This is a set of notes on the theory of δ-rings. Fix a prime p for the rest of the note. The running slogan is that δ-rings are "rings with a lift of Frobenius modulo p". In fact, if we take this slogan literally, it becomes a definition. Note that if A is a commutative ring equipped with a map $\phi : A \to A$ that is a lift of Frobenius on A/p, then for each $x \in A$ we have

$$\phi(x) = x^p + p\delta(x)$$

for some map of sets $\delta : A \to A$. If A is p-torsion free, then δ is *uniquely* determined by this formula. The definition relations for $\delta(-)$ then come from the relations encoding the fact that $\phi(-)$ is a ring homomorphism.

Definition. A δ-**ring** is a pair (A, δ) where A is a commutative ring, $\delta : A \to A$ a map of sets with $\delta(0) = \delta(1) = 0$ and satisfying the identities

$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}$$

and

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$$

In the literature, a δ-structure is sometimes called a p-derivation (especially by those who like p-adic differential equations). In computations it is often useful to write the multiplicative relation for δ in the asymmetric form

$$\delta(xy) = \phi(x)\delta(y) + y^p\delta(x)$$

A cute property of the Frobenius lift is that it is a homomorphism of δ-rings.

Lemma 1. *Let A be a δ-ring. Then $\phi : A \to A$ is a δ-map, that is, $\phi(\delta(x)) = \delta(\phi(x))$ for all $x \in A$.*

Proof. Suppose A is p-torsion free. Then we have $\delta(x) = \frac{1}{p}(\phi(x) - x^p)$. We can then verify directly that this is the case. Expanding,

$$\phi(\delta(x)) = \frac{1}{p}(\phi^2(x) - \phi(x)^p) = \delta(\phi(x))$$

When A has p-torsion, we can lift this argument to a p-torsionfree δ-ring using free functors. □

A p-torsionfree ring with a lift of Frobenius ϕ furnishes an example of a δ-ring. Since we have plenty of examples of this, we get plenty of examples of δ-rings.

Example. Some examples of δ-rings.

(1) $\mathbb{Z}$ with $\phi = \mathrm{id}$. Here, δ is pretty explicit, given by $\delta(n) = \frac{1}{p}(n - n^p)$. This is the initial object in the category of δ-rings.
(2) The polynomial ring $\mathbb{Z}[x]$ has Frobenius lift ϕ determined by $\phi(x) = x^p + pg(x)$ for any $g(x) \in \mathbb{Z}[x]$.
(3) For any perfect field k of characteristic $p > 0$, the ring of Witt vectors $W(k)$ with ϕ given by the standard (unique!) Frobenius lift. Since there is only one such lift, $W(k)$ only admits a δ-structure in one way. Geometrically, this corresponds to the p-cotangent space of $W(k)$ being trivial.

In an obvious way, δ-rings form a category, which we'll call Ring^δ. Here the p is implicit, and hence suppressed. A thing we would like is to understand whether Ring^δ has any nice categorical properties. It is cumbersome to prove them with all these relations floating about, so we want a more principled way to talk about δ-rings.

PROPOSITION. *A δ-structure on a ring A is the same as a ring section $w : A \to W_2(A)$ of the map $e : W_2(A) \to A$ that forgets the second component.*

PROOF. We can write the truncated Witt vector $W_2(A)$ as the fiber product of

$$\begin{tikzcd} & & A \arrow[d, "can"] \\ A \arrow[r, "can"] & A/p \arrow[r, "\phi"] & A/p \end{tikzcd}$$

Then it becomes clear that a ring map $A \to W_2(A)$ that splits the projection of $W_2(A)$ to A is the same data as a Frobenius lift on A. $\square$

The functoriality of $W_2(-)$ shows that Ring^δ is a bicomplete category, i.e., has all limit and colimits. Combining this with the adjoint functor theorem gives the useful fact that the forgetful functor $\mathrm{Ring}^\delta \to \mathrm{Ring}$ has both a left and right functor– the left adjoint is the **free δ-ring construction**, and the right adjoint is given by the Witt vector construction.

We give a description of the free δ-ring on a single variable: $\mathbb{Z}\{x\}$ is the polynomial ring $\mathbb{Z}[x_0, x_1, x_2, ...]$ with $x = x_0$ and $\delta(x_i) = x_{i+1}$. In particular, the free δ-rings are p-torsion free. This, combined with the existence of all colimits give us a way to build δ-rings using generators and relations. It's a good category indeed.

0.1. some ring-theoretic properties. δ-rings are stable under some natural ring-theoretic operations like localization and quotients.

