## 1. the prismatic site

Fix a base prism $(A, I)$ as well as a formally smooth $A/I$-algebra $R$. The goal of prismatic cohomology is to produce a complex $\Delta_{R/A}$ of $A$-modules with a "Frobenius" endomorphism $\phi_{R/A}$ such that

- $\Delta_{R/A}/I$ is related to differential forms on $R$ relative to $A/I$.
- $\Delta_{R/A}[1/p]$ is related to the $p$-adic etale cohomology of $R[1/p]$.

This complex will give a deformation between the algebraic de Rham cohomology of $R/(A/I)$ and the $p$-adic etale cohomology of $R[1/p]$ parameterized by $\operatorname{Spec} A$. To simplify things, we'll assume $(A, I)$ is a bounded prism and $I = (d)$ is generated by a distinguished element.

Example. Examples of base prisms to consider.

(1) (crystalline) Let $A$ be any $p$-torsionfree, $p$-complete $\delta$-ring and $I = (p)$.
(2) (Breuil-Kisin-Fargues) Let $A = \mathbb{Z}_p[[u]]$ with $\phi(u) = u^p$, $I = E(u)$ for $E(u)$ any Eisenstein polynomial (e.g. $E(u) = u^p - p$).
(3) ($A_{\text{inf}}$) Let $R$ be a perfectoid ring and $(A, I) = (A_{\text{inf}}(R), \ker \theta_R)$ be our perfect base prism.

The prismatic cohomology of $R$ will be defined akin to Grothendieckian crystalline cohomology– by studying probes of $R$ with prisms over $(A, I)$. Formalizing this is the prismatic site.

Definition. The **prismatic site** $(R/A)_\Delta$ of $R$ relative to $A$ is the category with objects given by prisms $(B, IB)$ over $(A, I)$ together with a map $R \to B/IB$ over $A/I$. Diagrammatically, an object of this category looks like

$$\begin{tikzcd} A \arrow[rr] \arrow[d] & & B \arrow[d] \\ A/I \arrow[r] & R \arrow[r] & B/IB \end{tikzcd}$$

We'll shorthand an element as $(R \to B/IB \leftarrow B) \in (R/A)_\Delta$ and endow it with the indiscrete Grothendieck topology, so that all presheaves are automatically sheaves.

We get the standard prismatic structure sheaves $\mathcal{O}_\Delta$ and $\bar{\mathcal{O}}_\Delta$ by sending $(R \to B/IB \leftarrow B) \in (R/A)_\Delta$ to $B$ and $B/IB$ respectively.

Remark. Note that we should really be talking about the opposite of $(R/A)_\Delta$ as we're working in the affine case. But it doesn't matter, there isn't too much confusion. Apparently, gluing sheaves in the affine case is easier, though I don't immediately see why.

Example. Three examples of the prismatic site.

(1) Let $R = A/I$. Then $(R/A)_\Delta$ is just then given as the category of prisms over $(A, I)$, and hence has an initial object given by the initial prism $(R \simeq A/I \leftarrow A)$.
(2) Let $R = A/I\langle x \rangle$ be the $p$-adic completion of $A/I[x]$. Then in this case $(R/A)_\Delta$ has no initial object. But we would like to single out some special elements of the site. There is a formal smooth lift $\tilde{R}$ of $R/(A/I)$ with $\delta$-structure (given by $\tilde{R} = A[x]^{\wedge}_{(p,x)}$). This gives us an object $(R \simeq B/IB \leftarrow B)$ in $(R/A)_\Delta$.

(3) (perfect prismatic site) Let $(A, I)$ be a perfect prism. As perfect prisms $\simeq$ perfectoid rings, we see that any map $R \to S$ between perfectoids has a lift of the composite $A/I \to R \to S$ to a unique map $(A, I) \to (A_{\mathrm{inf}}(S), \ker\theta_S)$ of prisms.

In particular, the resulting diagram is an object in the prismatic site

$$(R \to S \leftarrow A_{\mathrm{inf}}(S)) \in (R/A)_{\Delta}$$

This construction gives us a fully-faithful functor from the category of perfectoid $R$-algebras to $(R/A)_{\Delta}$ where the essential image $(R/A)_{\Delta}^{\mathrm{perf}} \subset (R/A)_{\Delta}$ comprises those $(R \to B/IB \leftarrow B)$ where $(B, IB)$ is a perfect prism. We call $(R/A)_{\Delta}^{\mathrm{perf}}$ the **perfect prismatic site**.

