Computational Finance
Math-472
Finite Difference methods for option pricing

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Outline

1. The generalized Black-Scholes equation
   - Some analysis of the BS equation
   - Truncation of the domain and boundary conditions
2. Finite difference approximation of the BS equation
   - Analysis of the FD approximation
   - Upwinding
   - Relation with binomial model
   - Barrier options – Computation of Greeks

Table of contents

The Generalized Black-Scholes equation

Consider the price dynamics of a financial asset

$$dS_\tau = \tilde{r}(\tau)S_\tau d\tau + \tilde{\sigma}(S_\tau,\tau)S_\tau dW_\tau, \quad 0 < \tau \leq T$$

where

- $\tilde{r}(\tau)$ denotes the instantaneous risk-free interest rate
- $\tilde{\sigma}(S_\tau,\tau)$ is the local volatility
- W_τ is a Brownian Motion

and a payoff function $\Psi(S_T)$ at time T (maturity)

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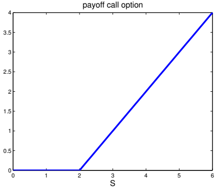
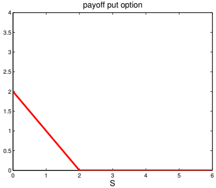
Typical payoffs: **call/put** options with strike price K



Call option

$$\Psi(S_T) = (S_T - K)_+$$



Put option

$$\Psi(S_T) = (K - S_T)_+$$

The Generalized Black-Scholes equation

The value of an European option at time $\tau \leq T$ and asset price $S_\tau = S$ is

$$v(S,\tau) = \mathbb{E}\left[\Psi(S_T) \mid S_\tau = S\right] e^{-\int_\tau^T \tilde{r}(\eta)d\eta}$$

The option price $v(S,\tau)$ satisfies the generalized Black-Scholes equation

$$\begin{cases} \dfrac{\partial}{\partial \tau} v(S,\tau) + \dfrac{\tilde{\sigma}^2(S,\tau)}{2} S^2 \dfrac{\partial^2}{\partial S^2} v(S,\tau) + \tilde{r}(t) S \dfrac{\partial}{\partial S} v(S,\tau) - \tilde{r}(t) v(S,\tau) = 0, \\ \hfill S \in \mathbb{R}_+, \quad \tau \in [0,T) \\ v(S,T) = \Psi(S) \qquad \text{(terminal condition)} \end{cases} \tag{1}$$

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This is a backward parabolic equation defined on an unbounded domain. One is typically interested in the option price at initial time $\tau = 0$ (today), i.e. $v(S,0)$.

Reformulation 1: changing the time

For convenience, we convert the backward BS equation into a forward one by performing the change of variable $t = T - \tau$ (time to maturity). Denoting $\sigma(S,t) = \tilde{\sigma}(S, T-t)$, $r(t) = \tilde{r}(T-t)$, and $u(S,t) = v(S, T-t)$ we have

$$\begin{cases} \dfrac{\partial}{\partial t}u(S,t) - \dfrac{\sigma^2(S,t)}{2}S^2\dfrac{\partial^2}{\partial S^2}u(S,t) - r(t)S\dfrac{\partial}{\partial S}u(S,t) + r(t)u(S,t) = 0, \\ \hfill S \in \mathbb{R}_+, \quad t \in (0,T] \\ u(S,0) = \Psi(S) \qquad \text{(initial condition)} \end{cases} \tag{2}$$

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This is now a forward parabolic equation and we are interested in the option price at final time $u(S,T)$.

Reformulation 2: change to log-asset price

Let us consider the further change of variable $x = \log(S)$. Then, the option price, still denoted $u = u(x,t)$, satisfies the equation

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) - \dfrac{\sigma^2(e^x,t)}{2}\dfrac{\partial^2}{\partial x^2}u(x,t) - \left(r(t) - \dfrac{\sigma^2(e^x,t)}{2}\right)\dfrac{\partial}{\partial x}u(x,t) \\ \qquad\qquad\qquad + r(t)u(x,t) = 0, \qquad x \in \mathbb{R}, \quad t \in (0,T] \\ u(x,0) = \Psi(e^x) =: \Psi_{log}(x) \qquad \text{(initial condition)} \end{cases} \tag{3}$$

which is now defined on the whole real line.

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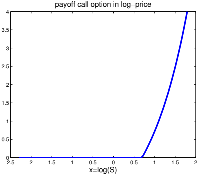
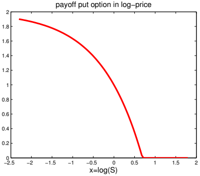


Call option

$\Psi_{log}(x) = (e^x - K)_+$



Put option

$\Psi_{log}(x) = (K - e^x)_+$

Preliminaries: constant volatility (cont.)

In the case of constant volatility, an exact solution can be found: consider the change of variables

$$\tilde{x} = x + R(t) - \frac{\sigma^2 t}{2}, \qquad \text{with } R(t) = \int_0^t r(\tau)d\tau.$$

and the new unknown $\tilde{u}(\tilde{x}, t) = u(x, t)e^{R(t)}$. It is easy to see (left as exercise) that $\tilde{u}$ satisfies the heat equation

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = 0, & \tilde{x} \in \mathbb{R},\ t \in (0, T] \\ u(\tilde{x}, 0) = \Psi_{log}(\tilde{x}), & \tilde{x} \in \mathbb{R} \end{cases}$$

and, using the fundamental solution of the heat equation

$$\tilde{u}(\tilde{x}, t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{y\in\mathbb{R}} e^{-\frac{y^2}{2\sigma^2 t}} \Psi_{log}(\tilde{x} - y)dy$$

Preliminaries: constant volatility

Hence, the option price is given by the formula

$$u(x,t) = \frac{e^{-R(t)}}{\sqrt{2\pi\sigma^2 t}} \int_{y\in\mathbb{R}} e^{-\frac{y^2}{2\sigma^2 t}} \Psi_{log}(x + R(t) - \frac{\sigma^2 t}{2} - y)dy \qquad (4)$$

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Remarks

- the solution is well defined, i.e. integral exists, as long as $\Psi_{log}(x) \sim e^{|x|^\alpha}$ with $0 \leq \alpha < 2$, as $|x| \to \infty$. This is fulfilled by both call and put payoff functions.

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- $\Psi_{log}(x) \geq 0$ for all $x \in \mathbb{R}$ implies $u(x,t) \geq 0$, $\forall x \in \mathbb{R}$, $t \in \mathbb{R}_+$.

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- Ψ_{log} convex implies $u(x,t)$ convex in x for all t.

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Exercise 1

Prove the previous statements. Use (4) to derive the Black-Scholes formula for option pricing in the case of a call or put option.

Existence results for the generalized BS equation

$$\begin{cases} \dfrac{\partial}{\partial t}u(S,t) - \dfrac{\sigma^2(S,t)}{2}S^2\dfrac{\partial^2}{\partial S^2}u(S,t) - r(t)S\dfrac{\partial}{\partial s}u(S,t) + r(t)u(S,t) = 0, \\ \hfill S \in \mathbb{R}_+, \quad t \in (0,T], \\ u(S,0) = \Psi(S), \quad S \in \mathbb{R}. \end{cases}$$

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Theorem 1 (see e.g. Friedman '64)

Assume that

- $\exists \sigma_{min}, \sigma_{max}$ *s.t.* $0 < \sigma_{min} \le \sigma(S,t) \le \sigma_{max}$, $\forall S \in \mathbb{R}_+,\ t \in [0,T]$,
- $\exists C > 0$ *s.t.* $\left|S\frac{\partial \sigma}{\partial S}\right| \le C$, $\forall S \in \mathbb{R}_+,\ t \in [0,T]$,
- $t \mapsto r(t)$ *is bounded and Lipschitz continuous in* $[0,T]$
- $S \mapsto \Psi(S)$ *is Lipschitz continuous and there exists* $C > 0$ *s.t.* $0 \le \Psi(S) \le C(1+S)$, $\forall S \in \mathbb{R}_+$.

Then, there exists a unique solution $u \in C^0(\mathbb{R}_+ \times [0,T])$, C^1*-regular in* t *and* C^2*-regular in* S *in* $\mathbb{R}_+ \times [0,T]$. *Moreover, there exists* $C' > 0$ *s.t.*

$$0 \le u(S,t) \le C'(1+S), \qquad \forall S \in \mathbb{R}_+,\ t \in [0,T].$$

Recovering the Put-Call parity

Let

- $p(S,t)$ be the unique solution of the BS eq. with initial datum $p(S,0) = (K-S)_+$ (put option)
- $c(S,t)$ be the unique solution of the BS eq. with initial datum $c(S,0) = (S-K)_+$ (call option)

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Define now $u(s,t) = c(s,t) - p(s,t)$, which is also a solution of the BS equation with initial datum

$$u(S,0) = c(S,0) - p(S,0) = (S-K)_+ - (K-S)_+ = S - K$$

It is immediate to check that such solution is given by

$$u(S,t) = S - Ke^{-R(t)}, \quad \text{with} \quad R(t) = \int_0^t r(\tau)d\tau.$$

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It follows the well known

put-call parity $\quad c(S,t) - p(S,t) = S - Ke^{-R(t)}$

Maximum principle

Theorem 2 (Maximum principle (see e.g. Achdou-Pironneau '05))

Assume that $u(S,t)$ is a smooth function that "does not grow too fast as $S \to \infty$" (polynomial growth of u and $\frac{\partial u}{\partial S}$ is enough).

If $u(S,0) \geq 0, \ \forall S \in \mathbb{R}_+$ and

$$\frac{\partial u}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 u}{\partial S^2} - rS \frac{\partial u}{\partial S} + ru \geq 0, \quad \forall S \in \mathbb{R}_+, \ t \in (0,T]$$

Then, $u(S,t) \geq 0, \ \forall (S,t) \in \mathbb{R}_+ \times [0,T]$.

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Then, $u(S,t) \geq 0,\ \forall (S,t) \in \mathbb{R}_+ \times [0,T]$.

In particular, if u is a solution of the BS equation and has positive initial condition, then u is positive everywhere.

Bounds for a put/call option

From the maximum principle, we can obtain bounds on the value of a put/call option at any time.

Proposition 1 (first bound on a put/call option)

$$\begin{aligned} &(Ke^{-R(t)} - S)_+ \leq p(S,t) \leq Ke^{-R(t)}, \\ &(S - Ke^{-R(t)})_+ \leq c(S,t) \leq S, \end{aligned} \qquad \forall S \in \mathbb{R}_+,\ t \in [0,T]$$

Proof: Let $u_1(S,t) = Ke^{-R(t)} - S$ and $u_2(S,t) = Ke^{-R(t)}$. They both are solutions of the BS equation and clearly

$$u_1(S,0) \leq p(S,0) \leq u_2(S,0), \quad \forall S \in \mathbb{R}_+$$

Hence $u_2(S,0) - p(S,0) \geq 0$ implies $u_2(S,t) - p(S,t) \geq 0$, $\forall S \in \mathbb{R}_+$, $t \in [0,T]$ and similarly for $p(S,0) - u_1(S,0) \geq 0$.
The bound on the call option follows from the put-call parity. □

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In particular, we have $p(0,t) = Ke^{-R(t)}$ and from the put-call parity, $c(0,t) = 0$, $\forall t \in [0,T]$.

