



Transformadas de Laplace

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8. $f(t) = \cos(t)$

Calculamos la transformada de Laplace de $f(t) = \cos(t)$:

$$\begin{aligned}\mathcal{L}\{\cos(t)\} &= \int_0^\infty e^{-st} \cos(t) dt \\ &= \lim_{b \rightarrow \infty} \int_0^b e^{-st} \cos(t) dt\end{aligned}$$

Integrando por partes, utilizando $u = \cos(t) \implies du = -\sin(t) dt$ y $dv = e^{-st} dt \implies v = -\frac{1}{s}e^{-st}$, obtenemos:

$$\begin{aligned}&= \lim_{b \rightarrow \infty} \left[\frac{-\cos(t)e^{-st}}{s} \Big|_0^b - \frac{1}{s} \int_0^b e^{-st} \sin(t) dt \right] \\ &= \lim_{b \rightarrow \infty} \left[-\frac{\cos(b)e^{-sb}}{s} + \frac{\cos(0)e^0}{s} \right] - \lim_{b \rightarrow \infty} \left[\frac{1}{s} \int_0^b e^{-st} \sin(t) dt \right] \\ &= \frac{1}{s} - \lim_{b \rightarrow \infty} \left[\frac{1}{s} \int_0^b e^{-st} \sin(t) dt \right] \\ &= \frac{1}{s} - \frac{1}{s} \lim_{b \rightarrow \infty} \int_0^b e^{-st} \sin(t) dt\end{aligned}$$

Una vez más, integrando por partes con $u = \sin(t) \implies du = \cos(t) dt$ y $dv = e^{-st} dt \implies v = -\frac{1}{s}e^{-st}$, tenemos:

$$\begin{aligned}&= \frac{1}{s} - \frac{1}{s} \lim_{b \rightarrow \infty} \left[\frac{-\sin(t)e^{-st}}{s} \Big|_0^b + \frac{1}{s} \int_0^b e^{-st} \cos(t) dt \right] \\ &= \frac{1}{s} - \frac{1}{s} \lim_{b \rightarrow \infty} \left[-\frac{\sin(b)e^{-sb}}{s} + \frac{\sin(0)e^0}{s} \right] + \lim_{b \rightarrow \infty} \left[\frac{1}{s} \int_0^b e^{-st} \cos(t) dt \right] \\ &= \frac{1}{s} - \frac{1}{s} \lim_{b \rightarrow \infty} \left[\frac{1}{s} \int_0^b e^{-st} \cos(t) dt \right] \\ &= \frac{1}{s} - \frac{1}{s^2} \lim_{b \rightarrow \infty} \int_0^b e^{-st} \cos(t) dt\end{aligned}$$

Observemos que:

$$\lim_{b \rightarrow \infty} \int_0^b e^{-st} \cos(t) dt = \frac{1}{s} - \frac{1}{s^2} \lim_{b \rightarrow \infty} \int_0^b e^{-st} \cos(t) dt$$

Estableciendo $a = \lim_{b \rightarrow \infty} \int_0^b e^{-st} \cos(t) dt$, tenemos:

$$\begin{aligned}
 a &= \frac{1}{s} - \frac{1}{s^2}a \\
 \implies a + \frac{1}{s^2}a &= \frac{1}{s} \\
 \implies a \left(\frac{1}{s^2} + 1 \right) &= \frac{1}{s} \\
 \implies a \left(\frac{1 + s^2}{s^2} \right) &= \frac{1}{s} \\
 \implies a &= \frac{s^2}{s(1 + s^2)} \\
 \implies a &= \frac{s}{s^2 + 1} \\
 \implies \lim_{b \rightarrow \infty} \int_0^b e^{-st} \cos(t) dt &= \frac{s}{s^2 + 1} \\
 \implies \int_0^\infty e^{-st} \cos(t) dt &= \frac{s}{s^2 + 1}
 \end{aligned}$$

