Computational Numerical Statistics
PROJECT 1

Group G2 – TP1

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References

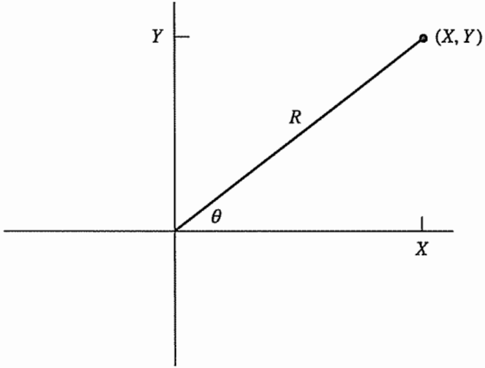
Sheldon M Ross.
Simulation.
Academic press, 2002.

Sheldon M Ross.
Introduction to Probability Models, ISE.
Academic press, 2006.

Random variable generation

Introduction

- Mervin Edgar Muller & George Edward Pelham Box, 19^{th} century
- Method for generating samples from two independent random variables X, Y $\sim N(0,1)$ from the $U(0,1)$ distribution
- Polar coordinates are created



[1]

- $R^2 = X^2 + Y^2$ and $\theta = tan^{-1}(\frac{x}{y})$
- Proposition: Let $Z_1, ..., Z_v$ be v independent random variables such that $Z_i \sim N(0,1)$. Then $\sum_{i=1}^{v} Z_i^2 \sim \chi_v^2$
so in our case $X^2 + Y^2 \sim \chi_2^2$

So we have the following p.d.f.

	χ_2^2	$\Gamma(1, \frac{1}{2})$	$Exp(\frac{1}{2})$
p.d.f.	$\frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} x^{\frac{k}{2}-1} e^{\frac{-x}{2}}$	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$	$\lambda e^{-\lambda x}$
	$\frac{1}{2} e^{-\frac{1}{2}x}$		

$\Rightarrow R^2 \sim \frac{1}{2} e^{-\frac{1}{2}x}$

Let's consider the joined density:

$$f(x, y) \overset{X, Y \sim N(0,1)}{=} f(x) * f(y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

$\rightarrow$ Joined density of $g(R^2, \theta)$?

given $J_{f(x,y)} = 2$ and $R^2 = X^2 + Y^2$:

$$g(R^2, \theta) = \frac{1}{|J_{f(x,y)}|} f(x, y) = \underbrace{\frac{1}{2\pi}}_{I} \underbrace{\frac{1}{2} e^{-\frac{1}{2}R^2}}_{II} \qquad [2]$$

Random variable generation

Derivation of the Box-Muller method: ITM

- Now we want to sample R^2 and θ from $U(0, 1)$
- **Inverse Transformation Method:** generate a random variable R^2 from the continuous distribution function F by generating a random number U and then setting $R^2 = F^{-1}(U)$

I. $u = \frac{1}{2}e^{-\frac{x}{2}} \Leftrightarrow x = -2log(\underbrace{2u}_{*})$ Remember: $R^2 \sim \frac{1}{2}e^{-\frac{1}{2}x}$

* $\sim U(0, 1)$ as well as -2log(u)

II. $g(R^2, \theta) = \frac{1}{2\pi}\frac{1}{2}e^{-\frac{1}{2}R^2} \Rightarrow \theta \in [0, 2\pi] \Rightarrow \theta = 2\pi u_2$

- once we have the radius R of the coordinates, the angle θ is uniformly distributed over the circumference with radius R.

Random variable generation

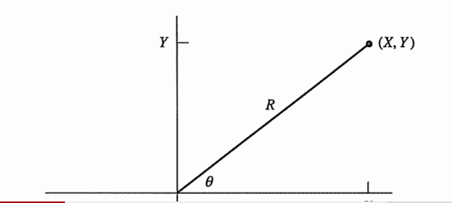
Derivation of the Box-Muller method: generating the samples

From I. and II. we get

$$R = \sqrt{-2log(u_1)} \qquad X = Rcos(\theta)$$
$$\theta = 2\pi u_2 \qquad Y = Rsin(\theta)$$

$$\Rightarrow X = \sqrt{-2log(u_1)}cos(2\pi u_2)$$
$$\Rightarrow Y = \sqrt{-2log(u_1)}sin(2\pi u_2)$$

with $u_1, u_2 \sim U(0, 1)$



Random variable generation

Transfer to other distribution

Q.: How to generate samples from $Z \sim N(\mu, \sigma^2)$?

$$y = ax + b \quad , \quad x \sim N(0, 1)$$

$E[Y] = E[a \cdot X + b] = aE[X] + E[b] = a \cdot \mu + b$
Var[Y] = Var[a $\cdot X + b] = a^2 * Var[X] = a^2 \cdot \sigma^2$
$\sigma[Y] = \sqrt{Var[Y]} = a \cdot \sigma$

Note: $Y \sim N(\mu^*, \sigma^{2*})$:

$$p.d.f._Y = \frac{1}{\sigma^* \cdot \sqrt{2\pi}} e^{-\frac{1}{2}(\frac{y-\mu^*}{\sigma^*})^2}$$

$\rightarrow$ Find transformation that converts it to a $N(\mu^*,\sigma^{2*})$:

$$y = \sigma^* \cdot X + \mu^*$$

Random variable generation
Algorithm

- Generate two observations u_1 and u_2 from U (0, 1).
- Set

$$\theta = 2\pi u_1$$
$$R = \sqrt{-2log(u_2)}$$

- Set

$$x = Rcos(\theta)$$
$$y = Rsin(\theta)$$

- Iterate the previous steps until you reach the desired sample size.

