A Microscopic Model for Traffic Flow

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Abstract

Traffic is an inconvenience surrounding large cities and popular destinations due to the increase in volume of vehicles on the road. In recent studies, models and simulations have been used to understand the characteristics and behavior of traffic from both a Lagrangian and Eulerian point of view. In this paper, we introduce a microscopic model, a form of a Eulerian point-of-view model, that simulates the relationship between spatial density and vehicular velocities. A baseline model with ideal driving conditions was first simulated to allow comparisons between different traffic scenarios. This model maintained desired velocities and safe distances. Next, an on-ramp was introduced to allow injection of cars into the system, thus increasing the overall car density in the model. Additionally, lanes were closed down after a specified time forcing all cars to merge into the remaining open lanes. Both of these scenarios resulted in an overall decrease in velocity compared to the baseline model until steady-state values were achieved. Lastly, we doubled the amount of cars in the system for the baseline model and both the on-ramp and closing lanes scenarios to analyze the effects of traffic.

1 Introduction

Traffic in large metropolitan areas is an inconvenience that drivers face on a regular basis when commuting to and from large cities (e.g., Los Angeles, CA). The leading cause of traffic buildup is a greater transit demand than the supply that road-capacity provides. As a result, roads become overcrowded, exceeding road capacity, and gridlocks begin to emerge. This primarily occurs during peak rush hours of the day when vehicles are traveling on the road at the same time of day to perform the various common daily activities such as work, school, or running errands [1]. Other factors contributing to the increase in traffic include road-side constructions and negligent driving habits and behavior [1].

To help better understand and improve traffic flow, mathematical models and simulations have been developed to identify critical points that trigger congestion [2, 3]. Additionally, mathematical models and simulations provide insight with regards to how vehicle densities correlate to average driving speeds and overall flow of traffic. Three types of models that have been used to study the effects of traffic are: microscopic, macroscopic, and mesoscopic models [3]. A microscopic approach models each vehicle's impact on traffic, whereas the macroscopic approach models the overall behavior [3]. The mesoscopic model is a fusion of both the microscopic and macroscopic approaches.

In this paper we aim to develop a microscopic model in order to simulate traffic flow on an interstate highway as discussed in Section 3. In Section 5, we apply our microscopic model to common traffic situations such as varying traffic density and closed lanes.

2 Previous Work

Mathematical modeling provides researchers and engineers a tool for understanding traffic dynamics. In the 1990s traffic flow was investigated by approaching the problem as a particle-system [4, 5]. Researchers believed that the same phenomena that appear in the interaction of particles would also appear in a socio-dynamical systems, such as traffic [6]. Traffic was and is often thought to occur only due to bottlenecks, or restrictions in flow, however the mathematical models used by Sugiyama et al. suggested that traffic jams could be created whenever average vehicle density of the road exceeded a certain critical value [6]. These theoretical results were then verified experimentally by placing 22 vehicles on a 230m circular track and having the vehicles drive along the circle at a speed of around 30km/h. Cars began evenly spaced along the track and traffic initially flowed smoothly. However, due to perturbations in car speed, a phenomenon known as a "phantom traffic jam" arose. Phantom traffic refers to the presence of traffic when no obstructions are present [6]. The presence of the phantom traffic jam verified the results of the study.

Macroscopic models provide an out-of-frame perspective for studying traffic. This modeling technique considers traffic as a stream or continuum fluid [7, 8, 9, 10, 11]. Additionally, such models assume that all information in the system is obtained from the vehicular density, averaged velocity, and spatial information. The authors in [7] derived a model that described traffic flow as a system of balance laws. Another method for deriving macroscopic models is by the Chapman-Enskog method [12, 13]. The Chapman-Enskog method allows

[1] Texting, rubber-necking, eating, checking email, road-rage.

the calculation of transportation terms: speed limit, mean car length, acceleration rate, and reaction time. Using this model gives insight on how traffic is affected when varying the transportation terms [14]. Researchers have also shown that traffic flow equations can be derived from the reduced Paveri-Fontana equation [15]. The Paveri-Fontana kinetic equation incorporates a Boltzmann-type treatment of traffic flow that takes into account individual drivers acceleration behavior in order to calculate the transportation terms. Additionally, the system considers the relaxation of the instantaneous velocity, desired velocity per vehicle, and a binary interaction term (collisions).