Por lo tanto, finalmente:

$$\mathcal{L}\{\cos(t)\} = \frac{s}{s^2 + 1}$$

11. $f(t) = e^{4t}$

Calculamos la transformada de Laplace de $f(t) = e^{4t}$:

$$\begin{aligned}
 \mathcal{L}\{e^{4t}\} &= \int_0^\infty e^{-st} e^{4t} dt \\
 &= \int_0^\infty e^{(4-s)t} dt \\
 &= \lim_{b \rightarrow \infty} \int_0^b e^{(4-s)t} dt
 \end{aligned}$$

Sustituyamos con $u = (4 - s)t \implies du = 4 - s dt$:

$$\begin{aligned}
 &= \frac{1}{4 - s} \lim_{b \rightarrow \infty} \int_0^{(4-s)b} e^u du \\
 &= \frac{1}{4 - s} \lim_{b \rightarrow \infty} (e^u) \Big|_0^{(4-s)b} \\
 &= \frac{1}{4 - s} \lim_{b \rightarrow \infty} (e^{(4-s)b} - e^0) \\
 &= \frac{1}{4 - s} \left[\lim_{b \rightarrow \infty} e^{(4-s)b} - 1 \right]
 \end{aligned}$$

Cuando $s > 4 \implies (4 - s) < 0$, por lo que podemos escribir:

$$\begin{aligned} &= \frac{1}{4 - s} \left[\lim_{b \rightarrow -\infty} e^b - 1 \right] \\ &= \frac{1}{4 - s} (0 - 1) \\ &= \frac{1}{s - 4} \end{aligned}$$

Así, finalmente obtenemos que:

$$\mathcal{L}\{e^{4t}\} = \frac{1}{s - 4} \quad s > 4$$

12. $f(t) = e^{-2t}$

Calculamos la transformada de Laplace de $f(t) = e^{-2t}$:

$$\begin{aligned} \mathcal{L}\{e^{-2t}\} &= \int_0^\infty e^{-st} e^{-2t} dt \\ &= \int_0^\infty e^{(-2-s)t} dt \\ &= \lim_{b \rightarrow \infty} \int_0^b e^{(-2-s)t} dt \end{aligned}$$

Sustituyamos con $u = (-2 - s)t \implies du = -2 - s dt$:

$$\begin{aligned} &= \frac{1}{-2 - s} \lim_{b \rightarrow \infty} \int_0^{(-2-s)b} e^u du \\ &= \frac{1}{-2 - s} \lim_{b \rightarrow \infty} (e^u) \Big|_0^{(-2-s)b} \\ &= \frac{1}{-2 - s} \lim_{b \rightarrow \infty} (e^{(-2-s)b} - e^0) \\ &= \frac{1}{-2 - s} \left[\lim_{b \rightarrow \infty} e^{(-2-s)b} - 1 \right] \end{aligned}$$

Cuando $s > -2 \implies (-2 - s) < 0$, por lo que podemos escribir:

$$\begin{aligned} &= \frac{1}{-2 - s} \left[\lim_{b \rightarrow -\infty} e^b - 1 \right] \\ &= \frac{1}{-2 - s} (0 - 1) \\ &= \frac{1}{s + 2} \end{aligned}$$

Así, finalmente obtenemos que:

$$\mathcal{L}\{e^{-2t}\} = \frac{1}{s + 2} \quad s > -2$$

14. $f(t) = \sinh(3t)$

Calculamos la transformada de Laplace de $f(t) = \sinh(3t)$:

$$\mathcal{L}\{\sinh(3t)\} = \int_0^{\infty} e^{-st} \sinh(3t) dt$$

Recordemos que $\sinh(u) = \frac{1}{2}(e^u - e^{-u})$, así que:

$$\begin{aligned} &= \frac{1}{2} \int_0^{\infty} e^{-st} (e^{3t} - e^{-3t}) dt \\ &= \frac{1}{2} \left[\int_0^{\infty} e^{(3-s)t} - e^{(-3-s)t} dt \right] \\ &= \frac{1}{2} \left[\int_0^{\infty} e^{(3-s)t} dt - \int_0^{\infty} e^{(-3-s)t} dt \right] \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\int_0^b e^{(3-s)t} dt - \int_0^b e^{(-3-s)t} dt \right] \end{aligned}$$

