Computational Finance
FIN-472
Homework 6

October 27, 2017

Exercise 1: Suppose that $S = \exp(X)$ is the price of a financial asset and that spot interest rate is equal to r. We define, for T given, the share measure $\mathbb{P}^S$ by

$$\mathbb{P}^S(A) = \frac{\mathbb{E}[S_T \mathbf{1}_A]}{\mathbb{E}[S_T]},$$

where $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and $\mathbf{1}_A(\omega) = 0$ otherwise. For any bounded random variable X we have that

$$\mathbb{E}^{\mathbb{P}^S}[X] = \frac{\mathbb{E}[S_T X]}{\mathbb{E}[S_T]}.$$

a) Deduce the following expression for the price of a put option $P(k)$ and a call option $C(k)$ with strike $K = \mathrm{e}^k$ and expiration date T:

$$\begin{aligned} P(k) &= \mathrm{e}^{k-rT}\mathbb{P}(X_T < k) - \mathrm{e}^{-rT}\mathbb{E}[S_T]\mathbb{P}^S(X_T < k), \\ C(k) &= \mathrm{e}^{-rT}\mathbb{E}[S_T]\mathbb{P}^S(X_T > k) - \mathrm{e}^{k-rT}\mathbb{P}(X_T > k). \end{aligned} \tag{1}$$

b) Let $\phi(\nu) = \mathbb{E}[\exp(i\nu X_T)]$ and $\phi^S(\nu) = \mathbb{E}^{\mathbb{P}^S}[\exp(i\nu X_T)]$, be the characteristic functions of X_T with respect to $\mathbb{P}$ and $\mathbb{P}^S$, respectively. Show that $\phi(\nu - i)$ is well-defined and

$$\phi^S(\nu) = \frac{\phi(\nu - i)}{\mathbb{E}[S_T]}.$$

Exercise 2: Consider a Variance Gamma model with the following parameters

$$S_0 = 100, \qquad \nu = 0.2, \qquad \theta = -0.14, \qquad r = 0.1, \qquad \sigma = 0.12$$

a) Make an appropriate choice of the discretization parameters η (for the numerical integration) and λ (for the log strikes) to compute the price of the European put options with maturities

$$T \in \{1/12, 1/6, 1/4, 1/2, 1, 2, 3, 5\}$$

and strikes

$$K \in \{50, 80, 90, 95, 100, 105, 110, 120, 150\},$$

using the FrFFT.

b) Compare your results with the ones obtained using the FFT in Exercise 3a) of Homework 3. What is the advantage of the FrFFT method?

Exercise 3: The saddle point method to price derivatives addresses the following points (circle all the valid statements):

a) It is faster than the FFT and FrFFT algorithms to price derivatives

b) It uses a damping factor such that the integrand in the pricing formula has rapid descent

c) It is a better method than FFT to price out-of-the-money options

Exercise 4: The *Cox-Ingersoll-Ross (CIR) process* is defined as a solution of the stochastic differential equation

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t. \tag{2}$$

Let $\mathrm{Pol}_n(\mathbb{R})$ be the space of univariate polynomials on $\mathbb{R}$ of maximal degree $n \in \mathbb{N}$, and let $\mathcal{H}_n = \{h_0, \dots, h_n\}$ be a basis for $\mathrm{Pol}_n(\mathbb{R})$. Write

$$H_n(x) = (h_0(x), \dots, h_n(x))$$

and denote by G_n the matrix representation with respect to $\mathcal{H}$ of the generator $\mathcal{G}$ associated to X_t and restricted to $\mathrm{Pol}_n(\mathbb{R})$, i.e.

$$\mathcal{G}p(x) = H_n(x)G_n\vec{p}, \tag{3}$$

where p is a polynomial in $\mathrm{Pol}_n(\mathbb{R})$ and has coordinate vector $\vec{p} \in \mathbb{R}^{n+1}$ with respect to $\mathcal{H}_n$. Then, the moments of X_T can be computed via the formula

$$\mathbb{E}[p(X_T)] = H_n(X_0)e^{G_nT}\vec{p}. \tag{4}$$

For this exercise we consider the monomial basis for $\mathrm{Pol}_n(\mathbb{R})$, i.e.

$$H_n(x) = (1, x, \cdots, x^n)^T.$$

a) Derive the generator $\mathcal{G}$ associated to the CIR process X_t, solution of (2). Write a Matlab code `GenCir.m` with input parameters $\kappa, \theta, \sigma, n$, that constructs the matrix G_n defined as in (3).

b) Write a Matlab code `MomCIR.m` which computes the moments of X_T as in formula (4). Input parameters: $G_n, X_0, T, \vec{p}$.

c) Compute the first three moments of X_T (using the previously implemented functions) for the following choice of model parameters:

$$\kappa = 2, \quad \theta = 0.4, \quad \sigma = 0.5, \quad X_0 = 0.4, \quad T = 1.$$

Check that your moments have been correctly computed, knowing that (see slides 9-10 of Lecture 5)

$$\frac{X_t}{\frac{\sigma^2}{4\kappa}(1 - e^{-\kappa t})}$$

has a non-central χ^2 distribution with parameters

$$k = \frac{4\kappa\theta}{\sigma^2}, \quad \alpha = \frac{e^{-\kappa t}X_0}{\frac{\sigma^2}{4\kappa}(1 - e^{-\kappa t})}.$$