Computational Finance
FIN-472
Transform methods for pricing III - Introduction to polynomial expansion methods / density approximation

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Outline

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Motivation to use FrFFT

- Recall the dependence between the parameters λ (discretization of log strikes), η (step size numerical integration over $[0, L]$) and N (number of points in the partition)

$$\lambda\eta = \frac{2\pi}{N}$$

- **Example:** As before, consider the pricing example with the VG model. The best results were obtained with $\eta = 0.25$ and $N = 2^{12} = 4096$. The following plot shows one tenth of the prices obtained with the FFT routine. Almost all the prices are irrelevant

Motivation to use FrFFT (cont.)

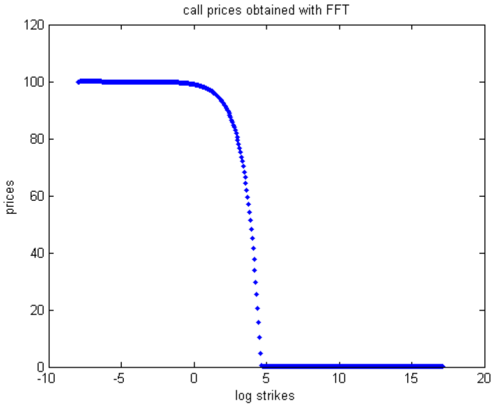

Figure: Call prices obtained in the VG model with FFT: $\eta = 0.25$; $N = 2^{12}$; $\alpha = 1$; $T = 1/12$; $S_0 = 100$; $r = 0.1$; $\nu = 0.2$; $\theta = -0.14$; $\sigma = 0.12$

Main idea

- The Fractional Fast Fourier Transform (FrFFT) is a method to calculate sums of the form

$$\widehat{x}_m = \sum_{j=1}^{N} x_j \exp\left(-i2\pi\gamma(j-1)(m-1)\right) \quad (1)$$

with γ arbitrary.

- Instead of the representation given in slide 25 of the previous lecture, we could use the following representation

$$C(k_m) \approx Re\left(\frac{\exp(-\alpha k_m)}{\pi} \sum_{j=1}^{N} x_j \exp\left(-i2\pi\gamma(j-1)(m-1)\right)\right) \quad (2)$$

with

$$x_j = w_j \widehat{C_\alpha}(\nu_j) \exp(-i\beta\nu_j)$$

and

$$\boxed{\lambda\eta = 2\pi\gamma}$$

Main idea (cont.)

- In the original FFT scheme

$$\gamma = \frac{1}{N}$$

- **Main idea:** Use the identity

$$2(j-1)(m-1) = (j-1)^2 + (m-1)^2 - (j-m)^2 \qquad (3)$$

to express the sum in terms of convolution and then we use that convolution becomes multiplication in the Fourier domain

DFT - Convolution

Suppose that $x = (x_i)_{i=1}^N, y = (y_i)_{i=1}^N$ are two arrays of numbers. We define their convolution $x * y = ((x * y)_m)_{m=1}^N$ through the formula

$$(x * y)_m = \sum_{i=1}^{N} x_i y_{m-i+1} \tag{4}$$

where the negative indices are interpreted mod N

Theorem 1

$$\widehat{x * y} = \widehat{x} \odot \widehat{y} \tag{5}$$

where $\odot$ is the Hadamard (componentwise) product of vectors. This is

$$(\widehat{x * y})_m = \widehat{x}_m \widehat{y}_m, \quad m = 1, \ldots, N$$

DFT - Inversion

For a fixed N and a size N array of numbers x we denote its DFT by

$$\mathcal{F}_N x = \widehat{x}$$

We define the Discrete Inverse Fourier Transform (DIFT) of x, $\mathcal{F}_N^{-1} x = ((\mathcal{F}_N^{-1} x)_m)_{m=1}^N$ by

$$(\mathcal{F}_N^{-1} x)_m = \frac{1}{N} \sum_{j=1}^{N} x_j \exp\left(i \frac{2\pi}{N} (j-1)(m-1) \right) \tag{6}$$