Another model that can be used to study the dynamics of traffic is agent-based modeling (ABM). ABMs provide a method for simulating hypothetical situations without having to waste valuable resources [2]. In the context of traffic, ABMs are implemented in both mesoscopic and microscopic traffic models. A mesoscopic traffic model can be implemented by imposing a rule-based modeling at the microscopic level and a fluid-flow modeling at the macroscopic level, which allows the characterization of the stop-and-go wave like behavior of traffic [3]. This mesoscopic model describes the traffic density between cars by the rule-based structure: safe distance between adjacent cars, instantaneous braking, and traffic velocity unaffected by surrounded events. In a previous study, researchers developed a microscopic agent-based model for evaluating traffic on freeway on-ramps [16]. The simulation incorporated a road network as a series of nodes, links, segments and lanes. In addition, the model was able to incorporate travel demand as a time-dependent origin to destination feature as an input. Next, we present our mathematical model based on the intuition of microscopic and mesocopic models.

3 Mathematical Model

A microscopic traffic model is a distance-based model in which a vehicle's speed, acceleration, and braking force is dependent on that of the surrounding vehicles [3]. A few key concepts serve as the foundation for the microscopic model: Desired Velocity, Safe Distance, Braking Force. The *Desired velocity* is the target speed that a car should maintain in their lane without any outside influence. For an interstate freeway model the desired velocity is highest in the lane furthest from the on-ramp and slowest in the lane closest to the on-ramp. Following this logic, a car will increase or decrease it's speed after performing a lane change in order to adjust to the target lane's desired velocity. The *Safe distance*, is described as the buffer between two cars. In practice a short safe-distance can be described as tail-gating. The *Braking force* refers to the amount of force required to decelerate the vehicle in motion.

The three microscopic rules can be modeled mathematically by the following differential equations of motion,

$$\frac{dx_a(t)}{dt} = v_\alpha(t), \tag{1}$$

$$\frac{dv_\alpha(t)}{dt} = a_\alpha(t), \tag{2}$$

$$
\begin{aligned}
a_\alpha(t) = \frac{\hat{v}_\alpha^\ell - v_\alpha(t)}{\tau_\alpha^0} * (1 - \Xi(\hat{v}_\alpha^\ell - V_\alpha^d(\Delta x_\alpha)) \\
+ \frac{V_\alpha^d(\Delta x_\alpha) - v_\alpha(t)}{\tau_\alpha^s} * \Xi(\hat{v}_\alpha^\ell - V_a^d(\Delta x_\alpha) \\
-\gamma \frac{{\Delta v_\alpha}^2}{\Delta x_\alpha - l_\beta} * \Xi(-\Delta v_\alpha) * \Xi(v_\alpha(t) - V_\alpha^d(\Delta x_\alpha))
\end{aligned} \tag{3}
$$

where $\Xi(X)$ is the Heaviside function defined as follows

$$
\Xi(X) = \begin{cases} 0 \text{ if } X \leq 0 \\ 1 \text{ if } X > 0 \end{cases} \tag{4}
$$

and x_α, v_α, and a_α are the position, velocity, and acceleration, respectively, of car α. The desired velocity, v_α^ℓ, is a constant that changes depending on lane ℓ, τ_α is the relaxation time, γ is the unit-less braking force proportionality factor, and l_α is half the length of the leading car β. Each of the three lines in the acceleration equation, Equation 3 above, represents a particular driving rule for modeling traffic which is described in detail below, where the first, second and third line of the acceleration equation represent rule 1, rule 2 and rule 3, respectively. However, the terms do not operate in complete independence; the useful properties of the Heaviside function act to "toggle" on or off each term allowing all the rules to be combined into one cohesive acceleration function.

3.1 Rule 1 - Desired Velocity

$$
\frac{\hat{v}_\alpha^\ell - v_\alpha(t)}{\tau_\alpha^0} * (1 - \Xi(\hat{v}_\alpha^\ell - V_\alpha^s(\Delta x_\alpha))
$$

The desired velocity acceleration rule arranges the behavior of the vehicle a without any outside influences on the vehicle [3]. Vehicle α adapts its current velocity, $\hat{v}_\alpha(t)$, to its desired lane velocity, v_α^ℓ, where ℓ is car's current lane. The relaxation time, τ_α^0, allows tuning the model to tend more or less towards the desired velocity by increasing or decreasing the value. For example, a Desired Velocity term with $\tau_\alpha^0 = 3$ would adjust its velocity towards 1/3 of the difference between its current velocity and the desired velocity at each step through time. As described earlier, the desired lane velocities are set to increase from lane one to lane ℓ in order to simulate a real freeway model where there exist fast lanes and slow lanes.

