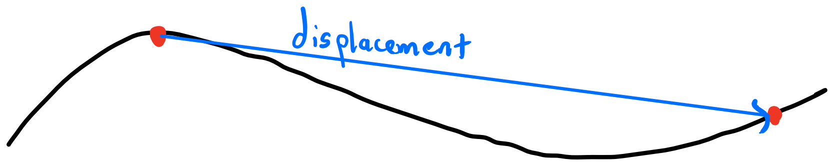
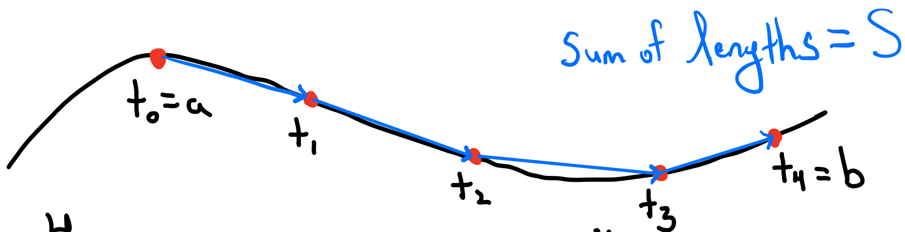
Consider a curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ on the interval $[a,b]$

Can we calculate this length?

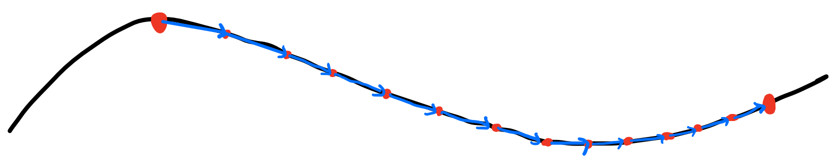


$$\|\vec{r}(b) - \vec{r}(a)\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$



$$S = \sum_{i=1}^{4} \|\vec{r}(t_i) - \vec{r}(t_{i-1})\| = \sum_{i=1}^{4} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

$\Delta x_i = x_i - x_{i-1}$
$\Delta y_i = y_i - y_{i-1}$
$\Delta z_i = z_i - z_{i-1}$



Imagine chopping up curve more and more and summing up tinier pieces. This means we'll $\Delta x, \Delta y$ and Δz all converge to 0. Equivalently, this means $\Delta t \to 0$.

$$S = \lim_{\Delta t \to 0} \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

Note: n denotes the amount of pieces we've chopped our curve into. $\boxed{\text{This depends on } \Delta t.}$ As $\Delta t \to 0$, $n \to \infty$.

$$= \lim_{\Delta t \to 0} \sum_{i=1}^{n} \frac{\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}}{\Delta t} \Delta t$$

bring Δt inside sqrt

$$= \lim_{\Delta t \to 0} \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2 + \left(\frac{\Delta z_i}{\Delta t}\right)^2}$$

take limit.

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

the limiting process allows us to relabel......

$\sum \rightsquigarrow \int$

$\frac{\Delta x_i}{\Delta t}, \frac{\Delta y_i}{\Delta t} \rightsquigarrow \frac{dx}{dt}, \frac{dy}{dt}$

$\Delta t \rightsquigarrow dt$

$$\boxed{S = \int_a^b \|\vec{r}\,'(t)\|\, dt}$$

is the arclength of the curve.

Ex 1. What is the arclength of the curve given by $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ on the interval $0 \leq t \leq 2\pi$

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$. Notice $\|\vec{r}\,'(t)\| = 1$.

$$S = \int_0^{2\pi} \sqrt{(\sin^2(t) + \cos^2(t))}\, dt = \int_0^{2\pi} dt = \boxed{2\pi}$$

When we have $\int_I 1 \cdot dt$, this measures size of the set we are integrating. For $I=[a,b]$, $\int_a^b 1\, dt = b-a$. So it's convenient to identify curves w/ $\|\vec{r}\,'(t)\|=1$. When $\vec{r}(t)$ has this property, we call the parameterization an arc length parameterization.

Ex. Give an arc length parameterization of the curve given by $\vec{r}(t)=\langle \cos(4t), \sin(4t), 3t\rangle$.

Step 1 Check $\|\vec{r}\,'(t)\| \neq 0$ for any t.

Step 2 Set $s=g(t)=\int_0^t \|\vec{r}\,'(u)\|\, du$. Since $g(t)$ is increasing, g has an inverse.

$$g(t)=\int_0^t \sqrt{16\cos^2(4t)+16\sin^2(4t)+9}\; dt = \int_0^t 5\, dt = 5t$$

$$g(t)=5t$$

Step 3 Determine the inverse of $g(t)$.

$$g^{-1}(s)=\frac{s}{5}.$$

Step 4 Take the new parameterization

$$\vec{r}(g^{-1}(s))$$

$$\vec{r}(g^{-1}(s)) = \left\langle \cos\left(\frac{4}{5}s\right), \sin\left(\frac{4}{5}s\right), \frac{3}{5}s \right\rangle$$

Step 5 Verify $\vec{r}(g^{-1}(s))$ has unit speed!