Computational Finance
FIN-472
Transform methods for pricing I

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Outline

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Definitions: Fourier Transform and Characteristic function

- Given $f \in L^1(\mathbb{R}^n)$, i.e. f is an integrable function, we denote by $\widehat{f}$ the Fourier transform of f, defined by

$$\widehat{f}(\nu) \triangleq \int_{\mathbb{R}^n} \exp(i\langle \nu, x\rangle) f(x)\, dx \tag{1}$$

- For a real random variable $X \in \mathbb{R}^n$ with distribution μ, we denote by ϕ_X, the Characteristic Function of X, defined by

$$\phi_X(\nu) \triangleq \mathbb{E}[\exp(i\langle \nu, X\rangle)] = \int_{\mathbb{R}^n} \exp(i\langle \nu, x\rangle)\mu(dx) \tag{2}$$

- **Observation:** If μ has *density function* f, i.e. $\mu(dx) = f(x)dx$, then $f \in L^1(\mathbb{R}^n)$ and

$$\phi_X = \widehat{f}$$

Example - From space to frequency domain

Suppose that

$$f(x) = \cos(6\pi x)\exp(-\pi x^2)$$

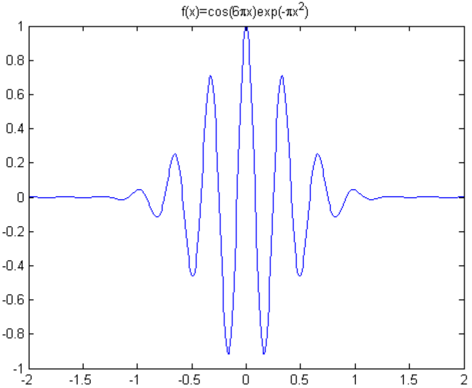

Figure: f oscillates 3 times per sec

Example - From space to frequency domain (cont.)

In this case the integrand in the Fourier transform takes the form

$$g(x,\nu) = \cos(6\pi x)\exp(-\pi x^2 + ix\nu)$$

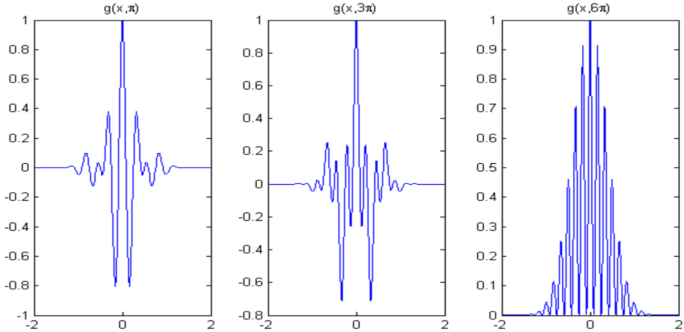

Figure: $Re(g(x,\nu))$ for different frequencies ν

Example - From space to frequency domain (cont.)

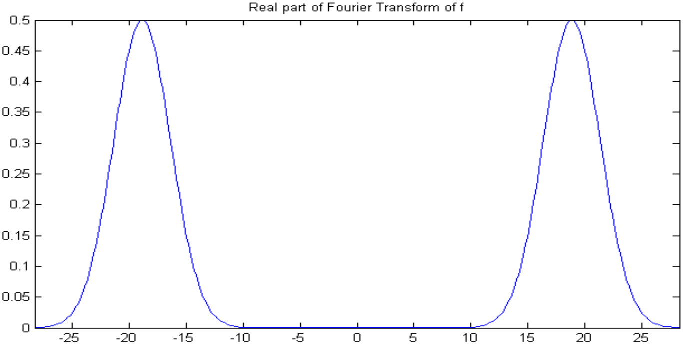

Figure: Graph of $Re(\widehat{f})$. Notice that the peaks occur at -6π and 6π. This is exactly when $f(x)$ and $\exp(i\nu x)$ are "synced"

The Inversion Formula

Theorem 1

Suppose that $f, \widehat{f}$ belong to $L^1(\mathbb{R}^n)$. Then the equality

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp(-i\langle \nu, x\rangle)\widehat{f}(\nu)\, d\nu \tag{3}$$

holds for almost all $x \in \mathbb{R}^n$

Remark: If $f \in L^1(\mathbb{R})$ is piecewise smooth and $\widehat{f} \in L^1(\mathbb{R})$ then