Theorem 2

$$\mathcal{F}_N^{-1}(\mathcal{F}_N x) = \mathcal{F}_N(\mathcal{F}_N^{-1} x) = x \tag{7}$$

FrFFT explained - Bailey and Swarztrauber (1991)

- Suppose that, as in (1) and (2), we have sums of the form

$$\widehat{x}_m = \sum_{j=1}^{N} x_j \exp\left(-i2\pi\gamma(j-1)(m-1)\right)$$

- Using the identity (3) we can write

$$\widehat{x}_m = \mathrm{e}^{\left(-i\pi\gamma(m-1)^2\right)} \sum_{j=1}^{N} x_j \mathrm{e}^{\left(-i\pi\gamma(j-1)^2\right)} \mathrm{e}^{\left(i\pi\gamma(m-j)^2\right)} \tag{8}$$

This expression is almost a convolution (see (4)) except that the sequences do not have the right periodicity to be coherent with the mod N interpretation of negative indices

FrFFT explained (cont.)

- However, by extending the sequences we can express (8) as a convolution as follows

$$\exp\left(i\pi\gamma(m-1)^2\right)\widehat{x}_m = (y * z)_m \tag{9}$$

where

$$\begin{aligned}
y_j &= x_j \exp\left(-i\pi\gamma(j-1)^2\right); 1 \leq j \leq N\\
y_j &= 0; N < j \leq 2N\\
z_j &= \exp(i\pi\gamma(j-1)^2); 1 \leq j \leq N\\
z_j &= \exp(i\pi\gamma(2N-j+1)^2); N < j \leq 2N
\end{aligned} \tag{10}$$

FrFFT algorithm

- By Theorems 1 and 2 we deduce with $\mathcal{F} = \mathcal{F}_{2N}$

$$\widehat{x}_m = \exp\left(-i\pi\gamma(m-1)^2\right)\left((\mathcal{F}^{-1})(\mathcal{F}y \odot \mathcal{F}z)\right)_m \tag{11}$$

- It should be highlighted that formula (11) only works for $1 \leq m \leq N$
- The algorithm (11) can be used in (2) to price options
- Although, two FFTs and one inverse FFT of size $2N$ have to be computed now, the parameters η, λ and N are independent

Illustration of the FrFFt method - VG model

K		80	90	100	110	120
Analytical		20.6702	10.8289	1.8150	0.0195	0.0007
$\eta = 0.25$	$N = 2^{12}$	20.6704	10.8288	1.8150	0.0195	0.0006

Table: Call prices obtained with the FrFFT method in the VG model. Parameters: $\eta = 0.25$; $N = 2^{12}$; $\lambda = 2/N$; $\alpha = 1$; $T = 1/12$; $S_0 = 100$; $r = 0.1$; $\nu = 0.2$; $\theta = -0.14$; $\sigma = 0.12$

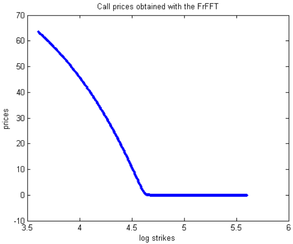

Figure: Plot of 1/10 of the prices obtained with the FrFFT method

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Motivation - Out-of-the-money options - Example

Strikes	Method			
	FrFFT	FrFFT/Saddle	Saddle	Simulation
150	4.8158e-04	4.6303e-04	4.5711e-04	4.28e-04
160	7.7876e-05	6.9294e-05	6.8764e-05	3.25e-05
230	-8.4688e-06	1.1154e-09	1.1211e-09	0

Table: European call option prices for Heston model with: $S_0 = 100$; $r = 0.03$; $T = 0.5$; $\kappa = 2$; $\theta = 0.04$; $\sigma = 0.5$; $\rho = -0.7$. For FrFFT take $\eta = 0.25$; $N = 2^8$; $\lambda = 2/N$; $\alpha = 1$. For FrFFT/Saddle take α to be the saddle point. Saddle point for $K = 150$ is 26.8242. Saddle point for $K = 160$ is 27.8844

- For deep-out-of-the-money options the Fourier method produces negative prices
- The saddle point method helps in the calculation of out-of-the-money options. It addresses two points: the right choice of alpha and an alternative numerical integration

Motivation - Out-of-the-money options - Example (cont.)

