$F = \langle F_1, F_2 \rangle \rightsquigarrow \operatorname{curl}(F) = \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$ scalar-valued function

Greens Theorem says $\displaystyle\oint_{\partial D} F \cdot d\vec{r} = \iint_D \operatorname{curl}(F)\, dA$

$F = \langle F_1, F_2, F_3 \rangle \rightsquigarrow \operatorname{curl}(F) = \left\langle \dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}, \dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x}, \dfrac{\partial F_2}{\partial y} - \dfrac{\partial F_1}{\partial x} \right\rangle$

vector-valued fctn i.e. vector field

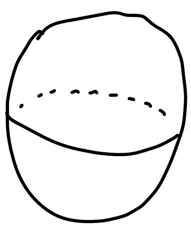
Stoke's Theorem says $\displaystyle\oint_{\partial S} F \cdot d\vec{r} = \oiint_S \operatorname{curl}(F) \cdot d\vec{S}$

vector-line integral of boundary

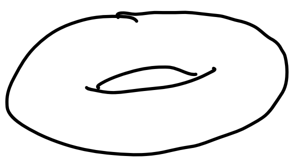
vector-surface integral of surface

A surface is closed if there is no boundary.

examples

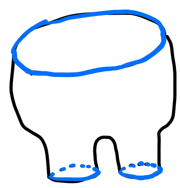


sphere

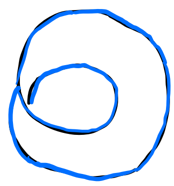


torus

non-examples



pants



möbius strip

If S has boundary, then the boundary of S is a curve, $\partial S = C$

If S has no boundary, then ∂S is empty and write $\partial S = \phi$.

$\displaystyle\oiint_{\partial S} \operatorname{curl}(F)\, d\vec{S} = 0$

Most surfaces we encounter will be orientable, so the boundary should also be orientable. The boundary orientation of surfaces will be such that walking along the boundary in positive direction means surface is to your left.

Ex 1. Let C be piece-wise linear loop from $(0,0,1)$ to $(2,0,2)$ to $(2,1,3)$ back to $(0,0,1)$ and $F = \langle e^x + 3y, 2xy + 5z, z^3 \rangle$

Calculate $\oint_C F \cdot d\vec{r}$

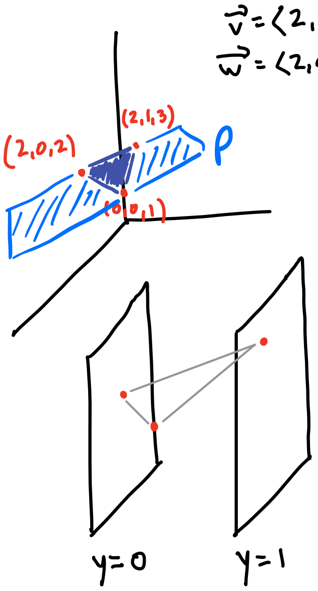
Can't use Green's cause F is a 3-dim'l vector field
Computing directly seems annoying.
Use Stokes, so we need

1) curl (F)
2) Pick a surface S, with a parameterization, whose boundary is C, with correct orientation
3) Set up integral and compute

$$\begin{array}{l} \qquad\qquad\quad x \qquad\quad y \qquad\quad z \\ \text{1) curl}(F) = \nabla \times \langle e^x + 3y, 2xy + 5z, z^3 \rangle \end{array}$$

$$= \langle -5, 0, 2y - 3 \rangle$$

② Surface we'll use is the plane containing that triangle.



$\vec{v} = \langle 2,1,2 \rangle$
$\vec{w} = \langle 2,0,1 \rangle$

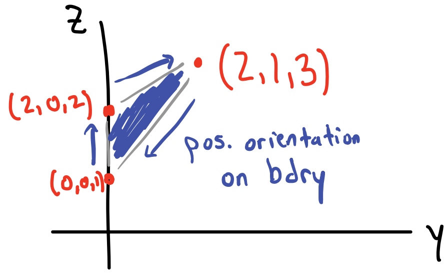
$\vec{v} \times \vec{w} = \langle 1, 2, -2 \rangle$

P: $1(x) + 2(y) - 2(z-1) = 0$

$x + 2y - 2z = -2$

$x = -2y + 2z - 2$

Project onto the $x=0$ plane



$0 \le y \le 1$

$2y+1 \le z \le y + 2$

$x = -2y + 2z - 2$

$G(y,z) = (-2y+2z-2,\ y,\ z)$ $\qquad 0 \le y \le 1,\ 2y+1 \le z \le y+2$

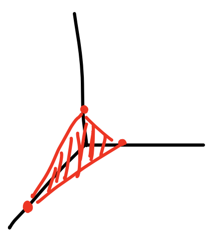
Eqn of plane gives normal vector $\langle 1,2,-2\rangle$ This is wrong orientation so take negative.

