The derivatives f_x and f_y give rate of change in x and y directions. What if we wanted to measure rate of change in another vector $\vec{v}$? That is, what if both x and y are changing at different rates?

Let $f(x,y)$ be a fctn and $P=(a,b)$ a pt. Then the gradient of f at P is

$$\nabla f_P = \langle f_x(a,b), f_y(a,b) \rangle.$$

Sometimes $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$.

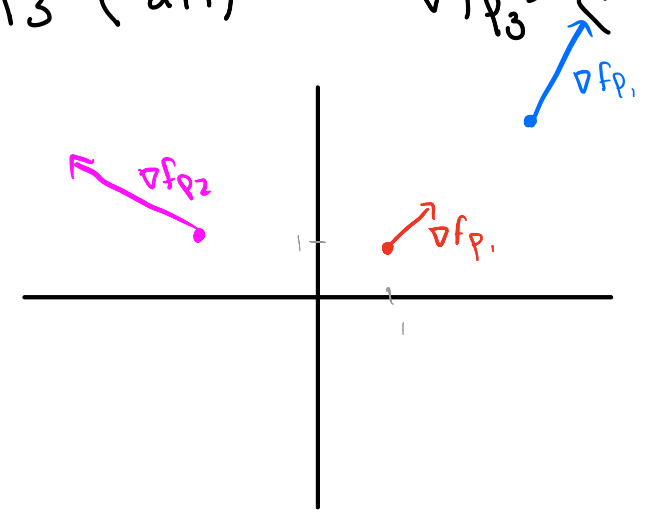
We can think of the gradient of f as a function, for each pt in the domain of f, we obtain a vector.

Ex! Let $f(x,y) = x^2+y^2$. Then $f_x = 2x$, $f_y = 2y$ and $\nabla f = \langle 2x, 2y \rangle$

$$P_1 = (1,1) \rightsquigarrow \nabla f_{P_1} = \langle 2,2 \rangle$$

$$P_2 = (3,4) \rightsquigarrow \nabla f_{P_2} = \langle 6,8 \rangle$$

$$P_3 = (-2,1) \rightsquigarrow \nabla f_{P_3} = \langle -4,2 \rangle$$



Properties

$f(x,y)=x^2$ $\quad \nabla f = (2x,0)$

$g(x,y)=y^2$ $\quad \nabla g = \langle 0, 2y \rangle$

1) linearity:

$$\nabla(f+g) = \nabla f + \nabla g$$
$$\nabla(cf) = c \cdot \nabla f$$

2) product rule:

$$\nabla(fg) = f\nabla g + g \nabla f$$

3) Chain rule (for gradients)

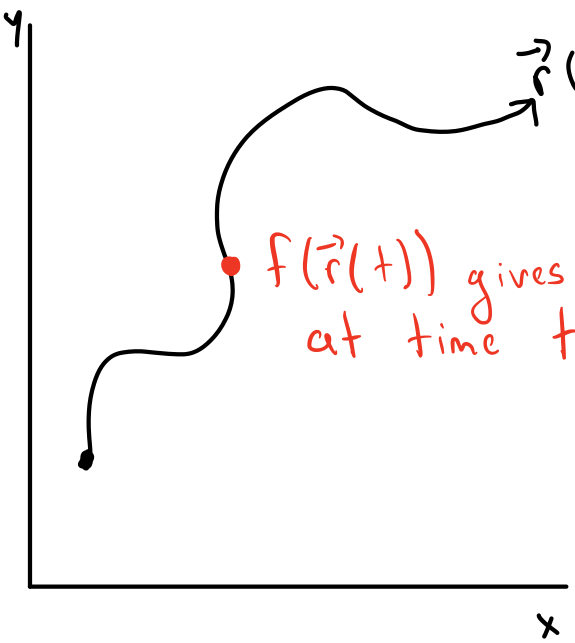
Let $F(t)$ be a single var. fctn. Then

$$\nabla(F(f(x,y,z))) = F'(f(x,y,z)) \cdot \nabla f$$

Ex2 Compute gradient of $g(x,y,z) = (x^2+y^2+z^2)^8$

Let $F(t) = t^8$ and $f(x,y,z) = x^2+y^2+z^2$.

