

## Notation

We sometimes write  $A \in \aleph_0$  to mean “ $A$  is countable”. and  $A \in \aleph_1$  to mean “ $A$  is uncountable”.

$\chi \sim N(0, 1)$  means  $\Xi$  has a normal distribution with mean 0 and std 1.

(TBD) means to be done (NPR) means ”needs peer review”. If you are not the original author and agree the problem is correct, you can remove this tag.

### 1.1.1

We show that  $(\mathbb{R}, \mathcal{F}, P)$  is a probability space, where  $\mathcal{F} = \{A \subset \mathbb{R} : A \in \aleph_0 \text{ or } A^c \in \aleph_0\}$  and  $P(A) = 0$  if  $A \in \aleph_0$  and 1 if  $A^c \in \aleph_0$ . We must first show that  $\mathcal{F}$  is a  $\sigma$ -algebra.

- Let  $A \in \mathcal{F}$ . If  $A \in \aleph_0$ , then  $A^c \in \mathcal{F}$  since  $(A^c)^c = A \in \aleph_0$ . If  $A^c \in \aleph_0$ , then  $(A^c)^c = A \in \mathcal{F}$  since  $A^c \in \aleph_0$ .
- Let  $(A_i) \subset \mathcal{F}$  be a countable sequence of sets. Then either  $A = \bigcup A_i$  is countable or not. If  $A$  is countable, then  $A \in \mathcal{F}$ . If  $A$  is uncountable, then at least one  $A_k$  must be uncountable. But  $A_k \in \mathcal{F}$ , so  $A_k^c \in \aleph_0$ . But then  $A^c = \bigcap A_i^c \subset A_k^c \in \aleph_0$ . Hence  $A \in \mathcal{F}$ .

It remains to check that  $P$  is a probability measure.

- Let  $A \in \mathcal{F}$ . Then  $P(A) = 0$  or 1, so  $P(A) \geq P(\emptyset) = 0$  since  $\emptyset \in \aleph_0$ .
- $P(\mathbb{R}) = 1$  since  $\mathbb{R}^c = \emptyset \in \aleph_0$ .
- Let  $(A_i) \subset \mathcal{F}$  be a countable sequence of disjoint sets. We must show  $P(\bigcup A_i) = \sum_i P(A_i)$ . Observe that if each  $A_i$  is countable, then  $\bigcup A_i$  is countable and so both sides of the equality are zero. We claim that in fact at most one  $A_i$  can be uncountable. In that case,  $\bigcup A_i$  is uncountable and  $\sum_i P(A_i) = 1$ , so the equality holds. Suppose  $A_i$  and  $A_j$  are two uncountable disjoint sets in  $\mathcal{F}$ . Then,

$$\mathbb{R} = \emptyset^c = (A_i \cap A_j)^c = A_i^c \cup A_j^c.$$

This is a contradiction, since  $A_i^c \cup A_j^c$  is countable, but  $\mathbb{R}$  is not.

### 1.1.2 (TBD)

Let  $\mathcal{S}_d$  be the collection of sets of the form

$$(a_1, b_1] \times \dots \times (a_d, b_d] \subset \mathbb{R}^d, -\infty \leq a_i < b_i \leq \infty$$

and in addition the empty set. We claim that  $\sigma(\mathcal{S}_d) = \mathcal{R}^d$ , the Borel subsets of  $\mathbb{R}$ .

### 1.1.3

We show that the  $\sigma$ -algebra  $\mathcal{R}^d$  of Borel sets in  $\mathbb{R}^d$  is countably generated. It suffices to provide a countable collection of sets in  $\mathcal{R}^d$  that generate any open subset of  $\mathbb{R}^d$ . Let  $Q$  be the following countable collection:

$$Q = \{B(x, 1/n) : x \in \mathbb{Q}^d, n \in \mathbb{N}\}$$

where  $B(x, 1/n) := \{y \in \mathbb{R}^d : |x - y| < 1/n\}$ . Let  $U \subset \mathbb{R}^d$  be any open set. Let  $Q' = Q \cap 2^U$  (i.e. the subcollection of  $Q$  consisting of subsets of  $U$ ). We claim that

