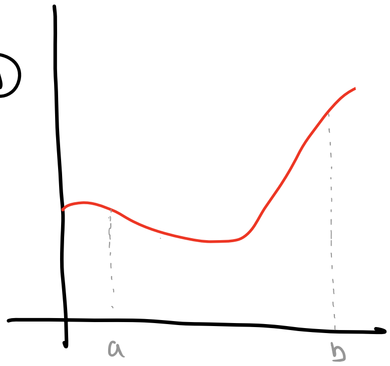
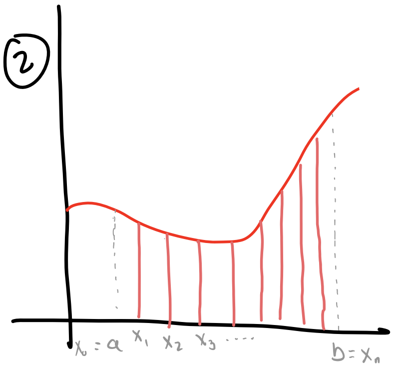
Snapshot of Calc 1 integration

1



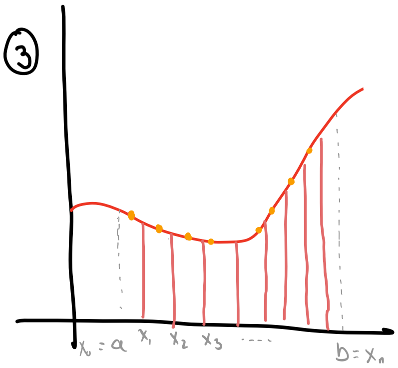
Consider domain of integration $[a,b]$

2



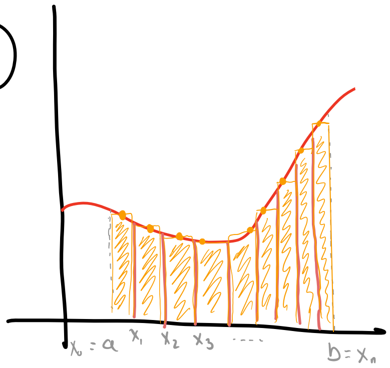
Subdivide the domain
$a=x_0<x_1<\dots<x_n=b$

3



Pick a pt x in (x_i, x_{i+1}) and draw the pt $(x, f(x))$ for every $i=0,1,\dots,n+1$

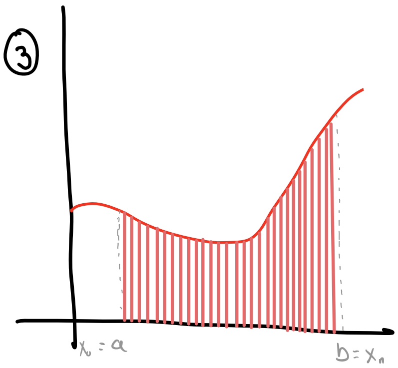
4



construct rectangles and compute sum of areas

$$\sum_{i=1}^{n} f(x)(x_i - x_{i+1})$$

3



take limit of $n \to \infty$

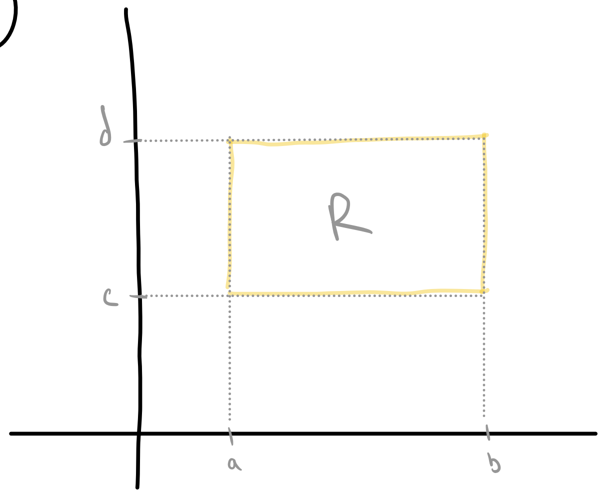
$$\int_a^b f(x)\,dx$$

<u>Interpretation</u>: $\int_a^b f(x)\,dx$ is the area under the curve $y=f(x)$ on the interval $[a,b]$.

<u>Goal</u>: generalize this to functions in two variables where the domain is a rectangle $R=[a,b]\times[c,d]$.