As with any situation involving the site-theoretic language, defining cohomology is then immediate.

Definition (Prismatic cohomology). The **prismatic cohomology** of $R$ is defined to be the cohomology complex of the prismatic sheaf $\mathcal{O}_{\Delta}$,

$$\Delta_{R/A} = \mathrm{R}\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta}) \in \mathbf{D}(A)$$

The Frobenius action on $\mathcal{O}_{\Delta}$ gives a $\phi$-semilinear map $\Delta_{R/A} \to \Delta_{R/A}$.

The **Hodge-Tate cohomology** of $R$ is defined to be the cohomology of $\overline{\mathcal{O}}_{\Delta}$

$$\overline{\Delta}_{R/A} = \mathrm{R}\Gamma((R/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta}) \in \mathbf{D}(R)$$

Note that in this case we no longer have a Frobenius action.

They both are commutative algebra objects in $\mathbf{D}(A)$ and $\Delta_{R/A} \otimes_A^{\mathbb{L}} A/I \simeq \overline{\Delta}_{R/A}$.

When $R = A/I$, the prismatic site $(R/A)_{\Delta}$ has an initial object so that $\Delta_{R/A} \simeq A$ and $\overline{\Delta}_{R/A} \simeq A/I$. Why is it called the prismatic site? One can visualize the prismatic site as a "prism", where $\operatorname{Spec} R$ goes through $\operatorname{Spec} A/I$ and out comes different cohomology theories. Ehh.

## 2. hodge-tate comparison

Let $B \to C$ be a map of commutative rings. The algebraic de Rham complex

$$\Omega^*_{C/B} = (C \xrightarrow{d} \Omega^1_{C/B} \xrightarrow{d} \Omega^2_{C/B} \to \cdots)$$

is a graded commutative $B$-dga with the universal property as the initial such object. We will use this universal property to set up a comparison morphism between this and our Hodge-Tate cohomology.

Note that as $\overline{\Delta}_{R/A}$ is a commutative algebra object in $\mathbf{D}(\mathrm{R})$, the cohomology object $H^*(\overline{\Delta}_{R/A})$ is a graded commutative $R$-algebra. To produce the comparison from the universal property, we need to endow $H^*(\overline{\Delta}_{R/A})$ with a differential.

Recall that $\mathcal{O}_{\Delta} \otimes_A^{\mathbb{L}} A/I \simeq \overline{\mathcal{O}}_{\Delta}$. This is a $A/I$-module. For any such module $M$, we define the **Breuil-Kisin twist** via $M\{n\} = M \otimes_{A/I} (I/I^2)^{\otimes n}$ (note that $I/I^2$ is an invertible $A/I$-module so this definition makes sense for all $n \in \mathbb{Z}$). The short exact sequence of $A/I$-modules

$$0 \to \overline{\mathcal{O}}_{\Delta} \otimes_{A/I} I^{n+1}/I^{n+2} \to \overline{\mathcal{O}}_{\Delta} \otimes_{A/I} I^{n}/I^{n+2} \to \overline{\mathcal{O}}_{\Delta} \otimes_{A/I} I^{n}/I^{n+1} \to 0$$

gives rise to the Bockstein for the cohomology exact sequence

$$\beta_I : H^i(\overline{\Delta}_{R/A})\{i\} \to H^{i+1}(\overline{\Delta}_{R/A})\{i+1\}$$

It is a nuisance, but possible to check that the pair $(H^*(\overline{\Delta}_{R/A})\{*\}, \beta_I)$ becomes a graded commutative $A/I$-dga and hence gets a comparison map

$$\eta_R^* : (\Omega^*_{R/(A/I)}, d_{\mathrm{dR}}) \to (H^*(\overline{\Delta}_{R/A})\{*\}, \beta_I)$$

THEOREM (Hodge-Tate comparison theorem). *The above* ***Hodge-Tate comparison map*** *is an isomorphism of actual complexes.*

## 3. computing prismatic cohomology

To prove the HT comparison theorem, we should know how to compute the cohomology of categories. Since in the indiscrete topology all presheaves are sheaves, our lives are a little bit easier.