Random variable generation

Algorithm

(a) Implement the Box-Muller method in R.

```
sim.norm<-function(n, mu, std){
  "Box-Muller Algorithm. This function generates samples
  from two independent random variables X,Y ~ N(mu,std)."

  x<-vector()
  y<-vector()

  # Box-Muller algorithm
  theta=2*pi*runif(n,0,1)
  R=sqrt(-2*log(runif(n,0,1)))

  # add N(0,1) observations to the samples
  x=c(x,R*cos(theta))
  y=c(y,R*sin(theta))

  # convert all observations from N(0,1) to N(mu,std)
  x=x*std+mu
  y=y*std+mu

  # return results in a table format
}
```

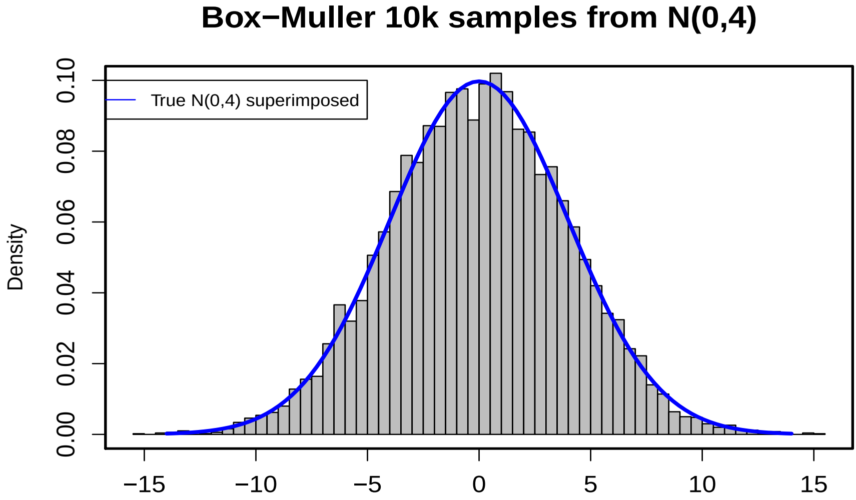
(b) Use routine sim.norm() to generate a sample of size m = 10000 from $X \frown N(0, 4)$.

```
#Exercise B - generate 10k samples from X~(0,4)
set.seed(123)
mu = 0
std = 4
n = 10 * 1000

# n/2 because we can merge X and Y since their both are independent.
samples = sim.norm(n/2,0,4)

# merge the 5k samples from X with the 5k samples from Y
samples = c(samples[,"X"], samples[,"Y"])
```

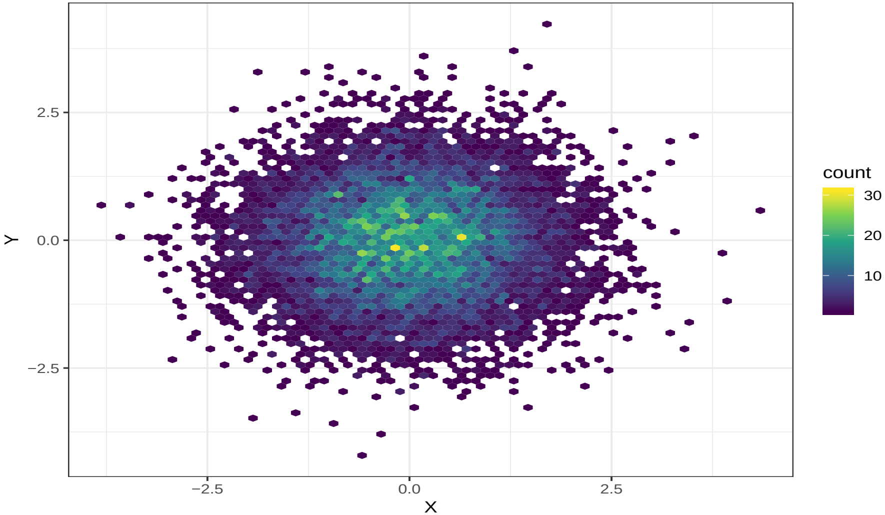
(c) Plot the histogram referring to the previous exercise.