$$U = \bigcup_{V \in Q'} V$$

Clearly  $\bigcup_{V \in Q'} V \subset U$ . For the other inclusion, let  $x \in U$ . Then, since  $U$  is open, there exists a number  $N \in \mathbb{N}$  such that  $B(x, 1/N) \subset U$ . By the density of  $\mathbb{Q}^d \subset \mathbb{R}^d$ , there exists a point  $p \in \mathbb{Q}^d$  such that  $d(x, p) < \frac{1}{2N}$ . But then  $x \in B(p, \frac{1}{2N})$  and this ball is a member of  $Q'$  since if  $z \in U^c$  is arbitrary then

$$\frac{1}{N} < d(x, z) \leq d(x, p) + d(p, z) < \frac{1}{2N} + d(p, z)$$

and hence  $d(p, z) > \frac{1}{2N}$ , implying  $d(p, U^c) = \inf_{z \in U^c} d(p, z) > \frac{1}{2N}$ . Thus  $x \in \bigcup_{V \in Q'} V$ .

#### 1.1.4 (TBD)

- (i) Suppose  $F_1 \subset F_2 \subset \dots$  are  $\sigma$ -algebras. We show  $F := \bigcup F_i$  is a  $\sigma$ -algebra. Let  $A \in F$ . Then  $A \in F_k$  for some  $k$ . Hence  $A^c \in F_k \subset F$ , since  $F_k$  is a  $\sigma$ -algebra. If also  $B \in F$  then  $B \in F_j$  for some  $j$ . and we have  $A \in F_N$  for all  $N \geq k$  and  $B \in F_M$  for all  $M \geq j$ . In particular  $A \in F_s$  and  $B \in F_s$ , where  $s = \max j, k$  so  $A \cup B \in F_s$  since  $F_s$  is a  $\sigma$ -algebra. Hence  $A \cup B \in F$ , and we are done.

(ii)

#### 1.1.5 (TBD)

#### 1.2.1 (NPR)

Suppose  $X$  and  $Y$  are random variables on  $(\Omega, F, P)$  and let  $A \in F$ . We show that if  $Z(\omega) := X(\omega)$  for  $\omega \in A$  and  $Z(\omega) := Y(\omega)$  for  $\omega \in A^c$ , then  $Z$  is a random variable. Clearly  $Z$  maps  $\Omega$  to  $\mathbb{R}$ . It remains to show that  $Z$  is  $F$ -measurable. Let  $B$  be a Borel subset of  $\mathbb{R}$ . Then:

$$\begin{aligned} Z^{-1}(B) &= \{\omega \in \Omega : Z(\omega) \in B\} \\ &= \{\omega \in A : Z(\omega) \in B\} \cup \{\omega \in A^c : Z(\omega) \in B\} \\ &= \{\omega \in A : X(\omega) \in B\} \cup \{\omega \in A^c : Y(\omega) \in B\} \\ &\subset \{\omega \in \Omega : X(\omega) \in B\} \cup \{\omega \in \Omega : Y(\omega) \in B\} \\ &= X^{-1}(B) \cup Y^{-1}(B) \\ &\in F, \end{aligned}$$

as desired.

#### 1.2.2 (NPR)

Suppose  $\chi \sim N(0, 1)$ . Using Theorem 1.2.6, we have the estimate:

$$\left(\frac{1}{4} - \frac{1}{4^3}\right) e^{-\frac{4^2}{2}} \leq \int_4^\infty e^{-\frac{y^2}{2}} dy \leq \frac{1}{4} e^{-\frac{4^2}{2}}.$$

Hence:

$$\frac{1}{\sqrt{2\pi}} \left( \frac{1}{4} - \frac{1}{4^3} \right) e^{\frac{-4^2}{2}} \leq P(\chi \geq 4) \leq \frac{1}{4\sqrt{2\pi}} e^{\frac{-4^2}{2}}.$$

i.e.

$$3.13664 \cdot 10^{-5} \leq P(\chi \geq 4) \leq 3.34575 \cdot 10^{-5}$$

Note that the true value is  $P(\chi \geq 4) \approx 3.16712 \cdot 10^{-5}$ .

### 1.2.3

We show that a distribution function has only countably many discontinuities. Suppose  $J \subset \mathbb{R}$  is the set of discontinuities of a distribution function  $F$ . Observe that, for each  $a \in J$ , we have  $F(a) - F(a-) = F(a) - \lim_{y \uparrow a} F(y) > 0$  since  $F$  is discontinuous at  $a$  and nondecreasing. Hence the open intervals  $(F(a-), F(a)) \subset \mathbb{R}$  have nonzero measure. The intervals are also disjoint, since  $F$  is nondecreasing. But then,

$$\{(F(a-), F(a)) : a \in J\}$$

is a collection of nonempty disjoint intervals in  $\mathbb{R}$ . We may (by the Axiom of Choice) define a function  $\phi : J \rightarrow \mathbb{Q}$  via  $\phi(a) = q_a$  where  $q_a$  is an arbitrary rational number in the interval  $(F(a-), F(a))$ . This function is injective because the intervals are disjoint. Hence the cardinality of  $J$  must be at most the cardinality of  $\mathbb{Q}$ . In particular  $J$  cannot be uncountable.

