Computational Finance FIN-472 Polynomial models and polynomial expansion methods

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Outline

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Introduction

- Polynomial models provide an analytically tractable and statistically flexible framework for financial modeling
- New factor process dynamics, beyond affine, enter the scene
- Definition of polynomial diffusions and basic properties
- **Polynomial models in finance:** option pricing, portfolio choice, risk management, economic scenario generation,..

Some Literature

- **Polynomial processes:** [Wong, 1964], [Mazet, 1997], [Forman and Sørensen, 2008],[Cuchiero, 2011], [Cuchiero et al., 2012], [Filipović and Larsson, 2016], and others
- **Polynomial models in finance:** [Zhou, 2003], [Delbaen and Shirakawa, 2002], [Larsen and Sørensen, 2007], [Gouriéroux and Jasiak, 2006], [Eriksson and Pistorius, 2011], [Filipović et al., 2016], [Filipović et al., 2014], [Ackerer and Filipović, 2015], [Ackerer et al., 2015], [Filipović and Larsson, 2017], [Biagini and Zhang, 2016], and others

These lectures are based on highlighted papers

Infinitesimal Generator

- Suppose $X_t = (X_t^i)_{i=1}^d$ on $\mathbb{R}^d$ solves a SDE of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

  for some **drift function** $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ and **volatility function** $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^{d\times n}$, where W is an n–dimensional Brownian motion
- Itô's formula with smooth f gives

$$df(X_t) = \underbrace{\left(\frac{1}{2}\mathrm{tr}(a\nabla^2 f) + b^\top \nabla f\right)(X_t)}_{\mathcal{G}f(X_t)} dt + \nabla f(X_t)^\top \sigma(X_t) dW_t$$

  for the **diffusion function** $a = \sigma\sigma^\top : \mathbb{R}^d \mapsto \mathbb{S}^d$ (symmetric matrices)
- $\mathcal{G}$ is the extended generator of X_t

Polynomials

- Multi-index notation

$$\boldsymbol{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d, \quad x^{\boldsymbol{k}} = x_1^{k_1} \cdots x_d^{k_d}, \quad |\boldsymbol{k}| = \textstyle\sum_{i=1}^d k_i$$

- A polynomial on $\mathbb{R}^d$ of degree n is a function of the form

$$p(x) = \sum_{|\boldsymbol{k}|=0}^{n} c_{\boldsymbol{k}} x^{\boldsymbol{k}}$$

with $c_{\boldsymbol{k}} \neq 0$ for some $\boldsymbol{k}$ with $|\boldsymbol{k}| = n$

- Space of polynomials of degree n or less

$$\mathrm{Pol}_n(\mathbb{R}^d) = \{p \text{ polynomial on } \mathbb{R}^d \text{ with } \deg p \leq n\}$$

has $\dim \mathrm{Pol}_n(\mathbb{R}^d) = \binom{n+d}{n}$

Definition of Polynomial Generator

Definition

An infinitesimal generator $\mathcal{G}$ of a diffusion X_t is polynomial if

$$\mathcal{G}\mathrm{Pol}_n(\mathbb{R}^d) \subseteq \mathrm{Pol}_n(\mathbb{R}^d) \text{ for all } n \in \mathbb{N}$$

In this case, we call X_t a polynomial difussion (PD)

Characterization of Polynomial Diffusions

There is a simple **characterization of polynomial diffusions**

Lemma: Charaterization of Polynomial Generators

The following are equivalent:

1. $\mathcal{G}$ is polynomial
2. $a(x)$ and $b(x)$ satisfy

$$b \in \mathrm{Pol}_1(\mathbb{R}^d)$$
$$a \in \mathrm{Pol}_2(\mathbb{R}^d)$$

Remark: Affine diffusions are polynomial diffusions

Proof of the lemma

Here is a **simple proof when $d = 1$:**

- Calculus gives

$$\mathcal{G}x^n = \frac{1}{2}a(x)n(n-1)x^{n-2} + b(x)nx^{n-1}$$

- Hence $\mathcal{G}x^n \in \mathrm{Pol}_n(\mathbb{R})$ for all $n \in \mathbb{N}$ if and only if $b \in \mathrm{Pol}_1(\mathbb{R})$ and $a \in \mathrm{Pol}_2(\mathbb{R})$