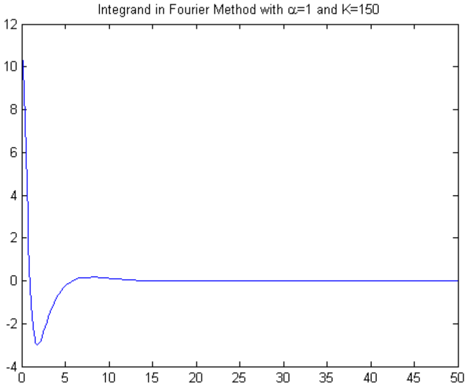

Figure: Plot of integrand for Fourier method with $\alpha = 1$ and $K = 150$

Saddle point setup - Prices as probabilities

- As usual we suppose that $S_T = \exp(X_T)$ and that the characteristic function of X_T, $\phi_{X_T} = \phi$, is known
- For a strike K we denote by $k = \log K$ the log strike. It is possible to express the put price as

$$P(k) = \mathrm{e}^{k-rT}\mathbb{P}(X_T < k) - \mathrm{e}^{-rT}\mathbb{E}[S_T]\mathbb{P}^S(X_T < k) \tag{12}$$

where the strike is $K = \log k$ and $\mathbb{P}^S$ is the share measure

Prices as probabilities (cont.)

- The share measure $\mathbb{P}^S$ is defined by

$$\mathbb{P}^S(A) = \frac{\mathbb{E}[S_T 1_A]}{\mathbb{E}[S_T]}$$

- Similarly for the call price we have

$$C(k) = e^{-rT}\mathbb{E}[S_T]\mathbb{P}^S(X_T > k) - e^{k-rT}\mathbb{P}(X_T > k) \tag{13}$$

- Hence the computation of option prices can be reduced to the computation of probabilities

Cumulant generating function

- For a random variable X we define the cumulant generating function w.r.t $\mathbb{P}$, $K_X^{\mathbb{P}}$, by
$$K_X^{\mathbb{P}}(z) = \log \mathbb{E}[\exp(zX)] \tag{14}$$
- Observe that
$$\phi_X(z) = \exp(K_X^{\mathbb{P}}(iz))$$
- In particular
$$\exp(K_X^{\mathbb{P}}(1)) = \mathbb{E}[\exp(X)]$$

Cumulant generating function (cont.)

- If we denote $K = K^{\mathbb{P}}_{X_T}$, we can write (12) and (13) as

$$\begin{aligned} P(k) &= \mathrm{e}^{k-rT}\mathbb{P}(X_T < k) - \mathrm{e}^{-rT+K(1)}\mathbb{P}^S(X_T < k) \\ C(k) &= \mathrm{e}^{-rT+K(1)}\mathbb{P}^S(X_T > k) - \mathrm{e}^{k-rT}\mathbb{P}(X_T > k) \end{aligned} \tag{15}$$

- Observe also that $\widetilde{K} = K^{\mathbb{P}^S}_{X_T}$ is given by

$$\widetilde{K}(z) = K(z+1) - K(1) \tag{16}$$

Problem

- Then it all boils down to the calculation of probabilities of the form

$$\mathbb{P}(X > a)$$

where the cumulant generating function of X

$$K(z) = \log \mathbb{E}[\exp(zX)]$$

is known

- This is completely analogous to the discussion around Proposition 2 in Lecture 4 / Slide 20

