# MATH 54 PRACTICE MIDTERM 2

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| Your Name<br><br>Student ID | |

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| Section number and leader | |

**Do not turn this page until you are instructed to do so.**

Show all your work in this exam booklet. No material other than simple writing utensils may be used. *In the event of an emergency or fire alarm leave your exam (closed) on your seat and meet with your GSI outside.*
If you need to use the restroom, leave your exam with your GSI while out of the room.

**Your grade is determined from all of the following 5 problems. Some extra credit problems are interspersed and can make up for up to 5 missed points within the same problem. However, only complete answers earn this credit, so check your other work before attempting these.**

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**1. (a)** Consider the subspace $H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, x_1 + 2x_2 + 3x_3 = 0 \right\}$ of $\mathbb{R}^3$.

(Without reasoning) find the dimension and a basis of $H$.

**(b)** Explain which vector(s) to drop from the spanning set $S = \{(t-1)^2, (t+1)^2, t^2 - 1, 2t\}$ of $\mathbb{P}_2$ to obtain a basis.

**(c)** Explain why the space of solutions to $\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{x}(t) = A(t)\mathbf{x}(t)$ is a vector space for any coefficient matrix $A : \mathbb{R} \to M_{n\times n}$. (You may use the fact that functions $\mathbb{R} \to \mathbb{R}^n$ form a vector space.)

**(extra credit)** Explain how one could approximate a basis for this vector space.

**(d)** Show that $t\cos t, t\sin t, \cos^2 t$ are linearly independent as functions on $[0, \frac{\pi}{2}]$. *Hint: You can avoid calculation of the Wronskian by working from the definition and plugging in convenient values of* $t$.

**2. (a)** Show that $S : M_{2\times 2} \to M_{2\times 2}, A \mapsto A - A^T$ is a linear transformation. Describe its kernel and range and determine their dimensions.

**(b)** What does the rank theorem assert about the kernel and range of a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$?

**(extra credit)** Show that the same assertion is true for the linear transformation in (a).

**(c)** Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(p) = \begin{bmatrix} p'(0) \\ p''(0) \end{bmatrix}$. Find its matrix $A$ with respect to the standard bases of $\mathbb{P}_2$ and $\mathbb{R}^2$, and explain how $Nul(A), Col(A)$ are related to the kernel and range of $T$.

**(d)** Given a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ and a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ of $\mathbb{R}^2$ so that $T(\mathbf{b}_1) = \mathbf{b}_1 - \mathbf{b}_2$ and $T(\mathbf{b}_2) = 5\mathbf{b}_1$, find the matrix for $T$ in $\mathcal{B}$-coordinates, $[T]_{\mathcal{B}}$.

**3. (a)** Find a change-of-coordinates that diagonalizes $A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$.

**(b)** Explain how each of $Nul(A)$, $Col(A)$ and $\mathrm{rank}(A)$ changes under a row operation on a matrix $A$.

**4. (a)** Find a basis for the kernel of the differential operator $L = (D^2 - 2D + 10)^2(D^2 + 3)(D^2 - D)^3$.

**(b)** Use the facts that $r^3 - 3r^2 + 10r + 14$ has roots $r = 1$ and $r = -2 \pm i\sqrt{10}$, and

$$\big(D^3 - 3D^2 + 10D + 14\big)\big[\ln(2+t)\big] = 14\ln(2+t) + \tfrac{10t^2+43t+48}{(2+t)^3},$$

to find the solution of

$$L[y] = y^{(3)} - 3y'' + 10y' + 14y = 14\ln(2+t) + \tfrac{10t^2+43t+48}{(2+t)^3}, \qquad y(0) = \ln 2, \quad y'(0) = 1, \quad y''(0) = 4.$$

Write the result as explicit function of $t$ and constants $C_i$ together with a linear system that determines the $C_i$. Indicate the interval on which the solution is valid. Don't solve for the constants!
*Hint: To simplify derivatives, use the identity* $Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t = \mathrm{Re}\big((A - iB)e^{(\alpha+i\beta)t}\big)$.

**5.** **(a)** Rewrite the following coupled differential equations for real functions $x, y, z$ as first order system in standard form:

$$\begin{aligned} 0 &= e^t \tfrac{\mathrm{d}}{\mathrm{dt}} x + t^2(x-y) + 3(x-z) \\ 0 &= \tfrac{\mathrm{d}}{\mathrm{dt}} y + \cos t(x - \sin t) \\ 0 &= \tfrac{\mathrm{d}}{\mathrm{dt}} z + e^{2t}(z-y) \end{aligned}$$

**(b)** Given a $3 \times 3$ matrix $A$ so that $A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $A \begin{bmatrix} 0 \\ 1+i \\ i \end{bmatrix} = i \begin{bmatrix} 0 \\ 1+i \\ i \end{bmatrix}$, find the general solution of the system $\mathbf{x}' = A\mathbf{x}$.

**(c)** Given the fundamental matrix $X(t) = \begin{bmatrix} \frac{1}{2}\cos \pi t & 6e^t \sin \pi t \\ \frac{1}{3}\sin \pi t & 4e^t \cos \pi t \end{bmatrix}$ for a (time-dependent) system $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$, find the solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ with $\mathbf{x}_1(2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.