Example: Scalar Polynomial Diffusions

For $d = 1$ any polynomial diffusion is of the form

$$dX_t = (b + \beta X_t)\, dt + \sqrt{a + \alpha X_t + A X_t^2}\, dW_t$$

Examples:

- **Brownian motion:** $dX_t = dW_t$
- **Ornstein–Uhlenbeck process:**

$$dX_t = \kappa(\theta - X_t)\, dt + \sigma\, dW_t$$

- **GARCH diffusions:**

$$dX_t = \kappa(\theta - X_t)\, dt + \sigma X_t\, dW_t$$

- **Square-root diffusions:**

$$dX_t = \kappa(\theta - X_t)\, dt + \sigma\sqrt{X_t}\, dW_t$$

- **Jacobi diffusions:**

$$dX_t = \kappa(\theta - X_t)\, dt + \sigma\sqrt{X_t(1 - X_t)}\, dW_t$$

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Coordinates on spaces of polynomials

- Let $\mathcal{G}$ be polynomial, so that X_t **is a Polynomial Diffusion**
- Fix $N \in \mathbb{N}$, denote $M = \dim \mathrm{Pol}_N(\mathbb{R}^d) = \binom{N+d}{N} < \infty$
- $\mathcal{G}$ **restricts to linear operator on** $\mathrm{Pol}_N(\mathbb{R}^d)$
- Fix polynomials $h_1(x), \ldots, h_M(x)$ that form a **basis of** $\mathrm{Pol}_N(\mathbb{R}^d)$**.** Denote

$$H_N : \mathbb{R}^d \to \mathbb{R}^M, \quad H_N(x) = (h_1(x), \ldots, h_M(x))$$

- **Coordinate representation** $\vec{p}$ of $p \in \mathrm{Pol}_N(\mathbb{R}^d)$:

$$p(x) = H_N(x)\vec{p}$$

- **Matrix representation** G_n of $\mathcal{G}$: $\mathcal{G}H_N(x) = H_N(x)G_N$

$$\mathcal{G}p(x) = H_N(x)G_N\vec{p}$$

Example: Scalar Polynomial Diffusions

- Generic scalar polynomial diffusion

$$dX_t = (b + \beta X_t)\,dt + \sqrt{a + \alpha X_t + A X_t^2}\,dW_t$$

- **Basis** $\{1, x, x^2, \cdots, x^N\}$ of $\mathrm{Pol}_N(\mathbb{R})$

- **Coordinate representation** of $p(x) = \sum_{k=0}^{N} p_k x^k$:

$$\vec{p} = (p_0, \ldots, p_N)^\top$$

- **Matrix representation** of $\mathcal{G}$: $(N+1) \times (N+1)$-matrix

$$G_N = \begin{pmatrix} 0 & b & 2\frac{a}{2} & 0 & \cdots & 0 \\ 0 & \beta & 2\left(b + \frac{\alpha}{2}\right) & 3 \cdot 2\frac{a}{2} & 0 & \vdots \\ 0 & 0 & 2\left(\beta + \frac{A}{2}\right) & 3\left(b + 2\frac{\alpha}{2}\right) & \ddots & 0 \\ 0 & 0 & 0 & 3\left(\beta + 2\frac{A}{2}\right) & \ddots & N(N-1)\frac{a}{2} \\ \vdots & & & 0 & \ddots & N\left(b + (N-1)\frac{\alpha}{2}\right) \\ 0 & & \cdots & & 0 & N\left(\beta + (N-1)\frac{A}{2}\right) \end{pmatrix}$$

Moment Formula

Theorem: Moment formula

For any $p \in \mathrm{Pol}_N(\mathbb{R}^d)$ the moment formula holds,

$$\mathbb{E}[p(X_T) \mid X_t] = H_N(X_t)\mathrm{e}^{(T-t)G_N}\,\vec{p}, \quad t \leq T$$

Remark: If the first polynomial in the basis is the constant polynomial 1, then the first column of G_N will be zero

Sketch of the proof of the moment formula

Proof.

