1. definition

The category of δ-pairs is formed by pairs (A, I) where A is a δ-ring and $I \subset A$ is an ideal. Morphisms are given by relative ring maps that respect the δ-structure.

Recall that we had an equivalence between the categories of perfect $\mathbb{F}_p$-algebras and perfect and p-adically complete δ-rings. Significantly, this relates objects living in characteristic p with characteristic 0 objects. The category of prisms will generalize this relationship to all such "perfect" objects, i.e. the perfectoid rings.

Definition. A **prism** is a δ-pair (A, I) satisfying

(1) $I \subset A$ defines a Cartier divisor on $\operatorname{Spec} A$.
(2) A is derived (p, I)-complete.
(3) $p \in (I, \phi(I))$.

As before, the last statement is equivalent to saying I is Zariski locally generated by distinguished elements.

Definition. A map $(A, I) \to (B, J)$ of prisms is **faithfully flat** if the map $A \to B$ is (p, I)-completely flat, i.e., $A/(p, I) \to B \otimes_A^{\mathbb{L}} A/(p, I)$ is flat, meaning the cotangent complex has cohomology only in degree zero.

To set terminology, we'll call a prism (A, I)

(1) *perfect* if A is perfect.
(2) *crystalline* if $I = (p)$.
(3) *bounded* if A/I has bounded p^∞-torsion.

Everything we'll see will be bounded.

Example. Some examples.

(1) Any p-torsionfree and p-adically complete δ-ring A gives a bounded crystalline prism $(A, (p))$.
(2) As we will see, perfect prisms = perfectoid rings.

This category has a nice rigidity property:

Lemma 1 (Rigidity). *Let $(A, I) \to (B, J)$ be a map of prisms. Then $I \otimes_A B \to J$ is an isomorphism, so a map of prisms is determined on the underlying δ-rings.*

Proof. Both sides are locally generated by distinguished elements. By the irreducibility of distinguished elements, we can locally conclude on $\operatorname{Spec} B$. $\square$

2. perfect prisms

The most interesting class of prisms are the perfect prisms. We will show these are equivalent to the theory of perfectoid rings, and are in fact quite a bit more usable.

To begin, we give a somewhat simplified definition of a perfectoid ring.

Definition. A commutative ring R is **perfectoid** if it has the form A/I for a perfect prism (A, I). The category of perfectoid rings is the full subcategory of all commutative rings spanned by perfectoid rings.

What gives? It turns out this definition coincides with that of integral perfectoids (though not with perfectoid Tate rings).

EXAMPLE. Some perfectoid rings.

(1) Let A be a perfect and p-adically complete δ-ring, $I = (p)$. Then by the structure map of such rings, $A \cong W(R)$ for a perfect $\mathbb{F}_p$-algebra R. As $(W(R), (p))$ is a crystalline prism, we have that $R \cong A/I$ is a perfectoid. So the category of perfect $\mathbb{F}_p$-algebras is equivalent to the category of crystalline prisms.

(2) $A = \mathbb{Z}_p[t^{1/p^\infty}]^\wedge_{(p,t)}$ with $\phi(t^{1/p^n}) = t^{1/p^{n-1}}$ is a p-torsionfree δ-ring, and $I = (t - p)$ is generated by a distinguished element. Hence $R = A/I = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge_{(p)}$ is a perfectoid ring.

The main theorem of this section is

THEOREM (Perfect prisms=perfectoid rings). *The mapping $(A, I) \mapsto A/I$ defines an equivalence of categories between perfect prisms and perfectoid rings.*

To prove this we need two helper lemmas.

LEMMA 2. *Let R be a perfect $\mathbb{F}_p$-algebra. Let $f \in R$. Then $R[f^\infty] = R[f^{1/p^n}]$ for any n (hence is killed by a small power of f, so its an almost zero module).*

PROOF. Suppose $x \in R$ such that $f^m x = 0$ for some $m \geq 0$. Then $f^m x^{p^n} = 0$ for all $n \geq 0$. By reducedness, $f^{m/p^n} x = 0$ for any $n \geq 0$. $\square$

LEMMA 3. *Let (A, I) be a perfect prism. Then $A/I[p^\infty] = A/I[p]$. In particular, perfect prisms are bounded.*

Now we can prove the theorem.

PROOF. We need to be able to recover the prism (A, I) from $R = A/I$. We claim that $A \simeq W(R^\flat)$ where $R^\flat$ is Fontaine's tilt of R. As both sides are p-complete and perfect, it's enough to show that $A/p \simeq R^\flat$.

Let d be a generator of I. Then $R = A/(d)$ so that $R/p = A/(p, d)$. As A/p is a perfect $\mathbb{F}_p$-algebra, we can identify the tower

$$\cdots \to R/p \xrightarrow{\phi} R/p \xrightarrow{\phi} R/p$$

with the tower

$$\cdots \to A/(p, d^{p^2}) \xrightarrow{\text{can}} A/(p, d^p) \xrightarrow{\text{can}} A/(p, d)$$

Taking inverse limits we get

$$(A/p)^\wedge_{(d)} = R^\flat$$

But A/p is d-complete (by the first lemma). $\square$