Computational Finance
FIN-472
Polynomial expansion methods: The Jacobi model

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Outline

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Stochastic volatility models

The volatility of stock price log-returns is stochastic

	Black-Scholes	Heston (affine)
volatility	constant	stochastic $\in \mathbb{R}_+$
calls and puts	closed-form	Fourier transform
exotic options	closed-form	...

Black-Scholes model $\subset$ $\boxed{\text{Jacobi model}}$ $\to$ Heston model

- **stochastic volatility** on a parametrized **compact support**
- vanilla and exotic **option prices** have a **series representation**
- **fast and accurate** price approximations

Jacobi Stochastic Volatility model

Fix $0 \leq v_{min} < v_{max}$. Define the quadratic function

$$Q(v) = \frac{(v - v_{min})(v_{max} - v)}{(\sqrt{v_{max}} - \sqrt{v_{min}})^2} \leq v$$

Jacobi Model

Stock price dynamics $S_t = e^{X_t}$ given by

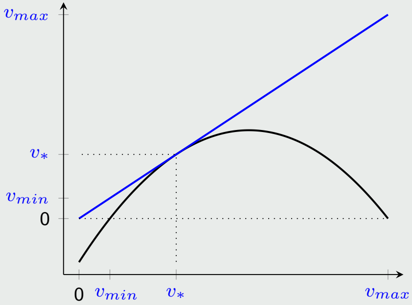
$$\begin{aligned} dV_t &= \kappa(\theta - V_t)\,dt + \sigma\sqrt{Q(V_t)}\,dW_{1t} \\ dX_t &= (r - V_t/2)\,dt + \rho\,\sqrt{Q(V_t)}\,dW_{1t} + \sqrt{V_t - \rho^2\,Q(V_t)}\,dW_{2t} \end{aligned} \tag{1}$$

for $\kappa, \sigma > 0$, $\theta \in (v_{min}, v_{max}]$, interest rate r, $\rho \in [-1, 1]$, and 2-dimensional BM $W = (W_1, W_2)$

Some properties

The function $Q(v)$

$v \geq Q(v)$, $v = Q(v)$ if and only if $v = \sqrt{v_{min} v_{max}}$, and $Q(v) \geq 0$ for all $v \in [v_{min}, v_{max}]$



Instantaneous variance

$d\langle X, X\rangle_t = V_t \in [v_{min}, v_{max}]$ is a **Jacobi process**

Some properties (cont.)

Polynomial model

(V_t, X_t) is a polynomial diffusion – **efficient calculation of moments**

Black-Scholes model nested

Take $V_0 = \theta = v_{max} = \sigma^2_{\mathrm{BS}}$

Heston model as a limit case

If $v_{min} = 0$ and $v_{max} \to \infty$ then (V_t, X_t) converges to the Heston model

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Log-price density

We define

$$C_T = \int_0^T \left(V_t - \rho^2 Q(V_t)\right) dt$$

Theorem 1

Let $\epsilon < 1/(2v_{max}T)$. Then the distribution of X_T admits a density $g_T(x)$ on $\mathbb{R}$ that satisfies

$$\int_{\mathbb{R}} \mathrm{e}^{\epsilon x^2} g_T(x)\, dx < \infty \tag{2}$$

If

$$\mathbb{E}\left[{C_T}^{-1/2}\right] < \infty \tag{3}$$

then $g_T(x)$ and $\mathrm{e}^{\epsilon x^2} g_T(x)$ are uniformly bounded and continuous on $\mathbb{R}$. A sufficient condition for (3) to hold is

$$v_{min} > 0 \text{ and } \rho^2 < 1$$

Remark: The Heston model does not satisfy (2) for any $\epsilon > 0$

A crucial corollary

Corollary 2

Assume (3) *holds. Then* $\ell(x) = \frac{g_T(x)}{w(x)} \in L^2_w$, *where*

$$L^2_w := \left\{ h : \int_{\mathbb{R}} |h(x)|^2 w(x)\, dx \right\}$$

and $w(x)$ *is any Gaussian density with variance* σ^2_w *satisfying*

$$\sigma^2_w > \frac{v_{max} T}{2} \tag{4}$$

- **(Filipovic, Mayerhofer, Schneider 2013)** For the Heston model we have that $\ell(x) = \frac{g_T(x)}{w(x)} \in L^2_w$, where $w(x)$ **is a (bilateral) Gamma density**
- **Corollary 2 is a consequence of Theorem 1 because any uniformly bounded and integrable function is square integrable**

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Weighted L^2–space

- **The weighted Hilbert space**

$$L^2_w = \left\{ f(x) \mid \|f\|^2_w = \int_{\mathbb{R}} f(x)^2\, w(x) dx < \infty \right\}$$

where $w(x)$ is a Gaussian density with mean μ_w and variance σ^2_w. This is a Hilbert space with scalar product

$$\langle f, g \rangle_w = \int_{\mathbb{R}} f(x) g(x)\, w(x) dx$$

- **Orthonormal basis – Generalized Hermite polynomials**

$$H_n(x) = \frac{1}{\sqrt{n!}} \mathcal{H}_n \left(\frac{x - \mu_w}{\sigma_w} \right)$$

where $\mathcal{H}_n(x)$ are the **standard (probabilists' / not physicists') Hermite polynomials**

