Chapter 1

Bad Proofs

Here, we explore examples of what **not** to do, and what consistutes bad form and holes in our reasoning. Wherever possible, we correct the mistakes made in our first try.

1.1 Types of Bad Form

As we have previously stated, proofs are the power of reasoning and so our proofs are only as strong as our reason. No matter what proof style we use or how eloquently we present our proof, if one deduction is faulty then the whole proof is rejected. The mathematical court holds zero reservations.

The issue boils down to logical validity and circularity. An argument that is logically valid means that every deduction is justified by the definitions or deductions *preceding* it.[1] On the other hand, a circular argument is one that assumes that its conclusion is true. Though a circular argument is logically valid (of course, if A is true then A is true), its discoveries hold no value until the assumed condition is independently proven.

We will explore these cases of bad proofs:

1. Non-exhaustive Cases,
2. Deductive Inconsistency,
3. Circular Logic, and
4. Ill-definiteness.

Note that an invalid argument for some claim does not necessarily permit us to conclude that the claim is false. To show that an argument is faulty is an argument - a proof - in itself. So by studying bad form we are participating in a logical exercise.

1.2 Non-exhaustive Cases

Consider the following (erroneous) claim:

[1] I will not attempt to define the term "justified" except for requiring that the deduction makes sense, taking prior information into consideration.

Claim 1. *Let $n \leq 10$, $n \in \mathbb{N}$. The number $n! = n \cdot (n-1) \cdots 2 \cdot 1$ is* ***not divisible*** *by 81.*

Proof. Since all the factors of $n!$ for $n < 10$ are factors in $10!$, it follows that $10!$ is not divisible by 81, then the claim is true.

Notice that

$$10! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10.$$

None of these integer factors are a multiple of 81, as wanted. □

This "proof" is invalid because it did not consider all the *possible* factors that come from combining the integer factors of $10!$. We show that, in fact, $9!$ is divisible by 81:

Proof.

$$\begin{aligned} 9! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 && (1.1)\\ &= 3 \cdot 3 \cdot 9 \cdot (2 \cdot 4 \cdot 5 \cdot 2 \cdot 7 \cdot 8), && \text{factoring} \quad (1.2)\\ &= 81 \cdot (2 \cdot 4 \cdot 5 \cdot 2 \cdot 7 \cdot 8) && (1.3) \end{aligned}$$

So, indeed, $9! = 81k$, for some $k \in \mathbb{N}$, and thus the above claim is false. □

1.3 Inconsistency

Consider the (incorrect) claim for numbers in $\mathbb{R}$:

Claim 2. $0 = 1$

Proof. Let $a = 1, b = a$. Then:

$$\begin{aligned} & a = b && \text{commutativity of } = \quad (1.4)\\ \implies & a^2 = b^2 && (1.5)\\ \implies & a^2 = ab && (1.6)\\ \implies & a^2 - b^2 = ab - b^2 && (1.7)\\ \implies & (a+b)(a-b) = b(a-b) && (1.8)\\ \implies & a + b = b && \text{dividing by } (a-b) \quad (1.9)\\ \implies & a = 0 && (1.10)\\ \implies & 1 = 0 && a = 1, \text{ by definition} \quad (1.11) \end{aligned}$$

□

The symbol $\implies$ loosely stands for "implies", or "it follows that". We discuss this in section ??.

Of course, the lines from equation (1.9) downward do not hold, as the operation performed is a division by zero. A division by zero by any number never yields a finite number, so the deduction is invalid in $\mathbb{R}$.

1.4 Circular Logic

We incorrectly prove the **correct** claim:

Claim 3. *Let* $x, y \in \mathbb{R}$ *such that* $0 \leq x \leq y$. *Then* $x^2 \leq y^2$.

Proof.

$$x \leq y \qquad \text{by assumption} \tag{1.12}$$
$$\implies x^2 \leq y^2 \qquad \text{squaring both sides.} \tag{1.13}$$

□

Perhaps because of its conciseness, this proof may seem to hold at first sight. But notice that the proof essentially restates the claim statement: equation (1.12) restates the first sentence (the condition), and equation (1.13) restates the second sentence (the consequent). The transition between these two equations *assumes* that the claim that we are trying to prove true, is true - hence circularity.

