Transformadas de Laplace

Camargo Badillo Luis Mauricio

11 de febrero de 2024

Ecuaciones Diferenciales II
Oscar Gabriel Caballero Martínez
Grupo 2602
Matemáticas Aplicadas y Computación

2. $f(t) = c$

Calculamos la transformada de Laplace de $f(t) = c$:

$$\begin{aligned}
\mathcal{L}\{c\} &= \int_0^\infty e^{-st}(c) \\
&= \int_0^\infty ce^{-st} \\
&= c\int_0^\infty e^{-st} \\
&= \lim_{b\to\infty} c\int_0^b e^{-st}\ dt
\end{aligned}$$

Sustiuimos con $u = -st \implies du = -s\ dt$, obteniendo:

$$\begin{aligned}
&= \lim_{b\to\infty} -\frac{c}{s}\left[\int_0^{-sb} e^u\ du\right] \\
&= -\frac{c}{s}\lim_{b\to\infty}\left[e^u\Big|_0^{-sb}\right] \\
&= -\frac{c}{s}\lim_{b\to\infty}\left[e^{-sb} - e^0\right] \\
&= -\frac{c}{s}\left[\lim_{b\to\infty} e^{-sb} - 1\right]
\end{aligned}$$

Cuando $s > 0 \implies -s < 0$, por lo que podemos escribir:

$$\begin{aligned}
&= -\frac{c}{s}\left[\lim_{b\to-\infty} e^b - 1\right] \\
&= -\frac{c}{s}[0 - 1] \\
&= \frac{c}{s}
\end{aligned}$$

Así, finalmente:

$$\mathcal{L}\{c\} = \frac{c}{s}$$

8. $f(t) = \cos(t)$

Calculamos la transformada de Laplace de $f(t) = \cos(t)$:

$$\begin{aligned}
\mathcal{L}\{\cos(t)\} &= \int_0^\infty e^{-st}\cos(t)\ dt \\
&= \lim_{b\to\infty}\int_0^b e^{-st}\cos(t)\ dt
\end{aligned}$$

Integrando por partes, utilizando $u = \cos(t) \implies du = -\sin(t)\ dt$ y $dv = e^{-st}\ dt \implies v = -\frac{1}{s}e^{-st}$, obtenemos:

$$
\begin{aligned}
&= \lim_{b\to\infty} \left[\left. \frac{-\cos(t)e^{-st}}{s} \right|_0^b - \frac{1}{s}\int_0^b e^{-st}\sin(t)\ dt \right] \\
&= \lim_{b\to\infty} \left[-\frac{\cos(b)e^{-sb}}{s} + \frac{\cos(0)e^0}{s} \right] - \lim_{b\to\infty} \left[\frac{1}{s}\int_0^b e^{-st}\sin(t)\ dt \right] \\
&= \frac{1}{s} - \lim_{b\to\infty} \left[\frac{1}{s}\int_0^b e^{-st}\sin(t)\ dt \right] \\
&= \frac{1}{s} - \frac{1}{s}\lim_{b\to\infty} \int_0^b e^{-st}\sin(t)\ dt
\end{aligned}
$$

Una vez más, integrando por partes con $u = \sin(t) \implies du = \cos(t)\ dt$ y $dv = e^{-st}\ dt \implies v = -\frac{1}{s}e^{-st}$, tenemos:

$$
\begin{aligned}
&= \frac{1}{s} - \frac{1}{s}\lim_{b\to\infty} \left[\left. \frac{-\sin(t)e^{-st}}{s} \right|_0^b + \frac{1}{s}\int_0^b e^{-st}\cos(t)\ dt \right] \\
&= \frac{1}{s} - \frac{1}{s}\lim_{b\to\infty} \left[-\frac{\sin(b)e^{-sb}}{s} + \frac{\sin(0)e^0}{s} \right] + \lim_{b\to\infty} \left[\frac{1}{s}\int_0^b e^{-st}\cos(t)\ dt \right] \\
&= \frac{1}{s} - \frac{1}{s}\lim_{b\to\infty} \left[\frac{1}{s}\int_0^b e^{-st}\cos(t)\ dt \right] \\
&= \frac{1}{s} - \frac{1}{s^2}\lim_{b\to\infty} \int_0^b e^{-st}\cos(t)\ dt
\end{aligned}
$$

Observemos que:

$$
\lim_{b\to\infty} \int_0^b e^{-st}\cos(t)\ dt = \frac{1}{s} - \frac{1}{s^2}\lim_{b\to\infty} \int_0^b e^{-st}\cos(t)\ dt
$$

