Computational Finance
FIN-472
Transform methods for pricing II

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Outline

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Black-Scholes model

- As we already noted in this model $S_T = \exp(X_T)$ where

$$X_T \sim \mathcal{N}\left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$

- Hence we have that

$$\mathbb{E}[\exp(i\nu X_T)|X_0 = x] = \exp\left(i\nu\left(x + \left(r - \frac{\sigma^2}{2}\right)T\right) - \frac{\sigma^2\nu^2}{2}T\right) \tag{1}$$

Affine models

- **Affine processes** are widely used to model stochastic factors, e.g. log prices, spot interest rates
- This is a large class of processes whose characteristic function is tractable
- In broad terms, a Markov process $(X_t)_{t\geq 0} \subset \mathbb{R}^d$ is an affine process if its characteristic function is exponentially affine in the initial state variable. This is

$$\begin{aligned}\mathbb{E}[\exp(i\langle \nu, X_T\rangle)|X_t = x] &= \mathbb{E}[\exp(i\langle \nu, X_T^{t,x}\rangle)] \\ &= \exp(\varphi(T-t,\nu) + \langle \psi(T-t,\nu), x\rangle)\end{aligned} \tag{2}$$

for some functions φ and ψ

Example - Back to Black-Scholes

- In this case: $dX_t = (r - \sigma^2/2)dt + \sigma dW_t$
- The characteristic function

$$v(t,x) := \mathbb{E}[\exp(i\nu X_T)|X_t = x] \tag{3}$$

satisfies the PDE

$$v_t + \mathcal{G}v = 0 \tag{4}$$

with terminal condition $v(T,x) = \exp(i\nu x)$, where

$$\mathcal{G}v = (r - \sigma^2/2)v_x + \frac{\sigma^2}{2}v_{xx}$$

- Assuming that v has the form (2), we obtain a system of ODEs for φ and ψ whose solution is

$$\psi(t,\nu) = i\nu$$
$$\varphi(t,\nu) = t\left(-\frac{\sigma^2\nu^2}{2} + i\nu\left(r - \frac{1}{2}\sigma^2\right)\right)$$

- This is coherent with formula (1)

Example - OU process

- **Ornstein Uhlenbeck process:** In this model

$$dX_t = \kappa(\theta - X_t)dt + \lambda dW_t$$

- The function v defined in (3) satisfies (4) with

$$\mathcal{G}v = \kappa(\theta - x)v_x + \frac{\lambda^2}{2}v_{xx}$$

- Assuming that v has the form (2), we obtain a system of ODEs for φ and ψ whose solution is

$$\psi(t,\nu) = i\nu \mathrm{e}^{-\kappa t}$$
$$\varphi(t,\nu) = -\frac{\nu^2\lambda^2(1-\mathrm{e}^{-2\kappa t})}{4\kappa} + i\nu\theta(1-\mathrm{e}^{-\kappa t})$$

Example - OU process (cont.)

- The characteristic function has the form

$$v(t,x) = \exp\left(i\nu(\mathrm{e}^{-\kappa(T-t)}x + \theta(1-\mathrm{e}^{-\kappa(T-t)})) - \frac{\nu^2\lambda^2}{4\kappa}(1-\mathrm{e}^{-2\kappa(T-t)}) \right)$$

- *Observation:* It can be shown that

$$X_t = \mathrm{e}^{-\kappa t}X_0 + \theta(1-\mathrm{e}^{-\kappa t}) + \int_0^t \lambda \mathrm{e}^{\kappa(s-t)}\, dW_s$$

  is normally distributed

- This type of processes is used in the Vasicek model to describe the evolution of spot interest rates

Example - CIR process

- **Cox-Ingersoll-Ross Model / Feller diffusion:** In this model

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t$$

- The function v defined in (3) satisfies (4) with

$$\mathcal{G}v = \kappa(\theta - x)v_x + \frac{\sigma^2 x}{2}v_{xx}$$

- Assuming that v has the form (2), we obtain a system of ODEs for φ and ψ whose solution is

$$\psi(t,\nu) = \frac{\mathrm{e}^{-\kappa t} i\nu}{1 - \frac{\sigma^2}{2\kappa}(1 - \mathrm{e}^{-\kappa t})i\nu}$$

$$\varphi(t,\nu) = -\frac{2\kappa\theta}{\sigma^2}\log\left(1 - \frac{\sigma^2}{2\kappa}(1 - \mathrm{e}^{-\kappa t})i\nu\right)$$

Example - CIR process (cont.)

