

This is a set of notes on the theory of  $\delta$ -rings. Fix a prime  $p$  for the rest of the note. The running slogan is that  $\delta$ -rings are "rings with a lift of Frobenius modulo  $p$ ". In fact, if we take this slogan literally, it becomes a definition. Note that if  $A$  is a commutative ring equipped with a map  $\phi : A \rightarrow A$  that is a lift of Frobenius on  $A/p$ , then for each  $x \in A$  we have

$$\phi(x) = x^p + p\delta(x)$$

for some map of sets  $\delta : A \rightarrow A$ . If  $A$  is  $p$ -torsion free, then  $\delta$  is *uniquely* determined by this formula. The definition relations for  $\delta(-)$  then come from the relations encoding the fact that  $\phi(-)$  is a ring homomorphism.

**DEFINITION.** A  $\delta$ -**ring** is a pair  $(A, \delta)$  where  $A$  is a commutative ring,  $\delta : A \rightarrow A$  a map of sets with  $\delta(0) = \delta(1) = 0$  and satisfying the identities

$$\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x + y)^p}{p}$$

and

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$$

In the literature, a  $\delta$ -structure is sometimes called a  $p$ -derivation (especially by those who like  $p$ -adic differential equations). In computations it is often useful to write the multiplicative relation for  $\delta$  in the asymmetric form

$$\delta(xy) = \phi(x)\delta(y) + y^p\delta(x)$$

A cute property of the Frobenius lift is that it is a homomorphism of  $\delta$ -rings.

**LEMMA 1.** *Let  $A$  be a  $\delta$ -ring. Then  $\phi : A \rightarrow A$  is a  $\delta$ -map, that is,  $\phi(\delta(x)) = \delta(\phi(x))$  for all  $x \in A$ .*

**PROOF.** Suppose  $A$  is  $p$ -torsion free. Then we have  $\delta(x) = \frac{1}{p}(\phi(x) - x^p)$ . We can then verify directly that this is the case. Expanding,

$$\phi(\delta(x)) = \frac{1}{p}(\phi^2(x) - \phi(x)^p) = \delta(\phi(x))$$

When  $A$  has  $p$ -torsion, we can lift this argument to a  $p$ -torsionfree  $\delta$ -ring using free functors.  $\square$

A  $p$ -torsionfree ring with a lift of Frobenius  $\phi$  furnishes an example of a  $\delta$ -ring. Since we have plenty of examples of this, we get plenty of examples of  $\delta$ -rings.

**EXAMPLE.** Some examples of  $\delta$ -rings.

- (1)  $\mathbb{Z}$  with  $\phi = \text{id}$ . Here,  $\delta$  is pretty explicit, given by  $\delta(n) = \frac{1}{p}(n - n^p)$ . This is the initial object in the category of  $\delta$ -rings.
- (2) The polynomial ring  $\mathbb{Z}[x]$  has Frobenius lift  $\phi$  determined by  $\phi(x) = x^p + pg(x)$  for any  $g(x) \in \mathbb{Z}[x]$ .
- (3) For any perfect field  $k$  of characteristic  $p > 0$ , the ring of Witt vectors  $W(k)$  with  $\phi$  given by the standard (unique!) Frobenius lift. Since there is only one such lift,  $W(k)$  only admits a  $\delta$ -structure in one way. Geometrically, this corresponds to the  $p$ -cotangent space of  $W(k)$  being trivial.

In an obvious way,  $\delta$ -rings form a category, which we'll call  $\text{Ring}^\delta$ . Here the  $p$  is implicit, and hence suppressed. A thing we would like is to understand whether  $\text{Ring}^\delta$  has any nice categorical properties. It is cumbersome to prove them with all these relations floating about, so we want a more principled way to talk about  $\delta$ -rings.

