Transformadas de Lagrange

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Ecuaciones Diferenciales II
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8. $f(t) = \cos(t)$

Calculamos la transformada de Lagrange de $f(t) = \cos(t)$:

$$\begin{aligned}
\mathcal{L}\{\cos(t)\} &= \int_0^\infty e^{-st}\cos(t)\ dt \\
&= \lim_{b\to\infty} \int_0^b e^{-st}\cos(t)\ dt
\end{aligned}$$

Integrando por partes, utilizando $u = \cos(t) \implies du = -\sin(t)\ dt$ y $dv = e^{-st}\ dt \implies v = -\frac{1}{s}e^{-st}$, obtenemos:

$$\begin{aligned}
&= \lim_{b\to\infty} \left[\frac{-\cos(t)e^{-st}}{s}\bigg|_0^b - \frac{1}{s}\int_0^b e^{-st}\sin(t)\ dt \right] \\
&= \lim_{b\to\infty} \left[-\frac{\cos(b)e^{-sb}}{s} + \frac{\cos(0)e^{0}}{s} \right] - \lim_{b\to\infty} \left[\frac{1}{s}\int_0^b e^{-st}\sin(t)\ dt \right] \\
&= \frac{1}{s} - \lim_{b\to\infty} \left[\frac{1}{s}\int_0^b e^{-st}\sin(t)\ dt \right] \\
&= \frac{1}{s} - \frac{1}{s}\lim_{b\to\infty} \int_0^b e^{-st}\sin(t)\ dt
\end{aligned}$$

Una vez más, integrando por partes con $u = \sin(t) \implies du = \cos(t)\ dt$ y $dv = e^{-st}\ dt \implies v = -\frac{1}{s}e^{-st}$, tenemos:

$$\begin{aligned}
&= \frac{1}{s} - \frac{1}{s}\lim_{b\to\infty} \left[\frac{-\sin(t)e^{-st}}{s}\bigg|_0^b + \frac{1}{s}\int_0^b e^{-st}\cos(t)\ dt \right] \\
&= \frac{1}{s} - \frac{1}{s}\lim_{b\to\infty} \left[-\frac{\sin(b)e^{-sb}}{s} + \frac{\sin(0)e^{0}}{s} \right] + \lim_{b\to\infty} \left[\frac{1}{s}\int_0^b e^{-st}\cos(t)\ dt \right] \\
&= \frac{1}{s} - \frac{1}{s}\lim_{b\to\infty} \left[\frac{1}{s}\int_0^b e^{-st}\cos(t)\ dt \right] \\
&= \frac{1}{s} - \frac{1}{s^2}\lim_{b\to\infty} \int_0^b e^{-st}\cos(t)\ dt
\end{aligned}$$

Observemos que:

$$\lim_{b\to\infty} \int_0^b e^{-st}\cos(t)\ dt = \frac{1}{s} - \frac{1}{s^2}\lim_{b\to\infty} \int_0^b e^{-st}\cos(t)\ dt$$

Estableciendo $a = \lim_{b\to\infty} \int_0^b e^{-st}\cos(t)\ dt$, tenemos:

$$
\begin{aligned}
a &= \frac{1}{s} - \frac{1}{s^2}a \\
\implies a + \frac{1}{s^2}a &= \frac{1}{s} \\
\implies a\left(\frac{1}{s^2}+1\right) &= \frac{1}{s} \\
\implies a\left(\frac{1+s^2}{s^2}\right) &= \frac{1}{s} \\
\implies a &= \frac{s^2}{s(1+s^2)} \\
\implies a &= \frac{s}{s^2+1} \\
\implies \lim_{b\to\infty}\int_0^b e^{-st}\cos(t)\ dt &= \frac{s}{s^2+1} \\
\implies \int_0^\infty e^{-st}\cos(t)\ dt &= \frac{s}{s^2+1}
\end{aligned}
$$

Por lo tanto, finalmente:

$$\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}$$

11. $f(t) = e^{4t}$

Calculamos la transformada de Lagrange de $f(t) = e^{4t}$:

$$
\begin{aligned}
\mathcal{L}\{e^{4t}\} &= \int_0^\infty e^{-st}e^{4t}\ dt \\
&= \int_0^\infty e^{(4-s)t}\ dt \\
&= \lim_{b\to\infty}\int_0^b e^{(4-s)t}\ dt
\end{aligned}
$$

Sustituyamos con $u = (4-s)t \implies du = 4 - s\ dt$:

$$
\begin{aligned}
&= \frac{1}{4-s}\lim_{b\to\infty}\int_0^{(4-s)b} e^u\ du \\
&= \frac{1}{4-s}\lim_{b\to\infty}\ (e^u)\big|_0^{(4-s)b} \\
&= \frac{1}{4-s}\lim_{b\to\infty}\left(e^{(4-s)b} - e^0\right) \\
&= \frac{1}{4-s}\left[\lim_{b\to\infty} e^{(4-s)b} - 1\right]
\end{aligned}
$$

Cuando $s > 4 \implies (4-s) < 0$, por lo que podemos escribir:

$$
\begin{aligned}
&= \frac{1}{4-s}\left[\lim_{b\to-\infty} e^{b} - 1\right] \\
&= \frac{1}{4-s}(0-1) \\
&= \frac{1}{s-4}
\end{aligned}
$$

Así, finalmente obtenemos que:

$$\mathcal{L}\{e^{4t}\} = \frac{1}{s-4} \qquad s > 4$$

12. $f(t) = e^{-2t}$

14. $f(t) = \sinh(3t)$

15. $f(t) = \cosh(6t)$

17. $f(t) = \cosh(at)$