### 1.2.4 (NPR)

We show that if  $F(x) = P(X \leq x)$  is continuous, then  $Y := F(X)$  has a uniform distribution on  $(0, 1)$ . Since  $F$  is continuous, for each  $y \in (0, 1)$  there exists by IVT an  $x \in \mathbb{R}$  such that  $F(x) = y$ . Define  $G : (0, 1) \rightarrow \mathbb{R}$  via  $G(y) = \sup\{x \in \mathbb{R} : F(x) = y\}$ . Observe that:

- (a)  $G$  is strictly increasing,
- (b)  $G(F(x)) \geq x$  for all  $x \in \mathbb{R}$ , and
- (c)  $F(G(y)) = y$  for all  $y \in (0, 1)$ .

Let  $y \in (0, 1)$  and consider the sets:

$$A = \{\omega \in \Omega : G(F(X(\omega))) \leq G(y)\}$$

$$B = \{\omega \in \Omega : X(\omega) \leq G(y)\}.$$

We claim  $A = B$  as follows. Let  $\omega \in A$ . Then  $X(\omega) \leq G(F(X(\omega))) \leq G(y)$  by (b). Hence  $\omega \in B$ . On the other hand, if  $\omega \in B$ , then  $G(F(X(\omega))) \leq G(F(G(y)))$  since both  $F$  and  $G$  are nondecreasing. But  $G(F(G(y))) = G(y)$  by (c). Hence  $\omega \in A$ . Therefore

$$(*) \quad P(G(F(X)) \leq G(y)) = P(X \leq G(y)).$$

Hence, we have for any  $y \in (0, 1)$ :

$$\begin{aligned} P(Y \leq y) &= P(F(X) \leq y) \\ &= P(G(F(X)) \leq G(y)) \quad \text{by (a)} \\ &= P(X \leq G(y)) \quad \text{by (*)} \\ &= F(G(y)) \\ &= y \quad \text{by (c)} \end{aligned}$$

as desired.

### 1.2.5 (TBD NPR)

Let  $X : \Omega \rightarrow \mathbb{R}$  be continuous with density  $f$ . Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be strictly increasing and differentiable on  $(a, b) \subset \mathbb{R}$ . We compute the density of  $g(X)$ .

Let  $a, b \in \text{range}(g(X(\Omega))) \subset \mathbb{R}$ . Since  $g$  is strictly increasing, there are unique values  $g^{-1}(a)$  and  $g^{-1}(b)$  such that  $g(g^{-1}(a)) = a$  and  $g(g^{-1}(b)) = b$ . Thus consider the sets:

$$A = \{\omega \in \Omega : a \leq g(X(\omega)) \leq b\}$$

$$A = \{\omega \in \Omega : g^{-1}(a) \leq X(\omega) \leq g^{-1}(b)\}$$

### 1.2.6 (TBD)

### 1.2.7 (TBD)

### 1.3.1 (TBD)

### 1.3.2 (TBD)

### 1.3.3

We show that if  $X_n \rightarrow X : \Omega \rightarrow \mathbb{R}$  almost surely and  $f$  is continuous on  $\mathbb{R}$ , then  $f(X_n) \rightarrow f(X)$  almost surely. Assume that Recall that  $X_n \rightarrow X$  a.s means  $P(\Omega_0) = 1$ , where

$$\Omega_0 = \{\omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) \text{ exists}\}.$$

Let

$$\Omega_1 = \{\omega \in \Omega : \lim_{n \rightarrow \infty} f(X_n(\omega)) \text{ exists}\}.$$

If  $\omega \in \Omega_0$ , then the limit  $X(\omega) := \lim_{n \rightarrow \infty} X_n(\omega)$  exists and so, by continuity, the limit

$$\lim_{x \rightarrow \infty} f(X_n(\omega)) = f(\lim_{n \rightarrow \infty} X_n(\omega)) = f(X(\omega))$$

exists. Hence  $\Omega_0 \subset \Omega_1$ . But then  $1 \geq P(\Omega_1) \geq P(\Omega_0) = 1$ , hence  $f(X_n) \rightarrow f(X)$  a.s.