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp(-i\langle \nu, x\rangle)\widehat{f}(\nu)\, d\nu = \frac{1}{2}(f(x-) + f(x+))$$

where $f(x-)$ and $f(x+)$ are the limits from the left and from the right, respectively

Plancherel - Parseval's Theorem

Theorem 2

If $f, g \in L^2(\mathbb{R}^n)$ *then*

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{f}(\nu)\overline{\widehat{g}(\nu)}\,d\nu \tag{4}$$

where the **Fourier transform for functions in** $L^2(\mathbb{R}^n)$ **is interpreted in an extended sense**

Remark: Formally one can connect this to the inversion formula. Take $g(x) = \delta_{x_0}(x)$ (the delta function concentrated at x_0). Then $\widehat{g}(\nu) = \exp(i\langle x_0, \nu\rangle)$ and $\overline{\widehat{g}(\nu)} = \exp(-i\langle x_0, \nu\rangle)$. Hence,

$$f(x_0) = \int_{\mathbb{R}^n} f(x)\delta_{x_0}(x)\,dx = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \exp(-i\langle \nu, x_0\rangle)\widehat{f}(\nu)\,d\nu$$

Of course, this is only a heuristic argument

The moments

Moments via the characteristic function:

- Assume that $n = 1$.
- The existence of moments is related to differentiability at 0 of the characteristic function

$$\mathbb{E}[X^n] = i^{-n}\phi_X^{(n)}(0) \tag{5}$$

- In particular

$$\mathbb{E}[X] = -i\phi'_X(0)$$

- Also, if $Y = X - \mathbb{E}[X]$

$$\phi_Y(\nu) = \exp(-\nu\phi'_X(0))\phi_X(\nu)$$

Analyticity of the characteristic function

- Suppose that
$$\mathbb{E}[\exp(-\alpha X)] < \infty$$
for $\alpha = a, b$ with $a < b$
- In this case it can be shown that the characteristic function ϕ_X is analytic on the open strip
$$\{\nu = \lambda + i\mu : \mu \in (a, b),\ \lambda \in \mathbb{R}\} \subset \mathbb{C}$$
and well-defined and continuous on the closure of the strip

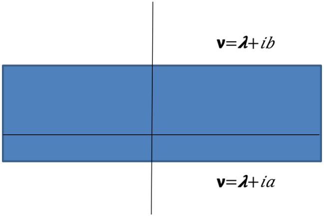

Figure: Region of analyticity of ϕ_X

Real valued functions

Proposition 1

Suppose that $n = 1$ and f is a real-valued function. Then $Re(\widehat{f}(\nu))$ for $\nu \in \mathbb{R}$ is an even function and $Im(\widehat{f}(\nu))$ for $\nu \in \mathbb{R}$ is an odd function. In particular, if $\widehat{f} \in L^1(\mathbb{R})$, the following inversion formula holds

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\nu) \exp(-i\nu x)\, d\nu \\ &= \frac{1}{\pi} \int_{0}^{\infty} Re\left(\widehat{f}(\nu) \exp(-i\nu x) \right)\, d\nu \end{aligned} \tag{6}$$

Real valued functions (cont.)

Sketch of the proof: If $\nu = \lambda + i\mu$ with $\lambda, \mu \in \mathbb{R}$

$$\begin{aligned}\widehat{f}(\nu) &= \int_{\mathbb{R}} f(x) \exp(-\mu x + ix\lambda)\, dx \\ &= \int_{\mathbb{R}} f(x) \exp(-\mu x) \cos \lambda x\, dx + i \int_{\mathbb{R}} f(x) \exp(-\mu x) \sin \lambda x\, dx\end{aligned}$$

Therefore

$$\begin{aligned}Re(\widehat{f}(\nu)) &= \int_{\mathbb{R}} f(x) \exp(-\mu x) \cos \lambda x\, dx \\ Im(\widehat{f}(\nu)) &= \int_{\mathbb{R}} f(x) \exp(-\mu x) \sin \lambda x\, dx\end{aligned}$$

If $\mu = 0$ ($\nu = \lambda$), we see that the real part is even and the imaginary part is odd in λ. Equation (6) follows

Convolution

- Given $f, g \in L^1(\mathbb{R}^n)$ one defines their convolution by the formula

$$(f * g)(y) = \int_{\mathbb{R}^n} f(x)g(y-x)\,dx \tag{7}$$

- **From convolution to multiplication:** We have that

$$\widehat{f * g}(\nu) = \widehat{f}(\nu)\widehat{g}(\nu) \tag{8}$$

Some complex analysis

Theorem 3 (Cauchy's integral theorem)

