Misc. Durrett Problems

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1.3.2. (NPR) We will prove that when $X_1, X_2, ..., X_n$ are random variables, then it is also true that $X_1 + X_2 + ... + X_n$ is a random variable. To do so, it is enough to verify that $X_1 + X_2$ is a random variable, and the general case will follow by induction.

By **Theorem 1.3.1**, it is enough to show that $(X_1 + X_2)^{-1}((-\infty, a)) \in \mathcal{F}$ for any $a \in \mathbb{Q}$, since we have seen that these sets generate the σ-algebra $\mathcal{R}$.

We claim

$$(X_1 + X_2)^{-1}((-\infty, a)) = \bigcup_{p \in \mathbb{Q}} \left[X_2^{-1}((-\infty, p)) \cap X_1^{-1}((-\infty, a - p)) \right]. \tag{1}$$

Indeed, one direction is immediate. If $X_2(\omega) < p$ for some $p \in \mathbb{Q}$ and $X_1(\omega) < a - p$, then $X_1 + X_2 < a$.

Conversely, if $X_1(\omega) + X_2(\omega) < a$, then by the density of the rational numbers in $\mathbb{R}$, we may pick $q \in \mathbb{Q}$ between $X_1(\omega) + X_2(\omega)$ and a. Since $a - q > 0$ by construction, we may again use the density of $\mathbb{Q}$ to pick $p \in \mathbb{Q}$ such that $X_2(\omega) < p < X_2(\omega) + (a - q)$. Rearranging the right-side inequality yields $p + q - a < X_2(\omega)$. Hence,

$$X_1(\omega) < q - X_2(\omega) < q - (p + q - a) = a - p,$$

as desired. Recall that $X_2(\omega) < p$ by construction. Hence, the sets are equal, and the right-hand side of (1) is a countable union of intersections of sets that are in $\mathcal{F}$, as X_1 and X_2 are assumed to be random variables. Hence, $(X_1 + X_2)^{-1}((-\infty, a)) \in \mathcal{F}$ by the axioms of a σ-algebra.

1.3.4 (i) Show that a continuous function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a measurable map from $(\mathbb{R}^d, \mathcal{R}^d)$ to $(\mathbb{R}, \mathcal{R})$.

Proof. Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be continuous. Recall that $\mathcal{R} = \sigma(\mathcal{T})$, where $\mathcal{T}$ represents the topology of open sets on $\mathbb{R}$. We have seen that in order to show f is measurable, it suffices to prove that $f^{-1}(U)$ is measurable for any $U \in \mathcal{T}$.

Of course, since f is continuous, $f^{-1}(U)$ is open by definition. Hence, $f^{-1}(U)$ is a Borel set in $\mathbb{R}^d$, meaning that $f^{-1}(U) \in \mathcal{R}^d$. Thus, since $f^{-1}(U) \in \mathcal{R}^d$ for every $U \in \mathcal{T}$, it follows from **Theorem 1.3.1** that $f^{-1}(A) \in \mathcal{R}^d$ for any $A \in \sigma(\mathcal{T})$. That is, f is measurable. $\square$

(ii) Show that $\mathcal{R}^d$ is the smallest σ-field that makes all the continuous functions measurable.

Proof. Let $\mathcal{A}$ be any σ-algebra such that any continuous function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is measurable as a function from $(\mathbb{R}^d, \mathcal{A})$ to $(\mathbb{R}, \mathcal{R})$. The goal is to show that $\mathcal{R}^d \subseteq \mathcal{A}$. One way to prove this is to show that any closed set $V \subseteq \mathbb{R}^d$ is contained in this σ-algebra $\mathcal{A}$. Indeed, since $\mathcal{R}^d$ is also generated by the closed subsets of $\mathbb{R}^d$, it is hence the *smallest* σ-algebra containing all the closed sets. This will prove that $\mathcal{R}^d \subseteq \mathcal{A}$.

To prove this goal, we let V be an arbitrary closed subset of $\mathbb{R}^d$. We may define a function $\delta_V : \mathbb{R}^d \longrightarrow \mathbb{R}$ by $\delta_V(x) := d(x, V) = \inf\{d(x, y) \mid y \in V\}$. It is not difficult to see that δ_V is a continuous function; and since V is closed, it will also have the convenient property that $\delta_V(x) = 0$ if and only if $x \in V$. Thus, we see that $\delta_V^{-1}(\{0\}) = V$.

Now, we have assumed that every continuous function must also be measurable with respect to the σ-algebra $\mathcal{A}$. Hence, $V = \delta_V^{-1}(\{0\}) \in \mathcal{A}$ since $\{0\} \in \mathcal{R}$. This completes the proof, as we have shown that any closed set in $\mathbb{R}^d$ must also be contained in the σ-algebra $\mathcal{A}$. $\square$