Chapter 1

Introduction

1.1 What are Proofs?

In life, we are given bits of information about the world around us. Statements like "it is raining in Toronto today", "if you are shorter than 147 centimeters, then you cannot enter", or "every person has two biological parents" is a statement that is either true, or false, exclusively. Presented with starting information, we are concerned with *reasoning* through whether another statement is true or false, consistent with our start.

Example: Suppose the second statement in the previous paragraph is true, and suppose that Ellie is 170 centimeters tall, Jack is taller than Ellie, and Sam is shorter than Jack.

We are concerned with whether it is true that (a) Ellie can enter (b) Jack can enter and (c) Sam can enter.

- Since Ellie is taller than 147 centimeters, she can enter.
- Since Jack is taller than Ellie, Jack's height is greater than Ellie's height. Therefore, Jack can enter.
- We cannot make any conclusion on whether Sam can enter.

Now, suppose that Ellie is exactly 147 centimeters tall. Then both Ellie and Jack can enter, but not Sam. If Ellie is any shorter, then Ellie and Sam may not enter, and we have no conclusion on whether Jack can enter.

To prove a statement is to *formalize* our reasoning about it. We assume a set of statements which we deem are true (assumptions), and we form a series of steps that logically follow from the previous step (deductions). In the next example, we will prefix our assumptions by **[A]** and our deductions by **[D]**.

Example: **[A]** In the land of Knights and Knaves, knights always tell the truth and knaves always tell lies. People are either knights, or they are knaves in this land.

You meet two people, Alice and Bob (henceforth referred to as A and B, respectively.) **[A]** A says "Both of us are knaves."

We wish to determine who is a knight and who is a knave.

[D] Suppose that A is a knight. If A is a knight, then the statement "Both of us are knaves" is a lie (since A is a knight and therefore cannot be a knave).

[D] Since knights cannot tell a lie, and A told a lie, then A must not have been a knight in the first place.

[D] Therefore, A cannot be a knight.

[D] Since A cannot be a knight, then A is a knave.

[D] Since knaves always tell lies, the statement "Both of us are knaves" is a false statement (a lie).

[D] The only way for this statement to be false is when B is a knight.

[D] Therefore, B must be a knight.

[D] Therefore, A is a knave and B is a knight.

Given a statement S, a proof for S is a convincing structured argument that shows S is true. The structure of a proof uses knowledge that we have to discover facts about that knowledge that we did not previously know. Naturally, we can use these facts to prove more results. The accumulation of these facts allows us to develop a body of statements known as a system.

⟨ **TODO: System Tree Diagram** ⟩

This document will discuss a handful of different ways to prove some statement S. Like tools in a toolbox, there are a variety of proof techniques wherein different scenarios are more effective in proving S than others; indeed, there is a craftsmanship and beauty in the way for which someone proves S.

The product of a (successful) proof allows us to definitively state whether some statement is true or false. This allows us to assert some perhaps counter-intuitive results:

Claim 1. *The infinite decimal number* $0.999...$ *(which we abbreviate to* $0.\bar{9}$*) is equal to* 1.

Proof. We examine properties of and manipulate the number $0.\bar{9}$ to achieve the desired equality. Let $n = 0.\bar{9}$. We deduce:

$$n = 0.\bar{9} \qquad \textbf{[Definition]} \tag{1.1}$$
$$10n = 9.\bar{9} \tag{1.2}$$
$$10n - (n) = 9.\bar{9} - (0.\bar{9}) \qquad \textbf{[Definition of } n\textbf{]} \tag{1.3}$$
$$9n = 9 \tag{1.4}$$
$$n = 1 \tag{1.5}$$

And thus, we see from the equivalence of the first and last equations that $0.999 = 1$, as claimed. □

Note that the term "convincing" stated is extremely important here. We provide an alternative proof to the claim:

Proof.

$$\begin{aligned}
3 \cdot 1/3 &= 1/3 + 1/3 + 1/3 && (1.6)\\
&= 0.\bar{3} + 0.\bar{3} + 0.\bar{3} \qquad \textbf{[Decimal representation of 1/3]} && (1.7)\\
&= 0.\bar{9} && (1.8)
\end{aligned}$$

But $3 \cdot 1/3 = 3/3 = 1$. □

Proofs are segregated into two groups: *Direct* and *Indirect*. We review some fundamental concepts of mathematics before diving in these techniques.