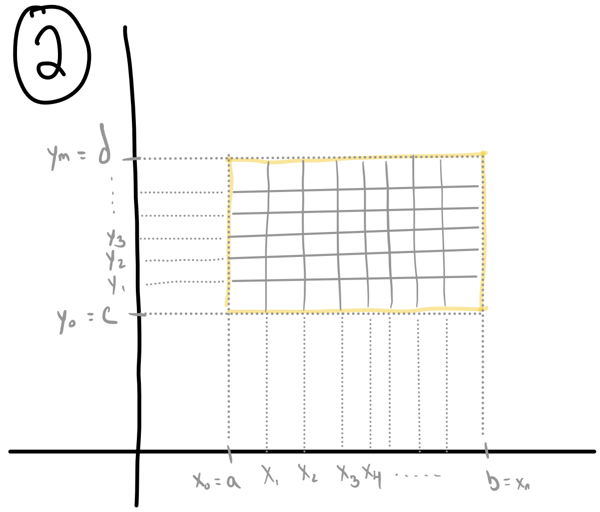
$\iint_R f(x,y)\,dA$ is the volume under the surface $z=f(x,y)$ on the rectangle R.

1



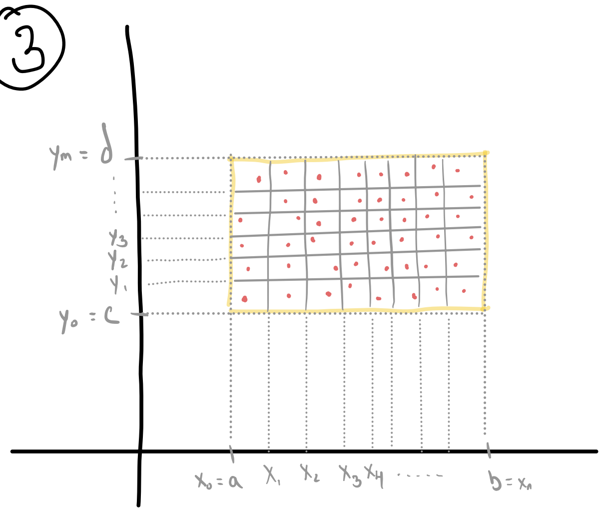
Consider the domain of integration R

2



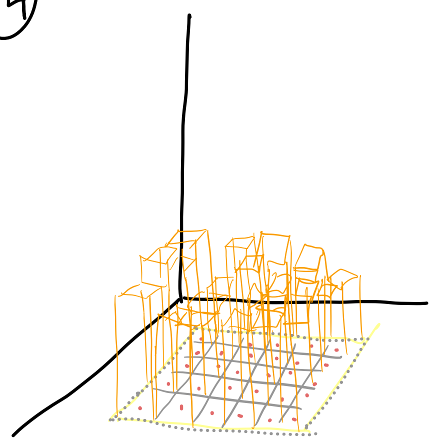
Subdivide R into smaller rectangles

3



Pick a pt in each sub-rectangle

4



In $\mathbb{R}^3$, for each subrectangle, draw a box with it's base the subrectangle and height $f(x,y)$, the picked sample pt in the subrectangle.

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \underbrace{f(x,y)}_{\text{height}} \underbrace{\Delta x_i \Delta y_i}_{\text{area of base}}$$

5 Take limit as $n,m \to \infty$

$$\lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x,y)\,\Delta x_i \Delta y_i = \iint_R f(x,y)\,dA.$$

This is the geometric meaning of a double integral. But how do we compute it? We do so by considering them as iterated integrals.

This is the equivalent of fixing a variable when taking partial derivatives. We'll hold one variable constant and integrate w/ respect to the other variable.

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

So a double integral may be computed as a sequence of regular intervals.

<u>Ex 1</u>. Integrate $f(x,y) = ye^x$ over $R = [2,4] \times [1,9]$.

