Optimization

Optimization focuses in finding the maximum or minimum optimum of a certain function. The optimum x in an arbitrary function $f(x)$ can be graphed when the curve is flat or when the value of x where $f'(x) = 0$. The value of $f''(x)$ determines whether the value is a maximum or a minimum. If $f''(x) < 0$, the point is a maximum, and if $f''(x) > 0$, the point is a minimum. The function $f(x)$ is being referred to as the objective function.

Constrained Optimization and Maximization Problems

In this method, constraints which limit the values of the optimum are being introduced. The objective function and the constraints are linear equations and the problems are formulated in a method known as linear programming. In linear programming, one must know if the problem is either a maximization or a minimization problem.

Maximization problems have objective functions of the form: $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$, where:

- Z is the variable to be maximized or (usually) the total cost;
- c_j is the cost of each unit of the j-th activity; and
- x_j is the magnitude of the j-th activity.

Constraints on the other hand have the form $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n \leq b_i$, where:

- a_{ij} is the amount of the i-th resource consumed for each unit of the j-th activity;
- and b_{ij} is the amount of the i-th resource that is available.

Placing constraints on a resource x_i means that it is limited. A second set of constraints with the form $x_i \geq 0$ means that all activities have a positive value.

Example

Suppose that a gas processing plant receives a fixed amount of raw gas each week. The raw gas is processed into two grades of heating gas: regular and premium quality. These grades of gas are in high demand (that is, they are guaranteed to sell) and yield different profits to the company. However, their production involves both time and on-storage constraints. For example, only one of the grades can be produced at a time, and the facility is open for only 80 hours/week. Further, there is a limited on-site storage for each of the products. All these factors are listed below.

Resource (Unit)	**Product**		**Resource Availability**
	Regular	**Premium**	
Raw gas (m^3/ton)	7	11	77 m^3/wk
Production time (hr/ton)	10	8	80 hr/wk
Storage (ton)	9	6	
Profit (price/ton)	150	175	

Develop/set up a linear programming formulation to maximize the profits for this operation.

We must decide how much of each gas to produce to maximize profits. If the amounts of regular and premium produced weekly are designated as x_1 and x_2 respectively, then the total weekly profit can be calculated as $P = 150x_1 + 175x_2$. Converting into a linear programming objective function, we write this as:

Maximize $Z = 150x_1 + 175x_2$.

Constraints are then being made. Since the total raw gas should not exceed 77 m^3/week, the constraint should be written as $7x_1 + 11x_2 \leq 77$. The other constraints should also be made in the same order. Therefore, the linear programming problem will be formulated as follows:

Maximize

$$Z = 150x_1 + 175x_2$$

subject to

$$7x_1 + 11x_2 \leq 77,$$
$$10x_1 + 8x_2 \leq 80$$
$$x_1 \leq 9$$
$$x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

Simplex Method[1]

The simplex method is a linear programming method which is based on the assumption that the optimum will be at an extreme point. The constraint inequalities in this method are reformulated as equalities by using slack variables.

Set-up Constraints: Slack Variables

Slack variables dictate how much of a particular resource is available or how much of the available resource will not be used. Introducing a slack variable on a constraint will change the form of the constraint into $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n + S_i = b_i$.

If $S_i > 0$, then there is still some remaining resources after the activity. If $S_i = 0$, then all of the resources are being used up, meeting the constraint. Otherwise, if $S_i < 0$, the production of a resource needs to exceed the demands of the constraint.

The slack variables should also meet the positivity constraint by placing $S_i \geq 0$ as a separate constraint. It should be taken note that the number of inequalities should also be the number of slack variables made. After introducing the slack variables for the constraints, we shall end up a system of linear algebraic equations.

Example

After formulating our linear programming problem, the slack variables will produce the following to the constraints:

Maximize

$$Z = 150x_1 + 175x_2$$

subject to

$$7x_1 + 11x_2 + S_1 = 77$$
$$10x_1 + 8x_2 + S_2 = 80$$
$$x_1 + S_3 = 9$$
$$x_2 + S_4 = 6$$
$$x_1, x_2, S_1, S_2, S_3, S_4 \geq 0$$

[1] http://www.zweigmedia.com/RealWorld/tutorialsf4/framesSimplex.html

Set-up: Initial Tableau

After the constraints have been lined up, the initial tableau can be set. Setting up the initial tableau column-wise includes all of the **coefficients** of the x_i's, the **slack** variables, the **variable** to be maximized and the **right hand side** of the equations in a matrix, in that order. Row-wise, the matrix shall be arranged **constraints** first, then the **objective function**.

Example

Doing the setup on our current maximization problem, we shall obtain the initial tableau below.

