Chapter 1

Direct Proofs

Just as the name implies, a **direct proof** is a method of proving that uses known facts to directly prove a given statement. We might synonymise the term “direct proof” with “straightforward reasoning”.

Example 1.1. *Cookies*

Alice, Bill, and Claire love chocolate-chip cookies. With their combined efforts, they are able to finish a cookie jar from full capacity in very short time. One morning, the three discover that one cookie remains in the jar and they decide to work out the logistics of sharing it later in the day. They come back that afternoon to find that the cookie jar has been releved of its treat. Alice asks with genuine curiousity: “who ate the cookie?”, to which Claire replies: “I don’t know!”. Our job is to figure out who ate the cookie.

Assume that in this scenario, (1) a person speaks with complete sincerity and (2) only Alice, Bill or Claire could have eaten the cookie.

We deduce that Alice couldn’t have been the one to eat the cookie, since she asked the question and we assumed sincerity. Thus either Bill or Claire had eaten the cookie. Just as quickly we find that Claire is innocent for the following reason: *if* she did eat the cookie, then she would have known who ate the cookie: namely herself.

Thus the only person left is Bill. Since either Alice, Claire or Bill could have eaten the cookie, and we know that neither Alice nor Claire did it, we conclude that Bill must have eaten the cookie.

Example 1.2. *Simplifying Taxes*

At the grocery, one finds that the total base price for thier items is multiplied by some scaling constant t. As of writing this document, in Toronto, Canada, $t = 1.13$. Suppose that our grocery list is:

Item	Price
Bag of Apples	1.75
French Baguette	1.25
Bag of Coffee Grounds	7.00

So, our base price is $b = 1.75 + 1.25 + 7.00 = 10.00$ dollars. Thus, our final price is $t \times b = 1.13(10.00)$ dollars.

What we really want to know is how much more we are going to pay after taxes, so we focus on the 13% corresponding to the decimal 0.13. Mentally computing 13% of 10.00 is not the easiest task. An easier task is computing 10% of 13.0 since it involves just moving the decimal place 1 place left. The numbers that we calculate are the same since by associatvity:

$$t \times b = 0.13 \times 10 = \frac{13}{100} \times 10 = 0.1 \times 13.$$

Next, suppose that we forget to buy items on our shipping list, so we buy each item seprately on multiple trips. So the tax is applied to each item:

$$(1.13 \times 1.75) + (1.13 \times 1.25) + (1.13 \times 7.00)$$

We are concerned with whether buying these items seperately actually costs us more than buying them all at once. But we know that by factoring out the common coefficient 1.13:

$$(1.13 \times 1.75) + (1.13 \times 1.25) + (1.13 \times 7.00) = 1.13(1.75 + 1.25 + 7.00)$$

So buying each item individually wastes us no more money than buying the entire list at once (time, however, is another story).

When we formalize our straightforward reasoning, we must also formalize our reasoning's structure. The proof structure, in order, is:

1. The statement to be proved (the claim),
2. Relevant definitions,
3. Logical deductions that follow from the definitions, and
4. The final conclusions.

So, applying this structure to the first example in this chapter, we have:

1. Bill is the one who ate the cookie,
2. (1) Everyone in this scenario speaks with sincerity, and (2) Either one of Alice, Bill or Claire could have eaten the cookie.
3. The deductions that lead us to show that neither Alice nor Claire ate the cookie.
4. Because of (2), Bill must have eaten the cookie.

In general, the structure will be assumed implicit so we won't enumerate the deductive structure in our proofs.

We start off by proving some naturally understood properties of the real numbers.

1.1 Proofs with Algebra

The tools of algebra give extreme expressive power in our proof structure.

Example 1.3. *Mutually Inclusive Definitions*

The definition of odd and even functions might lead you to believe that odd and even functions are mutually exclusive. We show that due to one special function, this is not the case.

Claim 1. *The function $z(x) = 0$ is the only function that is both an even function and an odd function.*

Proof. Let f be some function that has the property that it is both even and odd. Then, for all $x \in \text{Dom}(f)$, f must also satisfy the following:

$$\begin{aligned}
2f(x) &= f(x) + f(x) && && (1.1)\\
&= f(x) + f(-x) && f \text{ is even} && (1.2)\\
&= f(x) + (-f(x)) && f \text{ is odd} && (1.3)\\
&= f(x) - f(x) && && (1.4)\\
&= 0 && && (1.5)
\end{aligned}$$

By the transitive property of equality, we have that expressions (1.1) and (1.5) are equal. Thus $2f(x) = 0$, meaning that $f(x) = 0 = z(x)$ by dividing by 2 on both sides.