3.2 Rule 2 - Safe Distance and the Velocity-distance Relationship

$$
\frac{V_\alpha^d(\Delta x_\alpha) - v_\alpha(t)}{\tau_\alpha^d} * \Xi(\hat{v}_\alpha^\ell - V_a^d(\Delta x_\alpha))
$$

In order to understand the Safe Distance acceleration term, we must first define Velocity-Distance relation, $V_\alpha^d(X)$. The velocity-distance relation is a uniformly monotonic increasing function between 0 and the desired velocity $\hat{v}_a^\ell$. More specifically , $V_\alpha^d(0) = 0$ and $V_\alpha^d(X \geq d) = \hat{V}_a^l$. This constitutes a breaking sensitivity depending on distance to the car in front. Because breaking behavior is so radically different depending on the driver, an exact function for $V_a^d(X)$ cannot be determined from physical properties alone. Instead, we initially make the assumption that all of our drivers are moderately aggressive. Because of advancements in breaking technology, we see that vehicles can afford to (and often prefer to) break later. We choose to model this relationship by fitting the logarithmic function. Applying our necessary conditions yields the function in Equation 5. A visualization of the Velocity-Distance relation can be seen Figure 1.

$$V_\alpha^d(X) = \begin{cases} \frac{\hat{v}_a^\ell}{\log(d+1)}(log(X+1)) \ , \ X \leq d \\ \hat{v}_\alpha^\ell \ , \ X > d \end{cases} \tag{5}$$

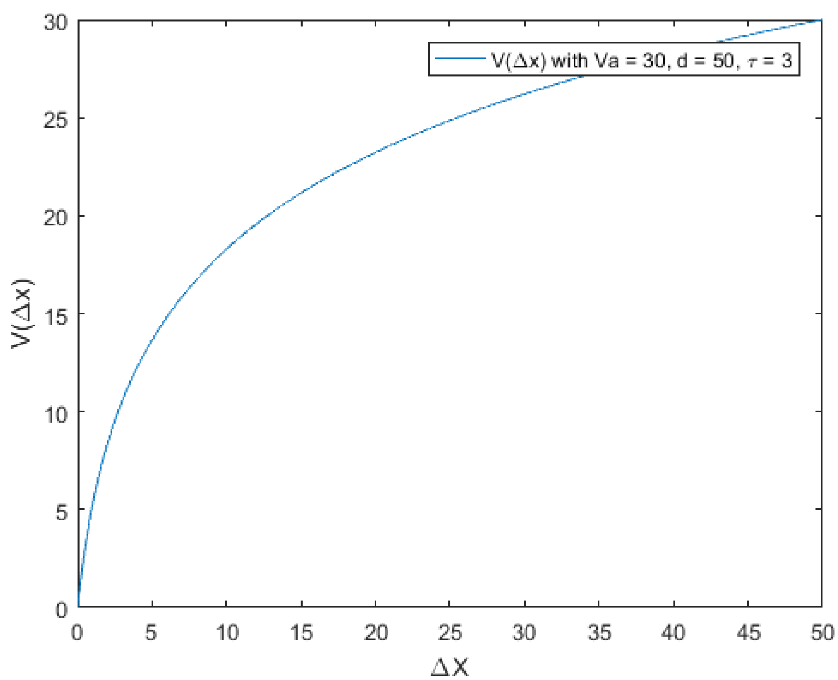

Figure 1: The Velocity-Distance relation given safe distance of 50 m and desired velocity of 30 m/s, over ΔX from 0 to 50 m.

It is clear that the function is monotonically increasing throughout the domain for $\Delta x \in 0$ to d, and satisfies the boundary conditions. Figure 2 is a visualization of the impact of our choice of Velocity-Distance relation on the Safe Distance for various Δx terms.