Notice that these "boundary conditions" are deduced from the BS equation and do not have to be imposed explicitly.

Bounds for a put/call option (cont.)

Notation: $\sigma_{max} = \max_{(S,t)} \sigma(S,t), \quad \sigma_{min} = \min_{(S,t)} \sigma(S,t)$

Proposition 2 (second bound on a put option)

Let $\overline{p}(S,t)$ be the solution of the BS equation with initial condition $\overline{p}(S,0) = (K-S)_+$ (put option) and constant volatility σ_{max}. Similarly, let $\underline{p}(S,t)$ be the solution with constant volatility σ_{min}. Then

$$\underline{p}(S,t) \le p(S,t) \le \overline{p}(S,t), \quad \forall (S,t) \in \mathbb{R}_+ \times [0,T].$$

Proof: The function $u(S,t) = \overline{p}(S,t) - p(S,t)$ satisfies

$$\frac{\partial u}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 u}{\partial S^2} - rS\frac{\partial u}{\partial S} + ru = \frac{\sigma_{max}^2 - \sigma^2}{2} S^2 \frac{\partial^2 \overline{p}}{\partial S^2} \ge 0, \quad \forall S \in \mathbb{R}_+,\ t \in (0,T]$$

the last inequality coming from the fact that $\overline{p}$ is a convex function in S (check with formula (4)). On the other hand, $u(S,0) = 0$, so by maximum principle, $u(S,t) \ge 0,\ \forall (S,t) \in \mathbb{R}_+ \times [0,T]$. □

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The same consideration applies to a call option:

$$\underline{c}(S,t) \le c(S,t) \le \overline{c}(S,t), \quad \forall (S,t) \in \mathbb{R}_+ \times [0,T].$$

Localization

When solving the BS equation with a numerical method as, e.g. Finite Differences, we need to truncate the computational domain to $[0, S_{max}]$.
Question

- What boundary conditions to enforce on the artificial boundary $S = S_{max}$?
- What is the error introduced by truncation?

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Artificial boundary condition at $S = S_{max}$ for a put/call option:

- Assuming $S_{max} \gg K$, for a put option it is reasonable to set

$$p(S_{max}, t) = 0, \quad \forall t \in [0, T].$$

- Using the put/call parity, for a call option we set

$$c(S_{max}, t) = S_{max} - Ke^{-R(t)}, \quad \forall t \in [0, T].$$

Truncated BS equation in asset price

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\sigma^2}{2} S^2 \dfrac{\partial^2 u}{\partial S^2} - rS \dfrac{\partial u}{\partial S} + ru = 0, & S \in [0, S_{max}),\ t \in (0, T] \\ u(S, 0) = \Psi(S), & S \in [0, S_{max}] \\ u(S_{max}, t) = \overline{g}(t), & t \in [0, T] \\ u(0, t) = \underline{g}(t), & t \in [0, T] \end{cases}$$

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	payoff $\Psi(S)$	left b.c. $\underline{g}(t)$	right b.c. $\overline{g}(t)$
put option	$(K - S)_+$	$Ke^{-R(t)}$	0
call option	$(S - K)_+$	0	$S - Ke^{-R(t)}$

Truncated BS equation in log-asset price

When working with the log-asset price $x = \log(S)$, the domain has to be truncated also at the left side:

$$S \in [S_{min}, S_{max}] \iff x \in [\log(S_{min}), \log(S_{max})] = [x_{min}, x_{max}],$$

clearly with $0 < S_{min} \ll K$.

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When working with the log-asset price $x = \log(S)$, the domain has to be truncated also at the left side:

$$S \in [S_{min}, S_{max}] \iff x \in [\log(S_{min}), \log(S_{max})] = [x_{min}, x_{max}],$$

clearly with $0 < S_{min} \ll K$.

Artificial boundary condition at $S = S_{min}$ for a put/call option:

- Assuming $S_{min} \ll K$, for a call option it is reasonable to set

$$c(S_{min}, t) = 0, \quad \forall t \in [0, T].$$

- Using the put/call parity, for a put option we set

$$p(S_{min}, t) = Ke^{-R(t)} - S_{min}, \quad \forall t \in [0, T].$$

Truncated BS equation in log-asset price (cont.)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} - \left(r - \dfrac{\sigma^2}{2}\right)\dfrac{\partial u}{\partial x} + ru = 0, & x \in (x_{min}, x_{max}),\ t \in (0, T] \\ u(x, 0) = \Psi_{log}(x), & x \in [x_{min}, x_{max}] \\ u(x_{min}, t) = \underline{g}_{log}(t), & t \in [0, T] \\ u(x_{max}, t) = \overline{g}_{log}(t), & t \in [0, T] \end{cases}$$

Truncated BS equation in log-asset price (cont.)

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} - \left(r - \dfrac{\sigma^2}{2}\right)\dfrac{\partial u}{\partial x} + ru = 0, & x \in (x_{min}, x_{max}),\ t \in (0, T] \\
u(x,0) = \Psi_{log}(x), & x \in [x_{min}, x_{max}] \\
u(x_{min}, t) = \underline{g}_{log}(t), & t \in [0, T] \\
u(x_{max}, t) = \overline{g}_{log}(t), & t \in [0, T]
\end{cases}
$$

	payoff $\Psi_{log}(x)$	left b.c. $\underline{g}_{log}(t)$	right b.c. $\overline{g}_{log}(t)$
put option	$(K - e^x)_+$	$Ke^{-R(t)} - e^{x_{min}}$	0
call option	$(e^x - K)_+$	0	$e^{x_{max}} - Ke^{-R(t)}$

Truncated BS equation in log-asset price (cont.)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\sigma^2}{2}\dfrac{\partial^2 u}{\partial x^2} - \left(r - \dfrac{\sigma^2}{2}\right)\dfrac{\partial u}{\partial x} + ru = 0, & x \in (x_{min}, x_{max}),\ t \in (0,T] \\ u(x,0) = \Psi_{log}(x), & x \in [x_{min}, x_{max}] \\ u(x_{min}, t) = \underline{g}_{log}(t), & t \in [0,T] \\ u(x_{max}, t) = \overline{g}_{log}(t), & t \in [0,T] \end{cases}$$

	payoff $\Psi_{log}(x)$	left b.c. $\underline{g}_{log}(t)$	right b.c. $\overline{g}_{log}(t)$
put option	$(K - e^x)_+$	$Ke^{-R(t)} - e^{x_{min}}$	0
call option	$(e^x - K)_+$	0	$e^{x_{max}} - Ke^{-R(t)}$

Notice that with this choice of boundary conditions, the put/call parity holds also for the truncated problem.

Maximum principle for the truncated problem

A maximum principle holds also for the truncated problem:

Theorem 3

Assume that

- $u(S,t)$ *is a smooth function*
- $\frac{\partial u}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 u}{\partial S^2} - rS\frac{\partial u}{\partial S} + ru \geq 0, \quad \forall S \in (S_{min}, S_{max}),\ t \in (0,T]$
- $u(S,0) \geq 0,\ \forall S \in [S_{min}, S_{max}]$
- $u(S_{min}, t) \geq 0$ *and* $u(S_{max}, 0) \geq 0, \quad \forall t \in [0,T]$.

Then, $u(S,t) \geq 0,\ \forall (S,t) \in [S_{min}, S_{max}] \times [0,T]$.

Truncation error

Let us consider a put option with payoff $\Psi(S) = (K - S)_+$ and define

- $\tilde{p}(S,t)$: solution of the truncated problem with initial cond. Ψ;
- $p(S,t)$: solution of the un-truncated problem with initial cond. Ψ;
- $\overline{p}(S,t)$: solution of the BS equation with initial cond. Ψ and constant volatility σ_{max}.

Proposition 3 (Bound on the truncation error for a put option)

Define

$$\varepsilon(S_{min}, S_{max}) = \max_{t\in[0,T]} \max\left\{\overline{p}(S_{min},t) - Ke^{-R(t)} + S_{min} \ , \ \overline{p}(S_{max},t)\right\}$$

Then

$$0 \le p(S,t) - \tilde{p}(S,t) \le \varepsilon(S_{min}, S_{max}), \quad \forall (S,t) \in [S_{min}, S_{max}] \times [0,T].$$

Truncation error (cont.)

Proof: define the function $u(S,t) = \varepsilon(S_{min}, S_{max}) - p(S,t) + \tilde{p}(S,t)$ which satisfies

$$\frac{\partial u}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 u}{\partial S^2} - rS \frac{\partial u}{\partial S} + ru = r\varepsilon(S_{min}, S_{max}) \geq 0, \quad \forall (S,t) \in [S_{min}, S_{max}] \times [0,T].$$

Let moreover $\Gamma = [S_{min}, S_{max}] \times \{0\} \cup \{S_{min}\} \times [0,T] \cup \{S_{max}\} \times [0,T]$.

Truncation error (cont.)

Proof: define the function $u(S,t) = \varepsilon(S_{min}, S_{max}) - p(S,t) + \tilde{p}(S,t)$ which satisfies

$$\frac{\partial u}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 u}{\partial S^2} - rS \frac{\partial u}{\partial S} + ru = r\varepsilon(S_{min}, S_{max}) \geq 0, \quad \forall (S,t) \in [S_{min}, S_{max}] \times [0,T].$$

Let moreover $\Gamma = [S_{min}, S_{max}] \times \{0\} \cup \{S_{min}\} \times [0,T] \cup \{S_{max}\} \times [0,T]$.

Since $p(S,t) \leq \overline{p}(S,t), \ \forall (S,t)$ it follows that

$$\text{on } \Gamma \quad p(S,t) - \tilde{p}(S,t) \leq \overline{p}(S,t) - (Ke^{-R(t)} - S)_+ \leq \varepsilon(S_{min}, S_{max}).$$

Hence $u(S,t) \geq 0$ on Γ which implies $u(S,t) \geq 0$, $\forall (S,t) \in [S_{min}, S_{max}] \times [0,T]$. This proves the right inequality.

Truncation error (cont.)

Proof: define the function $u(S,t) = \varepsilon(S_{min}, S_{max}) - p(S,t) + \tilde{p}(S,t)$ which satisfies

$$\frac{\partial u}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 u}{\partial S^2} - rS\frac{\partial u}{\partial S} + ru = r\varepsilon(S_{min}, S_{max}) \geq 0, \quad \forall (S,t) \in [S_{min}, S_{max}] \times [0,T].$$

Let moreover $\Gamma = [S_{min}, S_{max}] \times \{0\} \cup \{S_{min}\} \times [0,T] \cup \{S_{max}\} \times [0,T]$.

Since $p(S,t) \leq \overline{p}(S,t), \ \forall (S,t)$ it follows that

$$\text{on } \Gamma \quad p(S,t) - \tilde{p}(S,t) \leq \overline{p}(S,t) - (Ke^{-R(t)} - S)_+ \leq \varepsilon(S_{min}, S_{max}).$$

Hence $u(S,t) \geq 0$ on Γ which implies $u(S,t) \geq 0$, $\forall (S,t) \in [S_{min}, S_{max}] \times [0,T]$. This proves the right inequality.

