Computational Finance
FIN-472
Take-Home Exam 2
Transform and Polynomial expansion methods

November 10, 2017

- **Please hand in your solutions on Friday 17.11.2017 at the beginning of the lecture.**
- **For Exercises** $4.a$, $4.b$, $4.c$ **and** $4.d$ **do not forget to upload in moodle the Matlab codes. You should submit only one Matlab file for each part.**
- **Additionally, print out the Matlab code for Exercises** $4.a$, $4.b$, $4.c$ **and the plot for Exercise** $4.d$**. These have to be handed in together with the other solutions on Friday 17.11.2017.**

Exercise 1: (10/40) Suppose that $S = \exp(X)$ is the price of a financial asset and that the spot interest rate is equal to r. Assume that the *characteristic function of the log price* X_T at time T (with respect to the risk neutral measure) is known and equal to ϕ. For a given $K > 0$ consider a digital type derivative whose payoff is of the form

$$\Psi = 1_{\{S_T \geq K\}} = \begin{cases} 1 & \text{if } S_T \geq K, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by

$$C(k) = \exp(-rT)\mathbb{E}[1_{\{S_T \geq \exp(k)\}}] = \exp(-rT)\mathbb{P}(S_T \geq \exp(k))$$

the price of this derivative at time 0, where $k = \log K$. For $\alpha \in \mathbb{R}$, define

$$C_\alpha(k) := \exp(-\alpha k)C(k) = \exp(-\alpha k - rT)\mathbb{P}(S_T \geq \exp(k)).$$

a) **(6/40)** *Show* that if $\alpha < 0$ and $\phi(\nu + i\alpha)$ is well defined for all $\nu \in \mathbb{R}$, then the Fourier Transform of C_α exists and has the form

$$\widehat{C_\alpha}(\nu) = \exp(-rT)\frac{\phi(\nu + i\alpha)}{-\alpha + i\nu}. \tag{1}$$

b) **(2/40)** Suppose that $\widehat{C_\alpha}(\nu)$ is an integrable function. *Deduce* the following pricing formula for a digital option

$$C(k) = \frac{\exp(-rT + \alpha k)}{\pi} \int_0^\infty Re\left(\frac{\phi(\nu + i\alpha)}{-\alpha + i\nu}\exp(-i\nu k)\right) d\nu. \tag{2}$$

c) **(2/40)** *Why* is the assumption $\alpha < 0$ important for the derivation of (1)?

Exercise 2: (4/40) Suppose that $S = \exp(X)$ is the price of a financial asset and that spot interest rate is equal to r. We define, for T given, the share measure $\mathbb{P}^S$ by

$$\mathbb{P}^S(A) = \frac{\mathbb{E}[S_T 1_A]}{\mathbb{E}[S_T]},$$

where $1_A(\omega) = 1$ if $\omega \in A$ and $1_A(\omega) = 0$ otherwise. For any bounded random variable X we have that

$$\mathbb{E}^{\mathbb{P}^S}[X] = \frac{\mathbb{E}[S_T X]}{\mathbb{E}[S_T]}.$$

We recall the following facts:

a) The prices of a put option $P(k)$ and a call option $C(k)$ with strike $K = \mathrm{e}^k$ and expiration date T can be written as:

$$\begin{aligned} P(k) &= \mathrm{e}^{k-rT}\mathbb{P}(X_T < k) - \mathrm{e}^{-rT}\mathbb{E}[S_T]\mathbb{P}^S(X_T < k), \\ C(k) &= \mathrm{e}^{-rT}\mathbb{E}[S_T]\mathbb{P}^S(X_T > k) - \mathrm{e}^{k-rT}\mathbb{P}(X_T > k). \end{aligned} \tag{3}$$

b) Let $\phi(\nu) = \mathbb{E}[\exp(i\nu X_T)]$ and $\phi^S(\nu) = \mathbb{E}^{\mathbb{P}^S}[\exp(i\nu X_T)]$, be the characteristic functions of X_T with respect to $\mathbb{P}$ and $\mathbb{P}^S$, respectively. Then $\phi(\nu - i)$ is well-defined and

$$\phi^S(\nu) = \frac{\phi(\nu - i)}{\mathbb{E}[S_T]}.$$

Define the functions

$$C_\alpha^S(k) = \mathrm{e}^{-rT+\alpha k}\mathbb{P}^S(X_t > k); \quad C_\alpha(k) = \mathrm{e}^{-rT+(\alpha+1)k}\mathbb{P}(X_t > k).$$