The relaxation time for the Safe Distance term, τ_α^d is in general the same as the relaxation time for the Desired Velocity term, τ_α^0, however it is not necessary for this to be the case [17]. The relaxation term, τ_α^d, can be adjusted so long as $\tau_\alpha^d < \tau_\alpha^0$, as we would not want our vehicles to adjust their velocity relative to distance more slowly than their adjustments toward their desired velocity [17]. Finally, note that the Heaviside component of the term

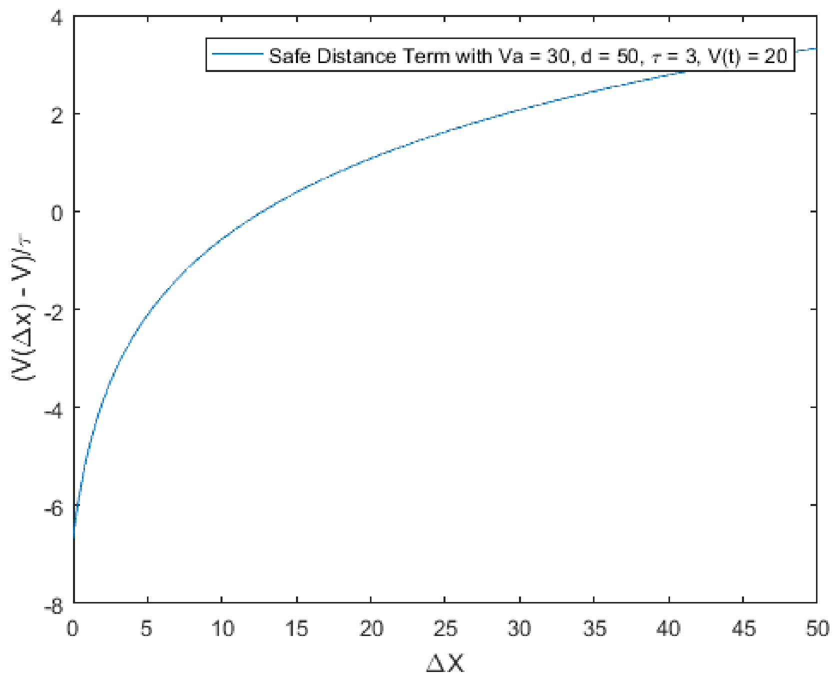

Figure 2: The Safe-Distance term given safe distance of 50 m and desired velocity of 30 m/s, and current velocity of 20 m/s over ΔX from 0 to 50 m.

allows for either the Safe Distance, or Desired Velocity to impact acceleration i.e. at any given time, only one of the terms is active while the other is dormant (equal to 0).

3.3 Rule 3 - Braking

$$- \quad \gamma \frac{\Delta v_\alpha{}^2}{\Delta x - l_\beta} * \Xi(-\Delta v_\alpha) * \Xi(v_\alpha(t) - V_\alpha^s(\Delta x_\alpha))$$

The braking acceleration rule arranges the behavior of the vehicle a when the safe distance to the leading vehicle b cannot be adhered to and vehicle a has to "slam" on the brakes. According to the kinetic laws the required braking braking is directly proportional to the square of the velocity difference $\Delta v_\alpha(t) = v_\beta(t) - v_\alpha(t)$ and indirectly proportional to the distance $\Delta x_a(t)$ between the vehicles.[3]. The l_β term in the denominator is half the distance of the leading car, which allows the deceleration term to increase more to avoid an accident. This is a direct result of Δx being calculated as the difference from the centers of the leading and trailing car. The Heaviside components dictate that there is no breaking if the leading car's velocity exceeds that of the following car, and likewise if the vehicle's current velocity is less than the given Velocity-Distance relation value, then no breaking is required.

3.4 Lane Changing

We have included additional conditions to the baseline model [3] by introducing a "lane-check" sensitivity parameter, ρ, as well as an Average Lane Velocity rule. The lane-check sensitivity parameter restricts a car from switching lanes if there is another car within ρ amount of units. The Average Lane Velocity Rule restricts lane changing unless the average velocity of cars in the desired lane is higher than the average velocity in the car's current lane. Additionally, our model includes a probabilistic aspect whereby cars have a 50-50 chance of desiring to change lanes left or right. This reflects differing desired paths in real traffic, i.e., certain cars looking to exit the freeway and certain cars preparing for prolonged travel on a single freeway. By introducing these lane changing rules, we are able to model both aggressive and passive drivers and a diverse range of driver behavior. The lane changing rules are described mathematically by

$$\begin{aligned} y(t+\Delta t) &= y(t) + \Delta y_{left} + \Delta y_{right} \\ \Delta y_{left} &= \Xi(p_\alpha^l) * \Xi(u) * \Xi(v_\alpha(t) - \bar{v}_\alpha^l) \\ \Delta y_{right} &= -1 * \Xi(p_\alpha^r) * (1 - \Xi(u)) * \Xi(v_\alpha(t) - \bar{v}_\alpha^r) \end{aligned} \tag{6}$$

where $u \sim U(-1, 1)$ is a uniformly distributed random variable between negative one and one, $\bar{v}_\alpha^i$ is the average velocity in the i (left or right) lane, and p_a^i is the left or right lane perimeter indicator defined by

$$p_a^i = \begin{cases} 1 \text{ if perimeter } \rho \text{ on side } i \text{ is not clear} \\ -1 \text{ if perimeter } \rho \text{ on side } i \text{ is clear} \end{cases} . \tag{7}$$