Lugannani-Rice approximation (1980)

- A similar argument as before shows that for any $\alpha > 0$

$$\begin{aligned}\mathbb{P}(X > a) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\phi(\nu - i\alpha)}{\alpha + i\nu}\exp(-a(\alpha + i\nu))\,d\nu \\ &= \int_{\alpha - i\infty}^{\alpha + i\infty}\frac{\exp(K(z) - za)}{z}\frac{dz}{2\pi i}\end{aligned} \tag{17}$$

- The idea of the saddle point method is to choose a **contour of integration where the integrand decreases rapidly.** Then **instead of using numerical integration, approximate the integrand** with another integrand for which the values are known, e.g. the integrand coming from a Gaussian model

Lugannani-Rice method - the "saddle point"

- Find α_* such that

$$K'(\alpha_*) = a \tag{18}$$

  so that the function $K(x) - ax$ is minimized at α_*

- The steepest descent of the function $K(z) - az$ happens if we integrate around the contour

$$\gamma = \{\alpha_* + i\nu : \nu \in \mathbb{R}\}$$

Lugannani-Rice formula

- If $\widehat{w} \neq 0$, we have the Lugannani-Rice approximation

$$\mathbb{P}(X > a) \approx (1 - F_{\mathcal{N}}(\widehat{\omega})) + f_{\mathcal{N}}(\widehat{\omega}) \left(\frac{1}{\alpha_*(K''(\alpha_*))^{\frac{1}{2}}} - \frac{1}{\widehat{\omega}} \right) \quad (19)$$

where $F_{\mathcal{N}}$ is the standard normal CDF, $f_{\mathcal{N}}$ is the standard normal PDF, α_* is the saddle point $(K'(\alpha_*) = a)$, and

$$\widehat{w} = sign(\alpha_*)\sqrt{2(\alpha_* a - K(\alpha_*))}$$

- If $\widehat{\omega} = \alpha_* = 0$ we have

$$\begin{aligned} \mathbb{P}(X > a) &\approx (1 - F_{\mathcal{N}}(0)) + f_{\mathcal{N}}(0) \left(-\frac{K'''(0)}{6[K''(0)]^{\frac{3}{2}}} \right) \\ &= \frac{1}{2} - \frac{K'''(0)}{6[2\pi(K''(0))^3]^{\frac{1}{2}}} \end{aligned} \quad (20)$$

Summary

- Over the strip of analyticity of K find the (real) saddle point α_* that solves (18). Notice then that by (16), $\alpha_* - 1$ solves the same equation for the cumulant generating function under the share measure
- Depending on whether $\alpha_* \neq 0$ or $\alpha_* = 0$ use equations (19) and (20) to approximate $\mathbb{P}(X_T > k)$
- Do the same with K replaced by $\widetilde{K}$ as in (16) to approximate $\mathbb{P}^S(X_T > k)$
- Use these approximations in (15)

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Polynomial expansion methods

- **Pricing problem:** Assume that X_T has a density $q(x)$ and the **discounted payoff of an European option is** $f(X_T)$. Then the price is

$$\pi_f = \mathbb{E}[f(X_T)] = \int_{\mathbb{R}} f(x)q(x)dx$$

- **Weighted Hilbert space:** Take an auxiliary density $w(x)$ and define

$$L^2_w = \left\{ f(x) : \|f\|^2_w = \int_{\mathbb{R}} f(x)^2\, w(x)dx < \infty \right\}$$

which is a Hilbert space with scalar product

$$\langle f, g \rangle_w = \int_{\mathbb{R}} f(x)g(x)\, w(x)dx$$

Assume $(H_n)_{n\geq 0}$ is an orthonormal (o.n.) basis of polynomials

Polynomial expansion methods (cont.)