Suppose that f is a complex-function that is analytic on a domain D. Let γ be a closed-contour in D (start and end points of γ are the same). Then

$$\oint_\gamma f(z)\, dz = 0 \tag{9}$$

Remark: Suppose that f, g are analytic complex functions on a domain D such that

$$g' = f$$

Let γ be a path in D with start point ω_1 and end point ω_2. Then

$$\int_\gamma f(z)\, dz = g(\omega_2) - g(\omega_1) \tag{10}$$

Characteristic function of some distributions

The exponential distribution: In this case the Probability Density Function (PDF) is of the form

$$f(x) = \lambda \exp(-\lambda x); \quad x \geq 0 \tag{11}$$

Then

$$\begin{aligned} \widehat{f}(\nu) &= \lambda \int_0^\infty \exp((-\lambda + i\nu)x)\, dx \\ &= \frac{\lambda}{\lambda - i\nu} \end{aligned} \tag{12}$$

Characteristic function of some distributions

The one-dimensional standard normal distribution: In this case the PDF is of the form

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{13}$$

Hence,

$$\begin{aligned} \widehat{f}(\nu) &= \frac{\exp(-\nu^2/2)}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{(x-i\nu)^2}{2}\right) dx \\ &= \exp\left(-\frac{\nu^2}{2}\right) \end{aligned} \tag{14}$$

Characteristic function of some distributions

Remark: In the last integral we used Cauchy's integral theorem and the fact that

$$\int_{\mathbb{R}} \exp\left(-\frac{z^2}{2}\right)\, dz = \sqrt{2\pi}$$

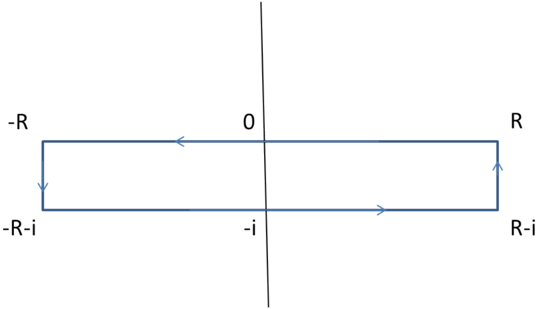

Figure: Contour used with $R \to \infty$

Characteristic function of some distributions

The normal distribution: If $X \sim N(\mu, \sigma)$, then

$$X \stackrel{d}{=} \mu + \sigma Z$$

with $Z \sim N(0, 1)$. Hence,

$$\begin{aligned} \phi_X(\nu) &= \exp(i\nu\mu)\mathbb{E}[\exp(i\sigma\nu Z)] \\ &= \exp\left(i\mu\nu - \frac{\sigma^2\nu^2}{2}\right) \end{aligned} \tag{15}$$

An illustrating example - CDF from Fourier Transform

Let

$$F_\alpha(x) = \exp(-\alpha x)F(x), \quad x \in \mathbb{R}$$

with $F(x) = Pr(X \leq x)$ the Cumulative Distribution Function (CDF) of a one-dimensional r.v. X

Proposition 2

Suppose that $\alpha > 0$ and $\mathbb{E}[\exp(-\alpha X)] < \infty$. Then $F_\alpha \in L^1(\mathbb{R})$, $\widehat{F_\alpha}(\nu)$ is well-defined for all $\nu \in \mathbb{R}$ and

$$\widehat{F_\alpha}(\nu) = \frac{\phi_X(\nu + i\alpha)}{\alpha - i\nu} \tag{16}$$

If in addition $\widehat{F_\alpha} \in L^1(\mathbb{R})$ then

$$F(x) = \frac{\exp(\alpha x)}{\pi}\int_0^\infty Re\left(\frac{\phi_X(\nu + i\alpha)}{\alpha - i\nu}\exp(-i\nu x)\right) d\nu \tag{17}$$

An illustrating example - CDF from Fourier Transform (cont.)

- This is the first example when we see the need to introduce a **damping factor** α
- Notice that without the damping factor $F \notin L^1(\mathbb{R})$ and $\widehat{F}$ would not necessarily be well-defined
- As we will see, this is related to a **change of integration contour and the saddle point method**
- The advantage of the calculation made in (18) is that it allows to express F in (17) using only *one integration*

An illustrating example - CDF from Fourier Transform (cont.)