Each vehicle in the model is assigned an integer valued property y, which indicates the lane that the vehicle is in. Via the use of the Heaviside components, we formulate our model so that at each time step a given vehicle is equally likely to change lanes left or right but does not do so until the perimeter and average velocity conditions are met. If the conditions are not met then the vehicle remains in it's current lane, i.e., $y(t + \Delta t) = y(t) + 0 + 0$.

4 Implementation

In order to implement our model a few assumptions were made. First we assumed that all vehicles are of equal size. Secondly, we assumed that all vehicles in the system are governed by the same of equations discussed in Sections 3.1-3. By doing so, we assume that all drivers exhibit the same driving habits. Lastly, we initialized all vehicles within equal distance from one another which is similar to the study conducted by [6].

In order to solve the coupled system of differential equations of motion, Equations 1-3, the method of reduction of order was used to account for the acceleration term which is a second order time derivative. This allows the coupled differential equations to be reduced to a first order system that can now easily be solved using iterative Runge-Kutta methods for an approximate solution.

After the foundational driver behavior is established, we introduce an on-ramp which allows us to conduct an analysis of how traffic is affected when introducing more cars into the system, thus increasing the overall car density. The on-ramp has been simulated as a metered single lane, where only one car is allowed to enter the system at time so the following car cannot proceed until the leading car has entered the system. Next, we close down lanes

3 and 4 after 25 seconds into the simulation, which effectively doubles the car density as the amount of lanes have been reduced by half the initial lanes.

A summary of the parameter descriptions and model algorithm is presented in Table 1 and Algorithm 1, respectively.

c	car	γ	brake force proportionality factor
α	trailing car	β	leading car
a	vehicle acceleration	τ	relaxation time
v	vehicle velocity	ρ	lane-check sensitivity
ℓ	lane number	Δx	distance between cars
d	safe distance	Δv	velocity between cars
x	vehicle position		
$\hat{v}$	desired velocity		

Table 1: Model parameters for the traffic model.

Algorithm 1 Traffic model

procedure Initialize: c, ρ, d, v, a, x.
  for each ℓ **do**
    for each $c \in \ell$ **do**
      Calculate distance between adjacent cars.
      Solve for v and x from Equations 1 - 5.
      Calculate a.
      if lane change **and** $\rho_{car} \notin \left(x^{\ell-},\ x^{\ell+}\right)$ **then**
        Move car to adjacent lane (Equations 6 and 7).

5 Results and Analysis

Choosing a microscopic model reveals individual car behavior such as position, velocity, and acceleration. Even though typically overall system information is more relevant to analyze, having resolution down to individual cars allows for more detailed analysis to be carried if needed. This type of analysis allows investigating the source of a changes in the system. Figure 3 is an example of a microscopic graphical representation of the velocity and acceleration for cars 18 and 37 over time.

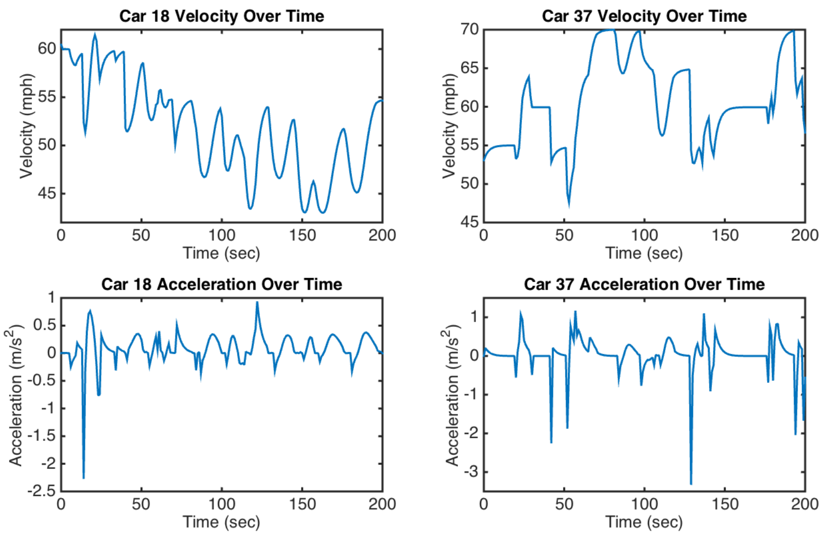

Figure 3: The velocity and acceleration plots over time for car's 18 and 37.