Estableciendo $a = \lim_{b\to\infty} \int_0^b e^{-st}\cos(t)\ dt$, tenemos:

$$
\begin{aligned}
a &= \frac{1}{s} - \frac{1}{s^2}a \\
\implies a + \frac{1}{s^2}a &= \frac{1}{s} \\
\implies a\left(\frac{1}{s^2} + 1\right) &= \frac{1}{s} \\
\implies a\left(\frac{1+s^2}{s^2}\right) &= \frac{1}{s} \\
\implies a &= \frac{s^2}{s(1+s^2)} \\
\implies a &= \frac{s}{s^2+1} \\
\implies \lim_{b\to\infty} \int_0^b e^{-st}\cos(t)\ dt &= \frac{s}{s^2+1} \\
\implies \int_0^\infty e^{-st}\cos(t)\ dt &= \frac{s}{s^2+1}
\end{aligned}
$$

Por lo tanto, finalmente:

$$\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}$$

11. $f(t) = e^{4t}$

Calculamos la transformada de Laplace de $f(t) = e^{4t}$:

$$\begin{aligned}
\mathcal{L}\{e^{4t}\} &= \int_0^\infty e^{-st}e^{4t}\ dt \\
&= \int_0^\infty e^{(4-s)t}\ dt \\
&= \lim_{b\to\infty}\int_0^b e^{(4-s)t}\ dt
\end{aligned}$$

Sustituyamos con $u = (4-s)t \implies du = 4-s\ dt$:

$$\begin{aligned}
&= \frac{1}{4-s}\lim_{b\to\infty}\int_0^{(4-s)b} e^u\ du \\
&= \frac{1}{4-s}\lim_{b\to\infty}\left.(e^u)\right|_0^{(4-s)b} \\
&= \frac{1}{4-s}\lim_{b\to\infty}\left(e^{(4-s)b} - e^0\right) \\
&= \frac{1}{4-s}\left[\lim_{b\to\infty} e^{(4-s)b} - 1\right]
\end{aligned}$$

Cuando $s > 4 \implies (4-s) < 0$, por lo que podemos escribir:

$$\begin{aligned}
&= \frac{1}{4-s}\left[\lim_{b\to-\infty} e^b - 1\right] \\
&= \frac{1}{4-s}(0-1) \\
&= \frac{1}{s-4}
\end{aligned}$$

Así, finalmente obtenemos que:

$$\mathcal{L}\{e^{4t}\} = \frac{1}{s-4} \qquad s > 4$$

12. $f(t) = e^{-2t}$

Calculamos la transformada de Laplace de $f(t) = e^{-2t}$:

$$\begin{aligned}
\mathcal{L}\{e^{-2t}\} &= \int_0^\infty e^{-st}e^{-2t}\ dt \\
&= \int_0^\infty e^{(-2-s)t}\ dt \\
&= \lim_{b\to\infty}\int_0^b e^{(-2-s)t}\ dt
\end{aligned}$$

Sustituyamos con $u = (-2-s)t \implies du = -2-s\ dt$:

$$
\begin{aligned}
&= \frac{1}{-2-s} \lim_{b\to\infty} \int_0^{(-2-s)b} e^u\ du \\
&= \frac{1}{-2-s} \lim_{b\to\infty} \left(e^u\right)\Big|_0^{(-2-s)b} \\
&= \frac{1}{-2-s} \lim_{b\to\infty} \left(e^{(-2-s)b} - e^0\right) \\
&= \frac{1}{-2-s} \left[\lim_{b\to\infty} e^{(-2-s)b} - 1\right]
\end{aligned}
$$

Cuando $s > -2 \implies (-2-s) < 0$, por lo que podemos escribir:

$$
\begin{aligned}
&= \frac{1}{-2-s} \left[\lim_{b\to-\infty} e^b - 1\right] \\
&= \frac{1}{-2-s}(0-1) \\
&= \frac{1}{s+2}
\end{aligned}
$$

Así, finalmente obtenemos que:

$$\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2} \qquad s > -2$$

14. $f(t) = \sinh(3t)$

Calculamos la transformada de Laplace de $f(t) = \sinh(3t)$:

$$\mathcal{L}\{\sinh(3t)\} = \int_0^\infty e^{-st} \sinh(3t)\ dt$$

Recordemos que $\sinh(u) = \frac{1}{2}(e^u - e^{-u})$, así que:

$$
\begin{aligned}
&= \frac{1}{2} \int_0^\infty e^{-st}(e^{3t} - e^{-3t})\ dt \\
&= \frac{1}{2} \left[\int_0^\infty e^{(3-s)t} - e^{(-3-s)t}\ dt\right] \\
&= \frac{1}{2} \left[\int_0^\infty e^{(3-s)t}\ dt - \int_0^\infty e^{(-3-s)t}\ dt\right] \\
&= \frac{1}{2} \lim_{b\to\infty} \left[\int_0^b e^{(3-s)t}\ dt - \int_0^b e^{(-3-s)t}\ dt\right]
\end{aligned}
$$

