Suppose we want to optimize a function. This involves finding maximum or minimum values in a certain domain such as the unit square $\{(x,y) \text{ in } \mathbb{R}^2 \mid 0 \le x,y \le 1\}$ or unit disk $x^2+y^2 \le 1$.

Similar to restricting our function $f(x,y)$ to a smaller domain, we can subject our function to a constraint $g(x,y)=0$. This means that the only points (a,b) we are allowed to consider are those satisfying $g(a,b)=0$. Hence to optimize f subject to g, we find maximum or minimum values of f along the curve $g(x,y)=0$.

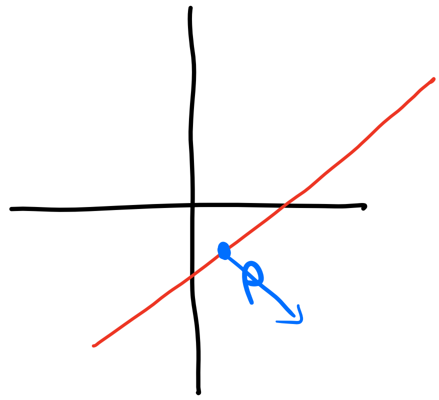
<u>Fact</u> Let $f(x,y), g(x,y)$ be fctns. If $f(x,y)$ has a local maximum or minimum value on the curve $g(x,y)=0$ at the pt $P=(a,b)$ and $\nabla g_P \neq 0$, then there is a scalar λ such that

$$\nabla f_P = \lambda \nabla g_P.$$

Lagrange Eqns.

$$f_x(a,b) = \lambda g_x(a,b)$$
$$f_y(a,b) = \lambda g_y(a,b)$$

Ex 1. Find the extreme value of $f(x,y) = x^2 + 2y^2$ subject to the constraint $g(x,y) = 4x - 6y = 25$



We know from inspection that f has a minimum at $(0,0)$ but $g(0,0) \neq 25$.

① Set up Lagrange eqns

$$\nabla f = \langle 2x, 4y \rangle \qquad \Rightarrow \qquad 2x = 4\lambda$$
$$\nabla g = \langle 4, -6 \rangle \qquad \qquad 4y = -6\lambda$$

② Solve for λ in terms of x and y

$$\frac{x}{2} = \lambda$$
$$\frac{-2}{3}y = \lambda$$

③ Set λ equal to each other

$$\frac{x}{2} = \frac{-2}{3}y \Rightarrow x = \frac{-4}{3}y$$

④ Substitute into $g(x,y) = 25$ and simplify

$$4\left(\frac{-4}{3}y\right) - 6y = 25$$
$$-16y - 18 = 75$$
$$\boxed{y = \frac{-75}{34}}$$

⑤ Back-substitute to obtain critical pt.

$x = \frac{-4}{3}\left(\frac{-75}{34}\right) = \frac{50}{17}$ and so $\lambda = \frac{25}{17}$

The pt $\left(\frac{50}{17}, \frac{-75}{34}\right)$ is a critical point of f subject to the constraint and f takes the value $\frac{625}{34} \approx 18.$

$\nabla f_p = \left\langle \frac{100}{17}, \frac{-150}{17} \right\rangle$ and observe $\left\langle \frac{100}{17}, \frac{-150}{17} \right\rangle = \frac{25}{17}\langle 4, -6 \rangle$

$\nabla g_p = \langle 4, -6 \rangle$

This is a minimum. //

Ex 2. Find max/min values of $f(x,y,z) = 3x + 2y + 4z$ subject to $g(x,y,z) = x^2 + 2y^2 + 6z^2 = 1$

$\nabla f = \langle 3, 2, 4 \rangle$

$\nabla g = \langle 2x, 4y, 12z \rangle$

$3 = 2x\lambda \qquad \lambda = \frac{3}{2x}$

$2 = 4y\lambda \qquad \lambda = \frac{1}{2y}$

$4 = 12z\lambda \qquad \lambda = \frac{1}{3z}$

$\frac{3}{2x} = \frac{1}{2y} \Rightarrow \boxed{3y = x}$

$\frac{1}{2y} = \frac{1}{3z} \Rightarrow 2y = 3z$

$\boxed{\frac{2}{3}y = z}$

$9y^2 + 2y^2 + \frac{8}{3}y^2 = 1$

$27y^2 + 6y^2 + 8y^2 = 3$

$41y^2 = 3$

$y = \pm\sqrt{\frac{3}{41}}$

① $\left(\frac{1}{3}\sqrt{\frac{3}{41}}, \sqrt{\frac{3}{41}}, \frac{3}{2}\sqrt{\frac{3}{41}}\right)$ max $\sqrt{\frac{41}{3}}$