On the other hand $p(S,t) \geq (Ke^{-R(t)} - S)_+, \ \forall (S,t)$, hence $p(S,t) - \tilde{p}(S,t) \geq 0$ on Γ and, by maximum principle, $p(S,t) - \tilde{p}(S,t) \geq 0$, $\forall [S_{min}, S_{max}] \times [0,T]$, which proves the left inequality. □

How to estimate the truncation error in practice

The function $\overline{p}(S,t)$ can be evaluated using the Black-Scholes formula, since it solves the BS equation with constant volatility

$$\overline{p}(S,t) = Ke^{-R(t)}\phi(-d_2) - S\phi(-d_1)$$

with

$$d_1 = d_1(S,t) = \frac{1}{\sigma_{max}\sqrt{t}}\left[\log\frac{S}{K} + R(t) + \frac{\sigma_{max}^2 t}{2}\right],$$

$$d_2 = d_2(S,t) = \frac{1}{\sigma_{max}\sqrt{t}}\left[\log\frac{S}{K} + R(t) - \frac{\sigma_{max}^2 t}{2}\right]$$

Simplified formulas for truncation error

Now, in $S = S_{max}$

$$\overline{p}(S_{max}, t) \leq K\phi(-d_2(S_{max}, t)) \leq K\phi(\alpha_2(S_{max}))$$

$$\text{with} \quad \alpha_2(S_{max}) = \frac{-\log(S_{max}/K) + \sigma_{max}^2 T/2}{\sigma_{max}\sqrt{T}} \tag{5}$$

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and in $S = S_{min}$

$$\begin{aligned}\overline{p}(S_{min}, t) - Ke^{-R(t)} - S_{min} &= S_{min}\phi(d_1(S_{min}, t)) - Ke^{-R(t)}\phi(d_2(S_{min}, t)) \\ &\le S_{min}\phi(d_1(S_{min}, t)) \le K\phi(\alpha_1(S_{min}))\end{aligned}$$

$$\text{with} \quad \alpha_1(S_{min}) = \frac{\log(S_{min}/K) + r_{max} T + \sigma^2_{max} T/2}{\sigma_{max}\sqrt{T}} \tag{6}$$

and $r_{max} = \max_{t\in[0,T]} r(t)$.

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and in $S = S_{min}$

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$$\text{with} \quad \alpha_1(S_{min}) = \frac{\log(S_{min}/K) + r_{max}T + \sigma_{max}^2 T/2}{\sigma_{max}\sqrt{T}} \tag{6}$$

and $r_{max} = \max_{t\in[0,T]} r(t)$.

Proposition 4

$$\varepsilon(S_{min}, S_{max}) \leq K \max\left\{\phi(\alpha_2(S_{max}))\,,\ \phi(\alpha_1(S_{min}))\right\}$$

with α_1 *and* α_2 *defined in* (6) *and* (5), *respectively.*

Table of contents

Finite Difference formulas – first derivative

Consider a smooth function $u : I \subset \mathbb{R} \to \mathbb{R}$ and $x, x+h \in I$.

To approximate $u'(x)$ we can use the following formulas

forward F.D. $$D_h^+ u(x) = \frac{u(x+h) - u(x)}{h}$$

backward F.D. $$D_h^- u(x) = \frac{u(x) - u(x-h)}{h}$$

centered F.D. $$D_{2h}^c u(x) = \frac{u(x+h) - u(x-h)}{2h}$$

Finite Difference formulas – first derivative

Proposition 5 (error estimates)

Assume $u \in C^2(\bar{I})$. *Then*

$$|D_h^+ u(x) - u'(x)| \leq \frac{h}{2}\|u''\|_{L^\infty(I)},$$
$$|D_h^- u(x) - u'(x)| \leq \frac{h}{2}\|u''\|_{L^\infty(I)}.$$

If $u \in C^3(\bar{I})$ *then* $\quad |D_{2h}^c u(x) - u'(x)| \leq \frac{h^2}{6}\|u'''\|_{L^\infty(I)}.$

Finite Difference formulas – first derivative

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If $u \in C^3(\bar{I})$ *then* $\quad |D_{2h}^c u(x) - u'(x)| \leq \dfrac{h^2}{6}\|u'''\|_{L^\infty(I)}.$

Proof: we prove only the first one as the others are analogous. By Taylor expansion

$$D_h^+ u(x) - u'(x) = \frac{u(x+h) - u(x)}{h} - u'(x)$$

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$$\begin{aligned} D_h^+ u(x) - u'(x) &= \frac{u(x+h) - u(x)}{h} - u'(x) \\ &= \frac{u'(x)h + u''(\xi)h^2/2}{h} - u'(x) \qquad \xi \in [x, x+h] \end{aligned}$$

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Finite Difference formulas – second derivative

To approximate the second derivative $u''(x)$ we can use the formula

$$D_h^2 u(x) = D_h^- D_h^+ u(x) = D_h^c D_h^c u(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

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Proposition 6 (error estimates)

Assume $u \in C^4(\bar{I})$. *Then*

$$|D_h^2 u(x) - u''(x)| \leq \frac{h^2}{12} \|u^{(iv)}\|_{L^\infty(I)}.$$

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Proof: again, using Taylor expansion we have

$$u(x+h) = u(x) + u'(x)h + u''(x)h^2/2 + u'''(x)h^3/6 + u^{(iv)}(\xi_1)h^4/24,$$
$$u(x-h) = u(x) - u'(x)h + u''(x)h^2/2 - u'''(x)h^3/6 + u^{(iv)}(\xi_2)h^4/24,$$

with $\xi_1 \in [x, x+h]$ and $\xi_2 \in [x-h, x]$. Replacing these expansions in the formula $D_h^2 u(x)$ the result follows. $\square$

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with $\xi_1 \in [x, x+h]$ and $\xi_2 \in [x-h, x]$. Replacing these expansions in the formula $D_h^2 u(x)$ the result follows. $\square$

Remark: If the function is not smooth enough, the rate $O(h^2)$ is not achieved. For example, if $u \in C^3(\bar{I})$, one will have $|D_h^2 u(x) - u''(x)| \leq \frac{h}{3} \|u'''\|_{L^\infty(I)}$.

Finite Difference approximation of BS equation in log-price

We consider the BS eq. in log-price

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - \frac{1}{2}\sigma^2(x,t)\frac{\partial^2}{\partial x^2}u(x,t) - \beta(x,t)\frac{\partial}{\partial x}u(x,t) + r(t)u(x,t) = 0, \\ \hfill x \in (x_{min}, x_{max}),\ t \in (0,T] \\ u(x,0) = \Psi_{log}(x), \hfill x \in [x_{min}, x_{max}] \\ u(x_{min},t) = \underline{g}_{log}(t), \quad u(x_{max},t) = \overline{g}_{log}(t), \hfill t \in [0,T] \end{cases}$$

where we have denoted $\beta(x,t) = r(t) - \frac{\sigma^2(x,t)}{2}$.

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where we have denoted $\beta(x,t) = r(t) - \frac{\sigma^2(x,t)}{2}$.

We divide now

- $[x_{min}, x_{max}]$ in N sub-intervals on equal length $h = \frac{x_{max} - x_{min}}{N}$
- $[0,T]$ in M sub-intervals of equal length $\Delta t = \frac{T}{M}$

and introduce the uniform grid

$$\begin{aligned} x_j &= x_{min} + jh, \qquad && j = 0, \dots, N, \\ t_m &= m\Delta t, && t = 0, \dots, M. \end{aligned}$$

Finite Difference approximation of BS equation in log-price

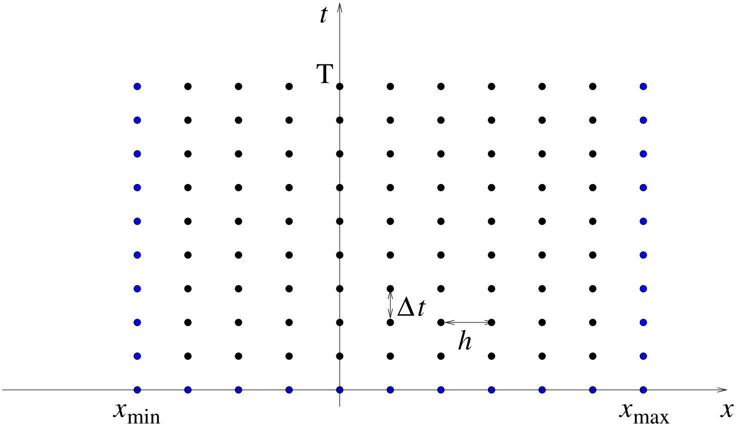

Figure: Computational uniform grid

Forward Euler (explicit) approximation

Next, we approximate the space derivatives by centered Finite Differences

$$\frac{\partial^2 u}{\partial x^2}(x_j, t_m) \to D_h^2 u(x_j, t_m), \qquad \frac{\partial u}{\partial x}(x_j, t_m) \to D_{2h}^c u(x_j, t_m),$$

and the time derivative by a forward Finite Difference

$$\frac{\partial u}{\partial t}(x_j, t_m) \to D_{\Delta t}^+ u(x_j, t_m),$$

We denote by u_j^m the F.D. approximation of the exact solution $u(x_j, t_m)$.

Notice that the "boundary values" u_j^0, $j = 0, \ldots, N$ and u_0^m, u_N^m, $m = 0, \ldots, M$ are known.

Forward Euler (explicit) approximation

Notation: $\sigma_j^m = \sigma(x_j, t_m)$; $\beta_j^m = \beta(x_j, t_m)$, $r^m = r(t_m)$.

$$\begin{cases} \dfrac{u_j^{m+1} - u_j^m}{\Delta t} - \dfrac{(\sigma_j^m)^2}{2h^2}(u_{j+1}^m - 2u_j^m + u_{j-1}^m) - \dfrac{\beta_j^m}{2h}(u_{j+1}^m - u_{j-1}^m) + r^m u_j^m = 0 \\ \qquad\qquad j = 1, \ldots, N-1, \quad m = 0, \ldots, M-1, \quad \text{(internal nodes only)} \\ u_j^0 = \Psi_{log}(x_j), \qquad j = 0, \ldots, N \qquad \text{(initial condition)} \\ u_0^m = \underline{g}_{log}(t_m), \qquad m = 0, \ldots, M \qquad \text{(left boundary cond.)} \\ u_N^m = \overline{g}_{log}(t_m), \qquad m = 0, \ldots, M \qquad \text{(right boundary cond.)} \end{cases}$$

Forward Euler (explicit) approximation

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$$
\begin{cases}
\dfrac{u_j^{m+1} - u_j^m}{\Delta t} - \dfrac{(\sigma_j^m)^2}{2h^2}(u_{j+1}^m - 2u_j^m + u_{j-1}^m) - \dfrac{\beta_j^m}{2h}(u_{j+1}^m - u_{j-1}^m) + r^m u_j^m = 0 \\
\qquad\qquad j = 1, \dots, N-1, \quad m = 0, \dots, M-1, \quad \text{(internal nodes only)} \\
u_j^0 = \Psi_{log}(x_j), \qquad j = 0, \dots, N \qquad \text{(initial condition)} \\
u_0^m = \underline{g}_{log}(t_m), \qquad m = 0, \dots, M \qquad \text{(left boundary cond.)} \\
u_N^m = \overline{g}_{log}(t_m), \qquad m = 0, \dots, M \qquad \text{(right boundary cond.)}
\end{cases}
$$

Define now the vector $U^m = (u_1^m, \dots, u_{N-1}^m)^T \in \mathbb{R}^{N-1}$. Then the Finite Difference formulation can be written compactly as

$$
\frac{1}{\Delta t}(U^{m+1} - U^m) + A^m U^m = F^m, \quad m = 0, \dots, M-1,
$$

with $A^m \in \mathbb{R}^{(N-1)\times(N-1)}$ and $F^m \in \mathbb{R}^{N-1}$. Notice that U^0 is known, $U^0 = (\Psi_{log}(x_1), \dots, \Psi_{log}(x_{N-1}))^T$.

