Transformadas de Lagrange

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11 de febrero de 2024

Ecuaciones Diferenciales II
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Grupo 2602
Matemáticas Aplicadas y Computación

8. $f(t) = \cos(t)$

Calculamos la transformada de Lagrange de $f(t) = \cos(t)$:

$$\begin{aligned}\mathcal{L}\{\cos(t)\} &= \int_0^\infty e^{-st}\cos(t)\ dt \\ &= \lim_{b\to\infty}\int_0^b e^{-st}\cos(t)\ dt\end{aligned}$$

Integrando por partes, utilizando $u = \cos(t) \implies du = -\sin(t)\ dt$ y $dv = e^{-st}\ dt \implies v = -\frac{1}{s}e^{-st}$, obtenemos:

$$\begin{aligned}&= \lim_{b\to\infty}\left[\left.\frac{-\cos(t)e^{-st}}{s}\right|_0^b - \frac{1}{s}\int_0^b e^{-st}\sin(t)\ dt\right] \\ &= \lim_{b\to\infty}\left[-\frac{\cos(b)e^{-sb}}{s} + \frac{\cos(0)e^0}{s}\right] - \lim_{b\to\infty}\left[\frac{1}{s}\int_0^b e^{-st}\sin(t)\ dt\right] \\ &= \frac{1}{s} - \lim_{b\to\infty}\left[\frac{1}{s}\int_0^b e^{-st}\sin(t)\ dt\right] \\ &= \frac{1}{s} - \frac{1}{s}\lim_{b\to\infty}\int_0^b e^{-st}\sin(t)\ dt\end{aligned}$$

Una vez más, integrando por partes con $u = \sin(t) \implies du = \cos(t)\ dt$ y $dv = e^{-st}\ dt \implies v = -\frac{1}{s}e^{-st}$, tenemos:

$$\begin{aligned}&= \frac{1}{s} - \frac{1}{s}\lim_{b\to\infty}\left[\left.\frac{-\sin(t)e^{-st}}{s}\right|_0^b + \frac{1}{s}\int_0^b e^{-st}\cos(t)\ dt\right] \\ &= \frac{1}{s} - \frac{1}{s}\lim_{b\to\infty}\left[-\frac{\sin(b)e^{-sb}}{s} + \frac{\sin(0)e^0}{s}\right] + \lim_{b\to\infty}\left[\frac{1}{s}\int_0^b e^{-st}\cos(t)\ dt\right] \\ &= \frac{1}{s} - \frac{1}{s}\lim_{b\to\infty}\left[\frac{1}{s}\int_0^b e^{-st}\cos(t)\ dt\right] \\ &= \frac{1}{s} - \frac{1}{s^2}\lim_{b\to\infty}\int_0^b e^{-st}\cos(t)\ dt\end{aligned}$$

Observemos que:

$$\lim_{b\to\infty}\int_0^b e^{-st}\cos(t)\ dt = \frac{1}{s} - \frac{1}{s^2}\lim_{b\to\infty}\int_0^b e^{-st}\cos(t)\ dt$$

Estableciendo $a = \lim_{b\to\infty} \int_0^b e^{-st}\cos(t)\ dt$, tenemos:

$$
\begin{aligned}
a &= \frac{1}{s} - \frac{1}{s^2}a \\
\implies a + \frac{1}{s^2}a &= \frac{1}{s} \\
\implies a\left(\frac{1}{s^2}+1\right) &= \frac{1}{s} \\
\implies a\left(\frac{1+s^2}{s^2}\right) &= \frac{1}{s} \\
\implies a &= \frac{s^2}{s(1+s^2)} \\
\implies a &= \frac{s}{s^2+1} \\
\implies \lim_{b\to\infty}\int_0^b e^{-st}\cos(t)\ dt &= \frac{s}{s^2+1} \\
\implies \int_0^\infty e^{-st}\cos(t)\ dt &= \frac{s}{s^2+1}
\end{aligned}
$$

Por lo tanto, finalmente:

$$\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}$$

11. $f(t) = e^{4t}$

Calculamos la transformada de Lagrange de $f(t) = e^{4t}$:

$$
\begin{aligned}
\mathcal{L}\{e^{4t}\} &= \int_0^\infty e^{-st}e^{4t}\ dt \\
&= \int_0^\infty e^{(4-s)t}\ dt \\
&= \lim_{b\to\infty}\int_0^b e^{(4-s)t}\ dt
\end{aligned}
$$

Sustituyamos con $u = (4-s)t \implies du = 4 - s\ dt$:

$$
\begin{aligned}
&= \frac{1}{4-s}\lim_{b\to\infty}\int_0^{(4-s)b} e^u\ du \\
&= \frac{1}{4-s}\lim_{b\to\infty}\left(e^u\right)\Big|_0^{(4-s)b} \\
&= \frac{1}{4-s}\lim_{b\to\infty}\left(e^{(4-s)b} - e^0\right) \\
&= \frac{1}{4-s}\left[\lim_{b\to\infty} e^{(4-s)b} - 1\right]
\end{aligned}
$$