**PROPOSITION.** *A  $\delta$ -structure on a ring  $A$  is the same as a ring section  $w : A \rightarrow W_2(A)$  of the map  $e : W_2(A) \rightarrow A$  that forgets the second component.*

**PROOF.** We can write the truncated Witt vector  $W_2(A)$  as the fiber product of

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow \text{can} & & \\ A & \xrightarrow{\text{can}} & A/p & \xrightarrow{\phi} & A/p \end{array}$$

Then it becomes clear that a ring map  $A \rightarrow W_2(A)$  that splits the projection of  $W_2(A)$  to  $A$  is the same data as a Frobenius lift on  $A$ .  $\square$

The functoriality of  $W_2(-)$  shows that  $\text{Ring}^\delta$  is a bicomplete category, i.e., has all limit and colimits. Combining this with the adjoint functor theorem gives the useful fact that the forgetful functor  $\text{Ring}^\delta \rightarrow \text{Ring}$  has both a left and right functor— the left adjoint is the **free  $\delta$ -ring construction**, and the right adjoint is given by the Witt vector construction.

We give a description of the free  $\delta$ -ring on a single variable:  $\mathbb{Z}\{x\}$  is the polynomial ring  $\mathbb{Z}[x_0, x_1, x_2, \dots]$  with  $x = x_0$  and  $\delta(x_i) = x_{i+1}$ . In particular, the free  $\delta$ -rings are  $p$ -torsion free. This, combined with the existence of all colimits give us a way to build  $\delta$ -rings using generators and relations. It's a good category indeed.

**0.1. some ring-theoretic properties.**  $\delta$ -rings are stable under some natural ring-theoretic operations like localization and quotients.

**LEMMA 2** (Localizations of  $\delta$ -rings). *Let  $A$  be a  $\delta$ -ring and  $S \subset A$  a multiplicative subset stable under the Frobenius lift. Then there is a unique  $\delta$ -structure on the localization  $S^{-1}A$  extending the one on  $A$ .*

**PROOF.** **TODO:**[prove this]  $\square$

**LEMMA 3** (Completions of  $\delta$ -rings). *Let  $A$  be a  $\delta$ -ring and  $I \subset A$  a finitely generated ideal that contains  $p$ . Then there is a unique  $\delta$ -structure on the  $I$ -adic completion  $A_I^\wedge$ .*

**PROOF.** **TODO:**[prove this]  $\square$

As a consequence,  $\mathbb{Z}_p$  with the  $\delta$ -structure given by the identity  $\phi = \text{id}$  is the initial object in the category of  $p$ -adically complete  $\delta$ -rings.

**LEMMA 4** (Etale extensions of  $\delta$ -rings). *Fix a map  $A \rightarrow B$  of  $p$ -adically complete and  $p$ -torsion free rings, where  $A$  is a  $\delta$ -ring. Suppose  $A \rightarrow B$  is etale modulo  $p$ . Then  $B$  has a unique  $\delta$ -structure compatible with the one on  $A$ .*

**PROOF.** By  $p$ -torsionfreeness of the two rings, we just need to exhibit a Frobenius lift on  $B$  compatible with the one on  $A$ . By  $p$ -adic completeness, we just need to exhibit this Frobenius lift modulo  $p^n$  for all  $n \geq 1$ .

But then we do this by induction. When  $n = 1$ , this is given by the pushout of the Frobenius on  $A$ , giving the relative Frobenius on  $B$ . For  $n > 1$ , we appeal to the topological invariance of the etale site.  $\square$

LEMMA 5 (Quotients of  $\delta$ -rings). *Let  $A$  be a  $\delta$ -ring and  $I \subset A$  an ideal such that  $\delta(I) \subset I$  (stable under  $\delta$ ). Then there is a unique  $\delta$ -structure on the quotient  $A/I$  compatible with the one on  $A$ .*

PROOF. Suffices to show that for  $x \in A$  and  $c \in I$ ,  $\delta(x + c) \equiv \delta(x) \pmod{I}$ . But this follows from the additivity relation that  $\delta$ -rings must satisfy.  $\square$

**0.2. perfect  $\delta$ -rings.** To get us closer to the theory of prismatic cohomology, we will study an important class of  $\delta$ -rings— the perfect ones.