Forward Euler (explicit) approximation

with

$$A^m = \begin{bmatrix} \alpha_1^m & \gamma_1^m & & & \\ \delta_2^m & \alpha_2^m & \gamma_2^m & & \\ & \delta_3^m & \ddots & \ddots & \\ & & \ddots & & \end{bmatrix}$$

$$\alpha_j^m = \frac{(\sigma_j^m)^2}{h^2} + r_j^m$$

$$\gamma_j^m = -\frac{(\sigma_j^m)^2}{2h^2} - \frac{\beta_j^m}{2h}$$

$$\delta_j^m = -\frac{(\sigma_j^m)^2}{2h^2} + \frac{\beta_j^m}{2h}$$

Forward Euler (explicit) approximation

with

$$A^m = \begin{bmatrix} \alpha_1^m & \gamma_1^m & & & \\ \delta_2^m & \alpha_2^m & \gamma_2^m & & \\ & \delta_3^m & \ddots & \ddots & \\ & & \ddots & & \end{bmatrix} \qquad \begin{aligned} \alpha_j^m &= \frac{(\sigma_j^m)^2}{h^2} + r_j^m \\ \gamma_j^m &= -\frac{(\sigma_j^m)^2}{2h^2} - \frac{\beta_j^m}{2h} \\ \delta_j^m &= -\frac{(\sigma_j^m)^2}{2h^2} + \frac{\beta_j^m}{2h} \end{aligned}$$

and

$$F^m = \begin{bmatrix} \left(\frac{(\sigma_1^m)^2}{2h^2} - \frac{\beta_1^m}{2h}\right) \underline{g}(t_m) \\ 0 \\ \vdots \\ 0 \\ \left(\frac{(\sigma_{N-1}^m)^2}{2h^2} + \frac{\beta_{N-1}^m}{2h}\right) \overline{g}(t_m) \end{bmatrix}$$

Forward Euler (explicit) approximation

This scheme is called explicit as the solution U^{m+1} can be obtained directly starting from the solution U^m by the formula

$$U^{m+1} = (I - \Delta t A^m)U^m + \Delta t F^m, \quad m = 0, \ldots, M-1$$

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Order of the method

Let us define the vector $U(t_m) = (u(x_0, t_m), \ldots, u(x_N, t_m))^T$ representing the exact solution evaluated on the grid at $t = t_m$.

Since we have used 2^{nd} order centered Finite Differences to discretize the space derivatives and only 1^{st} order Finite Difference to discretize the time derivative, we should expect and error

$$\max_{m=0,\ldots,M} \|U^m - U(t_m)\|_\infty \leq C(h^2 + \Delta t).$$

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$$\max_{m=0,\ldots,M} \|U^m - U(t_m)\|_\infty \leq C(h^2 + \Delta t).$$

We will see later that this is indeed the case, provided that Δt is taken sufficiently small.

Backward Euler (implicit) approximation

As we will see later, the forward Euler method has stability issues and requires a small Δt, namely: $\Delta t \sim h^2$, which could be over constraining.

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The backward Euler scheme overcomes this problem by using a backward finite difference in time: $\frac{\partial u}{\partial t} \to D^-_{\Delta t} u$.

This leads to the scheme

$$
\begin{cases}
\dfrac{u_j^{m+1} - u_j^m}{\Delta t} - \dfrac{(\sigma_j^{m+1})^2}{2h^2}(u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}) - \dfrac{\beta_j^{m+1}}{2h}(u_{j+1}^{m+1} - u_{j-1}^{m+1}) \\
\qquad + r^{m+1} u_j^{m+1} = 0, \qquad j = 1, \dots, N-1, \quad m = 0, \dots, M-1 \\
u_j^0 = \Psi_{log}(x_j), \qquad j = 0, \dots, N \qquad \text{(initial condition)} \\
u_0^m = \underline{g}_{log}(t_m), \qquad m = 0, \dots, M \qquad \text{(left boundary cond.)} \\
u_N^m = \overline{g}_{log}(t_m), \qquad m = 0, \dots, M \qquad \text{(right boundary cond.)}
\end{cases}
$$

Backward Euler (implicit) approximation

Introducing again the vectors $U^m, F^m \in \mathbb{R}^{N-1}$ and the matrix $A^m \in \mathbb{R}^{(N-1)\times(N-1)}$ (same as for forward Euler), the backward Euler scheme corresponds to

$$U^m + \Delta t A^m U^m = U^{m-1} + \Delta t F^m, \quad m = 1, \dots, M,$$

i.e. to the linear system of equations

$$B^m U^m = \tilde{F}^m, \quad m = 1, \dots, M,$$

$$\text{with} \quad B^m = I + \Delta t A^m, \quad \tilde{F}^m = U^{m-1} + \Delta t F^m.$$

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Being the matrix B^m tri-diagonal, this system is “easy to solve”, e.g. by the Thomas' algorithm, and costs $O(N)$ operations at each time step (complexity comparable to the forward Euler method).

Thomas' algorithm for tridiagonal matrices

A tridiagonal matrix $B \in \mathbb{R}^{(N-1)\times(N-1)}$ can be decomposed as $B = \mathcal{L}\mathcal{U}$ with $\mathcal{L}$ lower bi-diagonal and $\mathcal{U}$ upper bi-diagonal.

$$\underbrace{\begin{bmatrix} a_1 & c_1 & & \\ b_2 & a_2 & c_2 & \\ & b_3 & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 1 & & & \\ \beta_2 & 1 & & \\ & \beta_3 & \ddots & \\ & & \ddots & \ddots \end{bmatrix}}_{\mathcal{L}} \underbrace{\begin{bmatrix} \alpha_1 & c_1 & & \\ & \alpha_2 & c_2 & \\ & & \ddots & \ddots \\ & & & \ddots \end{bmatrix}}_{\mathcal{U}}$$

with $\alpha_1 = a_1$ and $\quad \beta_j = \dfrac{b_j}{\alpha_{j-1}}, \quad \alpha_j = a_j - \beta_j c_{j-1}, \quad j = 2, \ldots, N-1.$

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Once the $\mathcal{L}\mathcal{U}$ factorization computed, a linear system $Bx = f$ can be solved as

$\mathcal{L}y = f$ (forward subst.) $\quad y_1 = f_1,\ \ y_j = f_j - \beta_j y_{j-1},\ \ j = 2, \dots, N-1,$

$\mathcal{U}x = y$ (backward subst.) $\quad x_{N-1} = \dfrac{y_{N-1}}{\alpha_{N-1}},\ \ x_j = \dfrac{y_j - c_j x_{j+1}}{\alpha_j},\ \ j = N-2, \dots, 1.$

Backward Euler (implicit) approximation

In the case of a backward Euler approximation of the BS equation, we have to solve at each time step the linear system $B^m U^m = \tilde{F}^m$, $m = 1, \ldots, M$, by

$\mathcal{LU}$ factorization:	$B^m = \mathcal{L}^m \mathcal{U}^m$	$O(2N)$ multiplications
forward substitution:	$\mathcal{L}^m V^m = \tilde{F}^m$	$O(N)$ multiplications
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If the coefficients σ and r do not depend on time, hence $B^m = B$ does not change in time, the $\mathcal{LU}$ factorization of B can be computed only once, and the cost per iteration is $O(3N)$, similar to the forward Euler.

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We will see that the backward Euler method is much more stable than the forward Euler and does not have any restriction on Δt.

In terms of accuracy, we should expect the same as for forward Euler, namely:

$$\max_{m=0,\dots,M} \|U^m - U(t_m)\|_\infty \leq C(h^2 + \Delta t).$$

θ-method

We can actually take a linear combination of the forward and backward Euler methods

$$\begin{aligned}
&\text{forward Euler} && U^{m+1} = U^m - \Delta t A^m U^m + \Delta t F^m \\
&\text{backward Euler} && U^{m+1} = U^m - \Delta t A^{m+1} U^{m+1} + \Delta t F^{m+1}
\end{aligned}$$

This is known as the so called θ-method

$$U^{m+1} = U^m + \theta \underbrace{\Delta t(F^{m+1} - A^{m+1}U^{m+1})}_{\text{backward Euler}} + (1-\theta) \underbrace{\Delta t(F^m - A^m U^m)}_{\text{forward Euler}}$$

or equivalently

$$\underbrace{(I + \theta \Delta t A^{m+1})}_{B_\theta^m} U^{m+1} = \underbrace{(I - (1-\theta)\Delta t A^m)}_{C_\theta^m} U^m + \Delta t \underbrace{(\theta F^{m+1} + (1-\theta) F^m)}_{F_\theta^m}.$$

θ-method and Crank-Nicolson

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Of particular interest is the case $\theta = \frac{1}{2}$ known as Crank-Nicolson method. This case, indeed, can be seen as a centered finite difference in time

$$\frac{\partial u}{\partial t}(x_j, t_{m+\frac{1}{2}}) \approx D_{\Delta t}^c(x_j, t_{m+\frac{1}{2}})$$

and

$$\frac{\partial u}{\partial t}(x_j, t_{m+\frac{1}{2}}) \approx \frac{1}{2}\mathcal{L}_h^{m+1} u(x_j, t_{m+1}) + \frac{1}{2}\mathcal{L}_h^m u(x_j, t_m)$$

where

$$\mathcal{L}_h^m u(x_j, t_m) = \frac{(\sigma_j^m)^2}{2} D_h^2 u(x_j, t_m) + \beta_j^m D_{2h}^c u(x_j, t_m) - r_j^m u(x_j, t_m)$$

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As such, we do expect for the Cranck-Nicolson method an order

$$\max_{m=0,\dots,M} \|U^m - U(t_m)\|_\infty \le C(h^2 + \Delta t^2).$$

Analysis of the FD approximation

The θ-method can be written in the general form

$$B_\theta^m U^{m+1} = C_\theta^m U^m + \Delta t F_\theta^m, \qquad m = 0, 1, \dots, M-1, \qquad (7)$$

with

$$B_\theta^m = I + \theta \Delta t A^{m+1}, \quad C_\theta^m = I - (1-\theta) \Delta t A^m, \quad F_\theta^m = \theta F^{m+1} + (1-\theta) F^m$$

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We analyze the convergence of the θ-method in the norm $\|U^m\|_*$.
Typical examples are

- infinity norm: $\|U^m\|_* = \|U^m\|_\infty = \max_{j=1,\ldots,N-1} |U_j^m|$,
- ℓ^2 norm: $\|U^m\|_* = \|U^m\|_{\ell^2} = \sqrt{\frac{1}{N-1}\sum_{j=1}^{N-1}(U_j^m)^2}$ $(\leq \|U^m\|_\infty)$.

