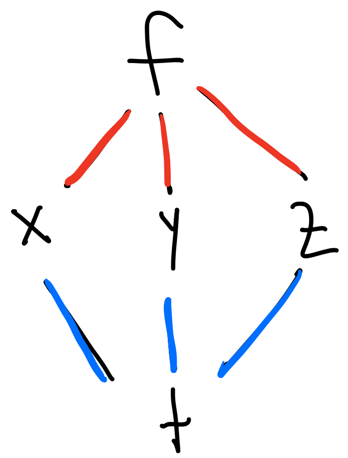
Chain rule is employed w/ composition of fctns. But it is also be used when several variables are at play.

Chain Rule for Paths If $f(x,y,z)$ and $\vec{r}(t)$ are diffable, then

$$\frac{d}{dt} f(\vec{r}(t)) = \underbrace{\nabla f_{\vec{r}(t)}}_{\text{partial derivatives of outside fctn w/ inside fctn plugged in}} \cdot \underbrace{r'(t)}_{\text{derivative of inside fctn.}}$$

$f(x,y,z): \mathbb{R}^3 \to \mathbb{R}$ $\vec{r}(t): \mathbb{R} \to \mathbb{R}^3$ defined by $x(t), y(t), z(t)$



$$\frac{d}{dt}\left(f(\vec{r}(t))\right) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

ex1 $f(x,y,z) = xyz + z^2$, $\vec{r}(t) = \langle \sin(t), \cos(t), t \rangle$

Compute $\left.\frac{d}{dt} f(\vec{r}(t))\right|_{t=\frac{\pi}{2}}$.

$= \nabla f \cdot r'(t)$

$f_x = yz$ $x_t = \cos(t)$

$f_y = xz$ $y_t = -\sin(t)$ $\Rightarrow$ $\frac{d}{dt} f(\vec{r}(t)) = yz\cos(t) - xz\sin(t) + xy + 2z$

$f_z = xy + 2z$ $z_t = 1$

$= \cos^2(t)\,t - \sin^2(t)\,t + \sin(t)\cos(t) + 2t$ } evaluate at $t = \frac{\pi}{2}$

$= -(1)\frac{\pi}{2} + \pi = \boxed{\frac{\pi}{2}}$

Alternatively, we may evaluate halfway in between $\nabla f_{\vec{r}(\frac{\pi}{2})} \cdot \vec{r}\,'(\frac{\pi}{2})$

1) $\vec{r}(\frac{\pi}{2}) = \langle \sin(\frac{\pi}{2}), \cos(\frac{\pi}{2}), \frac{\pi}{2} \rangle = \langle 1, 0, \frac{\pi}{2} \rangle$

$\nabla f = \langle yz, xz, xy + 2z \rangle$

$\nabla f_{\vec{r}(\frac{\pi}{2})} = \langle 0 \cdot \frac{\pi}{2}, 1 \cdot \frac{\pi}{2}, 1 \cdot 0 + \pi \rangle = \langle 0, \frac{\pi}{2}, \pi \rangle$

2) $r'(t) = \langle \cos(t), -\sin(t), 1 \rangle$

$r'(\frac{\pi}{2}) = \langle 0, -1, 1 \rangle$

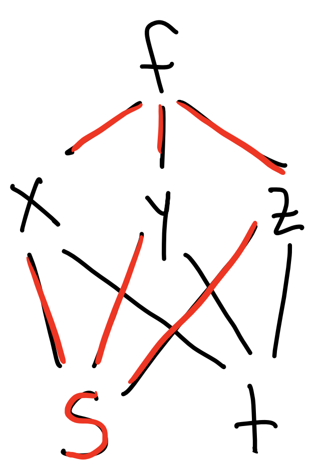
3) $\nabla f_{\vec{r}(\frac{\pi}{2})} \cdot r'(\frac{\pi}{2}) = \langle 0, \frac{\pi}{2}, \pi \rangle \cdot \langle 0, -1, 1 \rangle = \frac{-\pi}{2} + \pi = \boxed{\frac{\pi}{2}}$

//

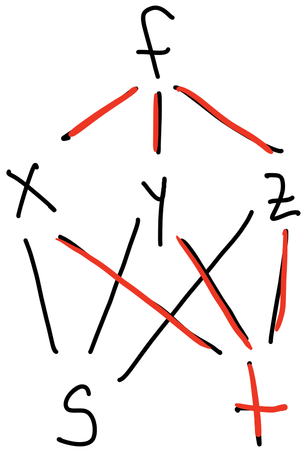
$f(x,y,z) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $x, y, z : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are fctns in vars. s, r

$x(r,s),\ y(r,s),\ z(r,s)$

We can differentiate w/ respect to s <u>or</u> t



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s}$$



$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial t}$$

Ex 2 $F(x,y,z) = xy - z^2$. $x = r\cos\theta$, $y = \cos^2\theta$, $z = r$

$$\frac{\partial F}{\partial r} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial r}$$

$$= (y)(\cos\theta) + (x)(0) + (-2z)\cdot(1)$$

$$= \cos^2\theta + r\cos\theta - 2r$$

$$\frac{\partial F}{\partial \theta} = (y)(-r\sin\theta) + (x)(-2\cos\theta\sin\theta) + (-2z)\cdot 0$$

$$= -r\sin\theta\cos^2\theta - 2r\sin\theta\cos^2\theta$$

$$= -3r\sin\theta\cos^2\theta$$

Ex 3. Let $f(x,y) = x^2y$ and $x = r\cos\theta$, $y = r\sin\theta$

Evaluate $\left.\dfrac{\partial f}{\partial \theta}\right|_{(x,y)=(1,1)}$