LEMMA 2 (Localizations of δ-rings). *Let A be a δ-ring and $S \subset A$ a multiplicative subset stable under the Frobenius lift. Then there is a unique δ-structure on the localization $S^{-1}A$ extending the one on A.*

PROOF. **TODO:**[prove this] $\square$

LEMMA 3 (Completions of δ-rings). *Let A be a δ-ring and $I \subset A$ a finitely generated ideal that contains p. Then there is a unique δ-structure on the I-adic completion $A^\wedge_I$.*

PROOF. **TODO:**[prove this] $\square$

As a consequence, $\mathbb{Z}_p$ with the δ-structure given by the identity $\phi = \mathrm{id}$ is the initial object in the category of p-adically complete δ-rings.

LEMMA 4 (Etale extensions of δ-rings). *Fix a map $A \to B$ of p-adically complete and p-torsion free rings, where A is a δ-ring. Suppose $A \to B$ is etale modulo p. Then B has a unique δ-structure compatible with the one on A.*

PROOF. By p-torsionfreeness of the two rings, we just need to exhibit a Frobenius lift on B compatible with the one on A. By p-adic completeness, we just need to exhibit this Frobenius lift modulo p^n for all $n \geq 1$.

But then we do this by induction. When $n = 1$, this is given by the pushout of the Frobenius on A, giving the relative Frobenius on B. For $n > 1$, we appeal to the topological invariance of the etale site. $\square$

LEMMA 5 (Quotients of δ-rings). *Let A be a δ-ring and $I \subset A$ an ideal such that $\delta(I) \subset I$ (stable under δ). Then there is a unique δ-structure on the quotient A/I compatible with the one on A.*

PROOF. Suffices to show that for $x \in A$ and $c \in I$, $\delta(x+c) \equiv \delta(x) \mod I$. But this follows from the additivity relation that δ-rings must satisfy. $\square$

0.2. perfect δ-rings. To get us closer to the theory of prismatic cohomology, we will study an important class of δ-rings– the perfect ones.

Just like for regular rings, we'll say a δ-ring A is **perfect** if the Frobenius $\phi : A \to A$ is an isomorphism. Perfect rings have incredible algebraic properties, but arguably the genesis of these traits come from the following derived fact:

THEOREM. *Let A be a perfect $\mathbb{F}_p$-algebra. Then the cotangent complex $L_{A/\mathbb{F}_p}$ vanishes.*

PROOF. Note that in characteristic $p > 0$, the derivative of the Frobenius on a polynomial ring is given by

$$d\phi(x) = d(x^p) = px^{p-1}dx = 0$$

The Frobenius is functorial, so taking a functorial simplicial resolution of A by polynomial $\mathbb{F}_p$-algebras gives that the Frobenius induces the 0 map on the cotangent complex. But since it is also simultaneously an isomorphism by perfection, the cotangent complex must vanish $L_{A/\mathbb{F}_p} \simeq 0$. $\square$

REMARK. As the deformation theory of A is controlled by its cotangent complex, the above result says that there are no obstructions to lifting over perfect $\mathbb{F}_p$-algebras. As a consequence, we get the equivalence of the following categories:

(1) Perfect $\mathbb{F}_p$-algebras.
(2) The category of flat $\mathbb{Z}/p^n$-algebras $\bar{A}$ with $\bar{A}/p$ perfect.
(3) p-adically complete and p-torsionfree $\mathbb{Z}_p$-algebras $\bar{A}$ with $\bar{A}/p$ perfect.

The equivalence of (1) and (2) follow from deformation theory, and that of (2) and (3) follow from the characterization of the category of p-adically complete and p-torsionfree $\mathbb{Z}_p$-algebras as the inverse limit of the categories of flat $\mathbb{Z}/p^n$-algebras.

To relate this to perfect δ-rings, we use the following algebraic fact.

LEMMA 6. *Let A be a δ-ring and let $x \in A$ with $px = 0$. Then $\phi(x) = 0$. In particular, if ϕ is injective, then A is p-torsionfree.*

PROOF. In $A[1/p]$, we would have trivially $x = 0$ so $\phi(x) = 0$, so we assume that A is a $\mathbb{Z}_{(p)}$-algebra. Expanding $\delta(px)$ we have

$$0 = \delta(px) = p^p\delta(x) + \phi(x)\delta(p)$$

Note that in $\mathbb{Z}_{(p)}$, we see that

$$\delta(p) = \frac{\phi(p) - p^p}{p} = \frac{p - p^p}{p} = 1 - p^{p-1} \in \mathbb{Z}_{(p)}^*$$

is a unit, so it suffices to show that $p^p\delta(x) = 0$. But

$$p^p\delta(x) = p^{p-1} \cdot p\delta(x) = p^{p-1}(\phi(x) - x^p) = p^{p-2}(\phi(px) - px \cdot x^{p-1}) = 0$$

where we used that $px = 0$ in the end. $\square$

This leads to a classification of perfect δ-rings: they just come from perfect $\mathbb{F}_p$-algebras!