Random variable generation

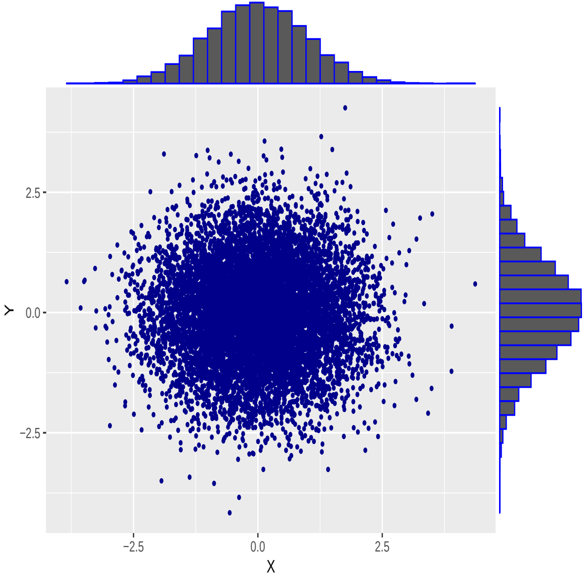
In-depth Visualization of the samples

Let´s see the density of the point cloud of X and Y.



Random variable generation

In-depth Visualization of the samples

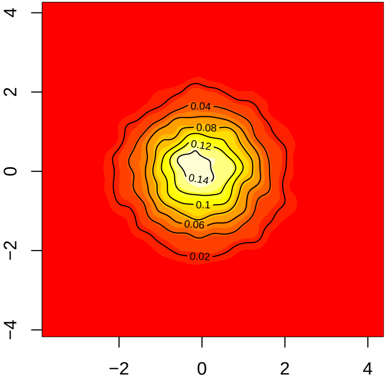


As most points are located near the origin they increase the density of N(0,1) around the mean of 0.
Outlier points are barely accounted for since there are 10000 samples.

Random variable generation

In-depth Visualization of the samples

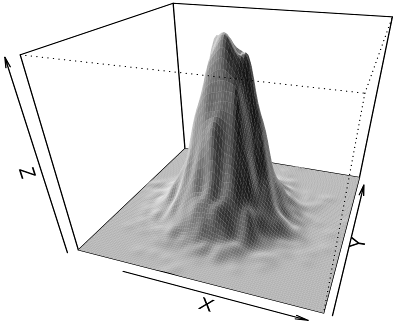
The different confidence regions, note that none of them are elliptical.



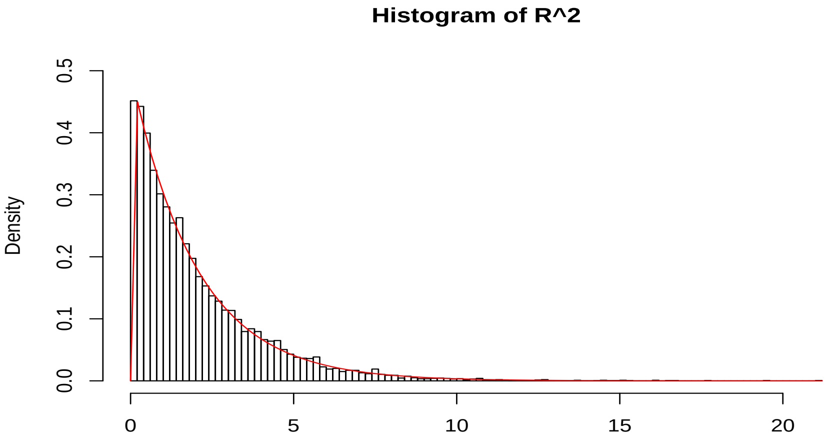
Calculating the covariance matrix and the mean of X and Y we get:

- for 10 samples, S= $\begin{bmatrix} 1.543 & -0.468 \\ -0.468 & 0.452 \end{bmatrix}$; mean = $\begin{bmatrix} 0.1303 \\ -0.4074 \end{bmatrix}$
- for 100 samples, S= $\begin{bmatrix} 0.8660 & -0.0221 \\ -0.0221 & 0.8487 \end{bmatrix}$; mean = $\begin{bmatrix} -0.1112 \\ 0.0059 \end{bmatrix}$
- for 1000 samples, S= $\begin{bmatrix} 1.02431 & 0.00116 \\ 0.00116 & 0.97704 \end{bmatrix}$; mean= $\begin{bmatrix} -0.0237 \\ -0.0341 \end{bmatrix}$
- for 10000 samples, S= $\begin{bmatrix} 1.00123 & 0.00103 \\ 0.00103 & 1.01359 \end{bmatrix}$; mean= $\begin{bmatrix} -0.0134 \\ -0.0041 \end{bmatrix}$

Bivariate Normal KDE:



Since $X, Y \frown N(\mu, \sigma^2)$, and $R^2 = X^2 + Y^2$ then $R^2 \frown \chi^2_2$
In the special case of two degrees of freedom, the chi-squared distribution coincides with the exponential distribution.



Questions?

If we finished the presentation on time, of course :-)