We conclude that if f is both even and odd, then it is necessarily equal to z, the additive function identity. □

Note a technicality: z must have the property that if $x \in \text{Dom}(z)$, then $-x \in \text{Dom}(z)$. For example, the function $z'(x) = 0$ defined on $x \in [1, 2]$ is neither even nor odd, while $z''(x) = 0$ defined on $x \in [-2, -1] \cup [1, 2]$ is.

The example above illustrates the power of a proof: out of the infinite possibilities and combinations of functions, we assert that only one function fits the criteria.

Example 1.4. *The sum of two odd numbers is even*

Theorem 1. *Let a and b be odd numbers. Then $a + b$ is an even number.*

Proof. We may express a and b as:

$$\begin{aligned}
a &= 2n + 1, && \text{and}\\
b &= 2m + 1; && \text{for } n, m \in \mathbb{Z}
\end{aligned}$$

This comes from the definitions of odd numbers. As a verifying example, $a = -7, b = 3$ if $n = -4$ and $m = 1$. Adding a and b, we immediately get the desired result:

$$\begin{aligned}
a + b &= (2n + 1) + (2m + 1) && \text{by definition of } a \text{ and } b\\
&= (2n + 2m) + (1 + 1) && \text{commutativity of } \mathbb{Z}\\
&= 2(n + m + 1) && \text{simplifying}
\end{aligned}$$

Since the sum of an integer and another integer will always be an integer, we conclude that $a + b$, by definition, is an even number, as wanted.

□

In this case, we used the known definitions of odd numbers and properties of simple algebra to achieve our desired result.

Aside from finding nice properties of various objects, proofs allow us to develop algebraic tools:

Theorem 2. *Let $a, b \in \mathbb{R}$ such that $a = b$. Then, for all $c \in \mathbb{R}$, $a + c = b + c$.*

Proof. We deduce:

$$\begin{aligned}
a + c &= a + 0 + c && (1.6)\\
&= a + (b - a) + c, \quad \text{since } b = a && (1.7)\\
&= (a - a) + b + c, \quad \text{commutativity of } + && (1.8)\\
&= b + c && (1.9)
\end{aligned}$$

So, by the transitivity of equality, we have $a + c = b + c$, as wanted. □

Theorem 3. *Let $a, b \in \mathbb{R}$ such that $a = b$. Then for all $c \in \mathbb{R}$, $ca = cb$.*

Proof.

$$\begin{aligned}
ca &= ca \cdot (1) && (1.10)\\
&= ca \cdot (b/a) \quad \text{because } b = a \text{ means } b = 1 \cdot a && (1.11)\\
&= cb \cdot (a/a) \quad \text{commutativity of } \cdot && (1.12)\\
&= cb && (1.13)
\end{aligned}$$

Thus, $ca = cb$, as wanted. □

Before we start the next section, it might be worthwhile to note in passing that, unless the proof is trivial, proofs require us to observe and describe a phenomenon in more than one way; to glean an additional perspective. And when we combine our knowledge, we lead ourselves to new truths. This is especially true when we prove Pythagoras' Theorem next.

1.2 Proofs with Geometry

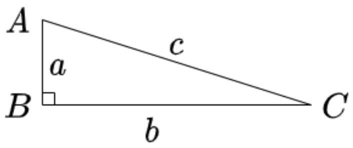
Theorem 4. *$\times$ distributes over $+$. That is, for $a, b, c \in \mathbb{R}$, $a(b+c) = ab+bc = (b+c)a$.*

Corollary 1.1. *Let $a, b, c, d \in \mathbb{R}$. Then $(a + b)(c + d) = ac + ad + bc + bd$.*

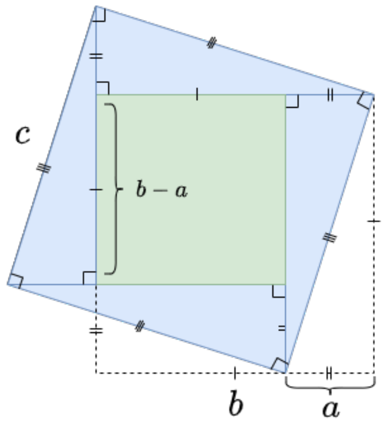
Proof. See questions. □

Theorem 5. *(Pythagoras' Theorem) Given any right-angled triangle $\triangle ABC$ where AC is the hypotenuse, we have:*

$$|AB|^2 + |BC|^2 = |AC|^2.$$



(a) An arbitrary right angle $\triangle ABC$ with side lengths $|AB| = a, |BC| = b, |AC| = c$.