THEOREM. *The category of perfect δ-rings that are p-adically complete is equivalent to the category of perfect $\mathbb{F}_p$-algebras.*

PROOF. The above lemma shows that perfect δ-rings are always p-torsionfree, and so we have a forgetful functor from the category of perfect p-adically complete δ-rings to the category of p-adically complete and p-torsionfree $\mathbb{Z}_p$-algebras $\bar{A}$ with $\bar{A}/p$ perfect. By the above remark, this is equivalent to the category of perfect $\mathbb{F}_p$-algebras. To go back to perfect δ-rings, fix a perfect $\mathbb{F}_p$ algebra, and by deformation theory lift it to a p-adically complete and p-torsionfree ring. As the cotangent complex vanishes, such a lift is unique and is equipped with a unique lift of Frobenius. By p-torsionfreeness of the lift, this defines the required δ-structure. $\square$

REMARK. We can be explicit about the equivalence above. One functor is given by modulo p, $A \mapsto A/p$, while the other is given by the Witt vector construction $A \mapsto W(A)$. In other words, every p-adically complete perfect δ-ring has the form $W(R)$ for some perfect $\mathbb{F}_p$-algebra R.

0.3. distinguished elements. From now on until specified otherwise, we assume that any commutative ring we see now are p-local, i.e. $p \in \mathrm{JRad}(A)$, where $\mathrm{JRad}(A)$ is the Jacobson radical of A. In particular, any p-adically complete ring is so.

In the previous sections, the fact that $\delta(p)$ was a unit came in handy in relating perfect δ-rings to perfect $\mathbb{F}_p$-algebras. It would do us well to give a name to elements with this property.

DEFINITION. Let R be a δ-ring. An element $d \in R$ is called **distinguished** if $\delta(d)$ is a unit. (This terminology dates back to Fontaine).

The slogan is that a distinguished element is a "deformation" of p. As ring homomorphisms preserve units, we note that any morphism of δ-rings preserves distinguished elements. In fact, it just needs to commute with δ: the Frobenius lift ϕ commutes with δ, and so we see that ϕ preserves distinguished elements. Even better, as our rings are p-local, if $\phi(d)$ is distinguished so is d.

EXAMPLE. Here are some examples of distinguished elements in δ-rings.

(1) (*Crystalline cohomology*) Let $A = \mathbb{Z}_{(p)}$ with $d = p$. More generally, for any δ-ring A with $p \in \mathrm{JRad}(A)$, the image of $p \in A$ is distinguished.
(2) (*Breuil-Kisin cohomology*) Fix a discretely-valued extension of $K/\mathbb{Q}_p$ with uniformizer π. Let $W(k) \subset \mathcal{O}_K$ be the maximal unramified subring. Let $A = W(k)[\![u]\!]$ with δ-structure induced by the canonical one on $W(k)$ and satisfying $\phi(u) = u^p$. There is a $W(k)$-equivariant surjection $A \to \mathcal{O}_K$ determined by $u \mapsto \pi$. Any generator of the kernel of this map is distinguished.

EXAMPLE. We construct a universal p-local δ-ring A with a distinguished element $d \in A$. Take the free δ-ring on an element $\mathbb{Z}\{d\}$. Maps out of this classify δ-rings with a choice of *any* element in it. To ensure that the image of $\delta(d)$ is unit, we can freely adjoin its inverse to our free δ-ring to get $\mathbb{Z}_{(p)}\{d, \delta(d)^{-1}\}$. But this can be written as the localization $S^{-1}\mathbb{Z}_{(p)}\{d\}$ along the multiplicative set $S = \{\delta(d), \phi(\delta(d)), \phi^2(\delta(d)), ...\}$. This ring may no longer be p-local, so for extra measure we p-localize again.

An important class of distinguished elements can be characterized when we are in the context of a perfect p-adically complete δ-ring. In this case we have seen that such rings are $W(R)$ for some perfect $\mathbb{F}_p$-algebra R, and so its elements have Teichmuller expansions.

LEMMA 7. *Let R be a perfect $\mathbb{F}_p$-algebra. Then an element $d \in W(R)$ is distinguished if and only if the coefficient of p in its Teichmuller expansion is a unit. All such distinguished elements are thus nonzero-divisors.*

PROOF. Fix an element d and its Teichmuller expansion $\sum_{i \geq 0} [a_i] p^i$. From the formula $\delta(d) = \frac{\phi(d) - d^p}{p}$ and the formula for the Frobenius $\phi([a_0]) = [a_0^p]$, we get that mod p, $\delta(d) \equiv [a_1^p]$. As $W(R)$ is p-adically complete, it follows that $\delta(d)$ is a unit precisely when $a_1 \in R$ is a unit.

To see the second claim, we consider the Teichmuller expansion of $xd = 0$, where x is an element killing d. $\square$