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Definition 1 (Stability in the norm $\|\cdot\|_*$)

We say that the scheme (7), applied to a problem with homogeneous boundary conditions, is stable in the norm $\|\cdot\|_*$ if there exists a constant $C_s > 0$, independent of Δt and h, such that for any $\{F_\theta^m\}_{m=0}^{M-1}$ it holds

$$\max_{m=1,\ldots,M} \|U^m\|_* \leq C_s \left(\max_{m=0,\ldots,M-1} \|F_\theta^m\|_* + \|U^0\|_* \right) \tag{8}$$

Analysis of the FD approximation (cont.)

Let $U_{ex}(t_m) = \{u(x_j, t_m),\ j = 1, \dots, N-1\}$ be the exact solution of the BS equation evaluated on the grid $\{x_1, \dots, x_{N-1}\}$ and define the local truncation error

$$\varepsilon^m = \frac{1}{\Delta t}\left[B_\theta^m U_{ex}(t_{m+1}) - C_\theta^m U_{ex}(t_m) - \Delta t F_\theta^m\right], \quad m = 0, \dots, M-1, \tag{9}$$

which measures to what extent the exact solution does not satisfy the discrete problem.

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which measures to what extent the exact solution does not satisfy the discrete problem.

Definition 2 (Consistency)

We say that the scheme (7) is consistent if

$$\varepsilon^m \xrightarrow{h,\Delta t \to 0} 0, \quad \forall\, m = 0, \dots, M-1.$$

and is consistent of order (p, q), in the norm $\|\cdot\|_*$, if there exists a constant $C_c > 0$, independent of h and Δt, such that

$$\max_{m=0,\dots,M-1} \|\varepsilon^m\|_* \le C_c(h^p + \Delta t^q).$$

Analysis of the FD approximation (cont.)

Theorem 4 (stability + consistency = convergence)

Assume that no error is introduced in the boundary conditions. If the scheme (7) is stable and consistent in the norm $\|\cdot\|_*$, then it is convergent in that norm. Moreover, if the scheme is consistent of order (p,q) then

$$\max_{m=1,\dots,M} \|U_{ex}(t_m) - U^m\|_* \leq C_s C_c (h^p + \Delta t^q) + C_s \|U_{ex}(0) - U^0\|_*$$

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Proof: The error $e^m = U_{ex}(t_m) - U^m$ satisfies the problem

$$B_\theta^m e^{m+1} = C_\theta^m e^m + \Delta t \varepsilon^m, \quad m = 0, \dots, M-1$$

with $e^0 = U_{ex}(0) - U^0$ and homogeneous boundary conditions. Hence, from the stability of the scheme we have

$$\max_{m=1,\dots,M} \|e^m\|_* \le C_s \left(\max_{m=0,\dots,M-1} \|\varepsilon^m\|_* + \|e^0\|_* \right),$$

and the result follows from the consistency assumption. □

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Hence, to analyze the convergence of a Finite Differences scheme we have to analyze its stability and consistency.

Consistency of the Forward Euler method

Proposition 7 (Consistency of Forward Euler)

The Forward Euler scheme is consistent of order $(2,1)$ *in the* ∞*-norm, i.e.*

$$\max_{m=0,\dots,M-1} \|\varepsilon^m\|_\infty \leq C_c(h^2 + \Delta t)$$

with constant C_c *that depends on* $\|\frac{\partial^2 u}{\partial t^2}\|_{L^\infty_Q}$, $\|\frac{\partial^4 u}{\partial x^4}\|_{L^\infty_Q}$, $\|\frac{\partial^3 u}{\partial x^3}\|_{L^\infty_Q}$.

Here we have denoted $\|v\|_{L^\infty_Q} = \sup_{x\in[x_{min},x_{max}]} \sup_{t\in[0,T]} |v(x,t)|$.

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Here we have denoted $\|v\|_{L_Q^\infty} = \sup_{x\in[x_{min},x_{max}]} \sup_{t\in[0,T]} |v(x,t)|$.

Proof: We have

$$\begin{aligned}
\varepsilon_j^m = & D_{\Delta t}^+ u(x_j, t_m) && \left(= \frac{\partial u}{\partial t}(x_j, t_m) + O(\Delta t)\right) \\
& - \frac{\sigma(x_j, t_m)^2}{2} D_h^2 u(x_j, t_m) && \left(= -\frac{\sigma^2(x_j, t_m)}{2} \frac{\partial^2 u}{\partial x^2}(x_j, t_m) + O(h^2)\right) \\
& - \beta(x_j, t_m) D_{2h}^c u(x_j, t_m) && \left(= -\beta(x_j, t_m) \frac{\partial u}{\partial x}(x_j, t_m) + O(h^2)\right) \\
& + r(x_j, t_m) u(x_j, t_m).
\end{aligned}$$

$\square$

Stability of the Forward Euler method

For simplicity we consider the case of constant coefficients (extend to the case of non-constant coefficients as an exercise)

Proposition 8 (Forward Euler – Sufficient conditions for stability)

Under the conditions

$$\Delta t \leq \frac{h^2}{\sigma^2 + rh^2}, \qquad h \leq \frac{\sigma^2}{|\beta|} \tag{10}$$

the forward Euler scheme is stable in the ∞*-norm and*

$$\max_{m=1,\dots,M} \|U^m\|_\infty \leq \|U^0\|_\infty + T \max_{m=0,\dots,M-1} \|F^m\|_\infty$$

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Notice that we have stability under the condition $\Delta t \approx h^2/\sigma^2$. This condition requires often a very small time step !

Stability of the Forward Euler method (cont.)

Proof: we have

$$u_j^{m+1} = u_j^m + \frac{\Delta t \sigma^2}{2h^2}(u_{j+1}^m - 2u_j^m + u_{j-1}^m) + \frac{\Delta t \beta}{2h}(u_{j+1}^m - u_{j-1}^m) - \Delta t r u_j^m + \Delta t f_j^m$$

Stability of the Forward Euler method (cont.)

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$$
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&= (1 - \Delta t\,\underbrace{(\frac{\sigma^2}{h^2} + r)}_{\alpha})u_j^m + \Delta t\,\underbrace{(\frac{\sigma^2}{2h^2} + \frac{\beta}{2h})}_{-\gamma}\,u_{j+1}^m + \Delta t\,\underbrace{(\frac{\sigma^2}{2h^2} - \frac{\beta}{2h})}_{-\delta}\,u_{j-1}^m + \Delta t f_j^m
\end{aligned}
$$

Stability of the Forward Euler method (cont.)

Proof: we have

$$u_j^{m+1} = u_j^m + \frac{\Delta t \sigma^2}{2h^2}(u_{j+1}^m - 2u_j^m + u_{j-1}^m) + \frac{\Delta t \beta}{2h}(u_{j+1}^m - u_{j-1}^m) - \Delta t r u_j^m + \Delta t f_j^m$$

$$= (1 - \Delta t \underbrace{(\frac{\sigma^2}{h^2} + r)}_{\alpha}) u_j^m + \Delta t \underbrace{(\frac{\sigma^2}{2h^2} + \frac{\beta}{2h})}_{-\gamma} u_{j+1}^m + \Delta t \underbrace{(\frac{\sigma^2}{2h^2} - \frac{\beta}{2h})}_{-\delta} u_{j-1}^m + \Delta t f_j^m$$

Under conditions (10), we have $(1 - \Delta t \alpha) > 0$ and $\gamma, \delta < 0$. Hence

$$|u_j^{m+1}| \leq (1 - \Delta t \alpha)|u_j^m| - \Delta t \gamma |u_{j+1}^m| - \Delta t \delta |u_{j-1}^m| + \Delta t |f_j^m|$$

Stability of the Forward Euler method (cont.)

Proof: we have

$$
\begin{aligned}
u_j^{m+1} &= u_j^m + \frac{\Delta t \sigma^2}{2h^2}(u_{j+1}^m - 2u_j^m + u_{j-1}^m) + \frac{\Delta t \beta}{2h}(u_{j+1}^m - u_{j-1}^m) - \Delta t r u_j^m + \Delta t f_j^m \\
&= (1 - \Delta t\,\underbrace{(\frac{\sigma^2}{h^2} + r)}_{\alpha})u_j^m + \Delta t\,\underbrace{(\frac{\sigma^2}{2h^2} + \frac{\beta}{2h})}_{-\gamma}\,u_{j+1}^m + \Delta t\,\underbrace{(\frac{\sigma^2}{2h^2} - \frac{\beta}{2h})}_{-\delta}\,u_{j-1}^m + \Delta t f_j^m
\end{aligned}
$$

Under conditions (10), we have $(1 - \Delta t\alpha) > 0$ and $\gamma, \delta < 0$. Hence

$$
\begin{aligned}
|u_j^{m+1}| &\le (1 - \Delta t\alpha)|u_j^m| - \Delta t\gamma|u_{j+1}^m| - \Delta t\delta|u_{j-1}^m| + \Delta t|f_j^m| \\
&\le (1 - \Delta t(\alpha + \gamma + \delta))\|U^m\|_\infty + \Delta t\|F^m\|_\infty
\end{aligned}
$$

Stability of the Forward Euler method (cont.)

Proof: we have

$$\begin{aligned}
u_j^{m+1} &= u_j^m + \frac{\Delta t \sigma^2}{2h^2}(u_{j+1}^m - 2u_j^m + u_{j-1}^m) + \frac{\Delta t \beta}{2h}(u_{j+1}^m - u_{j-1}^m) - \Delta t r u_j^m + \Delta t f_j^m \\
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Under conditions (10), we have $(1 - \Delta t \alpha) > 0$ and $\gamma, \delta < 0$. Hence

$$\begin{aligned}
|u_j^{m+1}| &\leq (1 - \Delta t \alpha)|u_j^m| - \Delta t \gamma |u_{j+1}^m| - \Delta t \delta |u_{j-1}^m| + \Delta t |f_j^m| \\
&\leq (1 - \Delta t(\alpha + \gamma + \delta))\|U^m\|_\infty + \Delta t \|F^m\|_\infty \\
&= \underbrace{(1 - \Delta t r)}_{\leq 1}\|U^m\|_\infty + \Delta t \|F^m\|_\infty
\end{aligned}$$

Stability of the Forward Euler method (cont.)

