Notation

We sometimes write $A \in \aleph_0$ to mean "A is countable". and $A \in \aleph_1$ to mean "A is uncountable".

$\chi \sim N(0,1)$ means Ξ has a normal distribution with mean 0 and std 1.

1.1.1

We show that $(\mathbb{R}, \mathcal{F}, P)$ is a probability space, where $\mathcal{F} = \{A \subset \mathbb{R} : A \in \aleph_0 \text{ or } A^c \in \aleph_0\}$ and $P(A) = 0$ if $A \in \aleph_0$ and 1 if $A^c \in \aleph_0$. We must first show that $\mathcal{F}$ is a σ-algebra.

- Let $A \in \mathcal{F}$. If $A \in \aleph_0$, then $A^c \in \mathcal{F}$ since $(A^c)^c = A \in \aleph_0$. If $A^c \in \aleph_0$, then $(A^c)^c = A \in \mathcal{F}$ since $A^c \in \aleph_0$.
- Let $(A_i) \subset \mathcal{F}$ be a countable sequence of sets. Then either $A = \bigcup A_i$ is countable or not. If A is countable, then $A \in \mathcal{F}$. If A is uncountable, then at least one A_k must be uncountable. But $A_k \in F$, so $A_k^c \in \aleph_0$. But then $A^c = \bigcap A_i^c \subset A_k^c \in \aleph_0$. Hence $A \in F$.

It remains to check that P is a probability measure.

- Let $A \in \mathcal{F}$. Then $P(A) = 0$ or 1, so $P(A) \geq P(\varnothing) = 0$ since $\varnothing \in \aleph_0$.
- $P(\mathbb{R}) = 1$ since $\mathbb{R}^c = \varnothing \in \aleph_0$.
- Let $(A_i) \subset \mathcal{F}$ be a countable sequence of disjoint sets. We must show $P(\bigcup A_i) = \sum_i P(A_i)$. Observe that if each A_i is countable, then $\bigcup A_i$ is countable and so both sides of the equality are zero. We claim that in fact at most one A_i can be uncountable. In that case, $\bigcup A_i$ is uncountable and $\sum_i P(A_i) = 1$, so the equality holds. Suppose A_i and A_j are two uncountable disjoint sets in $\mathcal{F}$. Then,

  $$\mathbb{R} = \varnothing^c = (A_i \cap A_j)^c = A_i^c \cup A_j^c.$$

  This is a contradiction, since $A_i^c \cup A_j^c$ is countable, but $\mathbb{R}$ is not.

1.1.2 (TBD)

Let $\mathcal{S}_d$ be the collection of sets of the form

$$(a_1, b_1] \times \dots \times (a_d, b_d] \subset \mathbb{R}^d, -\infty \leq a_i < b_i \leq \infty$$

and in addition the empty set. We claim that $\sigma(\mathcal{S}_d) = \mathcal{R}^d$, the Borel subsets of $\mathbb{R}$.

1.1.3

We show that the σ-alegbra $\mathcal{R}^d$ of Borel sets in $\mathbb{R}^d$ is countably generated. It suffices to provide a countable collection of sets in $\mathcal{R}^d$ that generate any open subset of $\mathbb{R}^d$. Let Q be the following countable collection:

$$Q = \{B(x, 1/n) : x \in \mathbb{Q}^d, n \in \mathbb{N}\}$$

where $B(x, 1/n) := \{y \in \mathbb{R}^d : |x - y| < 1/n\}$. Let $U \subset \mathbb{R}^d$ be any open set. Let $Q' = Q \cap 2^U$ (i.e. the subcollection of Q consisting of subsets of U). We claim that

$$U = \bigcup_{V \in Q'} V$$

Clearly $\bigcup_{V \in Q'} V \subset U$. For the other inclusion, let $x \in U$. Then, since U is open, there exists a number $N \in \mathbb{N}$ such that $B(x, 1/N) \subset U$. By the density of $\mathbb{Q}^d \subset \mathbb{R}^d$, there exists a point $p \in \mathbb{Q}^d$ such that $d(x,p) < \frac{1}{2N}$. But then $x \in B(p, \frac{1}{2N})$ and this ball is a member of Q' since if $z \in U^c$ is arbitrary then

