Examples of a change of coordinates are

- polar map $(x,y) \longmapsto (r\cos\theta, r\sin\theta)$ or $(r,\theta) \longmapsto \left(\sqrt{x^2+y^2}, \tan^{-1}\left(\frac{y}{x}\right)\right)$
- cylindrical map $(x,y,z) \longmapsto (r\cos\theta, r\sin\theta, z)$
- spherical map $(x,y,z) \longmapsto (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$

We usually will set up a change of variables where we treat x, y, z as functions of other variables

$$x = x(u,v)$$
$$y = y(u,v)$$

ex. $x = x(r,\theta) = r\cos\theta$

Therefore, we can treat a change of variables as a fctn

$$G : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$G(u,v) = (x(u,v), y(u,v))$$

another class of examples are linear changes of coordinates

$$G(u,v) = (au + bv, cu + dv)$$

where a, b, c, d are given constants. Linearity means it can do the following

1) break apart sums:

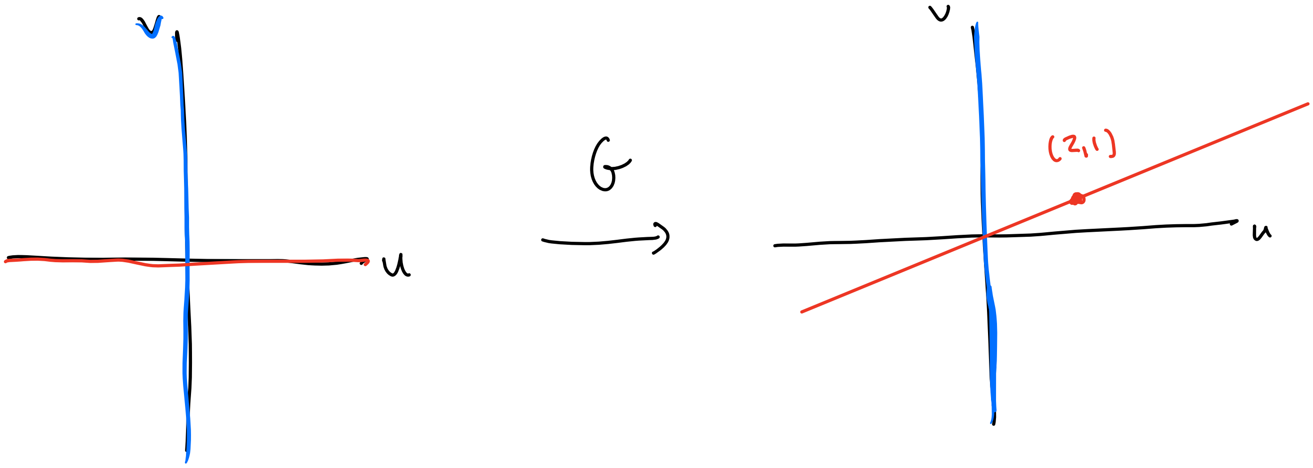
$$G(u_1+u_2, v_1+u_2) = G(u_1, v_1) + G(u_2, v_2)$$

2) Factor out scalars

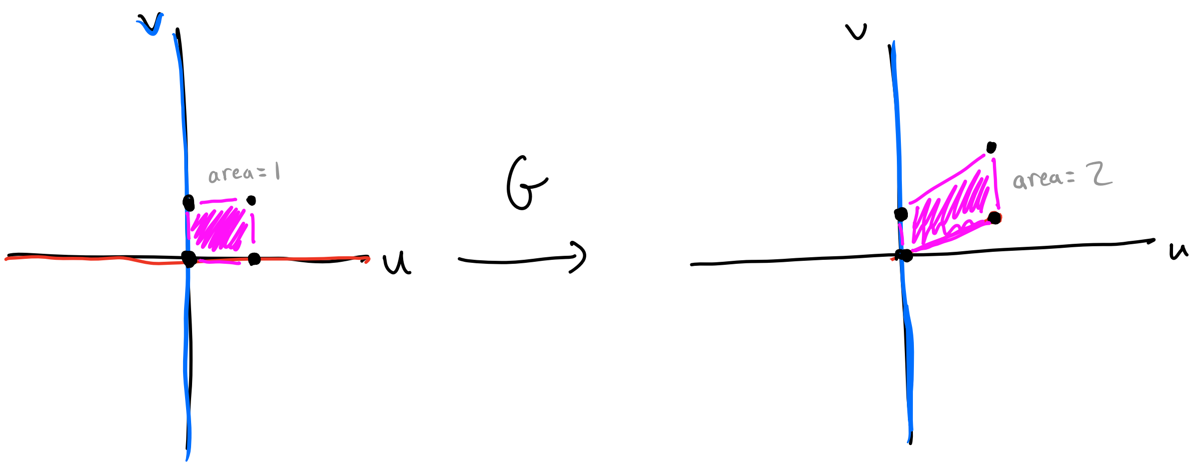
$$G(ku, kv) = k \cdot G(u,v) \quad \text{for } k \text{ a scalar}$$

ex 1. Consider $G(u,v) = (2u, u+v)$.

