, dy = \int_{\frac{1}{x}}^{\sqrt{x}} \left(\frac{x^3}{3} y\right)\Big|_{1}^{3} dy = \int_{\frac{1}{x}}^{\sqrt{x}} \frac{26}{3} \, dy$$

$$= \left(\frac{26}{3} y\right)\Big|_{\frac{1}{x}}^{y=\sqrt{x}}$$

$$= \frac{26}{3}\left(\sqrt{x} - \frac{1}{x}\right) \overset{?}{=} 9$$

Warning: outside variables cannot be dependent on inside variables

c) Can you fix the bounds?



Split into two regions (why?)

$\frac{1}{3} \le y \le 1$ $\qquad$ $1 \le y \le \sqrt{3}$

$\frac{1}{y} \le x \le 3$ $\qquad$ $y^2 \le x \le 3$

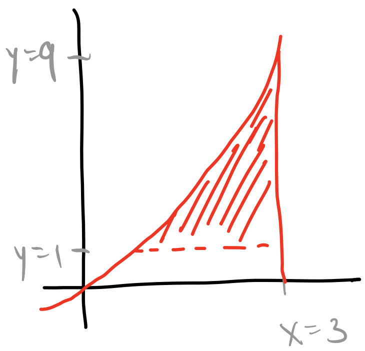
$$\iint_D x^2 y \, dA = \int_{\frac{1}{3}}^{1} \int_{\frac{1}{y}}^{3} x^2 y \, dx \, dy + \int_{1}^{\sqrt{3}} \int_{y^2}^{3} x^2 y \, dx \, dy = \int_{\frac{1}{3}}^{1} \left(\frac{x^3}{3} y\right)\Big|_{x=\frac{1}{y}}^{x=3} dy + \int_{1}^{\sqrt{3}} \left(\frac{x^3}{3} y\right)\Big|_{y^2}^{3} dy$$

$$= \int_{\frac{1}{3}}^{1} 9y - \frac{1}{3y^2} \, dy + \int_{1}^{\sqrt{3}} 9y - \frac{y^7}{3} \, dy = \left(\frac{9}{2} y^2 + \frac{1}{3y}\right)\Big|_{\frac{1}{3}}^{1} + \left(\frac{9}{2} y^2 - \frac{y^8}{24}\right)\Big|_{3}^{\sqrt{3}}$$

$$= \frac{10}{3} + \frac{17}{3} = \frac{27}{3} = 9$$

Remark Fubini's does work but you likely have to adjust the bounds.

Ex 3. Consider $\displaystyle\int_1^9 \int_{\sqrt{y}}^3 xe^y \, dx\, dy$. Change order of integration & evaluate.



$$1 \le y \le 9 \qquad \qquad 1 \le x \le 3$$
$$\sqrt{y} \le x \le 3 \quad \longleftrightarrow \quad 1 \le y \le x^2$$

$$\int_1^9 \int_{\sqrt{y}}^3 xe^y \, dx\, dy = \int_1^3 \int_1^{x^2} xe^y \, dy\, dx$$

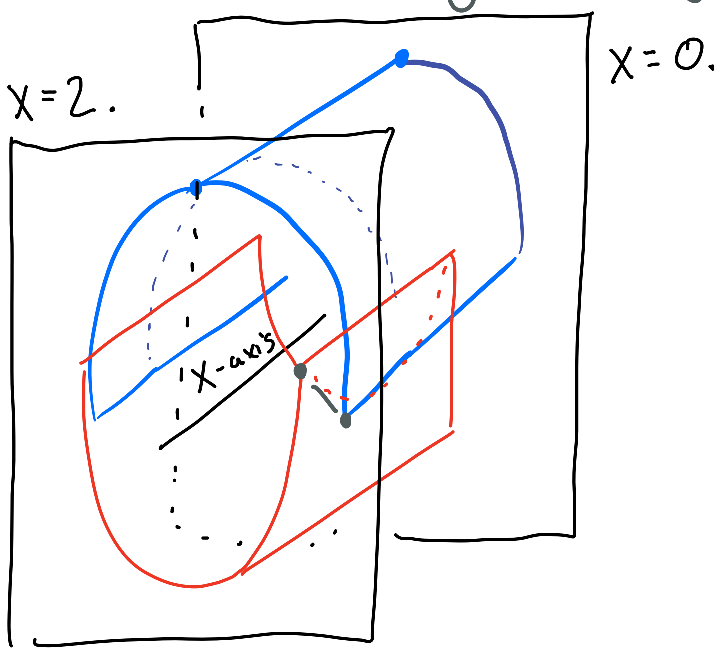
$$= \int_1^3 \left(xe^y\right)\Big|_{y=1}^{y=x^2} dx = \int_1^3 \left(xe^{x^2} - xe\right) dx$$

$$= \left(\frac{1}{2}e^{x^2} - \frac{ex^2}{2}\right)\Bigg|_1^3 = \frac{1}{2}e^9 - \frac{9}{2}e - \left(\frac{e}{2} - \frac{e}{2}\right)$$

$$= \boxed{\frac{1}{2}\left(e^9 - 9e\right)}$$

Ex 4. Find volume of the region enclosed by $z = 1 - y^2$ and $z = y^2 - 1$ for $0 \le x \le 2$.

$z = 1 - y^2$ is missing x, so graph is symmetric abt x.



$$1 - y^2 = y^2 - 1$$
$$1 = y^2$$
$$\pm 1 = y$$
$$-1 \le y \le 1$$

Take difference of volumes of these regions:

$$\int_0^2 \int_{-1}^1 (1 - y^2) - (y^2 - 1)\, dy\, dx = \int_0^2 \int_{-1}^1 2 - 2y^2\, dy\, dx$$

$$= \int_0^2 \left(2y - \frac{2y^3}{3}\right)\Bigg|_{y=-1}^{y=1} dx$$

$$= \int_0^2 \left(2 - \frac{2}{3}\right) - \left(-2 - \frac{-2}{3}\right) dx$$

$$= \int_0^2 \frac{4}{3} - \left(\frac{-4}{3}\right) dx = \int_0^2 \frac{8}{3}\, dx = \boxed{\frac{16}{3}}$$