Sketch of the proof: Let μ be the distribution of X. We have that

$$\begin{aligned}
\widehat{F_\alpha}(\nu) &= \int_{\mathbb{R}} \exp((i\nu - \alpha)x) F(x)\, dx \\
&= \int_{\mathbb{R}} \int_{-\infty}^{x} \exp((i\nu - \alpha)x) \mu(dy)\, dx \\
&= \int_{\mathbb{R}} \left(\int_{y}^{\infty} \exp((i\nu - \alpha)x)\, dx \right) \mu(dy) \qquad (18) \\
&= \frac{1}{\alpha - i\nu} \int_{\mathbb{R}} \exp(i(\nu + i\alpha)y) \mu(dy) \\
&= \frac{\phi_X(\nu + i\alpha)}{\alpha - i\nu}
\end{aligned}$$

Formula (17) thus follows from (6) and (18)

An illustrating example - CDF from Fourier Transform (cont.)

An alternative heuristic derivation: Suppose that there is a Probability Density Function (PDF) f. We can write

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(y)\, dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{x} \int_{\mathbb{R}} \exp(-i\nu y) \widehat{f}(\nu)\, d\nu\, dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{x} G(y)\, dy \end{aligned}$$

with

$$G(y) \triangleq \int_{\mathbb{R}} \exp(-i\nu y) \widehat{f}(\nu)\, d\nu$$

An illustrating example - CDF from Fourier Transform (cont.)

An alternative heuristic derivation (cont.): A change in integration contour allows to replace $G(y)$ by

$$\int_{\mathbb{R}} \exp(-i(\nu + i\alpha)y)\widehat{f}(\nu + i\alpha)\, d\nu = \int_{\mathbb{R}} \exp((\alpha - i\nu)y)\widehat{f}(\nu + i\alpha)\, d\nu$$

Hence,

$$\begin{aligned} F(x) &= \frac{1}{2\pi}\int_{-\infty}^{x}\int_{\mathbb{R}} \exp((\alpha - i\nu)y)\widehat{f}(\nu + i\alpha)\, d\nu\, dy \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\exp((\alpha - i\nu)x)\widehat{f}(\nu + i\alpha)}{\alpha - i\nu}\, d\nu \end{aligned}$$

which corresponds to (17)

An illustrating example - CDF from Fourier Transform (cont.)

Another derivation: As in Rogers and Zane (1999) we can write

$$F(x) = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \exp(\epsilon(y-x)) 1_{\{y \leq x\}} f(y)\, dy \tag{19}$$

Then

- Use Plancherel's Theorem and
- change the contour of integration to deduce (17)

The choice of α

- The identity (17) holds for all $\alpha > 0$ that satisfy the hypotheses of Proposition 2
- However, the behaviour of the integrand

$$Re\left(\frac{\phi_X(\nu + i\alpha)}{\alpha - i\nu}\exp(-i\nu x)\right) \tag{20}$$

 is different for different values of α

- If the integral in (17) is computed numerically the results depend on the choice of α

Example: Probabilities of a standard normal

In this case $\phi_X(\nu) = \exp(-\nu^2/2)$

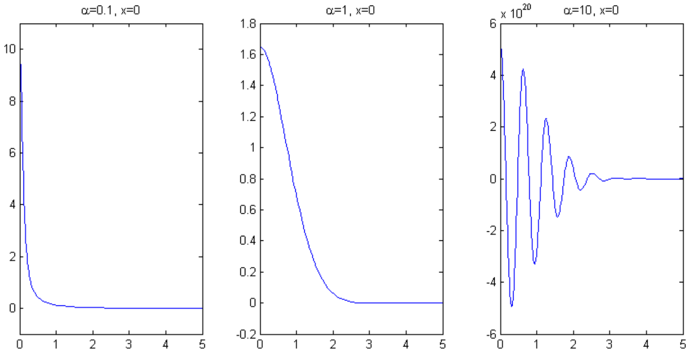

Figure: Plots of the integrand (20)

Example: Probabilities for a standard normal (cont.)