Proof: we have

$$\begin{aligned} u_j^{m+1} &= u_j^m + \frac{\Delta t \sigma^2}{2h^2}(u_{j+1}^m - 2u_j^m + u_{j-1}^m) + \frac{\Delta t \beta}{2h}(u_{j+1}^m - u_{j-1}^m) - \Delta t r u_j^m + \Delta t f_j^m \\ &= (1 - \Delta t\,\underbrace{(\frac{\sigma^2}{h^2} + r)}_{\alpha})u_j^m + \Delta t\,\underbrace{(\frac{\sigma^2}{2h^2} + \frac{\beta}{2h})}_{-\gamma}\,u_{j+1}^m + \Delta t\,\underbrace{(\frac{\sigma^2}{2h^2} - \frac{\beta}{2h})}_{-\delta}\,u_{j-1}^m + \Delta t f_j^m \end{aligned}$$

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$$\begin{aligned} |u_j^{m+1}| &\le (1 - \Delta t \alpha)|u_j^m| - \Delta t \gamma |u_{j+1}^m| - \Delta t \delta |u_{j-1}^m| + \Delta t |f_j^m| \\ &\le (1 - \Delta t(\alpha + \gamma + \delta))\|U^m\|_\infty + \Delta t \|F^m\|_\infty \\ &= \underbrace{(1 - \Delta t r)}_{\le 1}\|U^m\|_\infty + \Delta t \|F^m\|_\infty \end{aligned}$$

Hence

$$\|U^{m+1}\|_\infty \le \|U^m\|_\infty + \Delta t \|F^m\|_\infty \le \|U^0\|_\infty + \sum_{i=0}^{m} \Delta t \|F^i\|_\infty$$

Stability of the Forward Euler method (cont.)

Proof: we have

$$
\begin{aligned}
u_j^{m+1} &= u_j^m + \frac{\Delta t\sigma^2}{2h^2}(u_{j+1}^m - 2u_j^m + u_{j-1}^m) + \frac{\Delta t\beta}{2h}(u_{j+1}^m - u_{j-1}^m) - \Delta t r u_j^m + \Delta t f_j^m \\
&= (1 - \Delta t\,\underbrace{(\frac{\sigma^2}{h^2} + r)}_{\alpha})u_j^m + \Delta t\,\underbrace{(\frac{\sigma^2}{2h^2} + \frac{\beta}{2h})}_{-\gamma}\,u_{j+1}^m + \Delta t\,\underbrace{(\frac{\sigma^2}{2h^2} - \frac{\beta}{2h})}_{-\delta}\,u_{j-1}^m + \Delta t f_j^m
\end{aligned}
$$

Under conditions (10), we have $(1 - \Delta t\alpha) > 0$ and $\gamma, \delta < 0$. Hence

$$
\begin{aligned}
|u_j^{m+1}| &\leq (1 - \Delta t\alpha)|u_j^m| - \Delta t\gamma|u_{j+1}^m| - \Delta t\delta|u_{j-1}^m| + \Delta t|f_j^m| \\
&\leq (1 - \Delta t(\alpha + \gamma + \delta))\|U^m\|_\infty + \Delta t\|F^m\|_\infty \\
&= \underbrace{(1 - \Delta t r)}_{\leq 1}\|U^m\|_\infty + \Delta t\|F^m\|_\infty
\end{aligned}
$$

Hence

$$
\|U^{m+1}\|_\infty \leq \|U^m\|_\infty + \Delta t\|F^m\|_\infty \leq \|U^0\|_\infty + \sum_{i=0}^{m} \Delta t\|F^i\|_\infty
$$

and

$$
\max_{m=1,\dots,M} \|U^m\|_\infty \leq \|U^0\|_\infty + T \max_{m=0,\dots,M-1} \|F^m\|_\infty
$$

$\square$

Stability of the Forward Euler method (cont.)

One can show that a condition of the type $\Delta t \approx h^2$ is also necessary for stability. This analysis is based on the eigenvalues of the tridiagonal matrix

$$A = \begin{bmatrix} \alpha & \gamma & & \\ \delta & \ddots & \ddots & \\ & \ddots & \ddots & \end{bmatrix} \qquad \begin{aligned} \alpha &= \frac{\sigma^2}{h^2} + r, \\ \gamma &= \frac{\sigma^2}{2h^2} + \frac{\beta}{2h}, \quad \delta = \frac{\sigma^2}{2h^2} - \frac{\beta}{2h} \end{aligned}$$

Lemma 1

The eigenvalues of the matrix A are given by

$$\lambda_i(A) = \alpha + 2\sqrt{\beta\gamma}\cos\left(\frac{i\pi}{N+1}\right), \qquad i = 1, \ldots, N.$$

In particular, if $h \leq \sigma^2/|\beta|$, the eigenvalues are all real and positive and

$$\lambda_{min}(A) = r + \frac{\sigma^2}{h^2} + \sqrt{\frac{\sigma^4}{h^4} - \frac{\beta^2}{h^2}}\cos\left(\frac{N\pi}{N+1}\right) = O(h^{-2})$$

Stability of the Forward Euler method (cont.)

Proposition 9 (Forward Euler – necessary condition for stability)

Assume $h \leq \sigma^2/|\beta|$. *Then a necessary condition for stability of the forward Euler method is*

$$\Delta t \leq \frac{2}{\lambda_{min}(A)} = O(h^2).$$

Stability of the Forward Euler method (cont.)

Proposition 9 (Forward Euler – necessary condition for stability)

Assume $h \leq \sigma^2/|\beta|$. Then a necessary condition for stability of the forward Euler method is

$$\Delta t \leq \frac{2}{\lambda_{min}(A)} = O(h^2).$$

Proof: The forward Euler method reads

$$U^{m+1} = (I - \Delta t A)U^m + \Delta t F^m$$

Assume $F^m = 0$ for $m = 0, \ldots, M-1$ and U^0 an eigenvector of A corresponding to $\lambda_{min}(A)$. Then $U^m = (1 - \Delta t \lambda_{min}(A))^m U^0$ and

$$\max_{m=1,\ldots,M} \|U^m\|_\infty = \|U^0\|_\infty \max_{m=1,\ldots,M} |1 - \Delta t \lambda_{min}(A)|^m$$

so a necessary condition for stability is $|1 - \Delta t \lambda_{min}(A)| \leq 1$ which leads to the thesis. $\square$

Analysis of the backward Euler method

Backward Euler $\qquad (I + \Delta t A)U^{m+1} = U^m + \Delta t F^{m+1}$

Proposition 10 (Consistency of Backward Euler)

The backward Euler method is consistent of order $(2,1)$ *in the* ℓ^2 *and* ∞ *norms*

$$\max_{m=0,\dots,M-1} \|\varepsilon^m\|_{\ell^2} \le \max_{m=0,\dots,M-1} \|\varepsilon^m\|_{\infty} \le C_c(h^2 + \Delta t)$$

with $C_c = C_c(\|\frac{\partial^2 u}{\partial t^2}\|_{L^\infty_Q}, \|\frac{\partial^4 u}{\partial x^4}\|_{L^\infty_Q}, \|\frac{\partial^3 u}{\partial x^3}\|_{L^\infty_Q})$.

Proof: same as for forward Euler.

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Proof: same as for forward Euler.

For the stability analysis, we again limit ourselves to the case of constant coefficients and consider first the stability in the ℓ^2-norm

Proposition 11 (Backward Euler – stability in ℓ^2-norm)

The backward Euler scheme is stable in the ℓ^2*-norm with no conditions on* Δt *and* $\max_{m=1,\dots,M} \|U^m\|_{\ell^2} \leq \|U^0\|_{\ell^2} + T \max_{m=1,\dots,M} \|F^m\|_{\ell^2}$

Analysis of the backward Euler method (cont.)

Proof: The matrix A splits as $A = D + R + T$ with

$$D = \frac{\sigma^2}{2h^2}\begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & \end{bmatrix}, \quad R = rI, \quad T = \frac{\beta}{2h}\begin{bmatrix} 0 & -1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & \end{bmatrix}$$

For any $W \in \mathbb{R}^{N-1}$, it is easy to check that $W^T T W = 0$ and $W^T R W = r\|W\|^2$.

Analysis of the backward Euler method (cont.)

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For any $W \in \mathbb{R}^{N-1}$, it is easy to check that $W^T T W = 0$ and $W^T R W = r\|W\|^2$. Moreover, being D symmetric and positive definite

$$W^T D W \geq \lambda_{min}(D)\|W\|^2 = \frac{\sigma^2}{2h^2}\left(2 + 2\cos\left(\frac{N\pi}{N+1}\right)\right)\|W\|^2 > 0$$

Analysis of the backward Euler method (cont.)

Proof: The matrix A splits as $A = D + R + T$ with

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Hence $W^T(I + \Delta t A)W \geq \|W\|^2$ and

$$\|U^{m+1}\|^2 \leq (U^{m+1})^T(I + \Delta t A)U^{m+1} = (U^{m+1})^T(U^m + \Delta t F^{m+1})$$

Analysis of the backward Euler method (cont.)

Proof: The matrix A splits as $A = D + R + T$ with

$$D = \frac{\sigma^2}{2h^2}\begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & \end{bmatrix}, \quad R = rI, \quad T = \frac{\beta}{2h}\begin{bmatrix} 0 & -1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & \end{bmatrix}$$

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Hence $W^T(I + \Delta t A)W \geq \|W\|^2$ and

$$\|U^{m+1}\|^2 \leq (U^{m+1})^T(I + \Delta t A)U^{m+1} = (U^{m+1})^T(U^m + \Delta t F^{m+1})$$

from which we deduce

$$\|U^{m+1}\| \leq \|U^m\| + \Delta t\|F^{m+1}\| \leq \|U^0\| + (m+1)\Delta t \max_{i=1,\dots,m+1}\|F^i\|$$

and the thesis follows. $\square$

Analysis of the backward Euler method (cont.)

A similar stability result holds in the ∞ norm

Proposition 12 (Backward Euler – stability in ∞ norm)

The backward Euler scheme is stable in the ∞-norm with no conditions on Δt and under the condition $h < \sigma^2/|\beta|$

$$\max_{m=1,\dots,M} \|U^m\|_\infty \leq \|U^0\|_\infty + T \max_{m=1,\dots,M} \|F^m\|_\infty$$

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Proof: we have for $\alpha = \frac{\sigma^2}{h^2} + r$, $\gamma = -\frac{\sigma^2}{2h^2} - \frac{\beta}{2h}$, $\delta = -\frac{\sigma^2}{2h^2} + \frac{\beta}{2h}$

$$u_j^m = u_j^{m-1} - \Delta t \alpha u_j^m - \Delta t \gamma u_{j+1}^m - \Delta t \delta u_{j-1}^m + \Delta t f_j^m$$

Analysis of the backward Euler method (cont.)

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$$u_j^m = u_j^{m-1} - \Delta t \alpha u_j^m - \Delta t \gamma u_{j+1}^m - \Delta t \delta u_{j-1}^m + \Delta t f_j^m$$

Observe that under the condition $h < \sigma^2/|\beta|$ we have $\gamma, \delta < 0$ and $\alpha + \gamma + \delta = r$.

Analysis of the backward Euler method (cont.)

