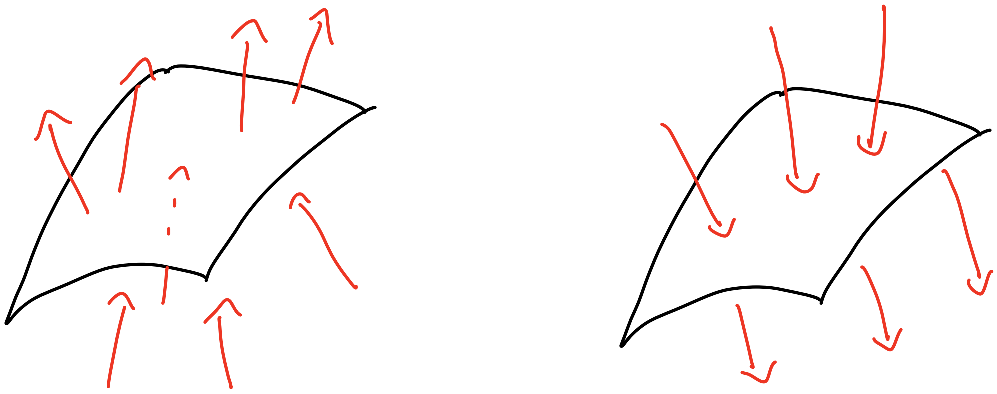
Flux is the flow from one side of the surface to another. Its helpful to have a notion of orientation.



(molecules passing thru some membrane)

We can declare either of these to be the positive direction of flow. A point on the surface has two normal directions, We will define positive orientation to be direction going outward. Given a vector field F and a point p on a surface S, the normal component is computed by

$$= F(p) \cdot \vec{n}(p)$$

, measure of how much vector field is flowing thru a surface perpendicularly

Doing this at every point, we obtain

$$\boxed{\text{flux of } F \text{ across } S = \iint_S (F \cdot \vec{n})\, dS}$$

If S is parameterized by $G(u,v)$ over D, recall then

$$dS = \|N(u,v)\|\,du\,dv = \underbrace{\|T_u \times T_v\|}_{\text{jacobian}}\,du\,dv \quad \text{and} \quad \vec{n} = \frac{N(u,v)}{\|N(u,v)\|}, \text{ giving}$$

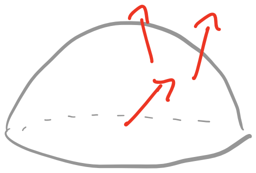
$$\iint_S F \cdot \vec{n}\, dS = \iint_S F(G(u,v)) \cdot \frac{N(u,v)}{\cancel{\|N(u,v)\|}} \cdot \cancel{\|N(u,v)\|}\,du\,dv$$

$$= \iint_S F(G(u,v)) \cdot N(u,v)\,du\,dv = \boxed{\iint_S F \cdot dS}$$

another notation you may see

$\cdot$ = dot product

Ex 1. Calculate the flux of $F(x,y,z) = \langle z, x, 1\rangle$ across upper hemisphere of unit sphere.



$$G(\theta,\phi) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)).$$

$$0 \le \theta \le 2\pi \qquad 0 \le \phi \le \frac{\pi}{2}$$

① The correct normal vector to S at p is given by $\boxed{T_\phi \times T_\theta}$ pts outward

$$T_\phi = \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi\rangle$$
$$T_\theta = \langle -\sin\theta\sin(\phi), \cos\theta\sin\phi, 0\rangle$$
$$\xrightarrow[\text{product}]{\text{cross}} N = \sin\phi\,\langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$

② $F(G(\theta,\phi)) = \langle z, x, 1\rangle = \langle \cos(\phi), \cos(\theta)\sin(\phi), 1\rangle$

③ $$\iint_S (F\cdot\vec{n})\,dS = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \cos\theta\cos\phi\sin^2\phi + \sin\theta\cos\theta\sin^3\phi + \cos\phi\sin\phi\,d\theta\,d\phi$$

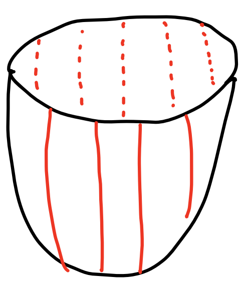
integral is 0.

$$= 2\pi\int_0^{\frac{\pi}{2}} \cos\phi\sin\phi\,d\phi = \pi\int_0^{\frac{\pi}{2}} \sin(2\phi)\,d\phi = -\frac{\pi}{2}\cos(2\phi)\Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{2}\big(\underbrace{\cos(\pi)}_{-1} - \underbrace{\cos(0)}_{1}\big)$$

$$= \boxed{\pi}$$

Ex 2. $\vec{F} = \langle z, y, x\rangle$ and S be portion of $z = x^2 + y^2$ where $z \le 9$, oriented towards $+z$ direction. Calculate flux of F thru S.



① $G(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$

$$0 \le r \le 3 \quad \text{and} \quad 0 \le \theta \le 2\pi$$

② $F(G(r,\theta)) = \langle r^2, r\sin\theta, r\cos\theta\rangle$

(3) $T_r = \frac{\partial G}{\partial r} = \langle \cos\theta, \sin\theta, 2r \rangle$, $T_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$N = \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle$

(4) $$\iint_S F \cdot \vec{n}\, dS = \int_0^{2\pi}\int_0^3 \langle r^2, r\sin\theta, r\cos\theta \rangle \cdot \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 -2r^4\underbrace{\cos\theta}_{0} - 2r^3\sin^2\theta + r^2\underbrace{\cos\theta}_{0} \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 -2r^3\sin^2\theta \, dr\, d\theta = \int_0^{2\pi} \sin^2\theta\, d\theta \int_0^3 -2r^3\, dr = \pi \cdot \frac{-81}{2}$$

Ex 3. Let H be part of unit sphere where $x, y, z \geq 0$ oriented towards origin. Let $F = \langle x^2, y^2, -z \rangle$.
Set up the flux of F through H.

(1) $H(\theta, \phi) = \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi \rangle$ $\quad 0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq \frac{\pi}{2}$

(2) $H_\phi = \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi \rangle$
$H_\theta = \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0 \rangle$

$N = \langle \cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \cos^2\theta\sin\phi\cos\phi + \sin^2\theta\sin\phi\cos\phi \rangle$

$= \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi \rangle$

This vector points in wrong direction, so take the negative.

$N = \langle -\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\sin\phi\cos\phi \rangle$

③ $F(H(r,\theta)) = \langle \cos^2\theta \sin^2\phi, \sin^2\theta \sin^2\phi, -\cos\phi \rangle$

$$\iint_H F \cdot \vec{n}\, dS = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} -\sin^4\phi \cos^3\theta - \sin^4\phi \sin^3\theta + \sin\phi \cos^2\phi \; d\theta \phi$$