Average Value

Recall that average value of $f(x)$ on $I=[a,b]$ is the value

$$\operatorname{avg}_I f = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$b-a$ is the length, or size, of I.

We can easily generalize this in 2 or 3 dimensions

average value of $f(x,y)$ on D, a domain in $\mathbb{R}^2$, is given by

$$\operatorname{avg}_D f = \frac{1}{\operatorname{area}(D)}\iint_D f(x,y)\,dA$$

average value of $f(x,y,z)$ on R, a domain in $\mathbb{R}^3$, is given by

$$\operatorname{avg}_R f = \frac{1}{\operatorname{vol}(R)}\iiint_R f(x,y,z)\,dV$$

Ex What is average distance of pts in $R\subseteq\mathbb{R}^3$ from origin?

Here, f is the function $f(x,y,z)=\sqrt{x^2+y^2+z^2}$. Then

$$\operatorname{avg}_R f = \frac{1}{\operatorname{vol}(R)}\iiint_R \sqrt{x^2+y^2+z^2}\,dV$$

Ex. What is the average y-coordinate in a domain $D\subseteq\mathbb{R}^2$?

Here, $f(x,y)=y$. Equivalent, we may interpret this as the function measuring distance from x-axis if we take abs. value.

Density

if a function ρ measuring density, in units/size^3, over a region R, in size^3,

we may use a triple integral to calculate mass in units

$$\text{mass} = \iiint_V \rho(x,y,z)\,dV$$

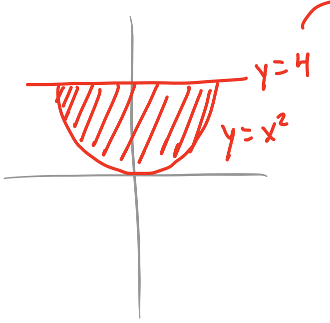
$$\text{units} = \frac{\text{units}}{\text{size}^3} \times \text{size}^3$$

- population = population density × area
  people people/sq ft sq ft

- mass = density × size

ex 1. An object occupies the space defined by $\begin{cases} x^2 \le y \le 4 \\ 0 \le z \le x \end{cases}$

and has density function $\rho(x,y,z) = xz$ (g/cm^3). What is mass of object?

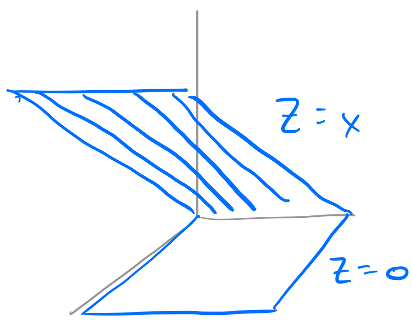


$x^2 \le y \le 4$

$\cancel{-x} \le x \le 2$

$0 \le z \le x$

$0 \le x \le 2$.



$$\text{mass} = \int_0^2\int_{x^2}^4\int_0^x \rho(x,y,z)\,dz\,dy\,dx = \int_0^2\int_{x^2}^4\int_0^x xz\,dz\,dy\,dx = \int_0^2\int_{x^2}^4 \left(\frac{xz^2}{2}\right)\Big|_{z=0}^{z=x} dy\,dx = \int_0^2\int_{x^2}^4 \frac{x^3}{2}\,dy\,dx$$

$$= \int_0^2 \left(\frac{x^3}{2}y\right)\Big|_{y=x^2}^{y=4} = \int_0^2 2x^3 - \frac{x^5}{2}\,dx = \left(\frac{x^4}{2} - \frac{x^6}{12}\right)\Big|_{x=0}^{x=2} = 8 - \frac{16}{3} = \frac{24}{3} - \frac{16}{3} = \boxed{\frac{8}{3}\,g}$$

Q: What is the center of mass?

A: a pt on the object where mass is most equally distributed.

$$\bar{x} = \frac{1}{M}\iiint_R x\,\rho(x,y,z)\,dV = \int_0^2\int_{x^2}^4\int_0^x x^2 z\,dz\,dy\,dx = \frac{48}{35}$$

$$\bar{y} = \frac{1}{M}\iiint_R y\,\rho(x,y,z)\,dV = \int_0^2\int_{x^2}^4\int_0^x xyz\,dz\,dy\,dx = 3$$

$$\bar{z} = \frac{1}{M}\iiint_R z\,\rho(x,y,z)\,dV = \int_0^2\int_{x^2}^4\int_0^x xz^2\,dz\,dy\,dx = \frac{32}{35}$$

Therefore the center of mass is $\left(\frac{48}{35}, 3, \frac{32}{35}\right)$