(b) Square A_{WXYZ} constructed from $\triangle ABC$.

Figure 1.1: Crucial Shapes of the Pythagorean Theorem proof.

Proof. Let $\triangle ABC$ be an arbitrary right-angled triangle, and let $a \equiv |AB|$, $b \equiv |BC|$ and $c \equiv |AC|$ We construct the square $\square WXYZ$ by connecting four similar triangles base side to hight side as in Figure 1.1.

The area of $WXYZ$, A_{WXYZ}, is equal to c^2 because $|WX| = |AC|$. However, we can compute its A_{WXYZ} a different way. By our construction, we find that A_{WXYZ} is equal to the four triangle areas plus the smaller square in the middle. The side of the square is equal to $|BC| - |AC| = b - a$. So, at last we get:

$$
\begin{aligned}
A_{WXYZ} &= 4(ba/2) + (b-a)^2 && \text{from above} && (1.14)\\
&= 2ba + (b^2 - 2ba + a^2) && \text{by Corollary 1.1} && (1.15)\\
&= b^2 + a^2 && && (1.16)
\end{aligned}
$$

So, $c^2 = A_{WXYZ} = b^2 + a^2$ and so our result follows by transitivity. □

With this powerful tool in hand, we prove the more general case:

Theorem 6 (Cosine Law)**.** *Given any triangle $\triangle ABC$, denote the longest side as AC and denote the angle that the line segment AB makes with BC. Then*

$$|AC|^2 = |AB|^2 + |BC|^2 - |AB||BC|\cos(\theta).$$

And so the cosine law leads us to this result.

Corollary 1.2. *Let $a, b, c \in \mathbb{R}$ be lengths such that $a \leq b \leq c$. Then, a triangle can be formed with lenghts a, b, c iff $a - b < c < a + b$.*

Proof. [$\Longrightarrow$] Suppose that $\triangle ABC$ is a triangle, where AC is the longest side with length c. Then, $\triangle ABC$ satisfies the cosine law. Since $-1 \leq cos(\theta) \leq 1$, we get:

$$c^2 = a^2 + b^2 - 2ab\cos(\theta) \tag{1.17}$$
$$> a^2 + b^2 - 2ab, \qquad \text{When } \theta = 0 \tag{1.18}$$
$$= (a-b)^2, \qquad \text{By Corollary 1.1} \tag{1.19}$$

and

$$c^2 < a^2 + b^2 + 2ab, \qquad \text{When } \theta = \pi \tag{1.20}$$
$$= (a+b)^2. \tag{1.21}$$

Taking square roots of the inequalities, we achieve our desired result.

[$\Longleftarrow$] Suppose that $a, b, c; a < b < c$ are positive lengths such that $|b-a| < c < a+b$. We wish to prove that we can make a triangle out of these lengths.

From the proof of the opposide direction, it is easy to see that $c^2 = a^2 + b^2 - 2ab\cos(\theta)$ for some $\theta \in (0, \pi)$. Construct a triangle $\triangle ABC$ such that AB is parallel on the x-axis with length a, and BC extends from AB's left-most point, turning π radians counter-clockwise as in Figure

Using Pythagoras' Theorem, we calculate $|AC|^2$:

$$|AC|^2 \tag{1.22}$$
$$= (a + b\sin(\theta - \pi/2))^2 - (b\cos(\theta - \pi/2))^2 \tag{1.23}$$
$$= a^2 - 2ab\cos(\theta) + b^2\cos^2(\theta) + b^2\sin^2(\theta) \qquad \text{Translating } \sin, \cos \text{ right} \tag{1.24}$$
$$= a^2 - 2ab\cos(\theta) + b^2(\sin^2(\theta) + \cos^2(\theta)) \tag{1.25}$$
$$= a^2 + b^2 - 2ab\cos(\theta) \tag{1.26}$$
$$= c^2 \tag{1.27}$$

So $|AC| = c$, as wanted. □