Let $\mathcal{C}$ be a small category, and let $\mathrm{PShv}(\mathcal{C})$ be the category of presheaves on $\mathcal{C}$ (the presheaf topos). Then $\mathrm{R}\Gamma(\mathcal{C}, -)$ is the derived functor $\mathbf{D}(\mathrm{Ab}(\mathrm{PShv}(\mathcal{C}))) \to \mathbf{D}(\mathrm{Ab})$ of

$$F \mapsto \varprojlim_{X \in \mathcal{C}} F(X)$$

In particular, if a final object existed in $\mathcal{C}$ it would be merely the cohomology of $F$ on that object.

Cech theory gives us a recipe for computing such cohomology via a cosimplicial complex:

LEMMA 1. *Let $\mathcal{C}$ be a small category admitting finite non-empty products. Let $F$ be an abelian presheaf on $\mathcal{C}$. Assume that there is a* ***weakly final*** *object $X \in \mathcal{C}$ such that* $\hom(Y, X) \neq \emptyset$ *for any $Y \in \mathcal{C}$. Then $\mathrm{R}\Gamma(\mathcal{C}, F)$ is computed by the cosimplicial complex*

$$F(X) \rightrightarrows F(X \times X) \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} F(X \times X \times X) \cdots$$

*which is $F$ applied to the simplicial Cech complex of $X$.*

To use the lemma to compute prismatic cohomology we need to find a weakly final object $X$ in $(R/A)_{\Delta}$ and to ensure that the prismatic site has finite non-empty coproducts (remember, our prismatic site is "opposite"). The analogous problem in crystalline cohomology is solved using divided-power envelopes. Here we need to create corresponding **prismatic envelopes** of $\delta$-pairs $(A, I)$.

LEMMA 2. *Let $(B, J)$ be a $\delta$-pair over $(A, I)$. Then there exists a universal map $(B, J) \to (C, IC)$ to a prism over $(A, I)$.*

Recall that to upgrade a $\delta$-pair into a prism, we need to turn the ideal into a Cartier divisor (relative to $(A, I)$). Inverting the generator of $I$ functorially as is done in the definition of the divided-power envelopes gives us our prismatic envelopes, which we denote by $C = B\{\frac{J}{I}\}^\wedge$ (here we think of the notation saying that $C$ is the universal prism where $J$ becomes divisible by $I$).

As a corollary, $(R/A)_{\Delta}$ has finite nonempty coproducts: given a pair of objects

$$(R \to B/IB \leftarrow B) \text{ and } (R \to C/IC \leftarrow C)$$

in $(R/A)_{\Delta}$, set $D_0 = B \otimes_A C$. There are two natural maps $R \to D_0/ID_0$ given by factoring through $B/IB$ and $C/IC$ respectively. These may not be the same map. To alleviate this, let

$$J = \ker(D_0 \to B/IB \otimes_{A/I} C/IC \to B/IB \otimes_R C/IC)$$

This is the ideal generated by $x \otimes 1 - 1 \otimes y$ where there exists $z \in R$ such that it reduces to $x$ in $B/IB$ and $y$ in $C/IC$.

The prismatic envelope of the $\delta$-pair $(D_0, J)$ is the prism $(D, ID) = D_0 \left\{\frac{J}{I}\right\}^\wedge$ over $(A, I)$. The induced maps $R \to D/ID$ coincide, so we get an object $(R \to D/ID \leftarrow D)$ in the prismatic site $(R/A)_{\Delta}$. This is the desired coproduct object.

**TODO:**[put in rest later]

## 4. derived prismatic cohomology

Our next goal is to use Quillen's formalism of non-abelian derived functors to extend the prismatic cohomology to (not necessarily smooth) $p$-adically complete $A/I$-algebras $R$.

Let $A$ be a commutative ring, and $\mathrm{CAlg}_A$ the category of commutative $A$-algebras. Let $\mathrm{Poly}_A \subset \mathrm{CAlg}_A$ be the full subcategory spanned by polynomial $A$-algebras in finitely many variables. Since any finitely-generated polynomial ring is a projective object in $\mathrm{CAlg}_A$, we can try to "derive" functors on $\mathrm{Poly}_A$ to functors on all of $\mathrm{CAlg}_A$ by applying our functors to "resolutions".

Consider a functor $F : \mathrm{Poly}_A \to \mathrm{Ab} \subset \mathbf{D}(\mathrm{Ab})$ where $\mathbf{D}(\mathrm{Ab})$ is the derived $\infty$-category of abelian groups. This admits all limits and colimits, and so we can perform left Kan extension along the inclusion $\mathrm{Poly}_A \subset \mathrm{CAlg}_A$.

