

## Math 130, 1.10 Problem 12

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Solve the following initial value problem:

$$y' = \frac{-2xy}{1+x^2} \quad y(0) = 1$$

We know this is a first-order linear ODE. But how can this be solved? Let's try to make it separable:

$$\frac{dy}{dx} = \frac{-2xy}{1+x^2}$$

$$\frac{dy}{dx}(1+x^2) = -2xy$$

$$\frac{1}{y} dy = \frac{-2x}{1+x^2} dx$$

Now that we have all the  $y$ 's on the left and the  $x$ 's on the right, we can integrate:

$$\int \frac{1}{y} dy = \int \frac{-2x}{1+x^2} dx$$

The left side is trivial:

$$\int \frac{1}{y} = \ln(y) + C_1$$

The right side can be solved with substitution:

$$\text{Let } u = x^2 + 1 \quad du = 2x dx \quad dx = \frac{du}{2x}$$

Now do the substitution:

$$-\int \frac{2x}{1+x^2} dx = -\int \frac{2x}{u} \frac{du}{2x} = -\int \frac{1}{u} du = -\ln(u) + C_2 = -\ln(x^2 + 1) + C_2$$

Putting the result of both integrations together:

$$\ln(y) + C_1 = -\ln(x^2 + 1) + C_2$$

$$\begin{aligned}\ln(y) &= -\ln(x^2 + 1) + C_3 \\ \ln(y) &= \ln((x^2 + 1)^{-1}) + C_3 \\ y &= e^{\ln((x^2+1)^{-1})+C_3} = e^{\ln((x^2+1)^{-1})} e^{C_3}\end{aligned}$$

So the general solution is:

$$y = \frac{e^{C_3}}{X^2 + 1}$$

Using the initial value  $y(0) = 1$ , we can calculate  $C_3$ .

$$y(0) = 1 = \frac{e^{C_3}}{0^2 + 1} = e^{C_3}$$

$$1 = e^{C_3}$$

$$\ln(1) = 0 = C_3$$

Therefore, the particular solution is

$$y = \frac{e^0}{X^2 + 1}$$

$$y = \frac{1}{X^2 + 1}$$