This is a set of notes on the theory of δ-rings. Fix a prime p for the rest of the note. The running slogan is that δ-rings are "rings with a lift of Frobenius modulo p". In fact, if we take this slogan literally, it becomes a definition. Note that if A is a commutative ring equipped with a map $\phi : A \to A$ that is a lift of Frobenius on A/p, then for each $x \in A$ we have

$$\phi(x) = x^p + p\delta(x)$$

for some map of sets $\delta : A \to A$. If A is p-torsion free, then δ is *uniquely* determined by this formula. The definition relations for $\delta(-)$ then come from the relations encoding the fact that $\phi(-)$ is a ring homomorphism.

Definition. A δ-**ring** is a pair (A, δ) where A is a commutative ring, $\delta : A \to A$ a map of sets with $\delta(0) = \delta(1) = 0$ and satisfying the identities

$$\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x + y)^p}{p}$$

and

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$$