Using (3) we can write

$$C(k) = \mathrm{e}^{-\alpha k}(\mathbb{E}[S_T]C_\alpha^S(k) - C_\alpha(k)).$$

Suppose that $\mathbb{E}[S_T^{\alpha+1}] < \infty$ and $\alpha > 0$. *Deduce*, with the help of Exercise 1 and the above mentioned facts, the Carr-Madan formula

$$C(k) = \frac{\mathrm{e}^{-rT-\alpha k}}{\pi}\int_0^\infty Re\left(\frac{\phi(\nu - i(\alpha+1))}{(\alpha + i\nu)(\alpha + 1 + i\nu)}\mathrm{e}^{-i\nu k}\right) d\nu.$$

Exercise 3: (12/40) The *Cox-Ingersoll-Ross (CIR) process* is defined as a solution of the Stochastic Differential Equation (SDE)

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t.$$

If we let

$$v(t,x) := \mathbb{E}[\exp(i\nu X_T)|X_t = x],$$

then v satisfies the PDE

$$v_t + \mathcal{G}v = 0$$

with terminal condition $v(T,x) = \mathrm{e}^{i\nu x}$, where

$$\mathcal{G}v = \kappa(\theta - x)v_x + \frac{\sigma^2 x}{2}v_{xx}.$$

Suppose that

$$v(t,x) = \exp(\varphi(T-t,\nu) + \psi(T-t,\nu)x). \tag{4}$$

a) **(4/40)** *Deduce* a system of Ordinary Differential Equations (ODEs) for the functions φ and ψ.

b) **(6/40)** *Solve* this system and write explicitly the form of the characteristic function of X_t given $X_0 = x$.

c) **(2/40)** Knowing that the characteristic function of a non-central $\chi^2_k(\alpha)$ distributed random variable is of the form

$$\frac{e^{\frac{i\nu\alpha}{1-2i\nu}}}{(1-2i\nu)^{\frac{k}{2}}},$$

deduce that $\frac{X_t}{\frac{\sigma^2}{4\kappa}(1-e^{-\kappa t})}$ is also non-central $\chi^2_k(\alpha)$ distributed. Give the parameters k and α explicitly.

Exercise 4: (14/40) Consider the Jacobi stochastic volatility model where the stock price dynamics $S_t = e^{X_t}$ and the squared volatility V_t are given by

$$\begin{aligned} dV_t &= \kappa(\theta - V_t)\,dt + \sigma\sqrt{Q(V_t)}\,dW_{1t}, \\ dX_t &= (r - V_t/2)\,dt + \rho\sqrt{Q(V_t)}\,dW_{1t} + \sqrt{V_t - \rho^2\,Q(V_t)}\,dW_{2t}, \end{aligned}$$

for $\kappa, \sigma > 0$, $\theta \in (v_{min}, v_{max}]$, interest rate r, $\rho \in [-1, 1]$, and a 2-dimensional Brownian motion $W = (W_1, W_2)$. Here, the function Q is defined as

$$Q(v) = \frac{(v - v_{min})(v_{max} - v)}{(\sqrt{v_{max}} - \sqrt{v_{min}})^2},$$

for some parameters v_{min} and v_{max} satisfying $0 \le v_{min} < v_{max}$.
The goal of this exercise is to implement the polynomial expansion method described in Lecture 8, which allows us to write the price $\mathbb{E}[f(X_T)]$ of a European option maturing at time T with payoff function f as

$$\pi_f := \mathbb{E}[f(X_T)] = \sum_{n\ge 0} f_n l_n, \tag{5}$$

where $\{l_n, n = 0, \cdots\}$ are the Hermite moments and $\{f_n, n = 0, \cdots\}$ are the Fourier coefficients. In the following, we consider f to be the payoff function of a European call with log strike k,

$$f(x) := e^{-rT}(e^x - e^k)^+.$$

a) **(6/40)** Let $\mu_w \in \mathbb{R}$ and $\sigma_w > 0$ be arbitrary parameters. Consider the basis vector of $\mathrm{Pol}_N(\mathbb{R}^2)$ defined as

$$B_N(v, x) = (1, v, H_1(x), v^2, vH_1(x), H_2(x), \cdots, v^n, v^{n-1}H_1(x), \cdots, H_N(x)),$$

where $H_n(x)$ denotes the generalized Hermite polynomials

$$H_n(x) = \frac{1}{\sqrt{n!}}\mathcal{H}_n\left(\frac{x - \mu_w}{\sigma_w}\right), \quad n \ge 1.$$

Here, $\mathcal{H}_n(x)$ are the standard Hermite polynomials.