Chain Rule for paths



$\vec{r}(t) = \langle x(t), y(t) \rangle$ and a function $f: \mathbb{R}^2 \to \mathbb{R}$ that measures a temperature at the pt (x,y).

$f(\vec{r}(t))$ gives temperature at time t.

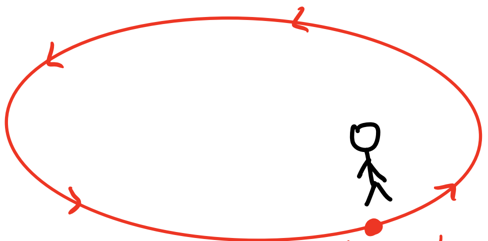
As t changes, we move along the path and so the temperature $f(\vec{r}(t))$ also changes.

We may take the derivative of $f(\vec{r}(t))$. This measures the rate of change of the temperature $f(\vec{r}(t))$ as t changes.

The chain rule for paths says that

$$\frac{d}{dt}\left(f(\vec{r}(t))\right) = \nabla f_{\vec{r}(t)} \cdot r'(t)$$

Ex 3



$\vec{r}(t) = \langle 4\cos(t), 2\sin(t) \rangle$

$$f(x,y) = 20 + 10e^{-.3(x^2+y^2)}$$

degrees celcius at the pt.

What is RoC of the temperature at $t = \frac{3\pi}{4}$?

① Set up gradient and $r'(t)$

$$\nabla f = \left\langle -6xe^{-.3(x^2+y^2)}, -6ye^{-.3(x^2+y)} \right\rangle$$

$$\vec{r}'(t) = \langle -4\sin(t), 2\cos(t) \rangle$$

② Plug in the pt.

$$\vec{r}\left(\frac{3\pi}{4}\right) = \left\langle 4 \cdot \frac{-\sqrt{2}}{2}, 2\frac{\sqrt{2}}{2} \right\rangle = \langle -2\sqrt{2}, \sqrt{2} \rangle \Rightarrow \begin{cases} \nabla f_{(-2\sqrt{2},\sqrt{2})} \\ r'\left(\frac{3\pi}{4}\right) \end{cases}$$

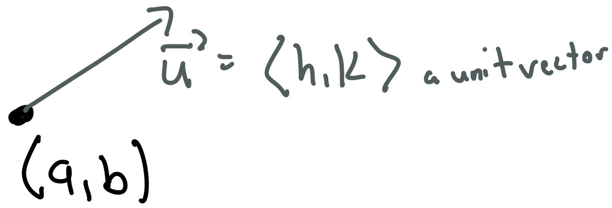
$$\nabla f_{\vec{r}\left(\frac{3\pi}{4}\right)} \cdot r'\left(\frac{3\pi}{4}\right) = \nabla f_{(-2\sqrt{2},\sqrt{2})} \cdot r'\left(\frac{3\pi}{4}\right)$$

$$= \left\langle -6(-2\sqrt{2})e^{-.3\left((-2\sqrt{2})^2+(\sqrt{2})^2\right)}, -6(\sqrt{2})e^{-.3\left((-2\sqrt{2})^2+(\sqrt{2})^2\right)} \right\rangle \cdot \langle -2\sqrt{2}, -2\sqrt{2} \rangle$$

≈ -1.19 celcisus per second.