An advantage of using a microscopic model is that information of the system as a whole (macroscopic) can be recovered by taking the average of all cars in the system at each time step. For the purposes of validating our model, we consider the model with no on-ramp or closed lanes as our baseline model. Figure 4 provides a macroscopic viewpoint of the system. The average velocity of the system oscillated around the mean of the desired velocities with an acceleration in a $\pm 0.2\ m/s^2$ band for all time. The small oscillations around zero acceleration imply that the traffic in the system was accelerating and decelerating almost equally during each time step. This behavior is expected in traffic; when leading cars begin to decelerate the distant trailing cars do not also instantaneously decelerate, and vice-versa, when leading cars begin to accelerate the distant trailing cars are still decelerating.

In the figure below the velocity and acceleration plots for the system can be seen. It can be observed that the average velocity oscillates around the average of the desired velocities, which is 62.5 mph.

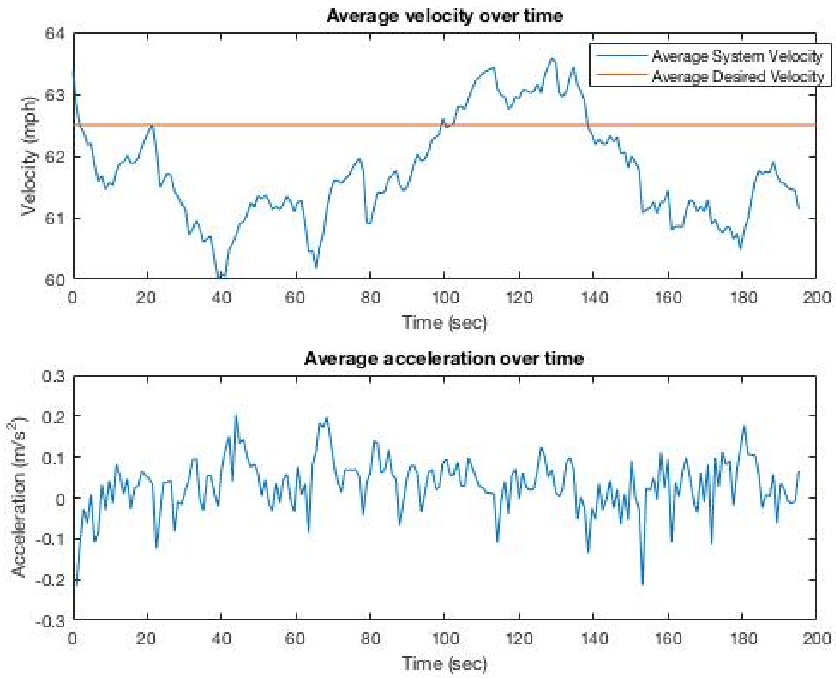

Figure 4: Results from the baseline simulation. Simulation ran with 40 cars and 4 lanes.

Observing the plots presented in Figure 5 and Figure 6, we can see that introducing an on-ramp and closing down a lane after 25 seconds had a decrease in system velocity as a whole in comparison to the baseline model. This is due to the fact that whenever a car enters the system, that car is forced to accelerate in order to achieve the desired velocity of $\hat{v}^1$. This also explains why the overall average acceleration in the system is slightly above zero for the majority of the time, indicating cars are accelerating more than decelerating.