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Observe that under the condition $h < \sigma^2/|\beta|$ we have $\gamma, \delta < 0$ and $\alpha + \gamma + \delta = r$. Let $i = \operatorname{argmax}_j |u_j^m|$ so that $|u_i^m| = \|U^m\|_\infty$. Then,

$$(1 + \Delta t \alpha)\|U^m\|_\infty = (1 + \Delta t \alpha)|u_i^m| \le |u_i^{m-1} - \Delta t \gamma u_{i+1}^m - \Delta t \delta u_{i-1}^m + \Delta t f_i^m|$$

Analysis of the backward Euler method (cont.)

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$$\begin{aligned}(1 + \Delta t \alpha)\|U^m\|_\infty = (1 + \Delta t \alpha)|u_i^m| &\leq |u_i^{m-1} - \Delta t \gamma u_{i+1}^m - \Delta t \delta u_{i-1}^m + \Delta t f_i^m| \\ &\leq \|U^{m-1}\|_\infty - \Delta t(\gamma + \delta)\|U^m\|_\infty + \Delta t \|F^m\|_\infty\end{aligned}$$

Analysis of the backward Euler method (cont.)

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$$u_j^m = u_j^{m-1} - \Delta t \alpha u_j^m - \Delta t \gamma u_{j+1}^m - \Delta t \delta u_{j-1}^m + \Delta t f_j^m$$

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$$\begin{aligned}(1+\Delta t\alpha)\|U^m\|_\infty = (1+\Delta t\alpha)|u_i^m| &\leq |u_i^{m-1} - \Delta t \gamma u_{i+1}^m - \Delta t \delta u_{i-1}^m + \Delta t f_i^m| \\ &\leq \|U^{m-1}\|_\infty - \Delta t(\gamma+\delta)\|U^m\|_\infty + \Delta t \|F^m\|_\infty\end{aligned}$$

Hence

$$(1+\Delta t\, r)\|U^m\|_\infty \leq \|U^{m-1}\|_\infty + \Delta t \|F^m\|_\infty$$

from which the thesis follows. $\square$

Analysis of the θ-method

For the θ-method we have the following general results:

Consistency

$$\max_{m=0,\dots,M-1} \|\varepsilon^m\|_\infty \le C_c(h^2 + \Delta t), \qquad \theta \neq 1/2,$$
$$\max_{m=0,\dots,M-1} \|\varepsilon^m\|_\infty \le C_c(h^2 + \Delta t^2), \qquad \theta = 1/2.$$

Stability in the ℓ^2 norm

unconditional for $\frac{1}{2} \le \theta \le 1$

conditional $\Delta t \sim h^2$ for $0 \le \theta < \frac{1}{2}$.

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Stability in the ℓ^2 norm

unconditional for $\frac{1}{2} \leq \theta \leq 1$

conditional $\Delta t \sim h^2$ for $0 \leq \theta < \frac{1}{2}$.

In particular, the Crank-Nicolson method is unconditionally stable and second order accurate in h and Δt.

Upwinding

- The stability results for both the Forward and Backward Euler schemes in the ∞-norm involve the condition $h \leq \sigma^2/|\beta|$.

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Let us consider the limit problem $\sigma = 0$, and $\beta = r$ constant (for simplicity we neglect here the reaction term ru):

$$\frac{\partial u}{\partial t} - \beta\frac{\partial u}{\partial x} = 0.$$

This is a pure transport problem, whose solution is $u(x,t) = \Psi_{log}(x + \beta t)$, with Ψ_{log} the initial condition (payoff).

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Traveling wave from right to left if $\beta > 0$ (from left to right if $\beta < 0$).

Upwinding (cont.)

In the case $\beta > 0$, since the information travels right-to-left, if is reasonable to approximate the space derivative $-\beta \frac{\partial u}{\partial x}$ by a one-sided forward (right) finite difference:

$$\frac{u_j^{m+1} - u_j^m}{\Delta t} - \beta \frac{u_{j+1}^m - u_j^m}{h} = 0 \tag{11}$$

By doing so, the solution u_j^{m+1} at the new time step, depends only on u_j^m and the upwinded value u_{j+1}^m.

Upwinding (cont.)

In the case $\beta > 0$, since the information travels right-to-left, if is reasonable to approximate the space derivative $-\beta\frac{\partial u}{\partial x}$ by a one-sided forward (right) finite difference:

$$\frac{u_j^{m+1} - u_j^m}{\Delta t} - \beta\frac{u_{j+1}^m - u_j^m}{h} = 0 \tag{11}$$

By doing so, the solution u_j^{m+1} at the new time step, depends only on u_j^m and the upwinded value u_{j+1}^m.

Similarly, in the case $\beta < 0$, we would rather use a one-sided backward (left) finite difference

$$\frac{u_j^{m+1} - u_j^m}{\Delta t} - \beta\frac{u_j^m - u_{j-1}^m}{h} = 0 \tag{12}$$

Upwinding (cont.)

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Proposition 13 (Stability of the upwinded scheme)

The upwinding scheme (11) *(for $\beta > 0$) and* (12) *(for $\beta < 0$) is stable in the ∞-norm under the CFL (Courant-Friedrichs-Lewy) condition $\Delta t \leq h/|\beta|$ and $\|U^{m+1}\|_\infty \leq \|U^m\|_\infty$.*

Upwinding (cont.)

Proof: We consider the case $\beta > 0$ (the case $\beta < 0$ is analogous)

$$u_j^{m+1} = u_j^m + \Delta t \beta \frac{u_{j+1}^m - u_j^m}{h} = \underbrace{(1 - \frac{\Delta t \beta}{h})}_{\geq 0 \text{under CFL cond.}} u_j^m + \frac{\Delta t \beta}{h} u_{j+1}^m$$

Upwinding (cont.)

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Let $i = \operatorname{argmax}_j |u_j^{m+1}|$ so that $\|U^{m+1}\|_\infty = |u_i^{m+1}|$. Then

$$\|U^{m+1}\|_\infty = |u_i^{m+1}| \leq (1 - \frac{\Delta t\beta}{h})|u_j^m| + \frac{\Delta t\beta}{h}|u_{j+1}^m| \leq \|U^m\|_\infty.$$

$\square$

Exercise 2

Compare with the stability result of an explicit centered scheme!)

Upwinding (cont.)

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Exercise 2

Compare with the stability result of an explicit centered scheme!)

- The upwinding strategy leads to stable discretizations in the ∞-norm under a mild CFL stability condition; Moreover, for a pure transport problem the ∞-norm is non-increasing (monotonicity property – no oscillations in the solution)
- Drawback: only first order accurate.

Forward Euler + upwinding for BS in log-price

Let us come back to the generalized BS equations in log-price and consider a forward Euler discretization with center FD for the second derivatives and upwinded FD for the drift term:

$$\begin{cases} \dfrac{u_j^{m+1}-u_j^m}{\Delta t} - \dfrac{(\sigma_j^m)^2}{2h^2}(u_{j+1}^m - 2u_j^m + u_{j-1}^m) - \beta_j^m D_h^{UW} u_j^m + r^m u_j^m = 0 \\ \qquad j=1,\dots,N-1, \quad m=0,\dots,M-1, \quad \text{(internal nodes only)} \\ u_j^0 = \Psi_{log}(x_j), \qquad j=0,\dots,N \qquad \text{(initial condition)} \\ u_0^m = \underline{g}_{log}(t_m), \qquad m=0,\dots,M \qquad \text{(left boundary cond.)} \\ u_{N+1}^m = \overline{g}_{log}(t_m), \qquad m=0,\dots,M \qquad \text{(right boundary cond.)} \end{cases}$$

where

$$D_h^{UW} u_j^m = \begin{cases} \frac{u_{j+1}^m - u_j^m}{h} & \text{if } \beta_j^m \geq 0, \\ \frac{u_j^m - u_{j-1}^m}{h} & \text{if } \beta_j^m < 0. \end{cases}$$

Forward Euler + upwinding for BS in log-price

Proposition 14

- *the forward Euler + upwinding scheme is stable in the ∞-norm under the condition $\Delta t \leq (\frac{\sigma^2}{h^2} + r + \frac{|\beta|}{h})^{-1}$ (and no cond. on h);*
- *the backward Euler + upwinding scheme is stable in the ∞-norm with no condition on Δt and h.*

Proof: left as exercise

Forward Euler + upwinding for BS in log-price

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Proof: left as exercise

Remarks:

- when approximating generalized BS equations for a single asset price, upwinding is rarely needed as the condition $h \leq \sigma^2/|\beta|$ is not too stringent and usually satisfied by the chosen spatial grid;

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- when approximating generalized BS equations for a single asset price, upwinding is rarely needed as the condition $h \leq \sigma^2/|\beta|$ is not too stringent and usually satisfied by the chosen spatial grid;
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- however, when looking at basket options or other exotic options (e.g. Asian) depending on multiple factors, if might well arrive that the volatility associated to one or more factors is very small (e.g. volatility of interest rate). Hence, upwinding might be needed;
- remember that upwinding is only first order accurate. Try to use it only when strictly needed. One could also think at a "blending" between centered and upwinded formulas depending on the size of β (see e.g. [Duffy, Finite Difference Methods in financial engineering])

FD approximation of the BS eq. in asset price

Consider now the truncated BS equation in asset price. Observe that in this case we do not need to truncate the domain on the left side (we can take $S_{min} = 0$).

$$\begin{cases} \frac{\partial u}{\partial t}(S,t) - \frac{\sigma^2(S,t)}{2} S^2 \frac{\partial^2 u}{\partial S^2}(S,t) - r(t) S \frac{\partial u}{\partial S}(S,t) + r(t) u(S,t) = 0, \\ \hfill S \in (0, S_{max}),\ t \in (0,T] \\ u(S,0) = \Psi(S), \hfill S \in [0, S_{max}] \\ u(S_{max}, t) = \overline{g}(t), \hfill t \in [0,T] \end{cases}$$

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What boundary conditions in $S = 0$? Observe that the equation in $S = 0$ reduces to

$$\frac{\partial u}{\partial t}(0,t) + r(t) u(0,t) = 0$$

whose solution is $u(0,t) = \Psi(0) e^{\int_0^t r(\tau) d\tau}$. Hence, the value $u(0,t)$ is uniquely determined by the equation and there is no need to impose any boundary condition.

We introduce a uniform grid in the asset price

$$S_j = jh, \;\; j = 0, \dots, N, \qquad h = \frac{S_{max}}{N}$$

$$t_m = m\Delta t, \;\; m = 0, \dots, M, \qquad \Delta t = \frac{T}{M}$$

In $S = 0$, we do not impose any boundary condition and let the numerical scheme find good solution. Hence, the vector of unknowns at each time step is

$$U^m = (u_0^m, u_1^m, \dots, u_{N-1}^m)$$

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Forward Euler method

$$\begin{cases} \frac{u_j^{m+1}-u_j^m}{\Delta t} - \frac{(\sigma_j^m)^2}{2h^2} S_j^2 (u_{j+1}^m - 2u_j^m + u_{j-1}^m) - \frac{r^m}{2h} S_j (u_{j+1}^m - u_{j-1}^m) + r^m u_j^m = 0 \\ \qquad\qquad j = 1, \dots, N-1, \;\; m = 0, \dots, M-1, \quad \text{(internal nodes only)} \\ \frac{u_0^{m+1}-u_0^m}{\Delta t} + r^m u_0^m = 0, \qquad m = 0, \dots, M \qquad \text{(left boundary cond.)} \\ u_N^m = \overline{g}(t_m), \qquad m = 0, \dots, M \qquad \text{(right boundary cond.)} \\ u_j^0 = \Psi(S_j), \qquad j = 0, \dots, N+1 \qquad \text{(initial condition)} \end{cases}$$

FD approximation of the BS eq. in asset price (cont.)

