Computational Finance
FIN-472
Take-Home Exam 3

December 15, 2017

- **Please hand in your solutions either on Friday 22.12.2017 at the beginning of the lecture, OR before Friday 29.12.2017 on the moodle or via e-mail.**
- **For Exercises** $1.b$ **and** $2.c$ **do not forget to upload in moodle the MATLAB codes. You should submit only one MATLAB file containing all the necessary functions and returning the prices at maturity as described in the following. (One file for the Exercise** $1.b$ **and on file for the Exercise** $2.c$**).**
- **Additionally, if you decide to submit on Friday 22.12.2017, for Exercises** $1.b$ **and** $2.c$**, print out the MATLAB codes and the plots with your results. These have to be handed in together with the other solutions.**

Exercise 1:(25 points /40) Consider the price dynamics of a financial asset S_τ and define the variable $I(\tau) = \int_0^\tau S_\eta d\eta$. Consider an Asian Put option with fixed strike K, then the payoff at expiry can be written as $\Psi(S_T, I_T) = (K - I_T/T)_+$, where

$$\begin{aligned} dS_\tau &= rS_\tau d\tau + \sigma S_\tau dW_\tau \\ dI_\tau &= S_\tau d\tau \end{aligned}$$

and the price is given by

$$u(S, I, \tau) = \mathbb{E}\left[\Psi(S_T, \frac{1}{T}I_T) |\, S_\tau = S, I_\tau = I\right] e^{-\int_\tau^T r d\eta}.$$

Consider the following truncated version of the two-dimensional pricing PDE satisfied by $u(S, I, \tau)$ in the variables (S, I) and time to maturity $t = T - \tau$:

$$\begin{cases} \partial_t U(S,I,t) - \frac{1}{2}\sigma^2 S^2 \partial_{SS} U(S,I,t) - rS\partial_S U(S,I,t) - S\partial_I U(S,I,t) + rU(S,I,t) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0, S_{max}) \times [0, I_{max}) \times (0,T], \\ \partial_{\mathbf{n}} U(S_{max}, I, t) = 0 \qquad\qquad \text{in } [0, I_{max}] \times (0,T], \\ U(S, I_{max}, t) = 0 \qquad\qquad \text{in } [0, S_{max}] \times (0,T], \\ U(S, I, 0) = G(S) = \max\{K - \dfrac{I}{T}, 0\}, \qquad \text{in } [0, S_{max}] \times [0, I_{max}]. \end{cases} \tag{1}$$

The parameters σ, r, S, T are defined as in the Black & Scholes model.

a) *Derive* a finite-difference discretization of (1), using the θ-method for the discretization of the time derivative. (Observe that equation (1) is a bi-dimensional time-depending problem in the variables S and I). In particular:

- Define a uniform computational grid with N intervals along both axes S and I ($N_S = N_I = N$). *Propose* **suitable** finite difference approximations of the differential operators $\partial_S, \partial_{SS}$ and ∂_I;
- *Write* the resulting equation to be solved in each internal node of the computational grid; *specify* how you treat the boundary conditions. (Do you need to impose any condition for $I = 0$ and $S = 0$? why?) *Detail* the resulting equations on all boundary nodes ($S = 0$, $S = S_{max}$, $I = 0$, $I = I_{max}$).
- The set of equations obtained after finite difference approximations can be written in compact form as

$$(\mathbb{I} + \theta\mathbb{A})U^{m+1} = (\mathbb{I} - (1-\theta)\mathbb{A})U^m + \Delta t F, \tag{2}$$

  Detail how you number the nodes of the computational grid and how you define the solution vector U. *Specify* the entries of the matrix A and *describe* how you would implement it in Matlab.

b) Taking inspiration from the 2D solver for the Heston model (cf. the file `heston_timestepping.m` on moodle), *write* a Matlab function to implement the θ-method time-stepping finite difference scheme chosen to solve (1). *Test* your Matlab solver with the following numerical values:

$$r = 0.02, \quad \sigma = 0.23, \quad K = 3.5, \quad T = 1 \quad S_{max} = I_{max} = 6 \tag{3}$$

and the following discretization parameters:

