Misc. Durrett Problems

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October 2023

1.3.2. (NPR) We will prove that when $X_1, X_2, ..., X_n$ are random variables, then it is also true that $X_1 + X_2 + ... + X_n$ is a random variable. To do so, it is enough to verify that $X_1 + X_2$ is a random variable, and the general case will follow by induction.

By **Theorem 1.3.1**, it is enough to show that $(X_1+X_2)^{-1}((-\infty, a)) \in \mathcal{F}$ for any $a \in \mathbb{Q}$, since we have seen that these sets generate the σ-algebra $\mathcal{R}$.

We claim

$$(X_1 + X_2)^{-1}((-\infty, a)) = \bigcup_{p \in \mathbb{Q}} \left[X_2^{-1}((-\infty, p)) \cap X_1^{-1}((-\infty, a-p)) \right]. \tag{1}$$

Indeed, one direction is immediate. If $X_2(\omega) < p$ for some $p \in \mathbb{Q}$ and $X_1(\omega) < a - p$, then $X_1 + X_2 < a$.

Conversely, if $X_1(\omega) + X_2(\omega) < a$, then by the density of the rational numbers in $\mathbb{R}$, we may pick $q \in \mathbb{Q}$ between $X_1(\omega) + X_2(\omega)$ and a. Since $a - q > 0$ by construction, we may again use the density of $\mathbb{Q}$ to pick $p \in \mathbb{Q}$ such that $X_2(\omega) < p < X_2(\omega) + (a-q)$. Rearranging the right-side inequality yields $p + q - a < X_2(\omega)$. Hence,

$$X_1(\omega) < q - X_2(\omega) < q - (p + q - a) = a - p,$$

as desired. Recall that $X_2(\omega) < p$ by construction. Hence, the sets are equal, and the right-hand side of (1) is a countable union of intersections of sets that are in $\mathcal{F}$, as X_1 and X_2 are assumed to be random variables. Hence, $(X_1 + X_2)^{-1}((-\infty, a)) \in \mathcal{F}$ by the axioms of a σ-algebra.