Q1 What is image of $u=0$ and $v=0$



Q2 What is image of unit square?



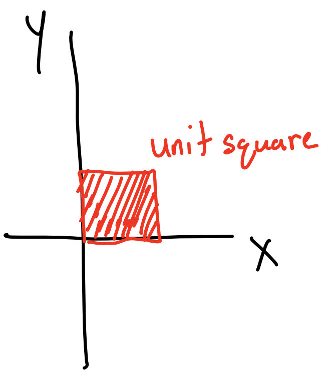
upshot about linear fct.s: takes lines to lines.

In general, change of variables $\boxed{\text{do not}}$ take lines to lines. They usually $\boxed{\text{do}}$ take curves to curves.

Integration when changing coordinates

Not sufficient to specify change of variables, also need to consider how much distance is being distorted and correct that distortion. This correction factor is what the Jacobian does. The Jacobian is determinant of first-order partial derivatives of G.

ex 2. Let's consider the unit square and $G(u,v)=(2u,2v)$.



$0\le x\le 1$
$0\le y\le 1$

$$\int_0^1\int_0^1 1\,dy\,dx=1$$

G scales "things" by 2!
So image of unit square should have area 4. This can also be seen by consider Jacobian.

Set $x=2u$ and $y=2v$. Then

$$\mathrm{Jac}(G)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}2 & 0\\ 0 & 2\end{vmatrix}=4$$

what are bounds on u,v?

$$0 \le x \le 1 \Rightarrow 0 \le 2u \le 1 \Rightarrow 0 \le u \le \tfrac{1}{2}$$

$$0 \le y \le 1 \Rightarrow 0 \le 2v \le 1 \Rightarrow 0 \le v \le \tfrac{1}{2}$$

Then

change of variables

Jac(G)

$$\underline{1} = \int_0^1 \int_0^1 1 \, dy \, dx = \int_0^{1/2} \int_0^{1/2} 1 \cdot 4 \, dv \, du = 1$$

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Change of variables Formula Let $G(u,v) = (x(u,v), y(u,v))$ be a change of variables map. Then the Jacobian of G is

$$\text{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

and if G maps D_0 to D, then

$$\iint_D f(x,y)\, dA = \iint_{D_0} f(x(u,v), y(u,v)) \, |\text{Jac}(G)| \, dv\, du$$

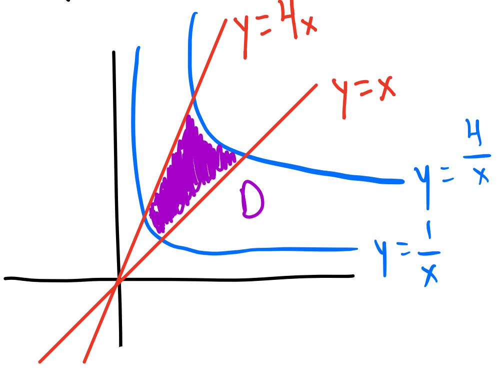
(1) D_0 is new region in u,v-coordinates

(2) plug $x(u,v)$ and $y(u,v)$ into $f(x,y)$

(3) multiply by Jacobian of G (abs. value if negative)

(4) integrating w.r.t. u and v!

Ex 3. Compute $\iint_D x^2+y^2\,dA$ where D is the domain given by $1 \le xy \le 4$ and $1 \le \frac{y}{x} \le 4$.



Can set $u=xy$ and $v=\frac{y}{x}$ and try to solve for x and y interms of u and v

Notice $uv = (xy)\left(\frac{x}{y}\right) = x^2 \Rightarrow x = \sqrt{uv}$

$\frac{u}{v} = \frac{(xy)}{\left(\frac{x}{y}\right)} = y^2 \Rightarrow y = \sqrt{\frac{u}{v}}$

Set $G(u,v) = \left(\sqrt{uv}, \sqrt{\frac{u}{v}}\right)$. So the Jacobian of G is

$$\begin{vmatrix} \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \\ \frac{1}{2\sqrt{uv}} & \frac{-\sqrt{u}}{2\sqrt[3]{v}} \end{vmatrix} = \frac{\sqrt{v}}{2\cancel{\sqrt{u}}} \cdot \frac{-\cancel{\sqrt{u}}}{2\sqrt[3]{v}} - \frac{1}{2\sqrt{u}\sqrt{v}} \cdot \frac{\sqrt{u}}{2\sqrt{v}} = \frac{-1}{4v} - \frac{1}{4v} = \frac{-1}{2v}$$

$\hookrightarrow = \frac{1}{2v}$ after abs value.

Therefore

$$\int_1^4 \int_1^4 \left[(\sqrt{uv})^2 + \left(\sqrt{\frac{u}{v}}\right)^2\right] \cdot \left(\frac{1}{2v}\right) dv\,du = \frac{1}{2} \int_1^4 \int_1^4 u + \frac{u}{v^2}\,dv\,du = \boxed{\frac{225}{16}}$$