Assume $t = 0$. Itô's formula yields

$$dp(X_t) = \mathcal{G}p(X_t)\,dt + \underbrace{\nabla f(X_t)^\top \sigma(X_t) dW_t}_{\text{zero expectation}}$$

Hence $\mathbb{E}[p(X_T)] = \mathbb{E}[H_N(X_T)]\vec{p}$ satisfies

$$\begin{aligned}\mathbb{E}[H_N(X_T)]\vec{p} &= p(X_0) + \int_0^T \mathbb{E}[\mathcal{G}p(X_s)]\,ds \\ &= \left(H_N(X_0) + \int_0^T \mathbb{E}[H_N(X_s)]G_N\,ds\right)\vec{p}\end{aligned}$$

This implies that $\frac{d}{dT}\mathbb{E}[H_N(X_T)] = \mathbb{E}[H_N(X_s)]G_N$. **The solution of this linear ODE is**

$$\mathbb{E}[H_N(X_T)] = H_N(X_0)\mathrm{e}^{TG_N}$$

First order moments of scalar polynomial diffusions

- For $N = 1$ and the basis $H_1(x) = (1, x)$ we obtain

$$G_1 = \begin{pmatrix} 0 & b \\ 0 & \beta \end{pmatrix}, \quad \mathrm{e}^{TG_1} = \begin{pmatrix} 1 & \frac{\mathrm{e}^{\beta T}-1}{\beta} b \\ 0 & \mathrm{e}^{\beta T} \end{pmatrix}$$

- Moment formula gives for $p(x) = x$, with $\vec{p} = (0, 1)^\top$

$$\mathbb{E}[X_T | X_t] = (1, X_t)\mathrm{e}^{(T-t)G_1}\vec{p} = (1, X_t) \begin{pmatrix} 1 & \frac{\mathrm{e}^{\beta(T-t)}-1}{\beta} b \\ 0 & \mathrm{e}^{\beta(T-t)} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= \frac{\mathrm{e}^{\beta(T-t)} - 1}{\beta} b + \mathrm{e}^{\beta(T-t)} X_t$$

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The Heston model

- We recall that in the **Heston model,** the dynamics of **squared volatility** V_t and the **log price** X_t are

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^{(1)}$$
$$dX_t = \left(r - \frac{V_t}{2}\right)dt + \sqrt{V_t}(\rho dW_t^{(1)} + \sqrt{1-\rho^2}dW_t^{(2)})$$

where $W^{(1)}, W^{(2)}$ are independent Brownian motions and ρ is a correlation parameter

- **The infinitesimal generator** in this case is

$$\mathcal{G}f(v,x) = \frac{1}{2}vf_{xx} + \rho\sigma v f_{xv} + \frac{\sigma^2}{2}vf_{vv} + (r - v/2)f_x + \kappa(\theta - v)f_v$$

- This generator is a polynomial generator. The **Heston model** (V_t, X_t) **is a polynomial model in two dimensions.** This is no longer true if we consider prices S_t instead of log prices X_t

The matrix of the generator - Heston model

- We first recall that $\dim \mathrm{Pol}_N(\mathbb{R}^2) = \binom{N+2}{N} = \frac{(N+1)(N+2)}{2} = M$
- Let $\pi : \mathcal{E} \to \{1, \ldots, M = (N+1)(N+2)/2\}$ be an enumeration of the set of pairs

$$\mathcal{E} = \{(m,n) : m,n \geq 0;\, m+n \leq N\}$$

- Let $\mu_w \in \mathbb{R}$ and $\sigma_w > 0$ be arbitrary parameters, the polynomials

$$h_{\pi(m,n)}(v,x) = v^m \left(\frac{x-\mu_w}{\sigma_w}\right)^n, \quad (m,n) \in \mathcal{E}$$

  form a basis of $\mathrm{Pol}_N(\mathbb{R}^2)$
- The $(M \times M)$–matrix G representing the infinitesimal generator of (V_t, X_t) on $\mathrm{Pol}_N(\mathbb{R}^2)$ is sparse

The matrix of the generator - Heston model (cont.)

