Math 130, 1.10 Problem 13

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Solve the following initial value problem:

$$y' = \frac{-2xy}{1+x^2} \quad y(0) = 2 \quad h = 0.05 \quad \text{and} \quad y(0.5)$$

We know this is a first-order linear ODE. But how can this be solved? Let's try to make it seprable:

$$\frac{dy}{dx} = \frac{-2xy}{1+x^2}$$

$$\frac{dy}{dx}(1+x^2) = -2xy$$
$$\frac{1}{y}dy = \frac{-2x}{1+x^2}dx$$

Now that we have all the y's on the left and the x's on the right, we can integrate:

$$\int \frac{1}{y}\,dy = \int \frac{-2x}{1+x^2}\,dx$$

The left side is trivial:

$$\int \frac{1}{y} = \ln(y) + C_1$$

The right side can be solved with substitution:

$$\text{Let} \quad u = x^2 + 1 \quad du = 2xdx \quad dx = \frac{du}{2x}$$

Now do the substitution:

$$-\int \frac{2x}{1+x^2}\,dx = -\int \frac{2x}{u}\,\frac{du}{2x} = -\int \frac{1}{u}\,du = -\ln(u) + C_2 = -\ln(x^2+1) + C_2$$

Putting the result of both integrations together:

$$\ln(y) + C_1 = -\ln(x^2+1) + C_2$$

$$\ln(y) = -\ln(x^2+1) + C_3$$
$$\ln(y) = \ln((x^2+1)^{-1}) + C_3$$
$$y = e^{\ln((x^2+1)^{-1})+C_3} = e^{\ln((x^2+1)^{-1})}e^{C_3}$$

So the general solution is:

$$y = \frac{e^{C_3}}{X^2+1}$$

Using the initial value $y(0) = 2$, we can calculate C_3.

$$y(0) = 2 = \frac{e^{C_3}}{0^2+1} = e^{C_3}$$
$$2 = e^{C_3}$$
$$\ln(2) = C_3$$

Therefore, the particular solution is

$$y = \frac{e^{\ln(2)}}{X^2+1}$$
$$y = \frac{2}{X^2+1}$$