We may express the double integral as

$$\iint_R ye^{xy}\,dA = \int_2^4 \int_1^9 ye^x\,dy\,dx = \int_2^4 \left(\int_1^9 ye^x\,dy \right) dx$$

$$= \int_2^4 \left(\frac{y^2}{2} e^x \right) \Bigg|_{y=1}^{y=9} dx$$

$$= \int_2^4 40e^x\,dx$$

$$= 40(e^4 - e^2)$$

Observe that we can reverse the order of integration

$$\iint_R f(x,y)\,dA = \int_1^9 \int_2^4 ye^x\,dx\,dy = \int_1^9 \left(ye^x\right)\Big|_{x=2}^{x=4} dy$$

$$= \int_1^9 y\left(e^4 - e^2\right) dy$$

$$= \left(e^4 - e^2\right) \int_1^9 y\,dy$$

$$= \left(e^4 - e^2\right)\left(\frac{y^2}{2}\right)\Big|_{y=1}^{y=9}$$

$$= 40\left(e^4 - e^2\right)$$

//

Fubini's Theorem (for rectangles) Let $f(x,y)$ be an integrable fctn and $R = [a,b] \times [c,d]$ is a rectangle. Then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Ex 2. Find volume of the region between graph of $f(x,y) = 16 - x^2 - 3y^2$ over $R = [0,3] \times [0,1]$.

$$\iint_R \left(16 - x^2 - 3y^2\right) dA = \int_0^3 \int_0^1 16 - x^2 - 3y^2\,dy\,dx$$

$$= \int_0^3 (16y - x^2y - y^3)\Big|_{y=0}^{y=1} dx$$

$$= \int_0^3 (16 - x^2 - 1)\, dx$$

$$= \int_0^3 15 - x^2\, dx$$

$$= \left(15x - \frac{x^3}{3}\right)\Big|_{x=0}^{x=3}$$

$$= 45 - 9 = \underline{\underline{36}}.$$

Ex 3 Compute $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x)\, dx$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x)\, dx = (-\cos(x))\Big|_{\frac{\pi}{2}}^{\frac{\pi}{2}} = 0.$$

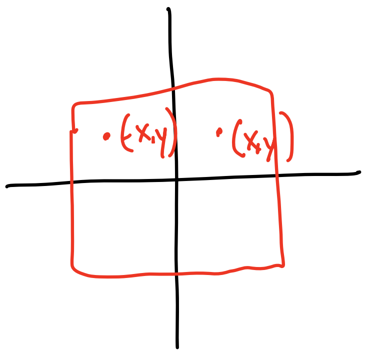
We can also argue by symmetry. Integrals gives a signed area, area above x-axis gives positive area and area below y-axis gives negative area.

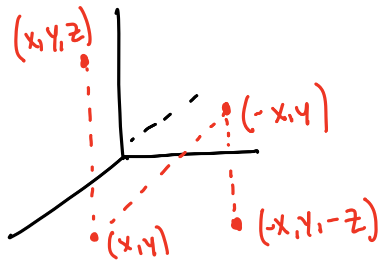
Check $\displaystyle -\int_{-\frac{\pi}{2}}^{0} \sin(x)\, dx = \int_0^{\frac{\pi}{2}} \sin(x)\, dx$

$$\begin{array}{ll} \| & \| \\ -\left(-\cos(0) + \cos\left(-\frac{\pi}{2}\right)\right) & (-\cos(x))\Big|_0^{\frac{\pi}{2}} \\ -(-1) & = \left(-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right) \\ = 1 & = 1. \end{array}$$

Ex 4. Compute $\iint_R xy^2\,dA$ where $R=[-1,1]\times[-1,1]$ by a symmetry arguement.

$$f(-x,y)=-f(x,y)$$





so box of height z will cancel with box of height "$-z$".

Verify by showing ① $-\int_{-1}^{0}\int_{-1}^{1} y^2x\,dy\,dx = $ ② $\int_{0}^{1}\int_{-1}^{1} xy^2\,dy\,dx$

①
$$-\int_{-1}^{0}\left(\frac{y^3}{3}x\right)\Big|_{y=-1}^{y=1}dx = -\int_{-1}^{0}\frac{1}{3}x-\left(\frac{-1}{3}x\right)dx = -\int_{-1}^{0}\frac{2}{3}x\,dx$$

$$= -\left(\frac{x^2}{3}\right)\Big|_{-1}^{x=0} = -\left(0-\frac{1}{3}\right)=\frac{1}{3}$$

✓

②
$$\int_{0}^{1}\int_{-1}^{1}xy^2\,dy\,dx=\int_{0}^{1}\left(\frac{y^3}{3}x\right)\Big|_{-1}^{1}dx=\int_{0}^{1}\frac{2}{3}x\,dx=\left(\frac{x^2}{3}\right)\Big|_{0}^{1}=\frac{1}{3}$$

Ex 5.
$$\int_{-a}^{a}\cos(x)\,dx = 2\int_{0}^{a}\cos(x)\,dx$$