Directional Derivative

Suppose f is differentiable at $p = (a,b)$

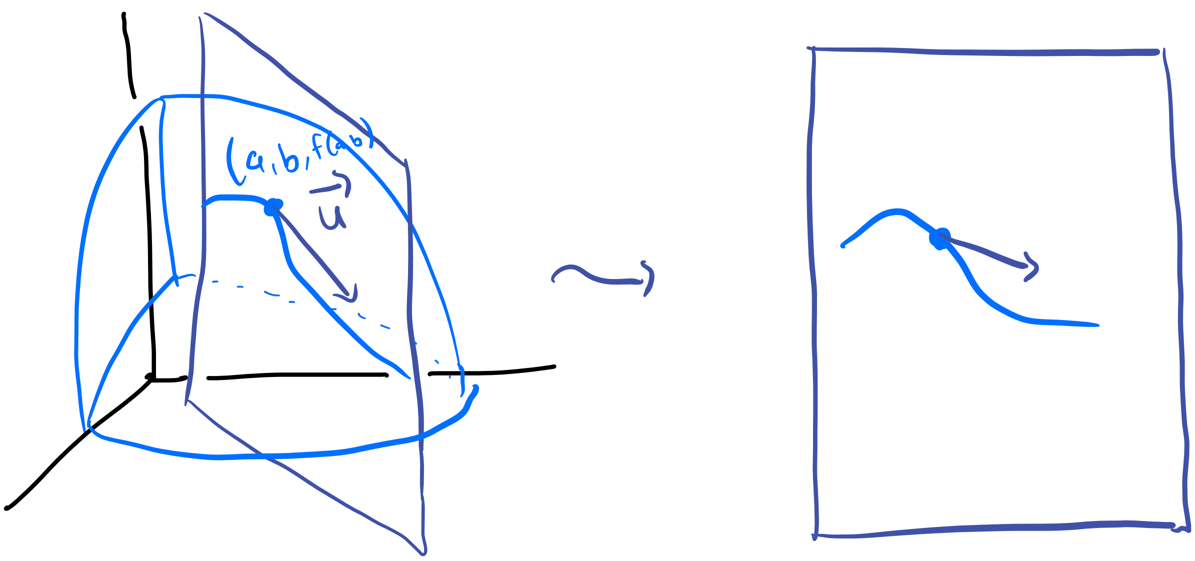


$\vec{r}(t) = \langle a+ht,\ b+kt \rangle$

The directional derivative of f with respect to $\vec{u}$ at P is defined as

$$D_u f(a,b) = \frac{d}{dt} f(\vec{r}(t))\Big|_{t=0} = \lim_{t \to 0} \frac{f(a+th,\ b+tk) - f(a,b)}{t}$$

Note that if $\vec{u} = \hat{\imath}$, we obtain $f_x(a,b)$ and if $\vec{u} = \hat{\jmath}$, then $f_y(a,b)$.



$D_u f(p)$ is RoC per unit change in the $\vec{u}$ direction at P.

$D_u f(p)$ is slope of tangent line at $(a,b,f(a,b))$ to trace curve obtained when intersected with vertical plane through P in the direction $\vec{u}$.

To compute the directional derivative, we use the dot product.

$$\boxed{D_{\vec{u}} f(p) = \nabla f_p \cdot \vec{u}}$$

Ex 4. Find RoC of pressure at $Q=(1,2,1)$ in the direction of $\vec{v} = \langle 0,1,1 \rangle$, where pressure is given by

$$f(x,y,z) = 1000 + .001(yz^2 + x^2z - xy^2)$$

millibars

xyz in kilometers.

(1) gradient at the pt

$$\nabla f = .001 \langle 2xz - y^2, z^2 - 2xy, 2yz + x^2 \rangle$$

$$\nabla f_Q = .001 \langle -2, -3, 5 \rangle =$$

(2) unit vector

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 0,1,1 \rangle}{\sqrt{2}} = \left\langle 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

(3) $D_{\vec{u}} f(Q)$

$$\nabla f_Q \cdot \vec{u} = .001 \langle -2,-3,5 \rangle \cdot \left\langle 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

$$= .001 \left(\frac{-3\sqrt{2}}{2} + \frac{5\sqrt{2}}{2} \right)$$

$$\approx .014 \text{ millibars/km}$$

(4) Interpretation?

As one moves in direction of $\vec{v}$ from Q, you expect pressure to increase by .014 millibars/km.

Ex 5. Altitude of a mountain is given by

$$f(x,y) = 2500 + 100(x+y^2)e^{-.3y^2}$$

x, y are in units of 100m.

a) Find $D_{\vec{u}}f((-1,-1))$ in direction $\vec{u}$ of making an angle of $\theta = \frac{\pi}{4}$ w/ gradient?

