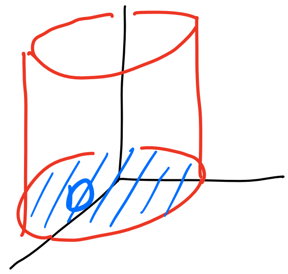
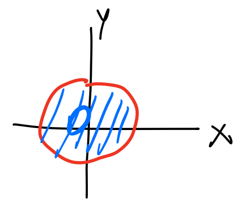
Cylindrical and spherical coordinates are another example of employing a <u>change of variables</u>.

Cylindrical coordinates are used if there is symmetry around an axis.

Suppose there is symmetry about the z-axis. Then we use

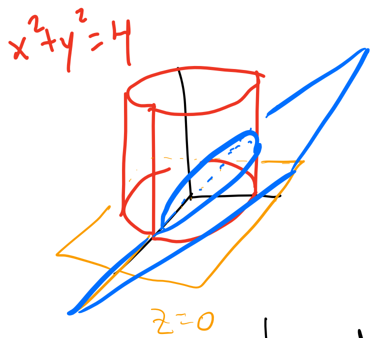
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$dV = r\,dz\,dr\,d\theta$$





In the domain D, we may express the domain using polar coords. The "cylindrical-ness" comes from the fact that every $z=c$ looks exactly like $z=0$ (the xy-plane)

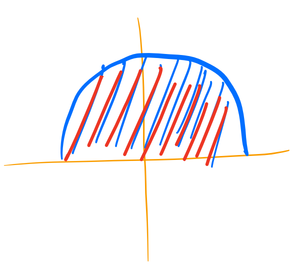
<u>ex 1</u>. Set up $\iiint_W z\,dV$ where W is region wedged in cylinder $x^2+y^2 \le 4$ between $z=0$ and $z=y$ in cylindrical coordinates.



$z=y$ cuts cylinder at an angle!

$$x^2+y^2=4 \Rightarrow r=2$$
$$z=0 \Rightarrow z=0$$
$$z=y \Rightarrow z=r\sin\theta$$

on domain

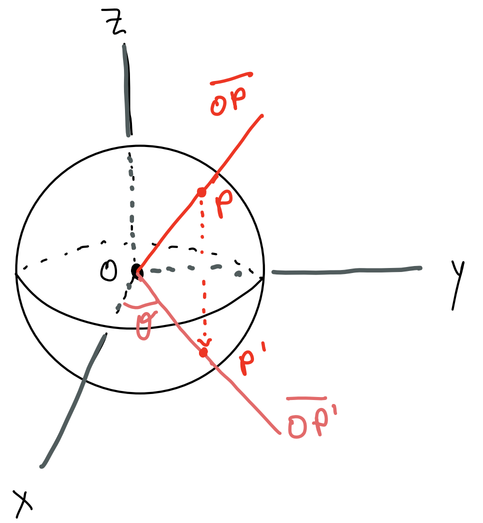


$\rightsquigarrow$ $0 \le r \le 2$, $0 \le \theta \le \pi$ $\qquad 0 \le z \le r\sin\theta$

$$\iiint_W z\,dV = \int_0^{\pi}\int_0^{2}\int_0^{r\sin\theta} z\,r\,dz\,dr\,d\theta \qquad = \pi$$

Spherical Coordinates are employed over cylindrical if the domain D varies on different slices $z=c$. A pt on a sphere is determined by 2 angles.

Consider a sphere centered at origin and a pt P on sphere



① θ is angle between $\overline{OP'}$ and positive x-axis, where P' is the projection of P onto xy-plane

② ϕ is angle between $\overline{OP}$ and positive z-axis. This is known as the angle of declination.

To specify distance away from origin (radius), we use ρ instead of r.

(x, y, z)

$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$

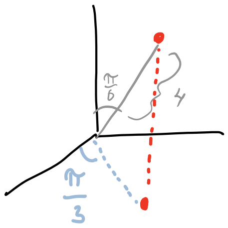
(ρ, θ, ϕ)

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

Ex 2. The pt $\left(4, \frac{\pi}{3}, \frac{\pi}{6}\right)$ is in C-coordinates. What is pt in rectangular coordinates?

$\rho = 4$
$\theta = \frac{\pi}{3}$
$\phi = \frac{\pi}{6}$



$x = 4\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right) = 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 1$

$y = 4\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right) = 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = 4\sqrt{3}$

$z = 4\cos\left(\frac{\pi}{6}\right) = \frac{4\sqrt{3}}{2} = 2\sqrt{3}$

__Ex 3__ The pt $(2, -2\sqrt{3}, 3)$ is in R-coords. What is this pt in S-coords?

$\rho = \sqrt{4+12+9} = \sqrt{25} = 5$

$\tan\theta = \frac{y}{x} = \frac{-2\sqrt{3}}{2} \Rightarrow \tan\theta = -\sqrt{3} \Rightarrow \theta = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$

$z = \rho\cos\phi \Rightarrow \cos\phi = \frac{3}{5} \Rightarrow \phi = \cos^{-1}\left(\frac{3}{5}\right)$

$$\left(5, \frac{5\pi}{3}, \cos^{-1}\left(\frac{3}{5}\right)\right)$$

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Integration in spherical coordinates helps when region is enclosed by and or drawn on a sphere. By change of variables formula, we have

$$dV = \underline{\rho^2 \sin\phi}\, d\rho\, d\phi\, d\theta$$

$x = x(\rho, \theta, \phi) = \rho\cos\theta\sin\phi$
$y = y(\rho, \theta, \phi) = \rho\sin\theta\sin\phi$
$z = z(\rho, \theta, \phi) = \rho\cos\phi$

$$\text{Jacobian} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$= \cos\phi\left[-\rho^2\sin^2\theta\sin\phi\cos\phi - \rho^2\cos^2\theta\sin\phi\cos\phi\right] - \rho\sin\phi\left[\rho\cos^2\theta\sin^2\phi + \rho\sin^2\theta\sin^2\phi\right]$

$= -\rho^2\sin\phi\cos^2\phi - \rho^2(1-\cos^2\phi)\sin\phi = -\rho^2\sin\phi \Rightarrow \rho^2\sin\phi.$

take abs. value!