Proposition. *There exists a unique extension $LF : \mathrm{CAlg}_A \to \mathbf{D}(\mathrm{Ab})$ such that*

1. *$LF$ commutes with filtered colimits.*
2. *$LF$ commutes with geometric realizations, i.e. if $P_\bullet \to B$ is a simplicial resolution of $B$ in $\mathrm{CAlg}_A$, then $|LF(P_\bullet)| \simeq LF(B)$.*

We call $LF$ the **left derived functor** of $F$. A historical reason why this construction turned up is in deriving the functor of Kahler differentials.

Definition. The **cotangent complex** $\mathbb{L}_{-/A} : \mathrm{CAlg}_A \to \mathbf{D}(\mathrm{Ab})$ is the left derived functor of $\mathrm{Poly}_A \to \mathrm{Mod}_A \subset \mathbf{D}(\mathrm{Ab})$ given by $B \mapsto \Omega^1_{B/A}$.

Example. The cotangent complex has immense importance in deformation theory and algebraic geometry. We'll highlight a useful example.

Suppose $A \to B$ is a map of perfect $\mathbb{F}_p$-algebras. In particular the relative Frobenius is simultaneously an isomorphism and zero on the level of cotangent complexes. It follows that the cotangent complex vanishes, $\mathbb{L}_{B/A} \simeq 0$. Now let $C \to D$ be a map of perfectoid rings. In this world, we care about phenomena after derived $p$-completion, so we ask what the derived $p$-completion of $\mathbb{L}_{D/C}$ is.

But by derived Nakayama, it suffices to understand $\mathbb{L}_{D/C} \widehat{\otimes}_{\mathbb{Z}} \mathbb{Z}/p$, which is 0 by the previous claim. Hence for perfectoid rings, the derived $p$-completion of $\mathbb{L}_{D/C}$ vanishes.

Now let $k$ be a base ring of characteristic $p$.

Definition. The **derived de Rham cohomology** functor $\mathrm{dR}_{-/k} : \mathrm{CAlg}_k \to \mathbf{D}(k)$ is the left derived functor of $\mathrm{Poly}_k \to \mathbf{D}(k)$ given by $R \mapsto \Omega^*_{R/k}$.

Let $\mathrm{Fun}(\mathbb{N}, \mathbf{D}(k))$ be the filtered derived $\infty$-category of diagrams of the form $\{F_n\} = \{F_0 \to F_1 \to F_2 \to \cdots\}$. We call the functor sending $\{F_n\}$ to $F_\infty = \mathrm{colim}_n F_n$ the "underlying object" functor.

Proposition (Derived Cartier isomorphism). *Derived de Rham cohomology* $\mathrm{dR} : \mathrm{CAlg}_k \to \mathbf{D}(\mathrm{k})$ *admits a lift to the filtered derived $\infty$-category, given by the conjugate filtration* $\mathrm{Fil}^{conj}$

$$\begin{tikzcd} & \mathrm{Fun}(\mathbb{N}, \mathbf{D}(k)) \arrow[d] \\ \mathrm{CAlg}_k \arrow[ur, dashed, "\mathrm{Fil}^{\mathrm{conj}}"] \arrow[r, "\mathrm{dR}_{-/k}"'] & \mathbf{D}(k) \end{tikzcd}$$

*with canonical identifications of the graded pieces* $\mathrm{gr}^i_{\mathrm{conj}} \mathrm{dR}_{R/k} \simeq \wedge^i \mathbb{L}_{R^{(1)}/k}[-i]$.

Proof. For smooth (or even more elementarily, polynomial) algebras, the de Rham complex $\Omega^*_{R/k}$ has a canonical filtration (the **conjugate filtration**) given by $\mathrm{Fil}^{\mathrm{conj}}_i \Omega^*_{R/k} = \tau^{\leq i}\Omega^*_{R/k}$. Note that this is an increasing filtration given by

$$\mathrm{Fil}^{\mathrm{conj}}_i \Omega^*_{R/k} = (R \to \Omega^1_{R/k} \to \cdots \to \Omega^{i-1}_{R/k} \to \ker(d : \Omega^i_{R/k} \to \Omega^{i+1}_{R/k}) \to 0 \to \cdots)$$

