## **MATH 54 PRACTICE MIDTERM 2**

Your Name		
Student ID		
Section number and leader		

Do not turn this page until you are instructed to do so.

Show all your work in this exam booklet. No material other than simple writing utensils may be used. In the event of an emergency or fire alarm leave your exam (closed) on your seat and meet with your GSI outside.

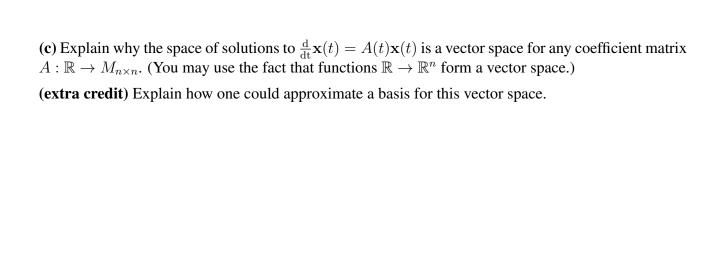
If you need to use the restroom, leave your exam with your GSI while out of the room.

Your grade is determined from all of the following 5 problems. Some extra credit problems are interspersed and can make up for up to 5 missed points within the same problem. However, only complete answers earn this credit, so check your other work before attempting these.

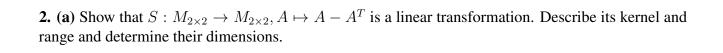
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**1. (a)** Consider the subspace  $H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 + 2x_2 + 3x_3 = 0 \right\}$  of  $\mathbb{R}^3$ . (Without reasoning) find the dimension and a basis of H.

(b) Explain which vector(s) to drop from the spanning set  $S = \{(t-1)^2, (t+1)^2, t^2-1, 2t\}$  of  $\mathbb{P}_2$ to obtain a basis.



(d) Show that  $t\cos t, t\sin t, \cos^2 t$  are linearly independent as functions on  $[0,\frac{\pi}{2}]$ . Hint: You can avoid calculation of the Wronskian by working from the definition and plugging in convenient values of t.



(b) What does the rank theorem assert about the kernel and range of a linear transformation  $\mathbb{R}^n \to \mathbb{R}^n$ ? (extra credit) Show that the same assertion is true for the linear transformation in (a).

(c) Define  $T: \mathbb{P}_2 \to \mathbb{R}^2$  by  $T(p) = \left[ \begin{array}{c} p'(0) \\ p''(0) \end{array} \right]$ . Find its matrix A with respect to the standard bases of  $\mathbb{P}_2$  and  $\mathbb{R}^2$ , and explain how Nul(A), Col(A) are related to the kernel and range of T.

(d) Given a linear transformation  $T: \mathbb{R}^2 \to \mathbb{R}^2$  and a basis  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$  of  $\mathbb{R}^2$  so that  $T(\mathbf{b}_1) = \mathbf{b}_1 - \mathbf{b}_2$  and  $T(\mathbf{b}_2) = 5\mathbf{b}_1$ , find the matrix for T in  $\mathcal{B}$ -coordinates,  $[T]_{\mathcal{B}}$ .

**3.** (a) Find a change-of-coordinates that diagonalizes  $A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$ .

(b) Explain how each of $Nul(A)$ , $Col(A)$ and $rank(A)$ changes under a row operation on a matrix $A$ .

4. (a) Find a basis for the kernel of the	differential operator	$L = (D^2 - 2D + 10)^2 (D^2 + 3)^2$	$)(D^2-D)^3.$

(b) Use the facts that  $r^3 - 3r^2 + 10r + 14$  has roots r = 1 and  $r = -2 \pm i\sqrt{10}$ , and

$$(D^3 - 3D^2 + 10D + 14) \left[ \ln(2+t) \right] = 14 \ln(2+t) + \frac{10t^2 + 43t + 48}{(2+t)^3},$$

to find the solution of

$$L[y] = y^{(3)} - 3y'' + 10y' + 14y = 14\ln(2+t) + \frac{10t^2 + 43t + 48}{(2+t)^3}, \qquad y(0) = \ln 2, \quad y'(0) = 1, \quad y''(0) = 4.$$

Write the result as explicit function of t and constants  $C_i$  together with a linear system that determines the  $C_i$ . Indicate the interval on which the solution is valid. Don't solve for the constants! Hint: To simplify derivatives, use the identity  $Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t = \text{Re}\big((A-iB)e^{(\alpha+i\beta)t}\big)$ . **5.** (a) Rewrite the following coupled differential equations for real functions x, y, z as first order system in standard form:

$$0 = e^{t} \frac{d}{dt}x + t^{2}(x - y) + 3(x - z)$$
$$0 = \frac{d}{dt}y + \cos t(x - \sin t)$$
$$0 = \frac{d}{dt}z + e^{2t}(z - y)$$

**(b)** Given a 
$$3 \times 3$$
 matrix  $A$  so that  $A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$  and  $A \begin{bmatrix} 0 \\ 1+i \\ i \end{bmatrix} = i \begin{bmatrix} 0 \\ 1+i \\ i \end{bmatrix}$ , find the general solution of the system  $\mathbf{x}' = A\mathbf{x}$ .

(c) Given the fundamental matrix  $X(t) = \begin{bmatrix} \frac{1}{2}\cos\pi t & 6e^t\sin\pi t \\ \frac{1}{3}\sin\pi t & 4e^t\cos\pi t \end{bmatrix}$  for a (time-dependent) system  $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ , find the solutions  $\mathbf{x}_1(t)$  and  $\mathbf{x}_2(t)$  with  $\mathbf{x}_1(2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  and  $\mathbf{x}_2(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .