Countability, Computability, Counting, Combinatorial Proofs

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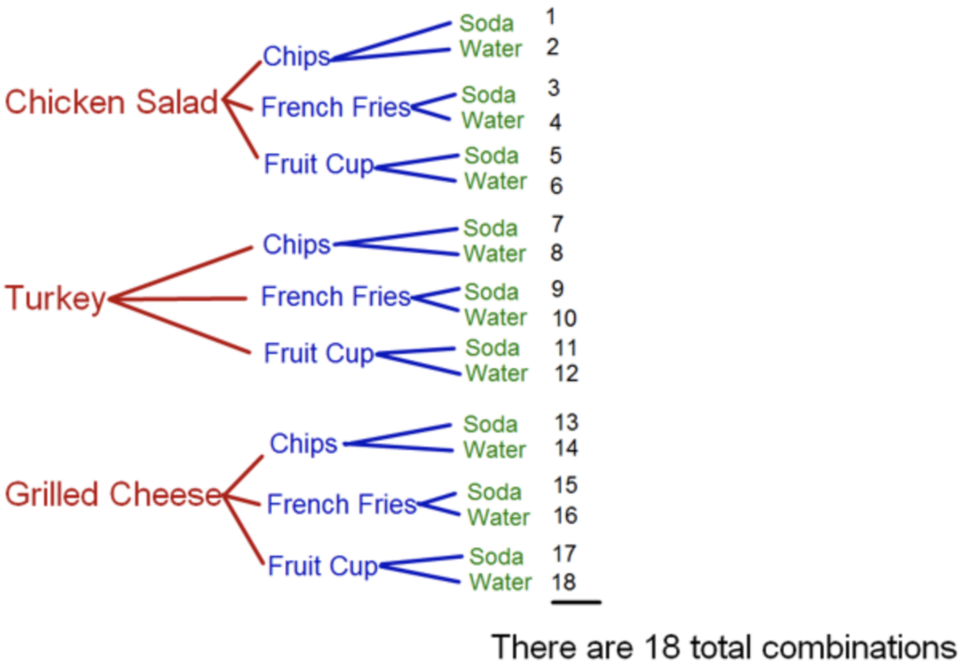
1 Counting

When we work with complex combinatorics and we are concerned with problems of selection, arrangement, and operation we seek to "count without counting". Right now, in terms of laying the foundations, our goal is to establish fundamentals. So we start with counting. Counting is just determining the number of ways to do something. The number of ways of making a certain number of choices.

The most basic counting principle is based upon a divide-and-conquer approach, where we will break the steps down into possible pathways along the branches of a tree.



At the first stage there are three possibilities, at the second stage another three, and at the final stage two. The different possibilities can be represented as tuples e.g. (Turkey, French Fries, Soda). If we label the different stages a, b, c respectively and index the options in each stage we can represent all possible combinations with tuples: (a_i, b_j, c_k) s.t $(i, j, k) \in$ [num choices at each stage]

Thus the number of possible combinations is the product of the different possible choices at each stage ijk. More generally, for any sequence of possible results at the first $i-1$ stages, there are n_i possible results at the ith stage. Thus the total number of possible results of the r-stage process is $n_1 n_2 ... n_r$

1.1 Various General Counting Techniques

- **Repeated Multiplication**: Use when choosing from a list where order matters (list is decreasing). For example, if you have ten flowers and you want to plant three of them in a row (where you count different orderings of the flowers), you can do this in 10*9*8 ways.
- **Addition**: Use when combining the results of disjoint cases. For example, if you can have three different cakes or four different kinds of ice cream (but not both), then there you have 3+4 choices of dessert.
- **Exponents**: Use when choosing from a list where order matters but the list is does not decrease. For example, if you had ample supply of each of your ten kinds of flowers, you could plant 10*10*10 different ways (because you can reuse the same kind of flower). *This technique is used frequently when calculating the number of subsets of a particular set*
- **Factorials/Permutations**: Extending off the flowers, the list is decreasing, but now you don't just plant three flowers you plant all 10.
- **Combinations**: Use this when selecting a group of items from a larger group, but the order does not matter. For example, if you have five different marbles and want to grab three of them to put in your pocket (so the order in which you choose them does not matter), this can be done in $\binom{5}{3}$ ways.

1.2 Balls and Bins

An insightful model to help look at counting problems intuitively. Consider m balls and n boxes/bins.

- Sampling with Replacement, order matters: for a given ball m_i there are n bins that m_i could land in. Since we are sampling with replacement, each ball can potentially land in any of the n bins. Thus to count the number of ways we can do this we use the first rule of counting and calculate the product of the number of choices for each of the m ball $\underbrace{n * n * n * ... * n}_{\text{m times}}$ which is just: n^m

We will revisit the balls and bins model when we look at the second rule of counting.