- The characteristic function has the form

$$v(t,x) = \left(1 - \frac{\sigma^2}{2\kappa}(1 - \mathrm{e}^{-\kappa(T-t)})i\nu\right)^{-\frac{2\kappa\theta}{\sigma^2}} \exp\left(\frac{\mathrm{e}^{-\kappa(T-t)} i\nu x}{1 - \frac{\sigma^2}{2\kappa}(1 - \mathrm{e}^{-\kappa(T-t)})i\nu}\right)$$

- This means that

$$\frac{X_t}{\frac{\sigma^2}{4\kappa}(1 - \mathrm{e}^{-\kappa t})}$$

has a non-central χ^2 – distribution with parameters

$$k = \frac{4\kappa\theta}{\sigma^2}, \quad \alpha = \frac{\mathrm{e}^{-\kappa t} X_0}{\frac{\sigma^2}{4\kappa}(1 - \mathrm{e}^{-\kappa t})}$$

Example - Heston model

- In this model we have

$$dX_t = (r - V_t/2)dt + \sqrt{V_t}dW_t^{(1)}$$
$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^{(2)}$$

where $W^{(1)}, W^{(2)}$ are Brownian motions with correlation ρ

- Assume

$$\begin{aligned} u(t,x,v) &:= \mathbb{E}[\exp(i(\nu_1 X_T + \nu_2 V_T))|X_t = x,\, V_t = v] \\ &= \exp(\varphi(T-t,\nu_1,\nu_2) + \psi_1(T-t,\nu_1,\nu_2)x + \psi_2(T-t,\nu_1,\nu_2)v) \end{aligned}$$

- Then u satisfies the PDE

$$u_t + \mathcal{G}u = 0,$$

with terminal condition $u(T,x) = \exp(i(\nu_1 x + \nu_2 v))$, where

$$\mathcal{G}u = \frac{1}{2}vu_{xx} + \rho\sigma v u_{xv} + \frac{\sigma^2}{2}vu_{vv} + (r - v/2)u_x + \kappa(\theta - v)u_v$$

Example - Heston model (cont.)

- This leads so a system of ODEs for φ, ψ_1, ψ_2. The solution of this system when $\nu_1 = \nu$, $\nu_2 = 0$ is

$$\varphi(t, \nu, 0) = \frac{\kappa\theta}{\sigma^2}\left[(\beta - \gamma)t - 2\ln\left(\frac{1 - \alpha e^{-\gamma t}}{1 - \alpha}\right)\right] + i\nu r t$$

$$\psi^1(t, \nu, 0) = i\nu$$

$$\psi^2(t, \nu, 0) = \frac{\beta - \gamma}{\sigma^2}\frac{1 - e^{-\gamma t}}{1 - \alpha e^{-\gamma t}}$$

where $\beta = \kappa - i\nu\rho\sigma$, $\gamma = \sqrt{\beta^2 + \sigma^2(\nu^2 + i\nu)}$ and $\alpha = \frac{\beta - \gamma}{\beta + \gamma}$

- This is consistent with the **form given in slide 36** of the previous lecture
- *Observation:* These formulas are only well defined on a finite interval of times

Example - Lévy processes

- For a Lévy process X (2) corresponds to the Lévy-Khintchine formula

$$\mathbb{E}[\exp(i\theta X_T)|X_t = x] = \exp(i\theta x + (T-t)\varphi(\theta))$$

where

$$\varphi(\theta) = ib\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (\mathrm{e}^{i\theta y} - 1 - i\theta y 1_{\{|y|\leq 1\}})\, d\nu(y)$$

- *Financial models:* Jumps are modeled using Lévy processes
    - Variance Gamma (VG) model
    - (CGMY) model
    - Normal Inverse Gaussian Model