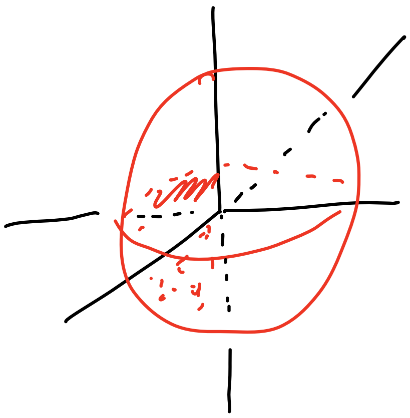
Ex 4. Set up $\iiint_R y\,dV$ where R is bounded by $x^2+y^2+z^2 \le 1$ and $x \le 0$, $y \le 0$, $z \le 0$

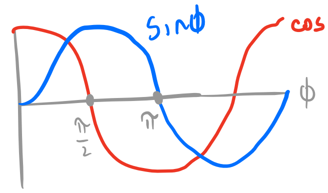
$x^2+y^2+z^2=1 \Rightarrow \rho^2=1,$
$\rho=1$

$x \le 0 \Rightarrow \rho\cos\theta\sin\phi \le 0 \Rightarrow \cos\theta\sin\phi \le 0$

$y \le 0 \Rightarrow \rho\sin\theta\sin\phi \le 0 \Rightarrow \sin\theta\sin\phi \le 0$

$z \le 0 \Rightarrow \rho\cos\phi \le 0 \Rightarrow \cos\phi \le 0 \Rightarrow \frac{\pi}{2} \le \phi \le \pi$





$\Rightarrow \sin(\phi) \ge 0$

$\Rightarrow$ So $\cos\theta \le 0$ and $\sin\theta \le 0$. This occurs between $\pi \le \theta \le \frac{3\pi}{2}$

Therefore $\iiint_R y\,dV = \int_{\pi}^{\frac{3\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} \int_0^1 (\rho\sin\theta\sin\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

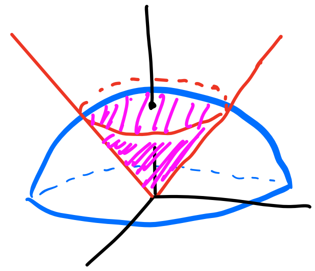
$= \int_{\pi}^{\frac{3\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} \int_0^1 (\sin\theta\sin^2\phi)\,\rho^3\,d\rho\,d\phi\,d\theta = \int_{\pi}^{\frac{3\pi}{2}} \frac{1}{4}\sin\theta \int_{\frac{\pi}{2}}^{\pi} \sin^2\phi\,d\phi\,d\theta$

$= \int_{\pi}^{\frac{3\pi}{2}} \frac{1}{4}\sin\theta \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}\left(1-\cos(2\phi)\right)d\phi\,d\theta = \int_{\pi}^{\frac{3\pi}{2}} \frac{1}{8}\sin\theta \left[\phi - \frac{\sin(2\phi)}{2}\right]\Big|_{\phi=\frac{\pi}{2}}^{\phi=\pi} d\theta$

$= \int_{\pi}^{\frac{3\pi}{2}} \frac{1}{16}\sin\theta\left[(\pi-0)-\left(\frac{\pi}{2}-0\right)\right]d\theta = \frac{\pi}{16}\int_{\pi}^{\frac{3\pi}{2}} \sin\theta = \frac{\pi}{16}\left[-\cos\theta\right]\Big|_{\theta=\pi}^{\theta=\frac{3\pi}{2}}$

$= \frac{-\pi}{16}\left(0-(-1)\right) = \boxed{\frac{-\pi}{16}}$

Ex 5. Integrate $\iiint_R z\,dV$ where R is region bounded below



$\underbrace{x^2+y^2+z^2=4}_{r=2}$ and above $\underbrace{x^2+y^2=z^2}$, $z\geq 0$.

$\rho^2\cos^2\theta\sin^2\phi+\rho^2\sin^2\theta\sin^2\phi=\rho^2\cos^2\phi$

$\Rightarrow \sin^2\phi=\cos^2\phi$

$\Rightarrow \phi=\frac{\pi}{4}$ or $\cancel{\frac{3\pi}{4}}$

$0\leq\phi\leq\frac{\pi}{4}$

$\Rightarrow 0\leq\theta\leq 2\pi$ and $0\leq\rho\leq 2$.

$$\iiint_R z\,dV=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2}\rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=4\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\cos\phi\sin\phi\,d\phi\,d\theta=8\pi\int_0^{\frac{\pi}{4}}\cos\phi\sin\phi\,d\phi$$

u-sub

$u=\sin\phi$

$du=\cos\phi\,d\phi$

$u=0$

$u=\frac{\sqrt{2}}{2}$

$$=8\pi\int_0^{\frac{\sqrt{2}}{2}}u\,du=8\pi\left(\frac{u^2}{2}\right)\Big|_0^{\frac{\sqrt{2}}{2}}=\frac{8\pi}{2}\left(\frac{\sqrt{2}}{2}\right)^2=4\pi\left(\frac{2}{4}\right)=\boxed{2\pi}$$