Write a Matlab function `HermiteMoments.m` that computes the first N Hermite moments using the moment formula given by

$$l_n = B_N(V_0, X_0)\,\mathrm{e}^{G_N T}\,\mathbf{e}_{\pi(0,n)}, \quad 0 \le n \le N,$$

where G_N is the matrix representation of the generator $\mathcal{G}$ of (V_t, X_t) restricted to $\mathrm{Pol}_N(\mathbb{R}^2)$, with respect to the basis B_N, and $\pi : \mathcal{E} \to \{1, \ldots, M = (N+2)(N+1)/2\}$ is an enumeration of the set of exponents

$$\mathcal{E} = \{(m, n) : m, n \geq 0;\ m + n \leq N\}.$$

Remark: You can use the same enumerating function π as defined in Exercise 2a of Homework 7.
In order to deal with the Hermite polynomials $\mathcal{H}_n$ you can use the built-in Matlab function `hermiteH.m`. Please also see the solutions of Homework 7 for a reference.

b) **(4/40)** In the case of the European call option, the Fourier coefficients can be recursively computed by

$$\begin{aligned} f_0 =& \mathrm{e}^{-rT+\mu_w} I_0\left(\frac{k-\mu_w}{\sigma_w}; \sigma_w\right) - \mathrm{e}^{-rT+k}\Phi\left(\frac{\mu_w - k}{\sigma_w}\right), \\ f_n =& \mathrm{e}^{-rT+\mu_w}\frac{1}{\sqrt{n!}}\sigma_w I_{n-1}\left(\frac{k-\mu_w}{\sigma_w}; \sigma_w\right), \quad n \geq 1. \end{aligned}$$

The functions $I_n(\mu; \nu)$ are defined recursively by

$$\begin{aligned} I_0(\mu; \nu) &= \mathrm{e}^{\frac{\nu^2}{2}}\Phi(\nu - \mu); \\ I_n(\mu; \nu) &= \mathcal{H}_{n-1}(\mu)\mathrm{e}^{\nu\mu}\phi(\mu) + \nu I_{n-1}(\mu; \nu), \quad n \geq 1 \end{aligned}$$

where $\mathcal{H}_n(x)$ are again the standard Hermite polynomials, $\Phi(x)$ denotes the standard Gaussian distribution function, and $\phi(x)$ its density.

Write a Matlab function `FourierCoefficients.m` that computes the first N coefficients following above recursions.

c) **(2/40)** *Write* a Matlab function `PriceApprox.m` that computes the approximation of the European call option price in the Jacobi model arising from cutting the sum in (5) after $N + 1$ terms, i.e.

$$\pi_f^{(N)} := \sum_{n=0}^{N} f_n l_n.$$

d) **(2/40)** Consider the parameters

$$\begin{aligned} &X_0 = 0,\ \ V_0 = 0.04,\ \ \kappa = 0.5,\ \ \theta = 0.04,\ \ \sigma = 1,\ \ r = 0,\ \ \rho = -0.5, \\ &T = 1/12,\ \ v_{min} = 10^{-4},\ \ v_{max} = 0.08,\ \ N = 10, \end{aligned}$$

together with $\mu_w = \mathbb{E}[X_T]$ and $\sigma_w^2 = \mathrm{Var}[X_T]$.

Using the function `PriceApprox.m`, plot the implied volatility smile as a function of the log strike k in the Jacobi model. In particular, for each value of k in `linspace(-0.1, 0.1, 50)` compute the corresponding implied volatility using the built-in Matlab function `blsimpv.m`. Moreover, plot on the same figure the implied volatility smile for the same call option computed in the Heston model, using the same model parameters.

Remarks:

- In order to compute the Heston price, please use the Fourier approach you have implemented in Exercise 3, Homework 4, with parameters $L = 100$ and $\alpha = 1$.
- For this part, please submit on the moodle the Matlab script/function that generates the needed plot.