Affine models - The Generator

For affine processes, the infinitesimal generator $\mathcal{G}$ is a integro-differential operator of the form

$$\begin{aligned}\mathcal{G}f(x) =& \frac{1}{2}Tr(A(x)\nabla^2 f(x)) + B(x)^T \nabla f(x) \\ &+ \int_{\mathbb{R}^n} (f(x+\xi) - f(x) - \xi^T \nabla f(x))\lambda(x, d\xi)\end{aligned} \tag{5}$$

with A, B, λ of the form

$$\begin{aligned}A(x) &= a + \sum_{i=1}^{d} x_i \alpha_i \in \mathbb{R}^{d\times d} \\ B(x) &= b + \sum_{i=1}^{d} x_i \beta_i = b + \mathcal{B}x \in \mathbb{R}^d \\ \lambda(x, d\xi) &= \nu(d\xi) + \langle x, m(d\xi)\rangle\end{aligned} \tag{6}$$

Here a and the α_i's are $d \times d$ matrices. b and the β_i's are vectors in $\mathbb{R}^d$; and $\mathcal{B} = (\beta_1, ..., \beta_d)$ is a $d \times d$ matrix

Affine models - Characteristics

- The parameters (B, A, λ) are known as the semimartingale characteristics of the process
- Similarly to the case of Lévy models:
    - B controls the drift of the continuous part
    - A controls the volatility of the continuous part
    - λ is related to the jumps of the process
- In Lévy models $\alpha = \beta = m = 0$

Affine models - The functions φ and ψ

- For $t \leq T$ let

$$\begin{aligned} v(t,x) &:= \mathbb{E}[\exp(i\langle \nu, X_T\rangle)|X_t = x] \\ &= \exp(\varphi(T-t,\nu) + \langle \psi(T-t,\nu), x\rangle) \end{aligned} \tag{7}$$

- Then v satisfies the following PIDE

$$v_t + \mathcal{G}v = 0$$

  with terminal condition $v(T,x) = \exp(i\langle \nu, x\rangle)$

- This leads us to a system of ODEs for the functions φ and ψ

Affine models - The functions φ and ψ - Diffusions

Assuming that $\lambda = 0$ (no jumps) we have the following result

Theorem 1

The functions φ and $\psi = (\psi^1, \dots, \psi^d)^T$ solve the system of Riccati equations

$$\begin{aligned} \varphi_t(t,\nu) &= \frac{1}{2}\psi(t,\nu)^T a \psi(t,\nu) + b^T \psi(t,\nu) \\ \varphi(0,\nu) &= 0 \\ \psi_t^i(t,\nu) &= \frac{1}{2}\psi(t,\nu)^T \alpha_i \psi(t,\nu) + \beta_i^T \psi(t,\nu) \quad 1 \le i \le d \\ \psi(0,\nu) &= i\nu \end{aligned} \tag{8}$$

In particular φ is given by the integration formula

$$\varphi(t,\nu) = \int_0^t \left(\frac{1}{2}\psi(s,\nu)^T a \psi(s,\nu) + b^T \psi(s,\nu) \right) ds \tag{9}$$

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Back to Carr-Madan - Truncation error

We recall the pricing formula

$$C(k) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty Re\left(\widehat{C_\alpha}(\nu) \exp(-i\nu k)\right) d\nu \tag{10}$$

where

$$\widehat{C_\alpha}(\nu) = e^{-rT} \frac{\phi(\nu - i(\alpha+1))}{(\alpha + i\nu)(\alpha + 1 + i\nu)} \tag{11}$$

ϕ is the characteristic function of the log price, k is the log strike and α is an appropriate damping factor

Proposition 1

Suppose that we truncate the integral in (10) *at a level* L. *Then the error from the truncation is bounded by*

$$\textit{error} \leq \frac{e^{-\alpha k}\mathbb{E}[S_T^{\alpha+1}]}{\pi L} \tag{12}$$

Back to Carr-Madan - Truncation error (cont.)

Proof:

- We have that $|\phi(\nu - i(\alpha+1))| \leq \mathbb{E}[S_T^{\alpha+1}]$
- Also
$$\nu^4 \leq |\alpha^2 + \alpha - \nu^2 + i\nu(2\alpha+1)|^2$$
- The bound follows from the previous inequalities because
$$\int_L^\infty \nu^{-2}\, d\nu = L^{-1}$$