$\theta = 0.5$,	(Crank-Nicolson method)
$N_S = N_I = 80$,	(number of intervals in S and I respectively)
$N_t = 100$,	(number of time step)

Plot the final value of the option versus the price S and the average mean I of the underlying asset. Check the sensitivity of the solution with respect to the discretization parameters N_S, N_I, N_t, S_{max}. Also, fill the following table with the price at maturity for $I = 0$:

	S= 1	S= 2	S=3
K=2	...	...	...
K=3	...	...	...

c) *State* the definition of stability for the finite differences scheme (2) with $\theta = 0$ (Forward Euler method). *Prove* the stability of the finite difference scheme proposed in point a) for $\theta = 0$ and derive precise stability conditions on discretization parameters.

Exercise 2:(**15 points /40**) Consider now an Asian Call option with average strike whose payoff at expiry is given by $\Psi(S_T, I_T) = (S_T - I_T/T)^+$. The price of the option is given by

$$u(S, I, \tau) = \mathbb{E}\left[\Psi(S_T, \frac{1}{T}I_T) |\, S_\tau = S, I_\tau = I\right] e^{-\int_\tau^T r d\eta}$$

and satisfies the same equation in (1):

$$\partial_t U(S,I,t) - \frac{1}{2}\sigma^2 S^2 \partial_{SS} U(S,I,t) - rS\partial_S U(S,I,t) - S\partial_I U(S,I,t) + rU(S,I,t) = 0 \qquad (4)$$

$\forall S, I \in \mathbb{R}^+$, $t \in (0,T]$, with initial condition $U(S,I,0) = \max(S - \frac{I}{T}, 0)$. Define the new variable $R_t = \frac{I_t}{S_t} = \frac{\int_0^t S_\tau d\tau}{S_t}$. Show that the solution $U(S,I,t)$ of equation (4) can be rewritten as $U(S,I,t) = S \cdot H(R,t)$ where $H(R,t)$ satisfies the following one-dimensional problem:

$$\begin{cases} \partial_t H(R,t) - \dfrac{\sigma^2 R^2}{2} \partial_{RR} H(R,t) - (1 - rR)\,\partial_R H(R,t) = 0, \;\; R \in \mathbb{R}^+, t \in (0,T] \\ H(R,0) = \max(1 - \dfrac{R}{T}, 0) \end{cases} \qquad (5)$$

a) *Propose* a suitable truncation of the problem (5) in $[0, R_{max}]$ and suitable boundary conditions for the truncated problem.

b) *Derive* a finite-difference discretization of the truncated version of (5), using the θ-method for the discretization of the time derivative. In particular:

- *Propose* **suitable** finite difference approximations of the differential operators $\partial_R, \partial_{RR}$.
- *Write* the resulting equation to be solved in each node of the computational grid. *Detail* the equation on R_{max}.
- *Show* that the set of equations can be written in compact form as

$$(\mathbb{I} + \theta\mathbb{A})H^{m+1} = (\mathbb{I} - (1-\theta)\mathbb{A})H^m + \Delta t F, \qquad (6)$$

  Specify the entries of the matrix A.

c) Starting from the solver for the Black & Scholes equation (cf. the file `bs_timestepping.m` on moodle), *write* a Matlab function to implement the θ-method time-stepping finite differences scheme chosen to solve (5). *Test* your Matlab solver with the following numerical values:

$$r = 0.02, \quad \sigma = 0.23, \quad T = 1, \quad R_{max} = 21, \qquad (7)$$

and the following discretization parameters:

$$\theta = 0.5 \text{ (Crank-Nicolson method)}, \qquad N_R = 200 \text{ (number of intervals in } [0, R_{max}]),$$

$$N_t = 100 \text{ (number of time step)}.$$

Consider $S_{max} = 6$ and discretize $[0, S_{max}]$ in 80 uniform intervals. Then *plot* the solution $U(S, 0, T) = S \cdot H(0, T)$.