- G has at most 5 nonzero elements in column $\pi(m,n)$

$$
\begin{aligned}
G_{\pi(m-1,n),\pi(m,n)} &= m\left(\kappa\theta + \frac{\sigma^2(m-1)}{2}\right), && m \geq 1 \\
G_{\pi(m,n-1),\pi(m,n)} &= \frac{n\left(\sigma\rho m + r\right)}{\sigma_w}, && n \geq 1 \\
G_{\pi(m+1,n-2),\pi(m,n)} &= \frac{n(n-1)}{2\sigma_w^2}, && n \geq 2 \\
G_{\pi(m,n),\pi(m,n)} &= -\kappa m \\
G_{\pi(m+1,n-1),\pi(m,n)} &= -\frac{n}{2\sigma_w}, && n \geq 1
\end{aligned}
$$

Calculation of moments in the Heston model

Theorem: Moments in the Heston model

The moments of the log-price are

$$\mathbb{E}\left[\left(\frac{X_T-\mu_w}{\sigma_w}\right)^n\right] = [h_1(V_0,X_0),\dots,h_M(V_0,X_0)]\,\mathrm{e}^{TG}\,\mathbf{e}_{\pi(0,n)},\ n\le N$$

where $\mathbf{e}_i$ is the i–th standard basis vector in $\mathbb{R}^M$. In particular,

$$\mathbb{E}\left[\frac{X_T-\mu_w}{\sigma_w}\right] = \frac{1}{\sigma_w}\left(rT-\frac{\theta}{2}\left(T+\frac{\mathrm{e}^{-\kappa T}-1}{\kappa}\right)+\frac{\mathrm{e}^{-\kappa T}-1}{2\kappa}V_0+X_0-\mu_w\right)$$

The bilateral Gamma distribution

- We will consider as weight function in the density approximation scheme, $w = \gamma_b$, a **bilateral Gamma density whose characteristic function is**

$$\Phi_{\gamma_b}(u) = \mathbb{E}_{\gamma_b}\left[\mathrm{e}^{iu\Gamma}\right] = \left(\frac{6}{6 + Cu^2}\right)^{3/C}, \quad \Gamma \sim \gamma_b$$

- **The distribution is normalized as**

$$\mathbb{E}_{\gamma_b}[\Gamma] = 0, \quad \mathrm{Var}_{\gamma_b}[\Gamma] = 1$$

- **The kurtosis** of this bilateral Gamma distribution is

$$\mathbb{E}_{\gamma_b}\left[\left(\frac{\Gamma - \mathbb{E}_{\gamma_b}[\Gamma]}{\sqrt{\mathrm{Var}_{\gamma_b}[\Gamma]}}\right)^4\right] = \mathbb{E}_{\gamma_b}[\Gamma^4] = 3 + C$$

- Hence the parameter C represents the excess kurtosis of the distribution ($=$ kurtosis - 3)

The bilateral Gamma density

- The previous **bilateral Gamma density function** has the following explicit form

$$\gamma_b(x) = \frac{2^{\frac{3(C-2)}{4C}} 3^{\frac{C+6}{4C}} C^{-\frac{C+6}{4C}} |x|^{\frac{3}{C}-\frac{1}{2}} K_{\frac{3}{C}-\frac{1}{2}}\left(|x|\frac{\sqrt{6}}{\sqrt{C}}\right)}{\Gamma\left(\frac{3}{C}\right)\sqrt{\pi}}$$

  where $K_n(\xi)$ **denotes the modified Bessel function of the second kind**

- **Recall:** For given mean and standard deviation parameters μ_w, σ_w

$$\widetilde{\gamma}_b(x) = \frac{1}{\sigma_w}\gamma_b\left(\frac{x-\mu_w}{\sigma_w}\right) \tag{1}$$

  is the density of a random variable with mean μ_w, variance σ_w^2 and excess kurtosis C