Back to Carr-Madan - Trapezoid rule

- Suppose that we use the trapezoid rule to evaluate the integral
- We get

$$C(k) \approx Re\left(\frac{\exp(-\alpha k)}{\pi}\sum_{j=1}^{N}\widehat{C_\alpha}(\nu_j)\exp(-i\nu_j k)w_j\right) \tag{13}$$

with $\widehat{C_\alpha}$ given by (11) and where $\eta = \frac{L}{N-1}$ is the size of the partition,

$$\nu_j = (j-1)\eta; \quad j = 1, \ldots, N$$

and $w_1 = w_N = \eta/2$, $w_j = \eta$ for $1 < j < N$

- Note that if instead of the trapezoid rule one uses Simpson's rule the only thing that changes in (13) is the definition of the weights w_i
- *Observation:* This sum has the structure of the Discrete Fourier Transform (DFT)

Trapezoid rule error

Proposition 2

Suppose that g is smooth on $[a, b]$. Then we have the following expression for the error of the trapezoid rule

$$\int_a^b g(x)\,dx - \frac{b-a}{N}\left\{\frac{g(a)+g(b)}{2} + \sum_{k=1}^{N-1} g\left(a + k\frac{b-a}{N}\right)\right\} = -\frac{(b-a)^3}{12N^2} g''(\xi) \tag{14}$$

for some $\xi \in (a, b)$

Observation: In (13) the function g corresponds to

$$g(\nu) = Re\left(\widehat{C_\alpha}(\nu)\exp(-ik\nu)\right)$$

Discrete Fourier Transform

Suppose that we have an array of numbers (data) of length N

$$x = (x_1, \ldots, x_N) \in \mathbb{R}^N$$

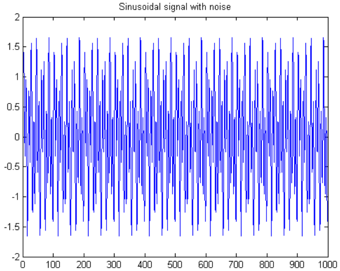
The Discrete Fourier Transform (DFT) of x is a vector $\widehat{x} \in \mathbb{R}^N$ defined by

$$\widehat{x}_m = \sum_{j=1}^{N} x_j \omega_N^{(j-1)(m-1)} \tag{15}$$

where $\omega_N = \mathrm{e}^{-\frac{2\pi}{N} i}$ (N root of 1)

- **Observation:** This can be seen as the Fourier Transform of a finitely supported measure

DFT - Example



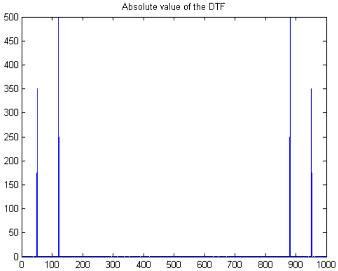


Figure: The DFT highlights the relevant frequencies of the original signal and discerns the noise

Back to Carr-Madan - FFT Scheme

- To create an "FFT situation" one discretizes the possible strikes as

$$k_m = \beta + (m-1)\lambda \quad m = 1, \ldots, N$$

where

$$\beta = \log S_0 + rT - \frac{\lambda(N-1)}{2}$$

- Then we can rewrite (13)

$$C(k_m) \approx Re\left(\frac{\exp(-\alpha k_m)}{\pi} \sum_{j=1}^{N} x_j \exp\left(-i\frac{2\pi}{N}(j-1)(m-1)\right)\right) \tag{16}$$

with

$$x_j = w_j \widehat{C_\alpha}(\nu_j) \exp(-i\beta\nu_j)$$

and

$$\boxed{\lambda\eta = \frac{2\pi}{N}}$$

Back to Carr-Madan - FFT Scheme - Constraints

- Here is the main constraint of this method: For N fixed, η and λ are inversely proportional. Hence
  - If λ is small, η is big and the numerical approximation of the integral is bad
  - If η is small, λ is big and we get many option prices that are irrelevant

FFT scheme

- For given data $x = (x_1, \ldots, x_N)$ we define $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_N)$ through the formula.

$$\widehat{x}_m = \sum_{j=1}^{N} x_j \exp\left(-i\frac{2\pi}{N}(j-1)(m-1)\right) \tag{17}$$

- In matrix form

$$\widehat{x} = W(N)x \tag{18}$$

with $W(N) = (W(N)_{jm})_{1\leq j,m\leq N}$ the Fourier matrix defined by

$$W(N)_{jm} = \exp\left(-i\frac{2\pi}{N}(j-1)(m-1)\right)$$

- Notice that these are the sums that appear in the scheme (16)