$$\frac{1}{N} < d(x,z) \le d(x,p) + d(p,z) < \frac{1}{2N} + d(p,z)$$

and hence $d(p,z) > \frac{1}{2N}$, implying $d(p,U^c) = \inf_{z \in U^c} d(p,z) > \frac{1}{2N}$. Thus $x \in \bigcup_{V \in Q'} V$.

1.1.4 (TBD)

(i) Suppose $F_1 \subset F_1 \subset \ldots$ are σ-algebras. We show $F := \bigcup F_i$ is a σ-algebra. Let $A \in F$. Then $A \in F_k$ for some k. Hence $A^c \in F_k \subset F$, since F_k is a σ-algebra. If also $B \in F$ then $B \in F_j$ for some j. and we have $A \in F_N$ for all $N \ge k$ and $B \in F_M$ for all $M \ge j$. In particular $A \in F_s$ and $B \in F_s$, where $s = \max j, k$ so $A \cup B \in F_s$ since F_s is a σ-algebra. Hence $A \cup B \in F$, and we are done.

(ii)

1.1.5 (TBD)

1.2.1 (NPR)

Suppose X and Y are random variables on (Ω, F, P) and let $A \in F$. We show that if $Z(\omega) := X(\omega)$ for $\omega \in A$ and $Z(\omega) := Y(\omega)$ for $\omega \in A^c$, then Z is a random variable. Clearly Z maps Ω to $\mathbb{R}$. It remains to show that Z is F-measurable. Let B be a Borel subset of $\mathbb{R}$. Then:

$$\begin{aligned} Z^{-1}(B) &= \{\omega \in \Omega : Z(\omega) \in B\} \\ &= \{\omega \in A : Z(\omega) \in B\} \cup \{\omega \in A^c : Z(\omega) \in B\} \\ &= \{\omega \in A : X(\omega) \in B\} \cup \{\omega \in A^c : Y(\omega) \in B\} \\ &\subset \{\omega \in \Omega : X(\omega) \in B\} \cup \{\omega \in \Omega : Y(\omega) \in B\} \\ &= X^{-1}(B) \cup Y^{-1}(B) \\ &\in F, \end{aligned}$$

as desired.

1.2.2 (NPR)

Suppose $\chi \sim N(0,1)$. Using Theorem 1.2.6, we have the estimate:

$$\left(\frac{1}{4} - \frac{1}{4^3}\right) e^{\frac{-4^2}{2}} \le \int_4^\infty e^{\frac{-y^2}{2}}\, dy \le \frac{1}{4} e^{\frac{-4^2}{2}}.$$

Hence:

$$\frac{1}{\sqrt{2\pi}} \left(\frac{1}{4} - \frac{1}{4^3}\right) e^{\frac{-4^2}{2}} \le P(\chi \ge 4) \le \frac{1}{4\sqrt{2\pi}} e^{\frac{-4^2}{2}}.$$

i.e.

$$3.13664 \cdot 10^{-5} \le P(\chi \ge 4) \le 3.34575 \cdot 10^{-5}$$

Note that the true value is $P(\chi \ge 4) \approx 3.16712 \cdot 10^{-5}$.