Sustituyendo con $u = (3-s)t \implies du = 3 - s\ dt$ y $v = (-3-s)t \implies dv = -3 - s\ dt$, obtenemos:

$$
\begin{aligned}
&= \frac{1}{2} \lim_{b\to\infty} \left[\frac{1}{3-s} \int_0^{(3-s)b} e^u\ du - \frac{1}{-3-s} \int_0^{(-3-s)b} e^u\ du \right] \\
&= \frac{1}{2} \lim_{b\to\infty} \left[\frac{1}{3-s} \left(e^u\right)\Big|_0^{(3-s)b} + \frac{1}{3+s} \left(e^u\right)\Big|_0^{(-3-s)b} \right] \\
&= \frac{1}{2} \lim_{b\to\infty} \left[\frac{e^{(3-s)b} - 1}{3-s} + \frac{e^{(-3-s)b} - 1}{3+s} \right] \\
&= \frac{1}{2} \lim_{b\to\infty} \left[\frac{(3+s)e^{(3-s)b} - 3 - s + (3-s)e^{(-3-s)b} - 3 + s}{9-s^2} \right] \\
&= \frac{1}{2} \lim_{b\to\infty} \left[\frac{(3+s)e^{(3-s)b} - 6 + (3-s)e^{(-3-s)b}}{9-s^2} \right] \\
&= \frac{1}{2(9-s^2)} \left[(3+s) \lim_{b\to\infty} e^{(3-s)b} + (3-s) \lim_{b\to\infty} e^{(-3-s)b} - 6 \right]
\end{aligned}
$$

Cuando $s > 3 \implies (3-s) < 0 \wedge (-3-s) < 0$, por lo que podemos escribir:

$$
\begin{aligned}
&= \frac{1}{2(9-s^2)} \left[(3+s) \lim_{b\to-\infty} e^b + (3-s) \lim_{b\to-\infty} e^b - 6 \right] \\
&= \frac{1}{2(9-s^2)} \left[(3+s)0 + (3-s)0 - 6 \right] \\
&= \frac{1}{2(9-s^2)} (-6) \\
&= \frac{3}{s^2-9}
\end{aligned}
$$

Así, finalmente:

$$\mathcal{L}\{\sinh(3t)\} = \frac{3}{s^2-9}$$

15. $f(t) = \cosh(6t)$

Calculamos la transformada de Laplace de $f(t) = \cosh(6t)$:

$$\mathcal{L}\{\cosh(6t)\} = \int_0^\infty e^{-st} \cosh(6t)\ dt$$

Recordemos que $\cosh(u) = \frac{1}{2}\left(e^u + e^{-u}\right)$, así que:

$$
\begin{aligned}
&= \frac{1}{2} \int_0^\infty e^{-st} \left(e^{6t} + e^{-6t}\right)\ dt \\
&= \frac{1}{2} \left[\int_0^\infty e^{(6-s)t} + e^{(-6-s)t}\ dt \right] \\
&= \frac{1}{2} \left[\int_0^\infty e^{(6-s)t}\ dt + \int_0^\infty e^{(-6-s)t}\ dt \right] \\
&= \frac{1}{2} \lim_{b\to\infty} \left[\int_0^b e^{(6-s)t}\ dt + \int_0^b e^{(-6-s)t}\ dt \right]
\end{aligned}
$$

Sustituyendo con $u = (6-s)t \implies du = 6 - s\ dt$ y $v = (-6-s)t \implies dv = -6 - s\ dt$, obtenemos:

$$
\begin{aligned}
&= \frac{1}{2} \lim_{b \to \infty} \left[\frac{1}{6-s} \int_0^{(6-s)b} e^u\ du + \frac{1}{-6-s} \int_0^{(-6-s)b} e^u\ du \right] \\
&= \frac{1}{2} \lim_{b \to \infty} \left[\frac{1}{6-s} \left(e^u\right)\Big|_0^{(6-s)b} - \frac{1}{6+s} \left(e^u\right)\Big|_0^{(-6-s)b} \right] \\
&= \frac{1}{2} \lim_{b \to \infty} \left[\frac{e^{(6-s)b} - 1}{6-s} - \frac{e^{(-6-s)b} - 1}{6+s} \right] \\
&= \frac{1}{2} \lim_{b \to \infty} \left[\frac{(6+s)e^{(6-s)b} - 6 - s - (6-s)e^{(-6-s)b} + 6 - s}{36 - s^2} \right] \\
&= \frac{1}{2} \lim_{b \to \infty} \left[\frac{(6+s)e^{(6-s)b} - 2s - (6-s)e^{(-6-s)b}}{36 - s^2} \right] \\
&= \frac{1}{2(36-s^2)} \left[(6+s) \lim_{b \to \infty} e^{(6-s)b} + (6-s) \lim_{b \to \infty} e^{(-6-s)b} - 2s \right]
\end{aligned}
$$