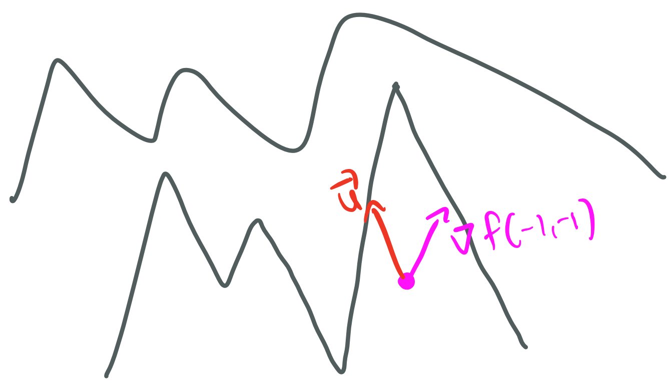
- $f_x(x,y) = 100e^{-.3y^2}$ $\quad f_x(-1,-1) = 100e^{-.3} \approx 74$
- $f_y(x,y) = 200ye^{-.03y^2} + 100(-.6)y(x+y^2)e^{-.3y^2}$

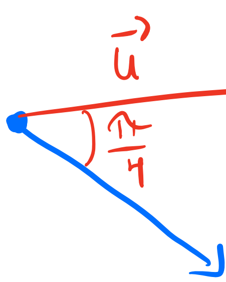
$$= 100ye^{-.03y^2}(2 + (-.6)(x+y^2)) \quad f_y(-1,-1) = -200e^{-.3}$$

$$\approx -148$$

- $\|\nabla f(-1,-1)\| = \sqrt{74^2 + (-148)^2} \approx 165.5$
- $D_{\vec{u}}f(-1,-1) = \|\nabla f(-1,-1)\| \cdot \cos(\frac{\pi}{4}) \approx 165.5 \cdot \left(\frac{\sqrt{2}}{2}\right) \approx 116.7$

b) Interpretation?



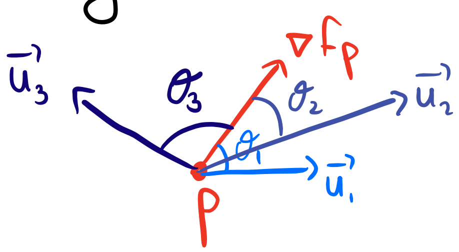


Moving in the direction of $\vec{u}$, altitude will increase at a rate of 116.7m per 100m.

Recall that $\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cos\theta$. dot product tells us abt angles for non-zero vectors.

$$D_{\vec{u}}f(P) = \nabla f_p \cdot \vec{u} = \|\nabla f_p\|\cdot\|\vec{u}\|\cdot\cos\theta = \|\nabla f_p\|\cdot\cos\theta$$

where θ is the angle between ∇f_p and $\vec{u}$.



As θ varies, so does the rate of change in a given direction.

Since

$$-1 \leq \cos\theta \leq 1$$

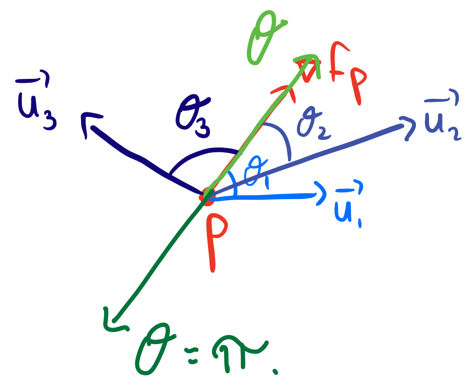
$$-\|\nabla f_p\| \leq \|\nabla f_p\|\cdot\cos\theta \leq \|\nabla f_p\|$$

$$-\|\nabla f_p\| \leq D_{\vec{u}}f(P) \leq \|\nabla f_p\|$$

When $\cos\theta = 1 \implies D_{\vec{u}}f(P) = \|\nabla f_p\|$

$\iff$

$\theta = 0$



When $\cos\theta = -1 \implies D_{\vec{u}}f(P) = -\|\nabla f_p\|$

$\iff$

$\theta = \pi$

Fact Let $\nabla f_p \neq 0$ and $\vec{u}$ a unit vector with θ the angle between them. Then

1) $D_{\vec{u}} f(p) = \|\nabla f_p\| \cdot \cos\theta$
2) ∇f_p points in direction of maximal increase and this rate of increase is $\|\nabla f_p\|$.
3) $-\nabla f_p$ points in direction of maximal decrease and this rate of decrease is $-\|\nabla f_p\|$.

Ex. In a field, the amt of light at a pt is given by

$$f(x,y) = 40 + 2x^2 + 18y - 12xy$$

Q1. An ant is at the pt $(1,-2)$ and heads toward to the $(-3,1)$. Does it get brighter?