- **Price expansion:** Weight the density with another density w such that $\ell(x) = q(x)/w(x) \in L^2_w$ and $f(x) \in L^2_w$. Then

$$\pi_f = \langle f, \ell \rangle_w = \sum_{n \geq 0} f_n \ell_n \tag{21}$$

for the **Fourier coefficients and Hermite moments**

$$f_n = \langle f, H_n \rangle_w, \quad \ell_n = \langle \ell, H_n \rangle_w = \int_{\mathbb{R}} H_n(x) q(x)\, dx$$

- **For some models (e.g. the affine models) we know how to calculate the moments ℓ_n!**

Polynomial expansion methods (cont.)

- **Price approximation:** Truncate the sum at a level N

$$\pi_f \approx \pi_f^{(N)} = \sum_{n=0}^{N} f_n \ell_n = \sum_{n=0}^{N} \langle f, \ell_n H_n \rangle_w = \int_{\mathbb{R}} f(x) q^{(N)}(x)\, dx \tag{22}$$

where

$$q^{(N)}(x) = \left(\sum_{n=0}^{N} \ell_n H_n(x) \right) w(x) \tag{23}$$

- The function $q^{(N)}(x)$ **is an approximation of the density** $q(x)$

Black-Scholes model - Normal distribution

- In the BS model X_T has a normal distribution with **mean** $\mu_w = \log S_0 + \left(r - \frac{1}{2}\sigma^2\right) T$ and **variance** $\sigma_w = \sigma^2 T$, hence

$$q(x) = \frac{\exp\left(-\frac{(x-\mu_w)^2}{2\sigma_w^2}\right)}{\sqrt{2\pi}\sigma_w}$$

- If we take $w(x) = q(x)$ then an o.n. basis of L^2_w is

$$H_n(x) = \frac{1}{\sqrt{n!}}\mathcal{H}_n\left(\frac{x-\mu_w}{\sigma_w}\right), \quad n \geq 0$$

  where $\mathcal{H}_n(x)$ are the standard "probabilists" Hermite polynomials
- As $\ell_0 = 1$ and $\ell_n = 0$ for $n \geq 1$ we have

$$\pi_f = \pi_f^{(N)} = f_0, \quad q(x) = q^{(N)}(x) = w(x), \quad N \geq 0$$

- For instance if f is the discounted payoff of a Call/Put option f_0 **is the BS price of the option**

CIR model - Density approximation

- **In some cases one cannot use w to be a normal distribution**
- Consider a **CIR model**

$$dX_t = \kappa(\theta - X_t)\,dt + \sigma\sqrt{X_t}\,dW_t$$

- Suppose that we take w as before. This is, w is a normal density $\mathcal{N}(\mu_w, \sigma_w^2)$ with

$$\mu_w = \mathbb{E}[X_T], \quad \sigma_w^2 = Var[X_T]$$

- As before

$$H_n = \frac{1}{\sqrt{n!}}\mathcal{H}_n\left(\frac{x-\mu_w}{\sigma_w}\right), \quad n \geq 0$$

constitutes an o.n. basis of L_w^2

- However, as the following graph shows, the density approximations $q^{(N)}(x)$ **diverge** as N grows

CIR model - Density approximation divergence

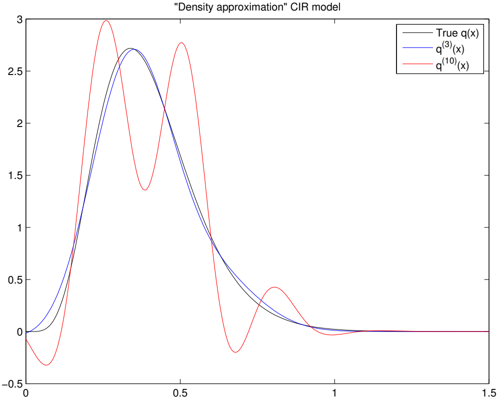

Figure: Functions $q^{(N)}(x)$ $(N = 3, 10)$ for the CIR model with parameters $\kappa = 2, \theta = 0.4, \sigma = 0.5, X_0 = 0.4, T = 1$