Cuando $s > 6 \implies (6-s) < 0 \wedge (-6-s) < 0$, por lo que podemos escribir:

$$
\begin{aligned}
&= \frac{1}{2(36-s^2)} \left[(6+s) \lim_{b \to -\infty} e^b + (6-s) \lim_{b \to -\infty} e^b - 2s \right] \\
&= \frac{1}{2(36-s^2)} \left[(6+s)0 + (6-s)0 - 2s \right] \\
&= \frac{1}{2(36-s^2)} (-2s) \\
&= \frac{s}{s^2 - 36}
\end{aligned}
$$

Así, finalmente:

$$
\mathcal{L}\{\cosh(6t)\} = \frac{s}{s^2 - 9}
$$

17. $f(t) = \cosh(at)$

Calculamos la transformada de Laplace de $f(t) = \cosh(at)$:

$$
\mathcal{L}\{\cosh(6t)\} = \int_0^\infty e^{-st} \cosh(at)\ dt
$$

Recordemos que $\cosh(u) = \frac{1}{2}\left(e^u + e^{-u}\right)$, así que:

$$
\begin{aligned}
&= \frac{1}{2} \int_0^\infty e^{-st} \left(e^{at} + e^{-at}\right)\ dt \\
&= \frac{1}{2} \left[\int_0^\infty e^{(a-s)t} + e^{(-a-s)t}\ dt \right] \\
&= \frac{1}{2} \left[\int_0^\infty e^{(a-s)t}\ dt + \int_0^\infty e^{(-a-s)t}\ dt \right] \\
&= \frac{1}{2} \lim_{b \to \infty} \left[\int_0^b e^{(a-s)t}\ dt + \int_0^b e^{(-a-s)t}\ dt \right]
\end{aligned}
$$

Sustituyendo con $u = (a-s)t \implies du = a - s\ dt$ y $v = (-a-s)t \implies dv = -a - s\ dt$, obtenemos:

$$
\begin{aligned}
&= \frac{1}{2}\lim_{b\to\infty}\left[\frac{1}{a-s}\int_0^{(a-s)b} e^u\ du + \frac{1}{-a-s}\int_0^{(-a-s)b} e^u\ du\right] \\
&= \frac{1}{2}\lim_{b\to\infty}\left[\frac{1}{a-s}\ (e^u)\big|_0^{(a-s)b} - \frac{1}{a+s}\ (e^u)\big|_0^{(-a-s)b}\right] \\
&= \frac{1}{2}\lim_{b\to\infty}\left[\frac{e^{(a-s)b}-1}{a-s} - \frac{e^{(-a-s)b}-1}{a+s}\right] \\
&= \frac{1}{2}\lim_{b\to\infty}\left[\frac{(a+s)e^{(a-s)b} - a - s - (a-s)e^{(-a-s)b} + a - s}{a^2-s^2}\right] \\
&= \frac{1}{2}\lim_{b\to\infty}\left[\frac{(a+s)e^{(a-s)b} - 2s - (a-s)e^{(-a-s)b}}{a^2-s^2}\right] \\
&= \frac{1}{2(a^2-s^2)}\left[(a+s)\lim_{b\to\infty} e^{(a-s)b} + (a-s)\lim_{b\to\infty} e^{(-a-s)b} - 2s\right]
\end{aligned}
$$

Cuando $s > a \implies (a-s) < 0 \wedge (-a-s) < 0$, por lo que podemos escribir:

$$
\begin{aligned}
&= \frac{1}{2(a^2-s^2)}\left[(a+s)\lim_{b\to-\infty} e^{b} + (a-s)\lim_{b\to-\infty} e^{b} - 2s\right] \\
&= \frac{1}{2(a^2-s^2)}\left[(a+s)0 + (a-s)0 - 2s\right] \\
&= \frac{1}{2(a^2-s^2)}(-2s) \\
&= \frac{s}{s^2-a^2}
\end{aligned}
$$

Así, finalmente:

$$\mathcal{L}\{\cosh(at)\} = \frac{s}{s^2-a^2}$$