Here is a $\boxed{\text{3rd vector}}$ operation, except it outputs a number.

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$
$$\vec{w} = \langle w_1, w_2, w_3 \rangle$$

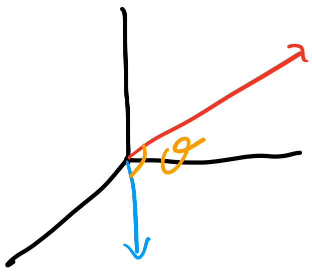
multiply

$$v_1w_1 \quad v_2w_2 \quad v_3w_3$$

add!

$$\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3$$

is the dot product of $\vec{v}$ and $\vec{w}$. The dot product helps us measure the angle θ between two vectors: $v \cdot w = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\theta$



$\vec{v} = \langle 3, 1, -2 \rangle$, $\quad \|\vec{v}\| = \sqrt{9+1+4} = \sqrt{14}$

$\vec{w} = \langle 4, -3, 0 \rangle$, $\quad \|\vec{w}\| = \sqrt{16+9} = 5$

$\vec{v} \cdot \vec{w} = 12 - 3 + 0 = 9.$

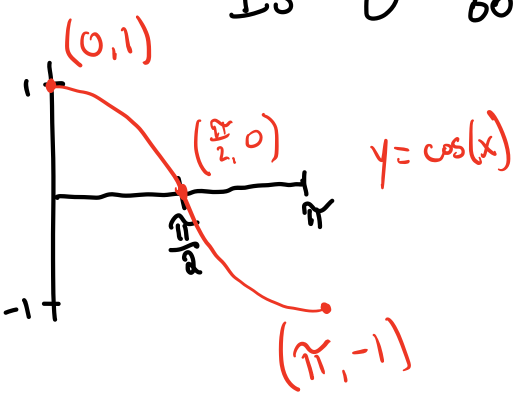
$$\underline{9} = \underline{5}\,\underline{\sqrt{14}}\cos\theta \Rightarrow \cos\theta = \frac{9}{5\sqrt{14}}$$

Is θ obtuse, acute or a right angle?

right angle: $\cos\theta = 0$

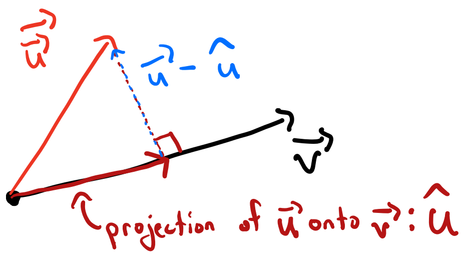
acute: $\cos\theta > 0$

obtuse: $\cos\theta < 0$



Projections

Given two vectors $\vec{u}$ and $\vec{v}$, we may project $\vec{u}$ onto $\vec{v}$



$$\hat{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

The vector $\hat{u}$ points in the direction of $\vec{v}$ but has length $\vec{u}$.
The vector $\vec{u}-\hat{u}$ is perpendicular to $\vec{v}$

$$(\vec{u}-\hat{u})\cdot\vec{v} = \left(\vec{u} - \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\right)\cdot\vec{v} = \vec{u}\cdot\vec{v} - \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}(\vec{v}\cdot\vec{v}) = 0$$

Note that $\vec{u}$ may be decomposed with respect to $\vec{v}$ as

$$\vec{u} = \underbrace{\hat{u}}_{\text{parallel to } \vec{u}} + \underbrace{(\vec{u}-\hat{u})}_{\text{perpendicular to } \vec{u}}$$

Ex 1. Find decomposition of $\hat{u} = \langle 5,1,-3\rangle$ with respect to $\vec{v} = \langle 4,4,2\rangle$

$\hat{u} = \langle 2,2,1\rangle$ $\quad$ $\vec{u}-\hat{u} = \langle 3,-1,-4\rangle$

Here is a $\boxed{\text{4th vector}}$ operation, it outputs a vector.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix. The determinant of this matrix is $ad - bc$. Denote it by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is a 3×3 matrix. The determinant is given by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \cdot \begin{vmatrix} a_2 & c_3 \\ a_3 & c_2 \end{vmatrix} + c_1 \cdot \begin{vmatrix} a_2 & b_2 \\ a_3 & c_3 \end{vmatrix}$$

<u>Ex 2</u> Compute det of $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \Rightarrow 0$

Cross Product

Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$.

Then the cross product is given by

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \vec{k}$$

<u>Ex 3</u>. Calculate $\vec{v} \times \vec{w}$ where $\vec{v} = \langle 1, 0, 0 \rangle$ and $\vec{w} = \langle 0, 1, 0 \rangle$

$\Rightarrow \langle 0, 0, 1 \rangle$

Why do we care?

1) the vector $\vec{v} \times \vec{w}$ is perpendicular to $\vec{v}$ <u>and</u> $\vec{w}$

2) $\vec{v}$, $\vec{w}$ and $\vec{v} \times \vec{w}$ form a right handed system.

<u>Q</u>: What would $\vec{w} \times \vec{v}$ be from previous example?

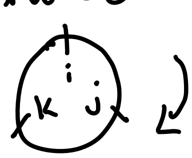
$\Rightarrow \langle 0, 0, -1 \rangle$

<u>Properties of cross product</u>

1) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

2) $\vec{v} \times \vec{v} = 0$

3) $\vec{v} \times \vec{w} = 0$ iff $\vec{v}$ and $\vec{w}$ are parallel

4) 

5) the parallelogram spanned by $\vec{v}$ and $\vec{w}$ has area $\|\vec{v} \times \vec{w}\|$.

6) the parallelopiped spanned by $\vec{v}, \vec{w}$ and $\vec{u}$ has volume $\|\vec{u} \cdot (\vec{v} \times \vec{w})\|$