② $\left(\frac{-1}{3}\sqrt{\frac{3}{41}}, -\sqrt{\frac{3}{41}}, \frac{-3}{2}\sqrt{\frac{3}{41}}\right)$ min. $-\sqrt{\frac{41}{3}}$

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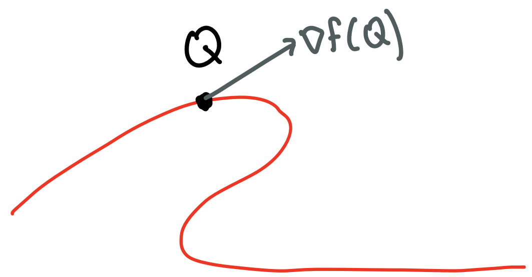
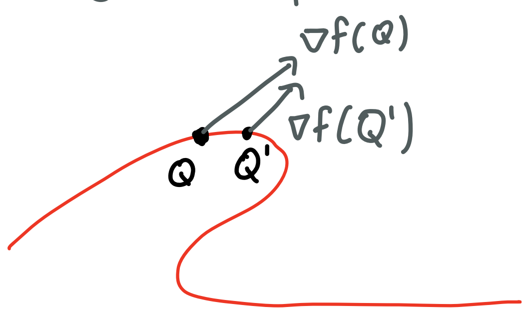
Why does $\nabla f(P) = \lambda \nabla g(P)$ work?



$\nabla f(Q)$ pts to where increase is maximal but moving in that direction takes us off $g(x,y)=0$. We can increase f slightly by still moving towards right on our curve.

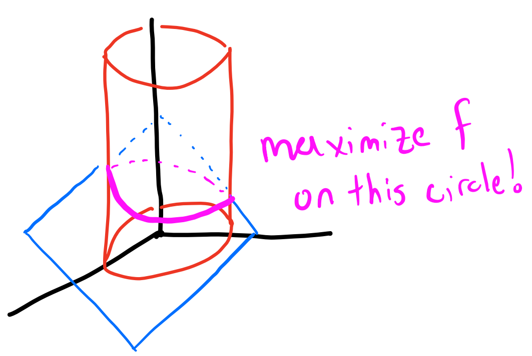


We continue to move on $g(x,y)=0$ until $\nabla f(P)$ is orthogonal to $g(x,y)=0$.

We can't increase f any further.

Since $\nabla g(P)$ is also orthogonal to $g(x,y)=0$ at P, $\nabla f(P)$ and $\nabla g(P)$ must be parallel.

<u>Ex 3</u>. Find the maximum of $f(x,y,z) = x+y+z$ subject to the constraints $\underset{g_1}{x^2+y^2=1}$ and $\underset{g_2}{x+2y+3z=6}$



$\nabla f = \langle 1,1,1 \rangle$

$\nabla g_1 = \langle 2x, 2y, 0 \rangle$

$\nabla g_2 = \langle 1, 2, 3 \rangle$

Lagrange eqns are $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$

x) $1 = \lambda_1 2x + \lambda_2$ y) $1 = \lambda_1 2y + 2\lambda_2$ z) $1 = 3\lambda_2$

$$\boxed{\frac{1}{3} = \lambda_2}$$

plug into x)

$$1 = \lambda_1 2x + \frac{1}{3}$$

$$\frac{2}{3} = \lambda_1 2x$$

$$\boxed{\frac{1}{3x} = \lambda_1}$$

plug into y)

$$1 = \left(\frac{1}{3x}\right) 2y + 2\left(\frac{1}{3}\right)$$

$$1 = \frac{2}{3} \cdot \frac{y}{x} + \frac{2}{3}$$

$$\boxed{\frac{1}{2} x = y}$$

plug into g_1

$$x^2 + \left(\frac{x}{2}\right)^2 = 1$$

$$5x^2 = 4$$

$$\boxed{x = \pm \frac{2}{\sqrt{5}}}$$

back substitute

$$\boxed{y = \frac{4}{\sqrt{5}}}$$

and

$$\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}} + 3z = 6$$

$$3z = 6 - \frac{4}{\sqrt{5}}$$

$$\boxed{z = 2 - \frac{4}{3\sqrt{5}}}$$

f is maximized at the pt $\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}, 2 - \frac{4}{3\sqrt{5}}\right)$ with a value of $\frac{1}{3}(6 + \sqrt{5})$

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