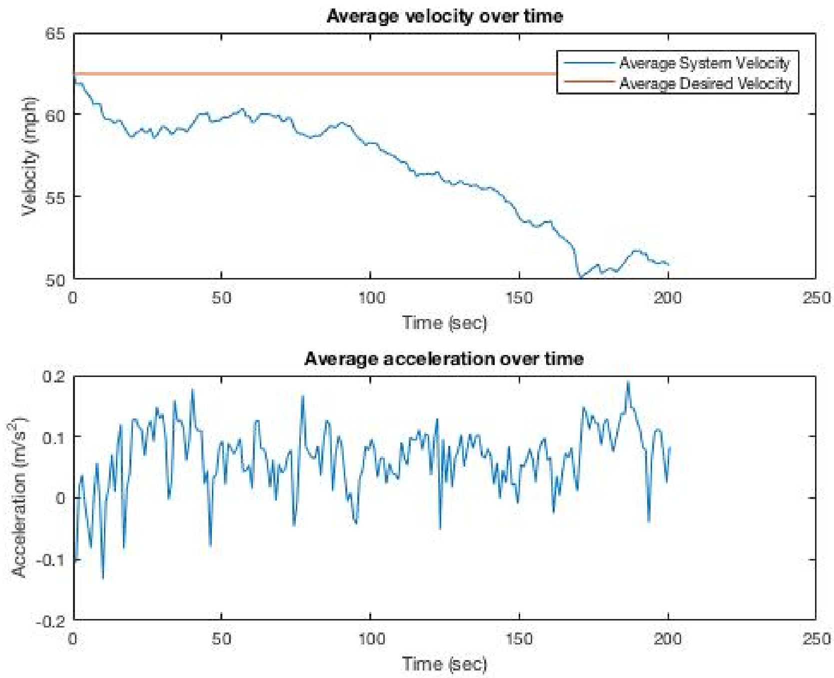

Figure 5: Results from the on-ramp simulation. Simulation ran on 4 lanes with 40 initial cars and injected 50 with periodic boundary conditions.

Furthermore, in the on-ramp simulation with the injection of cars into the system, vehicles have to accelerate from a lower velocity on the on-ramp to a higher velocity in lane one, which accounts for a higher acceleration than the baseline model. A similar yet opposite behavior is exhibited when closing down lanes. Vehicles are forced to slow down and at times come a complete standstill when a lane is shut down. In turn, this forces the vehicles to have to decelerate significantly which accounts for the closed lane scenario having the smallest average acceleration. This is not evident in the acceleration plot in Figure 6, however, it can be seen in Tables 2 and 3.

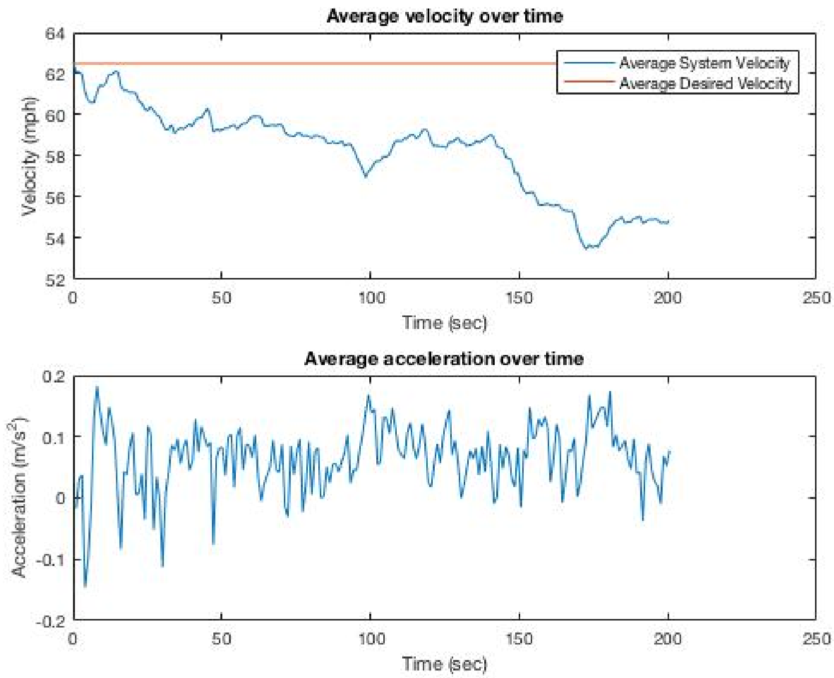

Figure 6: Results from the closed lanes simulation. Lanes 3 and 4 are closed after 25 seconds. Simulation ran with 40 cars and 4 lanes with periodic boundary conditions.

Combining both the on-ramp and closed lanes features had an interesting effect on the baseline model in regards to the decrease of the system's overall velocity. The velocity decreases which one would expect from combining two scenarios that decrease the average system velocity,however, observing Figure 7 shows that velocity in the system decreases with exponential behavior. Effectively, the car density in the system is being quadrupled as there are now half the lanes for cars to drive in and and over twice the amount of cars.