The non-derived Cartier isomorphism gives us natural isomorphisms $\mathrm{gr}^i_{\mathrm{conj}} \simeq \Omega^i_{R^{(1)}/k}[-i]$ for all $i \geq 0$. This gives a lift of the de Rham complex functor to $\mathrm{Poly}_k \to \mathrm{Fun}(\mathbb{N}, \mathbf{D}(k))$. As the functors involved are colimit-preserving, passing to left derived functors give the proposition. $\square$

As a corollary, by the natural isomorphism on graded parts we see that for smooth $k$-algebras $R$, the derived de Rham complex is classical, $\mathrm{dR}_{R/k} \simeq \Omega^*_{R/k}$.

Example. (Regular semiperfect rings) Let $k$ be a perfect ring, $S$ a $k$-algebra of the form $R/I$ where $R$ is a perfect $k$-algebra and $I \subset R$ is an ideal generated by a regular sequence. Such rings are called **regular semiperfect**. The point is that these rings are not smooth, but still have nice properties in the *derived world*.

By the transitivity triangle for $k \to R \to S$, we have $\mathbb{L}_{S/k} \simeq \mathbb{L}_{S/R}$ (since by perfection, $\mathbb{L}_{R/k} \simeq 0$). Since $I$ is generated by a regular sequence, it's cotangent complex is simple to describe: $\mathbb{L}_{S/R} \simeq I/I^2[1]$. By the Quillen shift formula, $\wedge^i \mathbb{L}_{S/R} \simeq \Gamma^i_R(I/I^2)[i]$. In particular, $\wedge^i \mathbb{L}_{S/R}[-i]$ is concentrated in degree 0.

Remark. In characteristic 0, there is no Cartier isomorphism and so the above filtration doesn't give us anything. In particular, for $k$ of characteristic 0, the Poincare lemma states for that for any polynomial $k$-algebra $R$, $k \simeq \Omega^*_{R/k}$, and so $k \simeq \mathrm{dR}_{R/k}$. To fix this in characteristic 0, we must "force the Hodge filtration to converge", i.e. we complete the derived de Rham complex with respect to the Hodge filtration.

Now we can turn our attention towards derived prismatic cohomology. Let $(A, I)$ be a bounded prism. Let $R$ be a formally smooth $A/I$-algebra. Let $\mathbf{D}_{\mathrm{comp}}(A)$ be the category of $(p, I)$-complete commutative algebra objects in $\mathbf{D}(A)$.

Definition (Derived prismatic cohomology). The **derived prismatic cohomology** functor

$$L\,\Delta_{-/A} : \mathrm{CAlg}_{A/I} \to \mathbf{D}_{\mathrm{comp}}(A)$$

is the left derived functor of $\mathrm{Poly}_{A/I} \to \mathbf{D}_{\mathrm{comp}}(A)$ given by $R \to \Delta_{\widehat{R}/A}$, where $\widehat{R}$ is the $p$-adic completion of $R$.

Similarly we have the **derived Hodge-Tate cohomology** $L\overline{\Delta}_{R/A} = L\,\Delta_{R/A} \otimes^{\mathbb{L}}_A A/I$.

The main point of the derived prismatic cohomology is that we get a derived version of the Hodge-Tate comparison.

Proposition. *For any $R \in \mathrm{CAlg}_{A/I}$, we have an increasing exhaustive filtration* $\mathrm{Fil}_*^{\mathrm{HT}}$ *on $L\overline{\Delta}_{R/A}$ such that*

$$\mathrm{gr}_i^{\mathrm{HT}}(L\overline{\Delta}_{R/A}) \simeq \wedge^i \mathbb{L}_{R/(A/I)}[-i]$$

Analogously as for the derived de Rham complex, we have an extension of the prismatic theory that works from (formally) smooth algebras to a larger class of semiperfectoids.

Example. Let $(A,(d))$ be a perfect prism, $S$ an $A/(d)$-algebra of the form $R/J$ for $R$ a perfectoid $A/(d)$-algebra and $J \subset R$ an ideal generated by a regular sequence. Such rings are called **regular semiperfectoid**. By the Hodge-Tate comparison, the graded pieces are described easily as the cotangent complex works well with such quotients. It follows that $\overline{\Delta}_{S/A}$ is concentrated in degree 0 and is given by a $p$-completely flat $A$-algebra.