Basic	x_1	x_2	S_1	S_2	S_3	S_4	Z	**Solution**
S_1	7	11	1	0	0	0	0	77
S_2	10	8	0	1	0	0	0	80
S_3	1	0	0	0	1	0	0	9
S_4	0	1	0	0	0	1	0	6
Z	-150	-175	0	0	0	0	1	0

Solve

The **pivot column** PC will then be selected by choosing the negative number with the highest magnitude in the bottom row, with the exception of the Solution Column. Break ties at random.

After which, the **pivot element** PE will then be selected from the pivot column. For each positive (non-zero) entry b in the pivot column, we shall compute for the test ratio $\frac{a}{b}$, where a is the rightmost value in its particular row. The entry with the **smallest positive test ratio** is the pivot element.

After getting the pivot element, we shall **clear** the pivot column by employing the steps of the **Gauss-Jordan Elimination** method, wherein the rows of the pivot column will be zero except for the pivot element. That is, for all rows except for the pivot row, we shall find a constant that when multiplied to the whole row will make the difference to the pivot element be equal to zero, replacing the difference.

The whole process continues until the **bottom row does not have negative answers**. The last column is permitted to have negative numbers.

Example

In our example, we shall iterate on the tableau with the iterations given below.

Iteration 1.

Variable	x_1	x_2	S_1	S_2	S_3	S_4	Z	**Solution**	**TR**
S_1	7	11	1	0	0	0	0	77	7
S_2	10	8	0	1	0	0	0	80	10
S_3	1	0	0	0	1	0	0	9	cannot be
S_4	0	1	0	0	0	1	0	6	6

Z	-150	-175	0	0	0	0	1	0	

Iteration 2:

Variable	x_1	x_2	S_1	S_2	S_3	S_4	Z	Solution	TR
S_1	7	0	1	0	0	-11	0	11	1.5174
S_2	10	0	0	1	0	-8	0	32	3.2
S_3	1	0	0	0	1	0	0	9	9
S_4	0	1	0	0	0	1	0	6	cannot be
Z	-150	0	0	0	0	175	1	1050	

Iteration 3:

Variable	x_1	x_2	S_1	S_2	S_3	S_4	Z	Solution	TR
S_1	1	0	0.142857	0	0	-1.57143	0	1.57143	-1
S_2	0	0	-1.42857	1	0	7.71429	0	16.2857	2.1111
S_3	0	0	-0.142857	0	1	1.57143	0	7.42857	4.7273
S_4	0	1	0	0	0	1	0	6	6
Z	0	0	21.4286	0	0	-60.7143	1	1285.71	

Iteration 4:

Variable	x_1	x_2	S_1	S_2	S_3	S_4	Z	Solution	TR
S_1	1	0	-0.148148	0.203704	0	0	0	4.88889	
S_2	0	0	-0.185185	0.12963	0	1	0	2.11111	
S_3	0	0	0.148148	-0.203704	1	0	0	4.11111	
S_4	0	1	0.185185	-0.12963	0	0	0	3.88889	
Z	0	0	10.1852	7.87037	0	0	1	1413.89	

Finding the Answers

We can find the answer in this process by dividing the last column by the rightmost number in the same row with the largest positive magnitude, except for the last row. The quotient will be the value of the variable heading that column.

For the last row, which is the row for the objective function, the result can be gathered on the immediate column on the left.

Example

In our example, the final tableau will be used to compute for the values of the variables.

Variable	x_1	x_2	S_1	S_2	S_3	S_4	Z	Solution	Value of
S_1	1	0	-0.148148	0.203704	0	0	0	4.88889	x_1
S_2	0	0	-0.185185	0.12963	0	1	0	2.11111	S_4
S_3	0	0	0.148148	-0.203704	1	0	0	4.11111	S_3
S_4	0	1	0.185185	-0.12963	0	0	0	3.88889	x_2
Z	0	0	10.1852	7.87037	0	0	1	1413.89	Z

Solving General Maximization Problems[2]

When given a constraint with the form $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n \geq b_i$, we need to introduce **surplus variables**. The constraint shall then be converted into the form $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n - S_i = b_i$, then proceeding with setting up the initial tableau.

It can then be observed that the surplus variables will then be **negative** once the tableau will be set up. Thus, these rows of the tableau will be **flagged**.

Once the initial tableau is set up, we shall choose the **pivot column** by selecting the largest positive number in the first flagged row. The **pivot element** shall then be selected and the **clearing** of the pivot column will be done.

We can find the answer in a tableau by dividing the last column by the rightmost number in the same row with the largest positive magnitude. The quotient will be the value of the variable heading that column.

If there are no numbers with the largest positive magnitude in the same row, its flag is not lost and one must repeat the process until all flags are lost.

If all flags are lost, the usual solving of the simplex method will take place of the above process.

Solving Minimization Problems[2]

When we have a minimization problem, we then **convert it into a maximization problem**. Minimizing a variable $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$ is the same as maximizing $Z = -c_1x_1 - c_2x_2 - ... - c_nx_n$, with all of the constraints unchanged.

[2] http://www.zweigmedia.com/RealWorld/tutorialsf4/framesSimplex2.html