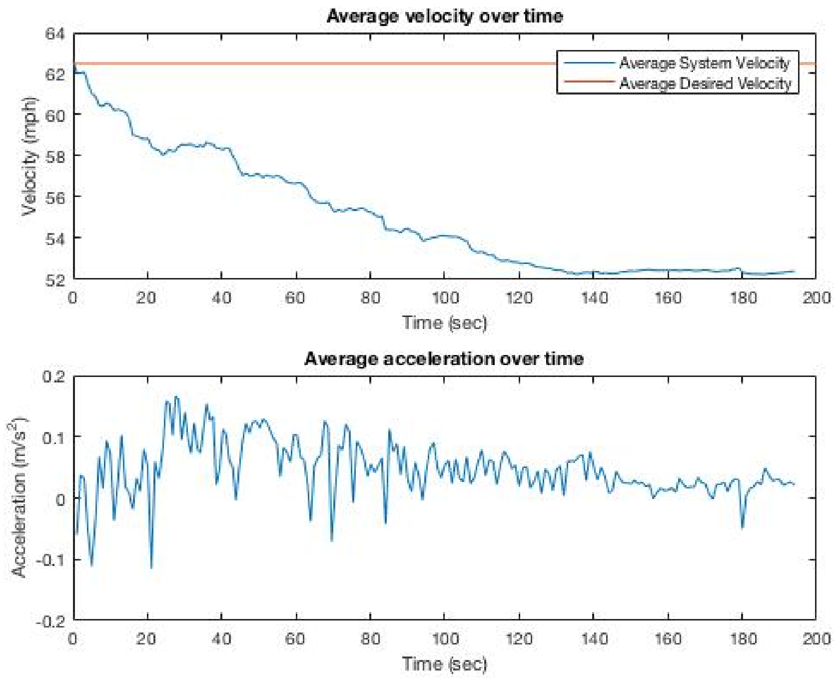

Figure 7: Results from the both on-ramp and closed lanes simulation. Simulation ran with 4 lanes, closing down lanes 3 and 4 after 25 seconds. A total of 40 initial cars with an injection of 50 cars with periodic boundary conditions.

The results of our simulations are presented in Tables 2 and 3. Table 2 shows results with 40 initial cars, whereas Table 3 shows results for doubling the amount of initial cars in the system to simulate more traffic and car density. Our simulations showed that increasing the density of cars resulted in an overall decrease in average system velocity, but an increase in average system acceleration. As stated previously, this implies that vehicles are forced to accelerate more often in order to reach the desired velocity. Furthermore, Table 2 and 3 show that closing down two lanes and introducing cars into the system had an impact in the overall velocity profile of the traffic, which can also be graphically observed in Figures 2 trough 5..

Model:	Baseline	On-ramp	Closed Lanes	Combined	Units
Average Vel.	61.505	55.2946	56.923	55.0921	mph
Average Accel.	0.032	0.0800	0.0033	0.0389	m/s^2

Table 2: Simulation results average over five runs for 40 initial cars and 50 added to the system. Lanes are closed after 25 seconds with periodic boundary conditions.

Model:	Baseline	On-ramp	Closed Lanes	Combined	Units
Average Vel.	58.079	53.4202	56.601	54.1856	mph
Average Accel.	0.0693	0.0870	0.0289	0.0541	m/s^2

Table 3: Simulation results average over five runs for 80 initial cars and 50 added to the system. Lanes are closed after 25 seconds with periodic boundary conditions.

6 Conclusion/Future Works

In this project we introduced a mathematical model that can be used to simulate traffic behavior. In turn, this allows researchers to experiment with different scenarios without having to waste valuable resources [6]. Our mathematical model provides a basis from which researchers can experiment with traffic density, obstructions, and overall vehicle behavior. The velocity profile from our baseline model shows that traffic can appear even if there is no obstruction on the road [6]. Future work to be conducted relating to this paper may include simulating off-ramps to represent cars getting off the freeway, which would reduce the car density. Modeling different car sizes and vehicles types would allow for a more realistic model. The introduction of motorcyclists would introduce lane splitting from motorcyclists. In turn, this would require multiple velocity-distance-relations, $V_s^s(X)$, to simulate different driving habits, such a sports cars versus eighteen-wheeler versus motorcycles. Additionally, different driving habits can be modeled via the velocity-distance-relation as well as assigning different relaxation times, and braking force proportionality factors to different cars to simulate aggressive, moderate, and safe drivers.

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