Computational Finance
FIN-472
Homework 7

November 3, 2017

Exercise 1: Consider a GARCH model $(X_t)_{0\leq t\leq T}$ of the form

$$dX_t = \kappa(\theta - X_t)\,dt + \sigma X_t\,dW_t,$$

where κ, θ and σ are model parameters.

a) Prove that X_t is a polynomial diffusion and write its infinitesimal generator $\mathcal{G}$.

b) For $u \in \mathbb{R}$, define v as

$$v(t,x) = \mathbb{E}[\exp(iuX_T)|X_t = x].$$

If the conditions of the Feyman-Kac theorem are satisfied, v solves the equation

$$v_t + \mathcal{G}v = 0, \quad v(T,x) = \exp(iux).$$

Can we write the solution of the PDE as

$$v(t,x) = \exp(\phi(T-t) + \psi(T-t)x)$$

for some function ϕ and ψ with $\phi(0) = 0$ and $\psi(0) = iu$?

c) Solve explicitly the differential equation for X_t.

Hint: Consider the Ansatz $L_t := e^{\left(\kappa+\frac{\sigma^2}{2}\right)t-\sigma W_t}X_t$.

d) For $N \in \mathbb{N}$, write the matrix representation G_N of the infinitesimal generator restricted to $\mathrm{Pol}_N(\mathbb{R})$, with respect to the monomial basis given by

$$H_N(x) = (1, x, x^2, \cdots, x^N).$$

e) Use the moment formula for polynomial diffusions to calculate the first moment $\mathbb{E}[X_T]$. Check that the obtained result is coherent with what one gets from the explicit formula derived in part c).

f) Consider the set of parameters

$$\kappa = 0.5, \ \ \theta = 0.4, \ \ \sigma = 0.2, \ \ X_0 = 1, \ \ T = 0.5.$$

Use the moment formula for polynomial diffusions to find the first 4 moments

$$\mathbb{E}[X_T], \ \mathbb{E}[X_T^2], \ \mathbb{E}[X_T^3], \ \mathbb{E}[X_T^4]$$

and calculate the 4-order "approximation" of the density of X_T with a Gaussian that matches the first two moments. Plot the density approximation for orders $1, 2, 3, 4$.

Exercise 2: Consider the Heston model where the squared volatility V_t and the log-asset price X_t are given by

$$\begin{aligned} dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^{(1)}, \\ dX_t &= (r - V_t/2)dt + \rho\sqrt{V_t}dW_t^{(1)} + \sqrt{V_t}\sqrt{1-\rho^2}dW_t^{(2)}. \end{aligned}$$

Here, $W^{(1)}$ and $W^{(2)}$ are independent Brownian motions and $\kappa, \theta, \sigma, \rho$ are model parameters.

a) Let $\mu_w \in \mathbb{R}$ and $\sigma_w > 0$ be arbitrary parameters. Consider the basis of $\mathrm{Pol}_N(\mathbb{R}^2)$ defined as

$$\mathcal{H}_N = \{1, v, \frac{x-\mu_w}{\sigma_w}, v^2, v\Big(\frac{x-\mu_w}{\sigma_w}\Big), \Big(\frac{x-\mu_w}{\sigma_w}\Big)^2, \cdots, v^n, v^{n-1}\Big(\frac{x-\mu_w}{\sigma_w}\Big), \cdots, \Big(\frac{x-\mu_w}{\sigma_w}\Big)^n\}$$

and write it in a row vector

$$H_N = (1, v, \frac{x-\mu_w}{\sigma_w}, v^2, v\Big(\frac{x-\mu_w}{\sigma_w}\Big), \Big(\frac{x-\mu_w}{\sigma_w}\Big)^2, \cdots, v^n, v^{n-1}\Big(\frac{x-\mu_w}{\sigma_w}\Big), \cdots, \Big(\frac{x-\mu_w}{\sigma_w}\Big)^n).$$

Note that the dimension of $\mathrm{Pol}_N(\mathbb{R}^2)$ is $M := \frac{(N+1)(N+2)}{2}$.

Define a bijective function

$$\pi : \{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0 \ | \ m, n \geq 0; \ \ m+n \leq N\} \to \{1, 2, \cdots, M\},$$

that describes the ordering for the basis H_N. In other words, each basis element of the form $v^m\Big(\frac{x-\mu_w}{\sigma_w}\Big)^n$ is stored in the position $\pi(m,n)$ in the vector H_N. Implement this function in Matlab and call it `Ind.m`.

b) Write a Matlab function `GenHeston.m` that constructs G_N, the matrix representation of the infinitesimal generator $\mathcal{G}$ restricted to $\mathrm{Pol}_N(\mathbb{R}^2)$ with respect to the basis H_N.

c) Consider the model parameters

$$X_0 = 5.1, \ \ V_0 = 0.04, \ \ \kappa = 1, \ \ \theta = 0.04, \ \ \sigma = 0.2, \ \ r = 0.03, \ \ \rho = -0.8, \ \ T = 1/52,$$

together with $\mu_w = \mathbb{E}[X_T]$ and $\sigma_w^2 = \mathrm{Var}[X_T]$. Using the moment formula for polynomial diffusions, compute

$$\mathbb{E}[\Big(\frac{X_T-\mu_w}{\sigma_w}\Big)], \ \mathbb{E}[\Big(\frac{X_T-\mu_w}{\sigma_w}\Big)^2], \ \mathbb{E}[\Big(\frac{X_T-\mu_w}{\sigma_w}\Big)^3], \ \mathbb{E}[\Big(\frac{X_T-\mu_w}{\sigma_w}\Big)^4].$$