1.2.3

We show that a distribution function has only countably many discontinuities. Suppose $J \subset \mathbb{R}$ is the set of discontinuities of a distribution function F. Observe that, for each $a \in J$, we have $F(a) - F(a-) = F(a) - \lim_{y\uparrow a} F(y) > 0$ since F is discontinuous at a and nondecreasing. Hence the open intervals $(F(a-), F(a)) \subset \mathbb{R}$ have nonzero measure. The intervals are also disjoint, since F is nondecreasing. But then,

$$\{(F(a-), F(a)) : a \in J\}$$

is a collection of nonempty disjoint intervals in $\mathbb{R}$. We may (by the Axiom of Choice) define a function $\phi : J \to \mathbb{Q}$ via $\phi(a) = q_a$ where q_a is an arbitrary rational number in the interval $(F(a-), F(a))$. This function is injective because the intervals are disjoint. Hence the cardinality of J must be at most the cardinality of $\mathbb{Q}$. In particular J cannot be uncountable.

1.2.4 (NPR)

We show that if $F(x) = P(X \le x)$ is continuous, then $Y := F(X)$ has a uniform distribution on $(0,1)$. Since F is continuous, for each $y \in (0,1)$ there exists by IVT an $x \in \mathbb{R}$ such that $F(x) = y$. Define $G : (0,1) \to \mathbb{R}$ via $G(y) = \sup\{x \in \mathbb{R} : F(x) = y\}$. Observe that:

(a) G is strictly increasing,

(b) $G(F(x)) \ge x$ for all $x \in \mathbb{R}$, and

(c) $F(G(y)) = y$ for all $y \in (0,1)$.

Let $y \in (0,1)$ and consider the sets:

$$A = \{\omega \in \Omega : G(F(X(\omega))) \le G(y)\}$$

$$B = \{\omega \in \Omega : X(\omega) \le G(y)\}.$$

We claim $A = B$ as follows. Let $\omega \in A$. Then $X(\omega) \le G(F(X(\omega))) \le G(y)$ by (b). Hence $\omega \in B$. On the other hand, if $\omega \in B$, then $G(F(X(\omega))) \le G(F(G(y)))$ since both F and G are nondecreasing. But $G(F(G(y))) = G(y)$ by (c). Hence $\omega \in A$. Therefore

$$(*) \qquad P(G(F(X)) \le G(y)) = P(X \le G(y)).$$

Hence, we have for any $y \in (0,1)$:

$$\begin{aligned} P(Y \le y) &= P(F(X) \le y) \\ &= P(G(F(X)) \le G(y)) \quad \text{by (a)} \\ &= P(X \le G(y)) \quad \text{by } (*) \\ &= F(G(y)) \\ &= y \quad \text{by (c)} \end{aligned}$$

as desired.

1.2.5 (TBD)

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1.2.6 (TBD)

1.2.7 (TBD)

1.3.1 (TBD)

1.3.2 (TBD)

1.3.3

We show that if $X_n \to X : \Omega \to \mathbb{R}$ almost surely and f is continuous on $\mathbb{R}$, then $f(X_n) \to f(X)$ almost surely. Assume that Recall that $X_n \to X$ a.s means $P(\Omega_0) = 1$, where

$$\Omega_0 = \{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) \text{ exists}\}.$$

Let

$$\Omega_1 = \{\omega \in \Omega : \lim_{n\to\infty} f(X_n(\omega)) \text{ exists}\}.$$

If $\omega \in \Omega_0$, then the limit $X(\omega) := \lim_{n\to\infty} X_n(\omega)$ exists and so, by continuity, the limit

$$\lim_{x\to\infty} f(X_n(\omega)) = f(\lim_{n\to\infty} X_n(\omega)) = f(X(\omega))$$

exists. Hence $\Omega_0 \subset \Omega_1$. But then $1 \geq P(\Omega_1) \geq P(\Omega_0) = 1$, hence $f(X_n) \to f(X)$ a.s.

1.3.4 (TBD)

1.3.5 (TBD)

1.3.6 (TBD)

1.3.7 (TBD)

1.3.8 (TBD)

1.3.9 (TBD)

1.4.1 (TBD)

1.4.2 (TBD)

1.4.3 (TBD)

1.4.4 (TBD)

1.5.1 (TBD)

1.5.2 (TBD)

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1.5.8 (TBD)

1.5.9 (TBD)

1.5.10 (TBD)