HW2 A.S.1, A.S.2, A.S.3

$|\psi\rangle, |\phi\rangle \in \mathbb{C}^2$ is arbitrary

$$\| \langle \psi | \phi \rangle \|^2 \leq \langle \psi | \psi \rangle \cdot \langle \phi | \phi \rangle$$

<u>Case I</u> $|\phi\rangle = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \langle \phi | \phi \rangle = 0$

$\Rightarrow \langle \psi | \psi \rangle \cdot \langle \phi | \phi \rangle = 0$

$$\| \langle \psi | \phi \rangle \|^2 = \left\| \begin{bmatrix} \psi_0 & \psi_1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\|^2 = 0^2 = 0$$

$0 \leq 0$ ✓

<u>Case II</u> $|\phi\rangle \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

let $|x\rangle = |\psi\rangle +$ ([illegible]) $|\phi\rangle$

Case II

$\langle\phi|\chi\rangle = \langle\phi|\psi\rangle - \langle\phi|\psi\rangle = 0$

$\Rightarrow \chi$ is orthonormal to ϕ

rearranging to

$$|\psi\rangle = \frac{\langle\chi|\phi\rangle}{\langle\phi|\phi\rangle}|\phi\rangle + |\chi\rangle$$

$$\||\psi\rangle\|^2 = \left\|\frac{\langle\chi|\phi\rangle}{\langle\phi|\phi\rangle}\right\|^2 \cdot \||\phi\rangle\|^2 + \|\chi\|^2$$

$$\geq \frac{\|\langle\chi|\phi\rangle\|^2}{\|\phi\|^2} \Rightarrow \|\langle\psi|\phi\rangle\|^2 + \|\chi\|^2 \geq \|\langle\chi|\phi\rangle\|$$

$$\Rightarrow \|\langle\psi|\phi\rangle\|^2 + |\chi| \geq (\|\langle\psi|\phi\rangle\| \cong \|\psi\|\|\phi\|$$

$$\cong \langle\psi|\psi\rangle \cdot \langle\phi|\phi\rangle)$$

A.5.2

If $U(2)$ is a vector space, vector addition must hold i.e. given that $I \in U(2)$, $(I - I)$ should be unitary, but it's not, because $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible, since its det = 0, and thus not unitary.

A.5.3 $|x\rangle, |w\rangle \in \mathbb{C}^2$

1. $\langle x | w \rangle = \begin{bmatrix} \bar{x}_0 & \bar{x}_1 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \bar{x}_0 w_0 + \bar{x}_1 w_1$

$\langle Ux | Uw \rangle = (U|x\rangle)^* (U|w\rangle) = |x\rangle^* U^* U |w\rangle$

because U is unitary, by defn $U^* U = I$

thus $= |x\rangle^* |w\rangle = \langle x | w \rangle$

2. $\| U|x\rangle \| = \| |x\rangle \|$

$\| |x\rangle \| = \sqrt{\langle x | x \rangle}$

from prior pf. we know $\langle Ux | Ux \rangle = \langle x | x \rangle$

$\Rightarrow \sqrt{\langle Ux | Ux \rangle} = \sqrt{\langle x | x \rangle}$

$\Rightarrow \| U|x\rangle \| = \| |x\rangle \|$

3. Given $\| V|\psi\rangle \| = \| |\psi\rangle \| \ \forall |\psi\rangle \in \mathbb{C}^2$

, prove V is unitary.

Case I $|\psi\rangle$ is a standard basis vector

i.e. $|\psi\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $|\psi\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

let $V = \begin{bmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{bmatrix}$ s.t. $v_{nm} \in \mathbb{C}$

$\Rightarrow \sqrt{\langle V\phi | V\phi \rangle} = \sqrt{\langle \phi | \phi \rangle}$

$= \sqrt{V^* |\phi\rangle^* V |\phi\rangle} \stackrel{?}{=} \sqrt{|\phi\rangle^* |\phi\rangle}$

in order for this to hold,

it must be true that equality $V^* V = 1$

$\Rightarrow V$ is unitary

4.

$\det(\bar{U}) \doteq \overline{\det(U)}$

given $M = \begin{bmatrix} x & \omega \\ \psi & \phi \end{bmatrix}$, $\bar{m} = \begin{bmatrix} \bar{x} & \bar{\omega} \\ \bar{\psi} & \bar{\phi} \end{bmatrix}$

$\det(\bar{m}) = \bar{x}\,\bar{\phi} - \bar{\omega}\,\bar{\psi}$

$\overline{\det(m)} = \overline{x\phi - \omega\psi} = \overline{x\phi} - \overline{\omega\psi} = \bar{x}\bar{\phi} - \bar{\omega}\bar{\psi}$

Thus $\det(\bar{m}) = \overline{\det(m)}$

also $\det(UU^*) = \det(I) = 1$ by unitarity

$\Rightarrow \det(U)\det(U^*) = 1$

$\det(U^T) = \det(U)$,

$\Rightarrow \det(U^*) = \overline{\det(U)} =$

$\det(U)\overline{\det(U)} = |\det(U)| = 1$

5.

$$A = \begin{bmatrix} x & 0 \\ 0 & \phi \end{bmatrix} \qquad |x| = 1 \qquad |\phi| = 1$$

$\det(A) = x\phi$

$\det(A^*) = \overline{x\phi}$

$\det(A)\det(A^*) = \det(AA^*)$

$= x\phi\,\overline{x\phi} = |x\phi|$

$= |x||\phi| = 1$ ✓

2. given $|\langle\psi|\phi\rangle|^2 \leq \langle\phi|\phi\rangle \cdot \langle\psi|\psi\rangle$

Prove $\|\psi\| + \|\phi\| \geq \|\psi + \phi\|$

$\cong \|\psi+\phi\|^2 \leq (\|\psi\| + \|\phi\|)^2$

$\Rightarrow \|\psi\|^2 + 2\psi\phi + \|\phi\|^2 \leq \|\psi\|^2 + 2\|\psi\|\|\phi\| + \|\phi\|^2$

$\Rightarrow |\psi\phi| \leq \|\psi\|\|\phi\|$ which we know is true from Cauchy Schwarz

```
### BEGIN PYTHON CODE ###
import cmath
import numpy
import math

# Cam Brown's HW2 Code

### Exercise A ###
def rect(r, phi):
    return r * (math.cos(phi) + math.sin(phi)*1j)

### Exercise B ###
def directSum(A, B):
    width = len(A[0]) + len(B[0])
    height = len(A) + len(B)
    C = numpy.zeros((height, width), dtype=numpy.array(0 + 0j).dtype)
    for row in range(height):
        for col in range(width):
            if row < len(A) and col < len(A[0]):
                C[row][col] = A[row][col]
            elif (row < len(B) + len(A) and row >= len(A)) and (col < len(A[0]) + le
n(B[0]) and col >= len(A[0])):
                C[row][col] = B[row-len(A)][col-len(A[0])]
    return C

def main():
    print(rect(2, math.pi))
    print(cmath.rect(2, math.pi))

    A = [[1, 1, 1], [2, 2, 2]]
    B = [[3, 3], [4, 4]]
    print(directSum(A, B))

if __name__ == '__main__':
    main()

### END PYTHON CODE ###
```