### 1.3.4 (TBD)

### 1.3.5

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and let  $S_a = \{x \in \mathbb{R} : f(x) \leq a\}$ . We show that  $f$  is lower semicontinuous (that is:  $\liminf_{y \rightarrow x} f(y) \geq f(x)$  for each  $x \in \mathbb{R}$ ) iff  $S_a$  is closed for each  $a \in \mathbb{R}$ . Suppose first that  $f$  is l.s.c. and let  $a \in \mathbb{R}$ . Let  $(x_n) \subset S_a$  be a sequence such that  $x_n \rightarrow y$ . Hence for all  $n$ , we have  $f(x_n) \leq a$ . Then:

$$f(y) \leq \liminf_{y \rightarrow x} f(y) = \liminf_{n \rightarrow \infty} f(x_n) \leq a.$$

Hence  $a \in S_a$ , so  $S_a$  is closed.

Conversely, suppose  $S_a$  is closed for each  $a \in \mathbb{R}$ . In particular, fixing  $x \in \mathbb{R}$  and  $\epsilon > 0$ , we have that

$$S = \{x \in \mathbb{R} : f(x) \leq f(x) - \epsilon\}$$

is closed. Now, observe that  $x \in S^c$  since  $f(x) > f(x) - \epsilon$ . Since  $S^c$  is open, there exists an  $N \in \mathbb{N}$  and a sequence  $y_k \rightarrow x$  such that  $y_k \in S^c$  for all  $k \geq N$ . Then if  $k \geq N$ , we have

$$\begin{aligned} f(y_k) &> f(x) - \epsilon && \text{since } y_k \in S^c \\ \inf_{k \geq N} f(y_k) &> f(x) - \epsilon && \text{since } f(x) - \epsilon \text{ is a lower bound} \\ \sup_N \inf_{k \geq N} f(y_k) &\geq f(x) && \text{since } \epsilon \text{ was arbitrary} \end{aligned}$$

as desired. Since  $x$  was arbitrary,  $f$  is l.s.c. on  $\mathbb{R}$ .

### 1.3.6 (TBD)

### 1.3.7 (TBD)

### 1.3.8 (TBD)

### 1.3.9 (TBD)

### 1.4.1

We show that if  $f \geq 0$  and  $\int f d\mu = 0$ , then  $f = 0$  a.e. Let  $n \in \mathbb{N}$  and consider the set  $A_n = \{x \in \mathbb{R} : f(x) \geq \frac{1}{n}\}$ . Observe that  $A_n \uparrow A := \{x \in \mathbb{R} : f(x) > 0\}$ . Hence, by continuity of measure from below we have  $\mu(A_n) \uparrow \mu(A)$ . We will show below that  $\mu(A_n) = 0$ , implying  $\mu(A) = 0$ . This gives the result because  $\mu(A) = 0$  (together with  $f \geq 0$ ) means precisely that  $f = 0$  except on a set of measure zero.

Observe that for each  $x \in \mathbb{R}$  we have  $f(x) \geq \frac{1}{n} \mathbb{I}_{A_n}(x)$ , where  $\mathbb{I}_{A_n}$  is the indicator function of  $A_n$ , since if  $x \in A_n$ , then  $f(x) \geq \frac{1}{n} \cdot 1 = \frac{1}{n} \mathbb{I}_{A_n}(x)$  and if  $x \notin A_n$ , then  $f(x) \geq 0 = \frac{1}{n} \mathbb{I}_{A_n}(x)$ . We then have by monotonicity:

$$0 = \int f d\mu \geq \int \frac{1}{n} \mathbb{I}_{A_n}(x) d\mu = \frac{1}{n} \mu(A_n) \geq 0,$$

which implies  $\mu(A_n) = 0$ , as desired.

**1.4.2 (TBD)**

**1.4.3 (TBD)**

**1.4.4 (TBD)**

**1.5.1 (TBD)**

**1.5.2 (TBD)**

**1.5.3 (TBD)**

**1.5.4 (TBD)**

**1.5.5 (TBD)**

**1.5.6 (TBD)**

**1.5.7 (TBD)**

**1.5.8 (TBD)**

**1.5.9 (TBD)**

**1.5.10 (TBD)**