	x=0		
α	L=1	L=2	L=3
0.1	0.4808	0.4979	0.4999
1	0.4006	0.4956	0.5002
10	-3.6724e+18	1.4872e+18	-1.5466e+17
	x=1		
α	L=1	L=2	L=3
0.1	0.7503	0.8371	0.8415
1	0.9187	0.8674	0.8407
10	-1.9849e+23	1.6048e+22	2.9270e+21

Table: Approximated probabilities using formula (17). L denotes the truncation bound in the numerical integration. Exact values for $x = 0$ and $x = 1$ are 0.5 and 0.8413, respectively

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Carr-Madan formula

Notation and assumptions:

- Suppose that $S_t = \exp(X_t)$ and that $\mathbb{E}[S_T^{\alpha+1}] < \infty$ with $\alpha > 0$
- Let $\phi := \phi_{X_T}$ be the characteristic function of X_T (under the risk neutral measure)
- Let $k = \log K$
- Define $C(k)$ as the price at time 0 of a call option with expiration T and strike $K = \mathrm{e}^k$, and let

$$C_\alpha(k) := \exp(\alpha k) C(k)$$

A common simplification

- The call option's price $C(k)$ can be written as

$$\begin{aligned} C(k) &= \mathrm{e}^{-rT}\mathbb{E}[(S_T - K)_+] \\ &= \mathrm{e}^{-rT} S_0 \mathbb{E}[(S_T/S_0 - K/S_0)_+] \\ &= S_0 \mathrm{e}^{-rT}\mathbb{E}\left[(\mathrm{e}^{X_T - X_0} - \mathrm{e}^{k - X_0})_+\right] \\ &= S_0 \widetilde{C}(k - X_0) \end{aligned}$$

  where $\widetilde{C}(k - X_0)$ **is the price of a call option with log strike** $k - X_0$ **and maturity** T **in a model with log returns** $\widetilde{X} = (X_t - X_0)_{0\leq t\leq T}$
- $\widetilde{X}$ has the same dynamics as X and

$$\phi_{\widetilde{X}_T}(\nu) = \mathrm{e}^{-i\nu X_0}\phi_X(\nu)$$

- The fact that $\widetilde{X}_0 = 0$ simplifies a lot of the formulas that will be discussed later. This explains why often for the implementations one calculates $\widetilde{C}(k - X_0)$
- In our presentation, to simplify notation, we keep working with the original log returns X

Carr-Madan formula (cont.)

Theorem 4

The characteristic function of C_α is well-defined on the real line and has the form

$$\widehat{C_\alpha}(\nu) = \mathrm{e}^{-rT} \frac{\phi(\nu - i(\alpha+1))}{\alpha^2 + \alpha - \nu^2 + i\nu(2\alpha+1)} \tag{21}$$

Moreover, we have the following representation of the call price

$$C(k) = \frac{\mathrm{e}^{-rT-\alpha k}}{\pi} \int_0^\infty Re\left(\frac{\phi(\nu - i(\alpha+1))}{\alpha^2 + \alpha - \nu^2 + i\nu(2\alpha+1)} \mathrm{e}^{-i\nu k} \right) d\nu \tag{22}$$

Carr-Madan formula (cont.)

Sketch of the proof: Let μ be the distribution of X_T. Then

$$\begin{aligned} C_\alpha(k) &= \mathrm{e}^{\alpha k} C(k) \\ &= \mathrm{e}^{-rT+\alpha k} \mathbb{E}[(S_T - K)_+] \\ &= \mathrm{e}^{-rT+\alpha k} \mathbb{E}[(\mathrm{e}^{X_T} - \mathrm{e}^{k})_+] \\ &= \mathrm{e}^{-rT} \int_k^\infty (\mathrm{e}^{x+\alpha k} - \mathrm{e}^{(\alpha+1)k})\, \mu(dx) \end{aligned}$$

Carr-Madan formula (cont.)