Sustituyendo con $u = (3-s)t \implies du = 3-s dt$ y $v = (-3-s)t \implies dv = -3-s dt$, obtenemos:

$$\begin{aligned} &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{1}{3-s} \int_0^{(3-s)b} e^u du - \frac{1}{-3-s} \int_0^{(-3-s)b} e^u du \right] \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{1}{3-s} (e^u)|_0^{(3-s)b} + \frac{1}{3+s} (e^u)|_0^{(-3-s)b} \right] \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{e^{(3-s)b} - 1}{3-s} + \frac{e^{(-3-s)b} - 1}{3+s} \right] \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{(3+s)e^{(3-s)b} - 3-s + (3-s)e^{(-3-s)b} - 3+s}{9-s^2} \right] \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{(3+s)e^{(3-s)b} - 6 + (3-s)e^{(-3-s)b}}{9-s^2} \right] \\ &= \frac{1}{2(9-s^2)} \left[(3+s) \lim_{b \rightarrow \infty} e^{(3-s)b} + (3-s) \lim_{b \rightarrow \infty} e^{(-3-s)b} - 6 \right] \end{aligned}$$

Cuando $s > 3 \implies (3-s) < 0 \wedge (-3-s) < 0$, por lo que podemos escribir:

$$\begin{aligned} &= \frac{1}{2(9-s^2)} \left[(3+s) \lim_{b \rightarrow -\infty} e^b + (3-s) \lim_{b \rightarrow -\infty} e^b - 6 \right] \\ &= \frac{1}{2(9-s^2)} [(3+s)0 + (3-s)0 - 6] \\ &= \frac{1}{2(9-s^2)} (-6) \\ &= \frac{3}{s^2-9} \end{aligned}$$

Así, finalmente:

$$\mathcal{L}\{\sinh(3t)\} = \frac{3}{s^2-9}$$

15. $f(t) = \cosh(6t)$

Calculamos la transformada de Laplace de $f(t) = \cosh(6t)$:

$$\mathcal{L}\{\cosh(6t)\} = \int_0^{\infty} e^{-st} \cosh(6t) dt$$

Recordemos que $\cosh(u) = \frac{1}{2} (e^u + e^{-u})$, así que:

$$\begin{aligned} &= \frac{1}{2} \int_0^{\infty} e^{-st} (e^{6t} + e^{-6t}) dt \\ &= \frac{1}{2} \left[\int_0^{\infty} e^{(6-s)t} + e^{(-6-s)t} dt \right] \\ &= \frac{1}{2} \left[\int_0^{\infty} e^{(6-s)t} dt + \int_0^{\infty} e^{(-6-s)t} dt \right] \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\int_0^b e^{(6-s)t} dt + \int_0^b e^{(-6-s)t} dt \right] \end{aligned}$$

Sustituyendo con $u = (6-s)t \Rightarrow du = 6-s dt$ y $v = (-6-s)t \Rightarrow dv = -6-s dt$, obtenemos:

$$\begin{aligned} &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{1}{6-s} \int_0^{(6-s)b} e^u du + \frac{1}{-6-s} \int_0^{(-6-s)b} e^u du \right] \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{1}{6-s} (e^u)|_0^{(6-s)b} - \frac{1}{6+s} (e^u)|_0^{(-6-s)b} \right] \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{e^{(6-s)b} - 1}{6-s} - \frac{e^{(-6-s)b} - 1}{6+s} \right] \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{(6+s)e^{(6-s)b} - 6-s - (6-s)e^{(-6-s)b} + 6-s}{36-s^2} \right] \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{(6+s)e^{(6-s)b} - 2s - (6-s)e^{(-6-s)b}}{36-s^2} \right] \\ &= \frac{1}{2(36-s^2)} \left[(6+s) \lim_{b \rightarrow \infty} e^{(6-s)b} + (6-s) \lim_{b \rightarrow \infty} e^{(-6-s)b} - 2s \right] \end{aligned}$$