$N = \langle -1,-2,2\rangle$

$$\oint_{\partial S} F\cdot d\vec{r} = \iint_S \text{curl}(F)\cdot d\vec{S} = \int_0^1 \int_{2y+1}^{y+2} \langle -5,0,2y-3\rangle \cdot \langle -1,-2,2\rangle \, dz\, dy$$

$$= \int_0^1 \int_{2y+1}^{y+2} 4y-1 \, dz\, dy \qquad \approx 5+\frac{1}{3}$$

Ex 2. Compute $\oint_C F\cdot d\vec{r}$ where C is boundary of Δ with vertices at $(3,0,0)$, $(0,2,0)$ and $(0,0,1)$ and $F = \langle \sin(x^2),\ e^{y^2}+x^2,\ z^4+2x^2\rangle$



The Δ lies on the plane $P: \frac{x}{3}+\frac{y}{2}+z=1$.

Project onto $y=0$, we have $0\le x\le 3$, $0 \le z \le 1-\frac{x}{3}$ and $y = 2-2z-\frac{2x}{3}$

The curl(F) is given by $\langle 0,-4x,2x\rangle$

The normal vector is $\langle \frac{1}{3}, \frac{1}{2}, 1\rangle$

But the dot product of these vectors is 0.

Hence by Stoke's,

$$\oint_C F\cdot d\vec{r} = \int_0^3 \int_0^{1-\frac{x}{3}} 0 \, dz\, dx = 0. \quad //$$

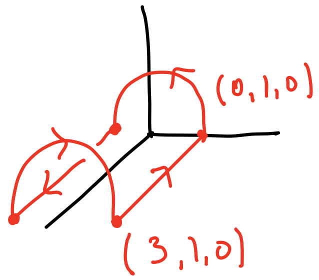
A related remark For $F = \langle F_1, F_2\rangle$ conservative and C a loop,

$$\oint_C F\cdot d\vec{r} = 0$$

For $F = \langle F_1, F_2, F_3\rangle$ solenoidal (there exists a v.f. A s.t. curl$(A) = F$) and S is a closed surface,

$$\iint_S F \cdot d\vec{S} = \oint_{\partial S} A \cdot d\vec{r} = 0$$ closure gives 0.

③ Let $F = \langle z^2, y^2, x^2 \rangle$ and C be the path given by



where the arcs are semi-circles of radius 1. Calculate $\oint_C F \cdot d\vec{r}$.

① $\text{curl}(F) = \langle 0, 2z - 2x, 0 \rangle$

② $S: \quad y^2 + z^2 = 1$

$$G(x,y) = (x, y, \sqrt{1-y^2}) \qquad \text{or} \qquad G(x,\theta) = (x, \cos(\theta), \sin(\theta))$$

$0 \leq x \leq 3 \qquad\qquad 0 \leq x \leq 3$

$-1 \leq y \leq 1 \qquad\qquad 0 \leq \theta \leq \pi$

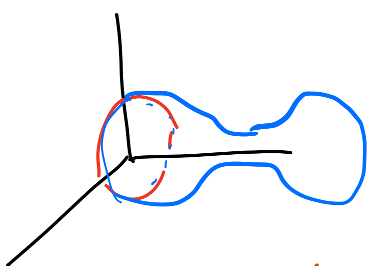
③ $\frac{\partial G}{\partial x} = \langle 1, 0, 0 \rangle$

$\frac{\partial G}{\partial y} = \langle 0, -\sin\theta, \cos\theta \rangle$

$N = \langle 0, -\cos\theta, -\sin\theta \rangle$

④ $\oint_C F \cdot d\vec{r} = 0$

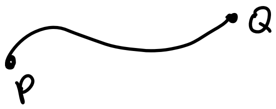
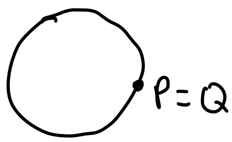
Ex 4. (Stoke's Backwards) Compute flux of $F = \text{curl}(A)$ where $A = \langle y+z, \sin(xy), e^{xyz} \rangle$ over S is



$\vec{r}(t) = \langle \cos(t), 0, \sin(t) \rangle$

Last Threads

Vector line integrals $\oint_C F\cdot d\vec{r}$: F a vector field, C a curve

	F not conservative	F conservative, $F=\nabla f$
C is not a loop (not closed) 	$\oint_C F\cdot d\vec{r} = \int_a^b F(r(t))\cdot r'(t)\,dt$ direct computation	$\oint_C F\cdot d\vec{r} = f(Q)-f(P)$ Fundamental thm of conservative vector fields
C is a loop (closed) 	2D: Green's Thm $\oint_C F\cdot d\vec{r} = \iint_D \mathrm{curl}(F)\,dA$ 3D: Stokes's Thm $\oint_C F\cdot d\vec{r} = \oiint_S \mathrm{curl}(F)\,d\vec{S}$	$\oint_C F\cdot d\vec{r} = 0$ By FTOCVF