Just like for regular rings, we'll say a  $\delta$ -ring  $A$  is **perfect** if the Frobenius  $\phi : A \rightarrow A$  is an isomorphism. Perfect rings have incredible algebraic properties, but arguably the genesis of these traits come from the following derived fact:

THEOREM. *Let  $A$  be a perfect  $\mathbb{F}_p$ -algebra. Then the cotangent complex  $L_{A/\mathbb{F}_p}$  vanishes.*

PROOF. Note that in characteristic  $p > 0$ , the derivative of the Frobenius on a polynomial ring is given by

$$d\phi(x) = d(x^p) = px^{p-1}dx = 0$$

The Frobenius is functorial, so taking a functorial simplicial resolution of  $A$  by polynomial  $\mathbb{F}_p$ -algebras gives that the Frobenius induces the 0 map on the cotangent complex. But since it is also simultaneously an isomorphism by perfection, the cotangent complex must vanish  $L_{A/\mathbb{F}_p} \simeq 0$ .  $\square$

REMARK. As the deformation theory of  $A$  is controlled by its cotangent complex, the above result says that there are no obstructions to lifting over perfect  $\mathbb{F}_p$ -algebras. As a consequence, we get the equivalence of the following categories:

- (1) Perfect  $\mathbb{F}_p$ -algebras.
- (2) The category of flat  $\mathbb{Z}/p^n$ -algebras  $\bar{A}$  with  $\bar{A}/p$  perfect.
- (3)  $p$ -adically complete and  $p$ -torsionfree  $\mathbb{Z}_p$ -algebras  $\bar{A}$  with  $\bar{A}/p$  perfect.

The equivalence of (1) and (2) follow from deformation theory, and that of (2) and (3) follow from the characterization of the category of  $p$ -adically complete and  $p$ -torsionfree  $\mathbb{Z}_p$ -algebras as the inverse limit of the categories of flat  $\mathbb{Z}/p^n$ -algebras.

To relate this to perfect  $\delta$ -rings, we use the following algebraic fact.

LEMMA 6. *Let  $A$  be a  $\delta$ -ring and let  $x \in A$  with  $px = 0$ . Then  $\phi(x) = 0$ . In particular, if  $\phi$  is injective, then  $A$  is  $p$ -torsionfree.*

PROOF. In  $A[1/p]$ , we would have trivially  $x = 0$  so  $\phi(x) = 0$ , so we assume that  $A$  is a  $\mathbb{Z}_{(p)}$ -algebra. Expanding  $\delta(px)$  we have

$$0 = \delta(px) = p^p \delta(x) + \phi(x) \delta(p)$$

Note that in  $\mathbb{Z}_{(p)}$ , we see that

$$\delta(p) = \frac{\phi(p) - p^p}{p} = \frac{p - p^p}{p} = 1 - p^{p-1} \in \mathbb{Z}_{(p)}^*$$

is a unit, so it suffices to show that  $p^p \delta(x) = 0$ . But

$$p^p \delta(x) = p^{p-1} \cdot p \delta(x) = p^{p-1} (\phi(x) - x^p) = p^{p-2} (\phi(px) - px \cdot x^{p-1}) = 0$$

where we used that  $px = 0$  in the end.  $\square$

This leads to a classification of perfect  $\delta$ -rings: they just come from perfect  $\mathbb{F}_p$ -algebras!

**THEOREM.** *The category of perfect  $\delta$ -rings that are  $p$ -adically complete is equivalent to the category of perfect  $\mathbb{F}_p$ -algebras.*

**PROOF.** The above lemma shows that perfect  $\delta$ -rings are always  $p$ -torsionfree, and so we have a forgetful functor from the category of perfect  $p$ -adically complete  $\delta$ -rings to the category of  $p$ -adically complete and  $p$ -torsionfree  $\mathbb{Z}_p$ -algebras  $\bar{A}$  with  $\bar{A}/p$  perfect. By the above remark, this is equivalent to the category of perfect  $\mathbb{F}_p$ -algebras. To go back to perfect  $\delta$ -rings, fix a perfect  $\mathbb{F}_p$  algebra, and by deformation theory lift it to a  $p$ -adically complete and  $p$ -torsionfree ring. As the cotangent complex vanishes, such a lift is unique and is equipped with a unique lift of Frobenius. By  $p$ -torsionfreeness of the lift, this defines the required  $\delta$ -structure.  $\square$

**REMARK.** We can be explicit about the equivalence above. One functor is given by modulo  $p$ ,  $A \mapsto A/p$ , while the other is given by the Witt vector construction  $A \mapsto W(A)$ . In other words, every  $p$ -adically complete perfect  $\delta$ -ring has the form  $W(R)$  for some perfect  $\mathbb{F}_p$ -algebra  $R$ .

### 0.3. distinguished elements.