Cuando $s > 6 \Rightarrow (6-s) < 0 \wedge (-6-s) < 0$, por lo que podemos escribir:

$$\begin{aligned} &= \frac{1}{2(36-s^2)} \left[(6+s) \lim_{b \rightarrow -\infty} e^b + (6-s) \lim_{b \rightarrow -\infty} e^b - 2s \right] \\ &= \frac{1}{2(36-s^2)} [(6+s)0 + (6-s)0 - 2s] \\ &= \frac{1}{2(36-s^2)} (-2s) \\ &= \frac{s}{s^2-36} \end{aligned}$$

Así, finalmente:

$$\mathcal{L}\{\cosh(6t)\} = \frac{s}{s^2-9}$$

17. $f(t) = \cosh(at)$

Calculamos la transformada de Laplace de $f(t) = \cosh(at)$:

$$\mathcal{L}\{\cosh(6t)\} = \int_0^{\infty} e^{-st} \cosh(at) dt$$

Recordemos que $\cosh(u) = \frac{1}{2} (e^u + e^{-u})$, así que:

$$\begin{aligned} &= \frac{1}{2} \int_0^{\infty} e^{-st} (e^{at} + e^{-at}) dt \\ &= \frac{1}{2} \left[\int_0^{\infty} e^{(a-s)t} + e^{(-a-s)t} dt \right] \\ &= \frac{1}{2} \left[\int_0^{\infty} e^{(a-s)t} dt + \int_0^{\infty} e^{(-a-s)t} dt \right] \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\int_0^b e^{(a-s)t} dt + \int_0^b e^{(-a-s)t} dt \right] \end{aligned}$$

Sustituyendo con $u = (a-s)t \implies du = a-s dt$ y $v = (-a-s)t \implies dv = -a-s dt$, obtenemos:

$$\begin{aligned} &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{1}{a-s} \int_0^{(a-s)b} e^u du + \frac{1}{-a-s} \int_0^{(-a-s)b} e^u du \right] \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{1}{a-s} (e^u)|_0^{(a-s)b} - \frac{1}{a+s} (e^u)|_0^{(-a-s)b} \right] \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{e^{(a-s)b} - 1}{a-s} - \frac{e^{(-a-s)b} - 1}{a+s} \right] \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{(a+s)e^{(a-s)b} - a-s - (a-s)e^{(-a-s)b} + a-s}{a^2 - s^2} \right] \\ &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{(a+s)e^{(a-s)b} - 2s - (a-s)e^{(-a-s)b}}{a^2 - s^2} \right] \\ &= \frac{1}{2(a^2 - s^2)} \left[(a+s) \lim_{b \rightarrow \infty} e^{(a-s)b} + (a-s) \lim_{b \rightarrow \infty} e^{(-a-s)b} - 2s \right] \end{aligned}$$

Cuando $s > a \implies (a-s) < 0 \wedge (-a-s) < 0$, por lo que podemos escribir:

$$\begin{aligned} &= \frac{1}{2(a^2 - s^2)} \left[(a+s) \lim_{b \rightarrow -\infty} e^b + (a-s) \lim_{b \rightarrow -\infty} e^b - 2s \right] \\ &= \frac{1}{2(a^2 - s^2)} [(a+s)0 + (a-s)0 - 2s] \\ &= \frac{1}{2(a^2 - s^2)} (-2s) \\ &= \frac{s}{s^2 - a^2} \end{aligned}$$

Así, finalmente:

$$\mathcal{L}\{\cosh(at)\} = \frac{s}{s^2 - a^2}$$