- More precisely

$$\mathcal{H}_n(x) = (-1)^n \mathrm{e}^{\frac{x^2}{2}} \frac{d^n}{dx^n} \mathrm{e}^{-\frac{x^2}{2}}$$

Price approximation

Pricing problem

Assume that X_T has a density $g_T(x)$

$$\pi_f = \mathbb{E}[f(X_T)] = \int_{\mathbb{R}} f(x) g_T(x) dx$$

Price series expansion

Suppose $\ell(x) = g_T(x)/w(x) \in L^2_w$ and $f(x) \in L^2_w$. Then

$$\pi_f = \langle f, \ell \rangle_w = \sum_{n \geq 0} f_n \ell_n \tag{5}$$

for the **Fourier coefficients and Hermite moments**

$$f_n = \langle f, H_n \rangle_w, \quad \ell_n = \langle \ell, H_n \rangle_w = \int_{\mathbb{R}} H_n(x) g_T(x)\, dx$$

Price and Density approximation

Price approximation

$$\pi_f \approx \pi_f^{(N)} = \sum_{n=0}^{N} f_n \ell_n = \sum_{n=0}^{N} \langle f, \ell_n H_n \rangle_w = \int_{\mathbb{R}} f(x) g_T^{(N)}(x)\, dx \qquad (6)$$

"Gram-Charlier A expansion"

$$g_T^{(N)}(x) = w(x) \sum_{n=0}^{N} \ell_n H_n(x)$$

European calls and puts - Fourier coefficients

- Consider the discounted payoff function for a call option with log strike k

$$f(x) = \mathrm{e}^{-rT}\left(\mathrm{e}^{x} - \mathrm{e}^{k}\right)^{+}$$

- Its **Fourier coefficients** f_n are given by

Fourier coefficients call option

$$f_0 = \mathrm{e}^{-rT+\mu_w} I_0\left(\frac{k-\mu_w}{\sigma_w};\sigma_w\right) - \mathrm{e}^{-rT+k}\Phi\left(\frac{\mu_w - k}{\sigma_w}\right)$$

$$f_n = \mathrm{e}^{-rT+\mu_w}\frac{1}{\sqrt{n!}}\sigma_w I_{n-1}\left(\frac{k-\mu_w}{\sigma_w};\sigma_w\right)\; n \geq 1$$

European calls and puts - Fourier coefficients (cont)

- The functions $I_n(\mu;\nu)$ are defined recursively by

$$I_0(\mu;\nu) = \mathrm{e}^{\frac{\nu^2}{2}}\Phi(\nu-\mu);$$
$$I_n(\mu;\nu) = \mathcal{H}_{n-1}(\mu)\mathrm{e}^{\nu\mu}\phi(\mu) + \nu I_{n-1}(\mu;\nu), \quad n \geq 1$$

  where $\mathcal{H}_n(x)$ are the standard Hermite polynomials, $\Phi(x)$ denotes the standard Gaussian distribution function, and $\phi(x)$ its density

- To prove this representation of Fourier coefficients it is sufficient to establish the following identity

$$I_n(\mu;\nu) = \int_\mu^\infty \mathcal{H}_n(x)\mathrm{e}^{\nu x}\phi(x)\,dx, \quad n \geq 0.$$

- This can be **shown using integration by parts and the identity**

$$\mathcal{H}_n(x) = x\mathcal{H}_{n-1}(x) - \mathcal{H}'_{n-1}(x).$$

Calculation of the Hermite-moments ℓ_n

- Let $\pi : \mathcal{E} \to \{1, \ldots, M = (N+2)(N+1)/2\}$ be an enumeration of the set of exponents

$$\mathcal{E} = \{(m,n) : m, n \geq 0;\, m+n \leq N\}$$

- The polynomials

$$h_{\pi(m,n)}(v,x) = v^m H_n(x), \quad (m,n) \in \mathcal{E}$$

form a basis of Pol_N

- The $(M \times M)$–matrix G representing the infinitesimal generator of (V_t, X_t) on Pol_N is sparse in view of the elementary property

$$H_n'(x) = \frac{\sqrt{n}}{\sigma_w} H_{n-1}(x)$$

which is a consequence of

$$\mathcal{H}_n'(x) = n\mathcal{H}_{n-1}(x)$$

The matrix of the generator

The matrix G has at most 7 nonzero elements in column $\pi(m,n)$

$$G_{\pi(m-2,n),\pi(m,n)} = -\frac{\sigma^2 m(m-1)v_{max}v_{min}}{2(\sqrt{v_{max}}-\sqrt{v_{min}})^2}, \quad m \geq 2$$

