Computational Finance
FIN-472
Homework 5

October 20, 2017

Exercise 1: The *Ornstein Uhlenbeck process* is defined as a solution of the Stochastic Differential Equation (SDE)

$$dX_t = \kappa(\theta - X_t)dt + \lambda dW_t. \tag{1}$$

If we let

$$v(t,x) := \mathbb{E}[\exp(i\nu X_T)|X_t = x]$$

then v satisfies the PDE

$$v_t + \mathcal{G}v = 0$$

with terminal condition $v(T,x) = \mathrm{e}^{i\nu x}$, where

$$\mathcal{G}v = \kappa(\theta - x)v_x + \frac{\lambda^2}{2}v_{xx}.$$

Suppose that

$$v(t,x) = \exp(\varphi(T-t,\nu) + \psi(T-t,\nu)x).$$

a) Deduce a system of Ordinary Differential Equations (ODEs) for the functions φ and ψ.

b) Solve this system and write explicitly the form of the characteristic function of X_t given $X_0 = x$.

c) Deduce that X_t is normally distributed. Write explicitly the mean and variance of X_t.

d) Using Itô's formula solve (1) explicitly and explain why this is consistent with the results in the previous part.

Exercise 2: This exercise proposes a technique to price out-of-the-money options. Suppose that $r = 0$ and fix a maturity T. Without loss of generality assume that $S_0 = 1$. This can always be achieved by modeling prices with S/S_0 and scaling up by S_0 option prices. For a given log strike k denote by $c(k)$ and $p(k)$ the call and put prices, respectively. Define the function

$$z(k) = p(k)1_{k<0} + c(k)1_{k>0}. \tag{2}$$

a) Write a formula for the Fourier transform of $z(k)$ in terms of the characteristic function of $X_T = \log(S_T)$ with respect to the risk-neutral measure. Propose, through Fourier inversion, a pricing formula. *Hint:* $\mathbb{E}[S_T] = e^{rT}S_0 = 1$.

b) Consider for $\alpha > 0$ the modified function

$$z_\alpha(k) = \sinh(\alpha k) z(k). \tag{3}$$

Write a formula for the Fourier transform of $z_\alpha(k)$ in terms of the characteristic function of $X_T = \log(S_T)$ with respect to the risk-neutral measure. Propose, through Fourier inversion, a pricing formula.

c) Choose a particular affine stochastic model. Is there a difference in the behavior of the integrands appearing in the pricing formulas in a) and b)?

Exercise 3: Using the Variance Gamma model and the following parameters

$$S_0 = 100, \qquad \nu = 0.2, \qquad \theta = -0.14, \qquad r = 0.1, \qquad \sigma = 0.12$$

a) Compute the price of the European put options with maturities

$$T \in \{1/12, 1/6, 1/4, 1/2, 1, 2, 3, 5\}$$

and strikes

$$K \in \{50, 80, 90, 95, 100, 105, 110, 120, 150\},$$

using the FFT approach proposed by Carr and Madan (1999) with $e^{\alpha k}$ as damping factor. Here e^k denotes the strike of the option.

b) Graph the implied volatility surface obtained in the previous part. This is the graph of the implied volatilities as a function of K and T. Recall that the implied volatility is the value $\sigma(K,T)$ such that

$$P(K,T) = P^{BS}(K,T;\sigma(K,T))$$

where $P(K,T)$ is the price of the put (in this case obtained by the FFT method) and $P^{BS}(K,T;\sigma^{BS})$ is the price of a put in the Black-Scholes model with parameter σ^{BS}. If you want, you can use the function `blsimpv` already implemented in Matlab.

c) Considering only the above options that are out-of-the-money, compute their price following the technique seen in Exercise 2b) with the sinh function as damping factor. Compare the results.

d) Redo a) and b), but this time use the Simpson rule instead of the trapezoidal rule. Compare the results.