$D_{\vec{u}} f((1,-2))$ where $\vec{u} = \dfrac{\langle -4,3\rangle}{5}$

$\nabla f(x,y) = \langle 4x - 12y, 18 - 12x\rangle$

$\nabla f(1,-2) = \langle 28, 6\rangle$

$\frac{1}{5}\langle -4,3\rangle \cdot \langle 28,6\rangle = \frac{-94}{5} < 0$

Q2: A plant is at $(3,2)$ but turns to direction for which light is brightest. What direction is this?

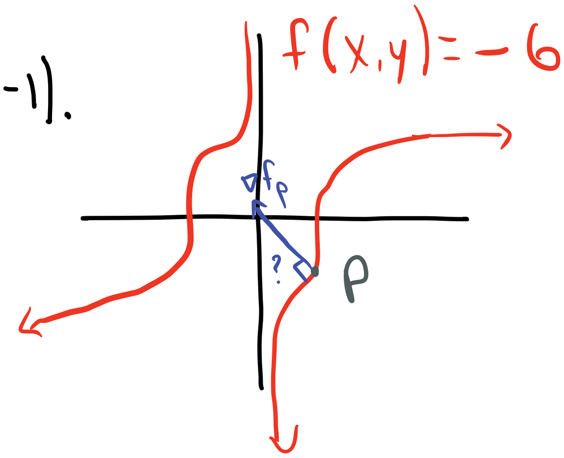
$\nabla f(3,2) = \langle -12, -18 \rangle$

Gradient and normality

Fact For $f(x,y)$, (or $f(x,y,z)$), the gradient at P, ∇f_P, is normal to the level curve (or surface respectively) at P.

Ex $f(x,y) = xy^3 - x^2$ at $P = (2,-1)$.

$f(2,-1) = -2 - 4 = -6$.



The pt P lies on the level curve $f(x,y) = k$.

1) $\nabla f = \langle y^3 - 2x, 3xy^2 \rangle$

$\nabla f(2,-1) = \langle -5, 6 \rangle$

! We calculate angle of ∇f w/ level curve $f(x,y) = -6$ using tangent vector at P.

2) Parameterize curve to get tangent direction.

$$xy^3 - x^2 = -6 \Rightarrow y^3 = -6x^{-1} + x$$

$x = t,$

$y = \left(-6x^{-1} + x\right)^{1/3}$

$t = 2 \Rightarrow (2,-1)$

$\vec{r}(t) = \left\langle t, \left(\frac{-6}{t} + t\right)^{1/3} \right\rangle$

$\vec{r}\,'(t) = \left\langle 1, \frac{1}{3}\left(\frac{-6}{t} + t\right)^{-2/3} \cdot \left(\frac{6}{t^2} + 1\right) \right\rangle$

$\vec{r}\,'(2) = \left\langle 1, \frac{1}{3}(-1)^{-2/3} \cdot \left(\frac{3}{2} + \frac{2}{2}\right) \right\rangle$

$= \left\langle 1, \frac{1}{3} \cdot \frac{5}{2} \right\rangle$

$= \left\langle 1, \frac{5}{6} \right\rangle$

Could also use $\langle 6,5 \rangle$!

3) Verify that $\nabla f_P \cdot r'(P) = 0$

$\langle -5, 6 \rangle \cdot \langle 6, 5 \rangle = 0$:)

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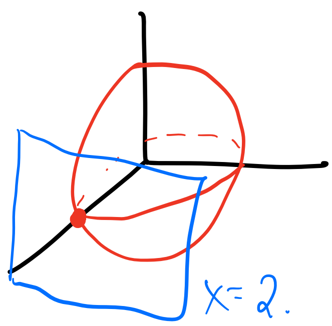
Recall we struggled w/ finding tangent plane to $f(x,y) = \sqrt{4 - x^2 - y^2}$ at $(2,0)$, since f is not diffable at the pt.

Ex. Let $f(x,y,z) = x^2 + y^2 + z^2$ and $P = (2,0,0)$. Consider the level surface $f(x,y,z) = 4$ at P.

$\nabla f = \langle 2x, 2y, 2z \rangle$

$\nabla f(2,0,0) = \langle 4,0,0 \rangle$ is normal to surface $f(x,y,z) = 4$.



$\Rightarrow 4(x-2) + 0 \cdot (y-0) + 0 \cdot (z-0) = 0$

$\Rightarrow 4x - 8 = 0 \Rightarrow \boxed{x = 2.}$