FFT algorithm

- The Fast Fourier Transform (FFT) is an algorithm to

  $$\boxed{\text{compute } \widehat{x} \text{ using } O(N \log N) \text{ operations}}$$

  instead of the N^2 operations that a regular matrix multiplication would require
- The main algebraic idea behind FFT is to

  $$\boxed{\text{split the sums into smaller sums}}$$

  that can be expressed in terms of DFTs of smaller arrays of data, or equivalently, to

  $$\boxed{\text{use a convenient factorization of the matrix } W}$$

Option prices with FFT - Example - VG model

- Consider a Variance Gamma model with parameters:

$$S_0 = 100;\ r = 0.1;\ \nu = 0.2;\ \theta = -0.14;\ \sigma = 0.12$$

- This is a model where log prices are modeled by

$$\log(S_0) + (r - \psi(-i))T + Y_T \tag{19}$$

where Y is a Lévy process with characteristic function (at time T)

$$\phi_{Y_T}(u) = \exp(T\psi(u)) \tag{20}$$

and

$$\psi(u) = -\frac{1}{\nu}\log(1 - iu\theta\nu + \sigma^2 u^2 \nu/2) \tag{21}$$

Option prices with FFT - Example (cont.)

- The characteristic function of $X_T := \log(S_T)$ is of the form

$$\phi_{X_T}(u) = \left(\frac{S_0 \mathrm{e}^{rT}}{\phi_{Y_T}(-i)}\right)^{iu} \phi_{Y_T}(u) \tag{22}$$

- One can represent Y as a time changed Brownian motion with drift $\theta t + \sigma W_t$ where the time change is given by a gamma process
- In this case we have a formula for the density function of Y_t

$$f(y;\sigma,\nu,\theta) = \int_0^\infty \frac{\exp\left(-\frac{(y-\theta g)^2}{2\sigma^2 g}\right)}{\sigma\sqrt{2\pi g}} \frac{g^{t/\nu-1}\mathrm{e}^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)}\, dg \tag{23}$$

- This (pure-jump) model was proposed by Madan, Carr and Chang (1998). They provide explicit analytic formulas for option prices

Option prices with FFT - Example (cont.)

Suppose that for $T = 1/12$ and $\alpha = 1$

K		80	90	100	110	120
Analytical		20.6702	10.8289	1.8150	0.0195	0.0007
$\eta = 0.15$	$N = 2^6$	23.5292	18.4794	13.9621	9.8758	6.1453
$\eta = 0.15$	$N = 2^8$	20.6002	10.6858	4.3707	-0.0827	-0.0520
$\eta = 0.15$	$N = 2^{10}$	20.6635	10.8229	2.0438	0.0244	-0.0008
$\eta = 0.15$	$N = 2^{12}$	20.6699	10.8283	1.8230	0.0195	0.0006
$\eta = 0.25$	$N = 2^6$	20.0634	13.8896	9.0771	4.7236	0.7492
$\eta = 0.25$	$N = 2^8$	20.6334	10.8051	3.0307	0.0181	-0.0139
$\eta = 0.25$	$N = 2^{10}$	20.6691	10.8274	1.8646	0.0197	0.0006
$\eta = 0.25$	$N = 2^{12}$	20.6701	10.8288	1.8168	0.0197	0.0006

Table: Call prices obtained by the FFT method in the VG model. Recall that for instance for $\eta = 0.25$ and $N = 2^{12}$, 4096 prices are calculated with the log distance between the strikes equal to $\lambda = 0.0061$

Option prices with FFT - Example (cont.)

Let

$$\psi(\nu) = \frac{e^{-rT-\alpha k}}{\pi} Re\left(\frac{\phi_{X_T}(\nu - i(\alpha+1))}{(\alpha+i\nu)(\alpha+1+i\nu)} e^{-i\nu k}\right)$$

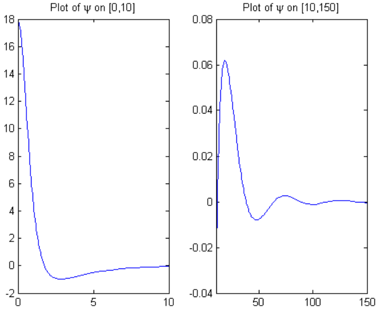


Figure: Plots of the integrand ψ and its tail. Here $K = 90$