Orthonormal polynomials

The first five orthonormal polynomials of the basis of $L^2_{\gamma_b}$ and $L^2_{\widetilde{\gamma}_b}$ are of the form

$$H_n^{\gamma_b}(x) = \frac{\widetilde{H}_n^{\gamma_b}(x)}{HO_n^{\gamma_b}}, \quad H_n^{\widetilde{\gamma}_b}(x) = H_n^{\gamma_b}\left(\frac{x-\mu_w}{\sigma_w}\right)$$

where

$$\begin{aligned}
&\widetilde{H}_0^{\gamma_b}(x) = 1, \quad HO_0^{\gamma_b} = 1 \\
&\widetilde{H}_1^{\gamma_b}(x) = x, \quad HO_1^{\gamma_b} = 1 \\
&\widetilde{H}_2^{\gamma_b}(x) = x^2 - 1, \quad HO_2^{\gamma_b} = \sqrt{C+2} \\
&\widetilde{H}_3^{\gamma_b}(x) = (-C-3)x + x^3, \quad HO_3^{\gamma_b} = \sqrt{\frac{7C^2}{3} + 9C + 6} \\
&\widetilde{H}_4^{\gamma_b}(x) = -\frac{2(5C^2+21C+18)(x^2-1)}{3(C+2)} - C + x^4 - 3 \\
&HO_4^{\gamma_b} = \sqrt{\frac{2(55C^4+363C^3+822C^2+756C+216)}{9(C+2)}}
\end{aligned} \tag{2}$$

Density approximation in the Heston model

- Suppose that

$$\mu_w = \mathbb{E}[X_T], \quad \sigma_w^2 = \mathrm{Var}[X_T], \quad C = \text{excess kurtosis of } X_T$$

- Let $w = \widetilde{\gamma}_b$ be as in (1) and let $q(x)$ be the density function of X_T
- It is shown in [Filipović et al., 2013] that $\frac{q(x)}{\widetilde{\gamma}_b(x)} \in L^2_{\widetilde{\gamma}_b}$
- Hence the following density approximation scheme converges:

$$q^{(N)}(x) = \left(\sum_{n=0}^{N} \ell_n H_n^{\widetilde{\gamma}_b}(x) \right) \widetilde{\gamma}_b(x)$$

where

$$\ell_n = \mathbb{E}\left[H_n^{\widetilde{\gamma}_b}(X_T) \right]$$

- Notice that if $N = 4$ then $\ell_1 = \ell_2 = \ell_4 = 0$ and

$$q^{(4)}(x) = \widetilde{\gamma}_b(x)(1 + \ell_3 H_3^{\widetilde{\gamma}_b}(x))$$

Polynomial expansion of prices in the Heston model

- Consider a call option with log strike k
- We can approximate the price of a call option as follows

$$\begin{aligned}\pi_{call} &= \mathrm{e}^{-rT}\left(\int_k^\infty (\mathrm{e}^x - \mathrm{e}^k)q(x)\,dx\right)\\ &\approx \mathrm{e}^{-rT}\left(\int_k^\infty (\mathrm{e}^x - \mathrm{e}^k)q^{(N)}(x)\,dx\right) = \pi_{call}^{(N)}\end{aligned}$$

- The last expression **only involves a numerical integration**
- In general if we consider a European derivative with discounted payoff $f(X_T)$ we can approximate its price by

$$\pi_f \approx \int_{-\infty}^{\infty} f(x)q^{(N)}(x)\,dx = \pi_f^{(N)}$$

Example - Polynomial expansion in the Heston model

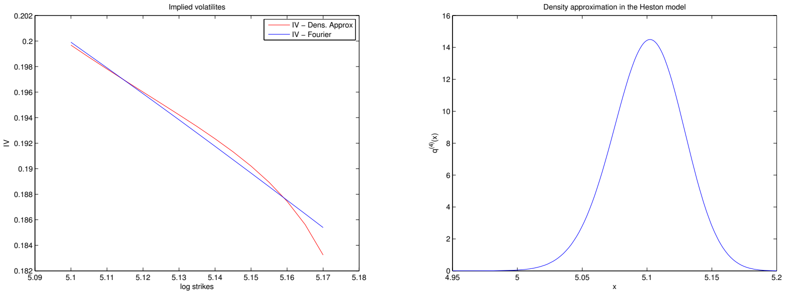

Figure: **Left:** Plots of the **implied volatilities** for the price approximation $\pi_{call}^{(4)}$ and the prices obtained with Fourier methods. **Right:** Graph of the density approximation $q^{(4)}(x)$. **Parameters:**
$r = 0.03, \kappa = 1, \theta = 0.04, \sigma = 0.2, \rho = -0.8, X_0 = 5.1, V_0 = 0.04, T = 1/52$

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