$$G_{\pi(m-1,n-1),\pi(m,n)} = -\frac{\sigma\rho m\sqrt{n}v_{max}v_{min}}{\sigma_w(\sqrt{v_{max}}-\sqrt{v_{min}})^2}, \quad m,n \geq 1$$

$$G_{\pi(m-1,n),\pi(m,n)} = \kappa\theta m + \frac{\sigma^2 m(m-1)(v_{max}+v_{min})}{2(\sqrt{v_{max}}-\sqrt{v_{min}})^2}, \quad m \geq 1$$

$$G_{\pi(m,n-1),\pi(m,n)} = \frac{r\sqrt{n}}{\sigma_w} + \frac{\sigma\rho m\sqrt{n}(v_{max}+v_{min})}{\sigma_w(\sqrt{v_{max}}-\sqrt{v_{min}})^2}, \quad n \geq 1$$

$$G_{\pi(m+1,n-2),\pi(m,n)} = \frac{\sqrt{n(n-1)}}{2\sigma_w^2}, \quad n \geq 2$$

$$G_{\pi(m,n),\pi(m,n)} = -\kappa m - \frac{\sigma^2 m(m-1)}{2(\sqrt{v_{max}}-\sqrt{v_{min}})^2}$$

$$G_{\pi(m+1,n-1),\pi(m,n)} = -\frac{\sqrt{n}}{2\sigma_w} - \frac{\sigma\rho m\sqrt{n}}{\sigma_w(\sqrt{v_{max}}-\sqrt{v_{min}})^2}, \quad n \geq 1$$

Calculation of the Hermite-moments ℓ_n (cont.)

Theorem 3

The coefficients ℓ_n are given by

$$\ell_n = [h_1(V_0, X_0), \ldots, h_M(V_0, X_0)]\, \mathrm{e}^{TG}\, \mathbf{e}_{\pi(0,n)}, \quad 0 \leq n \leq N$$

where $\mathbf{e}_i$ is the i–th standard basis vector in $\mathbb{R}^M$. In particular,

$$\ell_0 = 1;$$

$$\ell_1 = \frac{1}{\sigma_w}\left(rT - \frac{\theta}{2}\left(T + \frac{\mathrm{e}^{-\kappa T} - 1}{\kappa}\right) + \frac{\mathrm{e}^{-\kappa T} - 1}{2\kappa} V_0 + X_0 - \mu_w\right)$$

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Example: Call option pricing

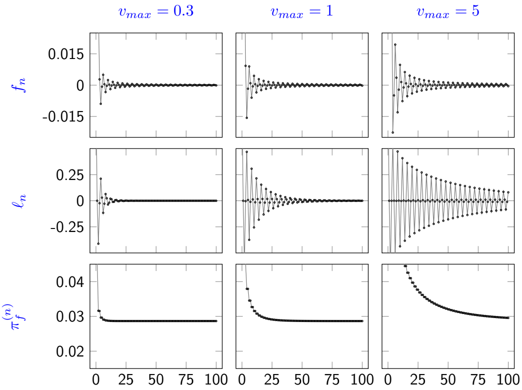
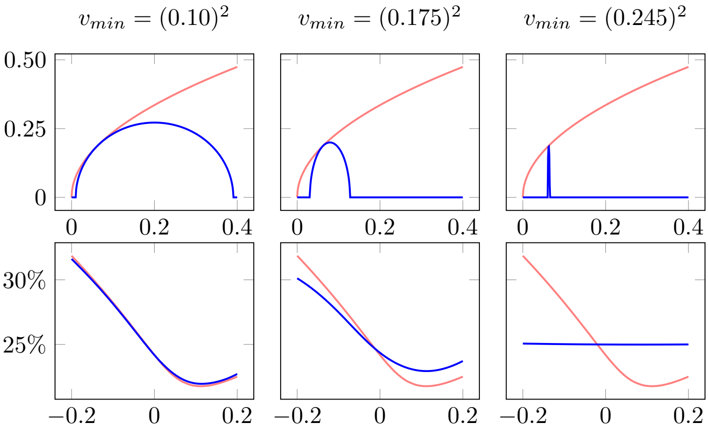


Figure: The Fourier coefficients (first row), the Hermite moments (second row), and the price expansion (third row) as a function of the order n. The parameters values are $T = 1/12$, $X_0 = k = 0$, $\kappa = 0.5$, $\theta = V_0 = (0.25)^2$, $\sigma = 0.25$, $v_{min} = (0.10)^2$, $\rho = -0.5$, and $v_{max} \in \{0.3, 1, 5\}$, $\mu_w = \mathbb{E}[X_T]$ and $\sigma_w^2 = v_{max}T/2 + \epsilon$

Volatility smiles - Call option

Fix $\theta = \sqrt{v_{min} v_{max}} = v_*$ and scale up v_{min}



Diffusion function $\sigma\sqrt{Q(v)}$ (1st row) and implied volatility smile (2nd row)