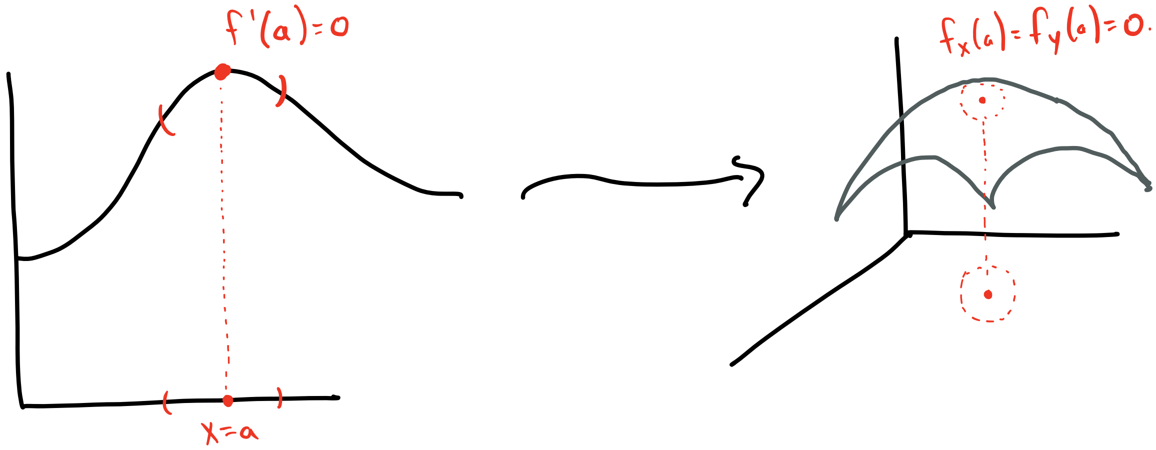
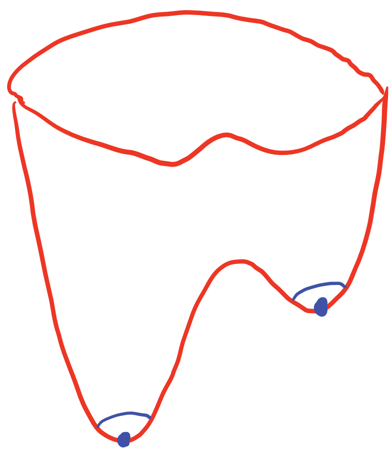
Derivatives in calc 1 tell us about max/min of a fctn.

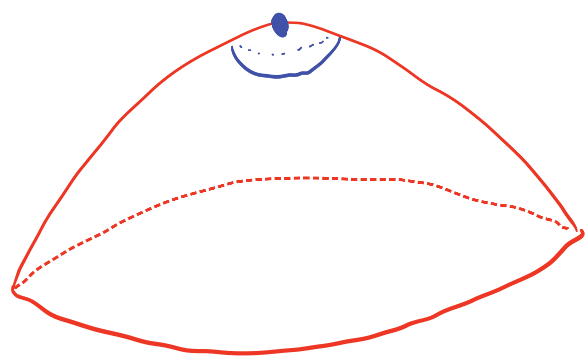


Exam 2!

For a fctn $f(x,y)$, a pt $P=(a,b)$ is a critical pt if
$f_x(a,b)=f_y(a,b)=0$ or if neither exist.

A critical pt can be a maximum or minimum.





local minima: a pt P is a local minima if there exists a disc around P such that $f(P)$ is minimum value of all values in disk.

local maxima: a pt P is a local maxima if there exists a disc around P such that $f(P)$ is maxima value of all values in disk.

Ex 1 How many crit pts does $f(x,y) = 11x^2 - 2xy + 2y^2 + 3y$ have?

$f_x = 22x - 2y \qquad f_y = -2x + 4y + 3$

$$\begin{matrix} f_x = 0 \\ f_y = 0 \end{matrix} \xrightarrow{\text{solve system!}} \left(\frac{-1}{14}, \frac{-11}{14}\right)$$

$$f\left(\frac{-1}{14}, \frac{-11}{14}\right) = \frac{11}{196} - \frac{22}{196} + \frac{242}{196} - \frac{33}{14} = \frac{-33}{28}.$$

local min! how to check?