Cuando $s > 4 \implies (4-s) < 0$, por lo que podemos escribir:

$$\begin{aligned}
&= \frac{1}{4-s}\left[\lim_{b\to -\infty} e^{b} - 1\right] \\
&= \frac{1}{4-s}(0-1) \\
&= \frac{1}{s-4}
\end{aligned}$$

Así, finalmente obtenemos que:

$$\mathcal{L}\{e^{4t}\} = \frac{1}{s-4} \qquad s > 4$$

12. $f(t) = e^{-2t}$

Calculamos la transformada de Lagrange de $f(t) = e^{-2t}$:

$$\begin{aligned}
\mathcal{L}\{e^{-2t}\} &= \int_0^\infty e^{-st}e^{-2t}\ dt \\
&= \int_0^\infty e^{(-2-s)t}\ dt \\
&= \lim_{b\to\infty} \int_0^b e^{(-2-s)t}\ dt
\end{aligned}$$

Sustituyamos con $u = (-2-s)t \implies du = -2 - s\ dt$:

$$\begin{aligned}
&= \frac{1}{-2-s} \lim_{b\to\infty} \int_0^{(-2-s)b} e^{u}\ du \\
&= \frac{1}{-2-s} \lim_{b\to\infty} \left(e^{u}\right)\Big|_0^{(-2-s)b} \\
&= \frac{1}{-2-s} \lim_{b\to\infty} \left(e^{(-2-s)b} - e^{0}\right) \\
&= \frac{1}{-2-s} \left[\lim_{b\to\infty} e^{(-2-s)b} - 1\right]
\end{aligned}$$

Cuando $s > -2 \implies (-2-s) < 0$, por lo que podemos escribir:

$$\begin{aligned}
&= \frac{1}{-2-s}\left[\lim_{b\to -\infty} e^{b} - 1\right] \\
&= \frac{1}{-2-s}(0-1) \\
&= \frac{1}{s+2}
\end{aligned}$$

Así, finalmente obtenemos que:

$$\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2} \qquad s > -2$$

14. $f(t) = \sinh(3t)$

Calculamos la transformada de Lagrange de $f(t) = \sinh(3t)$:

$$\mathcal{L}\{\sinh(3t)\} = \int_0^\infty e^{-st}\sinh(3t)\ dt$$

Recordemos que $\sinh(u) = \frac{1}{2}(e^u - e^{-u})$, así que:

$$\begin{aligned}
&= \frac{1}{2}\int_0^\infty e^{-st}(e^{3t} - e^{-3t})\ dt \\
&= \frac{1}{2}\left[\int_0^\infty e^{(3-s)t} - e^{(-3-s)t}\ dt\right] \\
&= \frac{1}{2}\left[\int_0^\infty e^{(3-s)t}\ dt - \int_0^\infty e^{(-3-s)t}\ dt\right] \\
&= \frac{1}{2}\lim_{b\to\infty}\left[\int_0^b e^{(3-s)t}\ dt - \int_0^b e^{(-3-s)t}\ dt\right]
\end{aligned}$$

Sustituyendo con $u = (3-s)t \implies du = 3 - s\ dt$ y $v = (-3-s)t \implies dv = -3 - s\ dt$, obtenemos:

$$\begin{aligned}
&= \frac{1}{2}\lim_{b\to\infty}\left[\frac{1}{3-s}\int_0^{(3-s)b} e^u\ du - \frac{1}{-3-s}\int_0^{(-3-s)b} e^u\ du\right] \\
&= \frac{1}{2}\lim_{b\to\infty}\left[\frac{1}{3-s}\ (e^u)\big|_0^{(3-s)b} + \frac{1}{3+s}\ (e^u)\big|_0^{(-3-s)b}\right] \\
&= \frac{1}{2}\lim_{b\to\infty}\left[\frac{e^{(3-s)b} - 1}{3-s} + \frac{e^{(-3-s)b} - 1}{3+s}\right] \\
&= \frac{1}{2}\lim_{b\to\infty}\left[\frac{(3+s)e^{(3-s)b} - 3 - s + (3-s)e^{(-3-s)b} - 3 + s}{9 - s^2}\right] \\
&= \frac{1}{2}\lim_{b\to\infty}\left[\frac{(3+s)e^{(3-s)b} - 6 + (3-s)e^{(-3-s)b}}{9 - s^2}\right] \\
&= \frac{1}{2(9-s^2)}\left[(3+s)\lim_{b\to\infty} e^{(3-s)b} + (3-s)\lim_{b\to\infty} e^{(-3-s)b} - 6\right]
\end{aligned}$$

Cuando $s > 3 \implies (3-s) < 0 \land (-3-s) < 0$, por lo que podemos escribir:

$$\begin{aligned}
&= \frac{1}{2(9-s^2)}\left[(3+s)\lim_{b\to-\infty} e^b + (3-s)\lim_{b\to-\infty} e^b - 6\right] \\
&= \frac{1}{2(9-s^2)}\left[(3+s)0 + (3-s)0 - 6\right] \\
&= \frac{1}{2(9-s^2)}(-6) \\
&= \frac{3}{s^2 - 9}
\end{aligned}$$

Así, finalmente:

$$\mathcal{L}\{\sinh(3t)\} = \frac{3}{s^2 - 9}$$

15. $f(t) = \cosh(6t)$

17. $f(t) = \cosh(at)$