In matrix form the forward Euler method reads

$$U^{m+1} = (I - \Delta t A^m) U^m + F^m$$

with $A^m \in \mathbb{R}^{N \times N}$

$$A^m = \begin{bmatrix} \alpha_0^m & \gamma_0^m & & \\ \delta_1^m & \alpha_1^m & \gamma_1^m & \\ & \delta_2^m & \ddots & \ddots \\ & & \ddots & \end{bmatrix}$$

$$\alpha_j^m = \frac{(\sigma_j^m)^2 S_j^2}{h^2} + r^m$$

$$\gamma_j^m = -\frac{(\sigma_j^m)^2 S_j^2}{2h^2} - \frac{r^m S_j}{2h}$$

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and $F^m \in \mathbb{R}^N$

$$F^m = \left[0 \quad 0 \quad \cdots \quad 0 \quad \left(\frac{(\sigma_N^m)^2 S_N^2}{2h^2} + \frac{r^m S_N}{2h}\right) \overline{g}(t_m)\right]^T$$

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Similarly, the θ-method will read

$$(I + \theta \Delta t A^{m+1})U^{m+1} = (I - (1-\theta)\Delta t A^m)U^m + \Delta t(\theta F^m + (1-\theta)F^{m-1}).$$

FD approximation of the BS eq. in asset price (cont.)

Exercise 3

In the case of constant coefficients σ and r, derive sufficient conditions for the stability of the forward Euler approximation of the BS equation in asset price. Analyze then the convergence of the scheme.

The Binomial model

Idea: approximate the price dynamics

$$dS_\tau = rS_\tau d\tau + \sigma S_\tau dW_\tau$$

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by a random walk. Given the asset price $S^m = s$ at time $\tau_m = m\Delta t$, then at time $(m+1)\Delta t$ we can only have

$$S^{m+1} = \begin{cases} u\,s & \text{with probability } p \qquad \text{(jump up)} \\ d\,s & \text{with probability } 1-p \quad \text{(jump down)} \end{cases}$$

Typical choice is $d = \frac{1}{u}$.

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Typical choice is $d = \frac{1}{u}$. The probability of jumping up is determined by matching moments:

$$\begin{aligned} \mathbb{E}[S^{m+1}|S^m = s] &= se^{r\Delta t} = usp + ds(1-p) \\ \mathbb{V}\mathrm{ar}[S^{m+1}|S^m = s] &= s^2 e^{2r\Delta t}(e^{\sigma^2\Delta t} - 1) \\ &= (us)^2 p + ds^2(1-p) - (usp + ds(1-p))^2 \end{aligned}$$

The Binomial model (cont.)

For the choice $d = \frac{1}{u}$, and setting $A = \frac{1}{2}(e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t})$, this leads to

$$u = A + \sqrt{A^2 - 1} = e^{\sigma\sqrt{\Delta t}} + O(\Delta t^{\frac{3}{2}})$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}} + O(\Delta t^{\frac{3}{2}})$$

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{1}{2} + \frac{(r - \sigma^2/2)\Delta t}{2\sigma\sqrt{\Delta t}} + O(\Delta t^{\frac{3}{2}})$$

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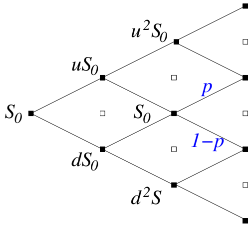
The **forward phase** of the method consists in building the *Binomial tree*:



Binomial model – backward phase

Denote by $s_i^m = u^i S_0$, $i = -m, \dots, m$. The approximate process S^m that starts at S_0 at time $\tau = 0$, will visit at time $\tau_m = m\Delta t$ any of the values $\{s_{-m}^m, s_{-m+2}^m, \dots, s_{m-2}^m, s_m^m\}$.

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Backward phase: calculate the option price

- The value of the option at time $t_M = M\Delta t$ is known and given by the payoff: $V_i^M = \Psi(s_i^m)$.
- The value of the option at time τ_m and asset price s_i^m is given by

$$
\begin{aligned}
V_i^m &= e^{-r\Delta t}\mathbb{E}[V(S^{m+1}, \tau_{m+1})|S^m = s_i^m] \\
&= e^{-r\Delta t}(V_{i+1}^{m+1} p + V_{i-1}^{m+1}(1-p))
\end{aligned}
$$

Observe that we have only to compute it for
$i = -m, -m+2, \ldots, m-2, m$.

Binomial model – relation with finite differences

Observe that:

- the grid $\{s_i^m, \;\; i=-m,\dots,m\}$ is a uniform grid in log-price

$$x_i^m = \log(s_i^m) = \log(u^i S_0) = S_0 + i\log(u), \quad i=-m,\dots,m$$

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- we have

$$\begin{aligned} e^{-r\Delta t} p &= \frac{e^{-r\Delta t}}{2} + \frac{e^{-r\Delta t}(r - \sigma^2/2)\Delta t}{2\sigma\sqrt{\Delta t}} + O(\Delta t^{\frac{3}{2}}) \\ &= \frac{(1 - r\Delta t)\sigma^2 \Delta t}{2\sigma^2 \Delta t} + \frac{\sqrt{1 - r\Delta t}(r - \sigma^2/2)\Delta t}{2\sigma\sqrt{\Delta t}} + O(\Delta t^{\frac{3}{2}}) \end{aligned}$$

$$\log(u) = \sigma\sqrt{\Delta t} + O(\Delta t^{\frac{3}{2}}) = \frac{\sigma\sqrt{\Delta t}}{\sqrt{1 + r\Delta t}} + O(\Delta t^{\frac{3}{2}})$$

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- Setting now $h^2 = \frac{\sigma^2 \Delta t}{1 - r\Delta t}$ and up to $O(\Delta t^{\frac{3}{2}})$ the option price is

$$V_i^m = \Delta t \left(\frac{\sigma^2}{2h^2} + \frac{(r - \sigma^2/2)}{2h} \right) V_{i+1}^{m+1} + \Delta t \left(\frac{\sigma^2}{2h^2} - \frac{(r - \sigma^2/2)}{2h} \right) V_{i-1}^{m+1}$$

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We recognize a forward Euler FD discretization of the (backward) BS equation in log-price with the particular choice of time step $\Delta t = \frac{h^2}{\sigma^2 + r h^2}$.

Barrier options

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Let us consider an option with payoff $\Psi(S_T)$, lower barrier L, and upper barrier H, i.e. the option becomes worthless if $S_\tau \leq L$ or $S_\tau \geq H$ for some $\tau < T$.

Let $u(S,t)$ be the value of the option for the asset price S and time to maturity t. Clearly

$$u(L,t) = 0, \qquad u(H,t) = 0, \qquad \forall t \in [0,T]$$

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$$u(L,t) = 0, \qquad u(H,t) = 0, \qquad \forall t \in [0,T]$$

Hence $u(S,t)$ satisfies the BS equation

$$\begin{cases} \frac{\partial u}{\partial t}(S,t) - \frac{\sigma^2(S,t)}{2} S^2 \frac{\partial^2 u}{\partial S^2}(S,t) - r(t) S \frac{\partial u}{\partial S}(S,t) + r(t) u(S,t) = 0, & \\ \qquad S \in (L,H),\ t \in (0,T] & \\ u(S,0) = \Psi(S), & S \in [L,H] \\ u(L,t) = u(H,t) = 0, & t \in [0,T] \end{cases}$$

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A knock-out barrier option is an option that becomes worthless if the asset price S_τ hits the barrier H before expiration.

Let us consider an option with payoff $\Psi(S_T)$, lower barrier L, and upper barrier H, i.e. the option becomes worthless if $S_\tau \leq L$ or $S_\tau \geq H$ for some $\tau < T$.

Let $u(S,t)$ be the value of the option for the asset price S and time to maturity t. Clearly

$$u(L,t) = 0, \qquad u(H,t) = 0, \qquad \forall t \in [0,T]$$

Hence $u(S,t)$ satisfies the BS equation

$$\begin{cases} \frac{\partial u}{\partial t}(S,t) - \frac{\sigma^2(S,t)}{2} S^2 \frac{\partial^2 u}{\partial S^2}(S,t) - r(t) S \frac{\partial u}{\partial S}(S,t) + r(t) u(S,t) = 0, \\ \hfill S \in (L,H),\ t \in (0,T] \\ u(S,0) = \Psi(S), \hfill S \in [L,H] \\ u(L,t) = u(H,t) = 0, \hfill t \in [0,T] \end{cases}$$

Barrier options are easy to treat with a PDE approach (more cumbersome with a Monte Carlo approach)

Computation of Greeks

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Types of Greeks for an option price $u(S,t)$ assuming constant parameters σ and r:

δ (delta):	$\partial_S u$
γ (gamma):	$\partial_{SS} u$
Θ (time-decay):	$-\partial_t u$
κ (vega):	$\partial_\sigma u$
ρ (rho):	$\partial_r u$
η (eta):	$\partial_K u$ (for a call or put option with strike K)

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The Greeks δ, γ, Θ, can be computed numerically using Finite Difference formulas on the approx. solution $\{u_j^m, \; j=0,\ldots,N, \; m=0,\ldots,M\}$

Computation of Greeks (cont.)

To compute the other Greeks (κ, ρ, η), one can differentiate the BS equation (and boundary conditions) with respect to the corresponding parameter.

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Example: equation for vega ($\kappa = \partial_\sigma u$) for a double barrier knock-out option:

$$\frac{\partial}{\partial \sigma}\left(\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\sigma^2}{2} S^2 \dfrac{\partial^2 u}{\partial S^2} - rS\dfrac{\partial u}{\partial S} + ru = 0, & S \in (L,H),\ t \in (0,T] \\ u(S,0) = \Psi(S), & S \in [L,H] \\ u(L,t) = u(H,t) = 0, & t \in [0,T] \end{cases}\right)$$

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$$\Longrightarrow \begin{cases} \dfrac{\partial \kappa}{\partial t} - \dfrac{\sigma^2}{2} S^2 \dfrac{\partial^2 \kappa}{\partial S^2} - rS\dfrac{\partial \kappa}{\partial S} + r\kappa = \sigma S^2 \dfrac{\partial^2 u}{\partial S^2}, & S \in (L,H),\ t \in (0,T] \\ \kappa(S,0) = 0, & S \in [L,H] \\ \kappa(L,t) = \kappa(H,t) = 0, & t \in [0,T] \end{cases}$$

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One has first to compute u (with some FD scheme) and then compute κ (again with some FD scheme) using the computed u as right hand side.