We have a test to classify critical pts.

for a fctn $f(x,y)$, the Hessian matrix is the matrix

$$\begin{pmatrix} f_{xx}(x,y) & f_{yx}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y) \end{pmatrix}$$

Symmetric b.c. of Clairaut's thm.

the determinant of the hessian is

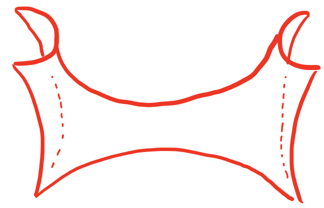
$$H = f_{xx} f_{yy} - f_{xy} f_{yx} = f_{xx} f_{yy} - f_{xy}^2$$

To test what kind of critical pt P is, we plug P into H

Second derivative test Let $p=(a,b)$ be a critical pt of f.

1) if $H(a,b)>0$ and $f_{xx}(a,b)>0$, $f(p)$ is a local min.
2) if $H(a,b)>0$ and $f_{xx}(a,b)<0$, $f(p)$ is a local max.
3) if $H(a,b)<0$, $f(p)$ is a saddle pt (analogue of inflection pt)
4) if $H(a,b)=0$, inconclusive.

Ex 2. $f(x,y)=x^2-y^2$ (hyperbolic paraboloid)



$f_x=2x \quad f_{xx}=2$

$f_y=-2y \quad f_{yy}=-2$

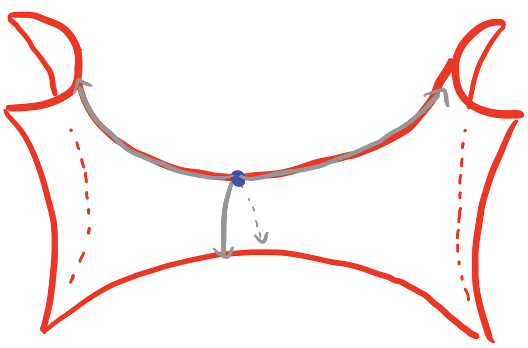
$f_{xy}=0$

$$\begin{aligned} f_x &= 2x = 0 \\ f_y &= -2y = 0 \end{aligned} \Rightarrow p=(0,0)$$

$$H = f_{xx}f_{yy}-(f_{xy})^2$$

$$H(p) = 2\cdot(-2)-0=-4<0$$

Then p is a saddle pt //



At the saddle pt, moving in one direction results in a function decrease; while moving in other direction results in function increase

Ex 2. Compute distance from the pt $(1,0,-2)$ to $x+2y+z=4$.

Want to minimize $d^2=(x-1)^2+y^2+(z+2)^2$

$$d^2=(x-1)^2+y^2+(4-x-2y+2)^2$$

$$f(x,y)=(x-1)^2+y^2+(6-x-2y)^2$$

$$f_x=2(x-1)-2(6-x-2y)=4x+4y-14=0$$

$$f_y=2y-4(6-x-2y)=4x+10y-24=0$$

$f_y-f_x=6y-10=0$ $\quad\left(\frac{11}{6},\frac{5}{3}\right)$ is only critical pt.

① $f_{xx}=4$

$f_{yy}=10$ $\quad$ ② $H\left(\frac{11}{6},\frac{5}{3}\right)=40-16=24>0$

$f_{xy}=4$ $\quad$ So $\left(\frac{11}{6},\frac{5}{3}\right)$ is a global minima.

$$\boxed{f\left(\frac{11}{6},\frac{5}{3}\right)=\frac{5}{6}\sqrt{6}}$$

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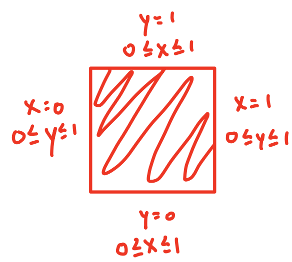
Sometimes a critical pt can occur on the boundary of a region.

Ex 3. $f(x,y)=2x+y-3xy$ on unit square

What's maximum value f takes on unit square?



$f_x=2-3y$ $\quad\Rightarrow\left(\frac{1}{3},\frac{2}{3}\right)$ is a critical pt.

$f_y=1-3x$

$$\Rightarrow f\left(\frac{1}{3},\frac{2}{3}\right)=\frac{4}{3}-\frac{2}{3}=\boxed{\frac{2}{3}}$$

left edge: $f(0,y) = y$

right edge: $f(1,y) = 2-2y$

bottom edge: $f(x,0) = 2x$

top edge: $f(x,1) = 1-x$

$\Rightarrow$ achieve value $2 > 2/3$ @ $(1,0)$ and $(1,0)$