We retry the proof of the claim:

Proof.

$$x \leq y \tag{1.14}$$
$$\implies x - y \leq 0 \tag{1.15}$$
$$\implies (x-y)(x+y) \leq 0 \qquad - \text{ multiplied by } + \text{ is } - \tag{1.16}$$
$$\implies x^2 - y^2 \leq 0 \qquad \text{distributing} \tag{1.17}$$
$$\implies x^2 \leq y^2 \qquad \text{adding } y^2 \text{ on both sides} \tag{1.18}$$

□

1.5 Ill-definiteness

Ill-definiteness refers to some definition or condition for an object that has no satisfying member, or whose value is undefined. To assign x to be an even odd number or an odd even number, for example, is a ill-defined assignment as there is no number that is both even and odd at the same time. Another example is letting b to be the largest number in the interval $[0, 1)$.

Of course, if one is doubtful, then they must *prove* that the definition is ill-defined.

Example 1.1. *There is no largest number in the interval* $[0, 1)$.

Proof. Let $a \in [0, 1)$. Then $0 \leq a < 1$. Consider the value $A = (a + 1)/2$. We show that $a < A < 1$, proving that for any choice of a, we can always choose a larger number in the interval.

$$\begin{aligned}
A &= \frac{a+1}{2} && && (1.19)\\
&> \frac{a+a}{2}, && \text{since } a < 1, \text{ so } a + a < a + 1. && (1.20)\\
&= \frac{2a}{2} && && (1.21)\\
&= a && && (1.22)
\end{aligned}$$

So $a < A$. Similarly:

$$\begin{aligned}
A &= \frac{a+1}{2} && && (1.23)\\
&< \frac{1+1}{2}, && 1 > a. && (1.24)\\
&= \frac{2}{2} && && (1.25)\\
&= 1 && && (1.26)
\end{aligned}$$

We conclude that $a < A < 1$, as wanted. □

An argument that claims and uses the existence of an ill-defined object is a contradiction, because no such object exists.

When we explore *Proofs by Contradiction* in Chapter **??**, we will actually leverage these concepts of ill-definiteness to prove our claims. The reason why this is a valid method of proof is because we *want* to reach a contradiction and one way is showing that a definition that we use is ill-defined.

1.6 Disproving

Questions

1. Consider the following argument for the theorem: If $a, b \in \mathbb{R}$ such that $a = b$, then $a + c = b + c$

   Proof. We deduce the following:

   $$\begin{aligned}
   &a = b\\
   &a - b = 0\\
   &(a - b) + (c - c) = 0 && \text{since } c - c = 0\\
   &(a + c) + (-b + -c) = 0 && \text{commutativity of } \mathbb{R}\\
   &a + c = b + c && \text{adding } b + c \text{ on both sides}
   \end{aligned}$$

   □

   What is wrong with this proof?

2. In Example 1.1 we asserted that there is no largest value in $[0, 1)$. One might raise an objection with the counterexample that the number $0.999... = 0.\bar{9}$ is the largest number in the set. Show the surprising fact that $0.\bar{9}$ is actually equal to 1. (Hint: $1/3 = 0.\bar{3}$).

3. Show that there is no smallest member of $(0, 1]$.

4. Suppose someone gives you the following proof that there is a integer that is both even and odd:

   Proof. Let $a = 2n + 1$ and $a = 2m$ for some integers $m, n \in \mathbb{Z}$. Then $2n + 1 = 2m$ means $n = (2m - 1)/2$. Thus, a is both even and odd when $(2m - 1)/2$ is an integer. As wanted. □

   What is wrong with this proof?

5. Suppose someone gives you the following proof that for any two numbers $a, b \in \mathbb{Z}$, $(a + b)^2 = a^2 + b^2$:

   Proof. We have two cases to check:

   Case 1: $a = b$

   Case 2: $a \neq b$ □