Sketch of the proof (cont.): We have then

$$\begin{aligned}
\widehat{C_\alpha}(\nu) &= \int_{\mathbb{R}} C_\alpha(k) \mathrm{e}^{ik\nu}\, dk \\
&= \mathrm{e}^{-rT} \int_{\mathbb{R}} \int_k^\infty (\mathrm{e}^{x+\alpha k} - \mathrm{e}^{(\alpha+1)k}) \mathrm{e}^{ik\nu}\, \mu(dx)\, dk \\
&= \mathrm{e}^{-rT} \int_{\mathbb{R}} \mu(dx) \int_{-\infty}^x (\mathrm{e}^{x+\alpha k} - \mathrm{e}^{(\alpha+1)k}) \mathrm{e}^{ik\nu}\, dk \\
&= \mathrm{e}^{-rT} \int_{\mathbb{R}} \left(\frac{\mathrm{e}^{ix(\nu - i(\alpha+1))}}{\alpha + i\nu} - \frac{\mathrm{e}^{ix(\nu - i(\alpha+1))}}{(\alpha+1) + i\nu} \right) \mu(dx) \\
&= \mathrm{e}^{-rT} \phi(\nu - i(\alpha+1)) \left(\frac{1}{\alpha + i\nu} - \frac{1}{(\alpha+1) + i\nu} \right) \\
&= \mathrm{e}^{-rT} \frac{\phi(\nu - i(\alpha+1))}{\alpha^2 + \alpha - \nu^2 + i\nu(2\alpha+1)}
\end{aligned}$$

Carr-Madan (cont.)

Remarks:

- Once again this computation has the advantage that it involves only one integration
- It is possible to write the call price in terms of tails probabilities. Formula (22) can be derived in a similar manner as (17)
- Alternatively, one can use Plancherel's theorem to derive this formula

Example: The choice of α

- As before the behavior of the integrand in (22) depends on the choice of α
- For the Heston model with parameters $(\kappa, \theta, \sigma, \rho)$, we have that

$$\begin{aligned}\phi(u) &= \mathbb{E}[\exp(iu\ln(S_t))] \\ &= \frac{\exp\left(iu\ln S_0 + iurt + \frac{\kappa\theta t(\kappa - i\rho\sigma u)}{\sigma^2}\right)}{\left(\cosh\frac{\gamma t}{2} + \frac{\kappa - i\rho\sigma u}{\gamma}\sinh\frac{\gamma t}{2}\right)^{\frac{2\kappa\theta}{\sigma^2}}} \exp\left(\frac{-(u^2+iu)V_0}{\gamma\coth\frac{\gamma t}{2} + \kappa - i\rho\sigma u}\right)\end{aligned} \tag{23}$$

where $\gamma = \sqrt{\sigma^2(u^2+iu) + (\kappa - i\rho\sigma u)^2}$, r is the risk free rate and, S_0 and V_0 are the initial values of the price process and the volatility process

Example: The choice of α - The Integrand

Let

$$\psi(\nu) = \frac{e^{-rT-\alpha k}}{\pi} Re\left(\frac{\phi(\nu - i(\alpha+1))}{\alpha^2+\alpha-\nu^2+i\nu(2\alpha+1)} e^{-i\nu k}\right)$$

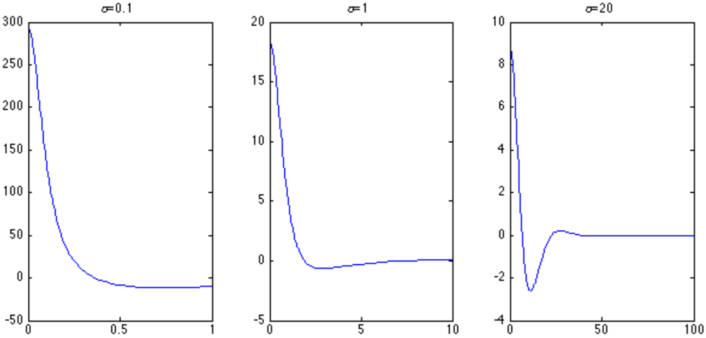

Figure: Plots of the integrand ψ for different values of α. Here $\kappa = 2$, $\theta = V_0 = 0.04$, $\sigma = 0.5$, $\rho = -0.7$, $r = 0.03$, $S_0 = 100$, $K = 90$, $T = 0.5$

Example: The choice of α - Prices

We truncate the integral in (22) at a level L

$$C(k) \approx \frac{e^{-rT-\alpha k}}{\pi} \int_0^L Re\left(\frac{\phi(\nu - i(\alpha+1))}{\alpha^2 + \alpha - \nu^2 + i\nu(2\alpha+1)} e^{-i\nu k} \right) d\nu$$

α	L=5	L=10	L=50
0.1	**0.0328**	**0.0282**	**1.4322e-05**
1	0.0341	0.0404	2.1461e-05
20	1.4324	1.1903	0.0010

Table: **Relative error** of call option prices for different values of α and L. The parameters are the same as in the previous figure. **True price** ≈ 13.2023