Computational Finance
FIN-472
Take-Home Exam 1

October 6, 2017

- **Please hand in your solutions on Friday 13.10.2017 at the beginning of the lecture.**
- **For Exercises** $4.a$**,** $4.b$ **and** $4.c$ **do not forget to upload in moodle the Matlab codes. You should submit only one Matlab file for each part.**
- **Additionally, print out the Matlab code for Exercises** $4.a$ **and** $4.b$ **and the plot for Exercise** $4.c$**. These have to be handed in together with the other solutions on Friday 13.10.2017.**

Exercise 1: Black Scholes model (8 points /40)
In the Black Scholes model, the dynamics of the price process $(S_t)_{0\in[0,T]}$ with respect to the real-world measure $\mathbb{P}$ are given by

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t), \tag{1}$$

for parameters $\sigma > 0$ and μ being constant and for a standard Brownian motion W.

a) *Show* that the process

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

is s solution of (1) and compute $\mathbb{E}[S_T]$.

b) Let W be a Brownian motion. *Show* that for any $\lambda \in \mathbb{R}$

$$\mathbb{E}\left[\mathrm{e}^{-\lambda W_T - \frac{\lambda^2}{2}T}\right] = 1$$

c) Suppose that $\sigma \neq 0$. Define

$$\lambda = \frac{\mu - r}{\sigma}$$

and

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathrm{e}^{-\lambda W_T - \frac{\lambda^2}{2}T}.$$

Use Bayes formula to *show* that

$$\mathbb{E}^{\mathbb{Q}}[S_T] = \mathrm{e}^{rT}.$$

Explain why this result is consistent with Girsanov's theorem.

Exercise 2: OU process (12 points /40)
An Ornstein-Uhlenbeck process is defined as solution of the stochastic differential equation

$$dX_t = \kappa(\theta - X_t)\,dt + \lambda dW_t, \quad \text{for } 0 \le t \le T \tag{2}$$

for real parameters $\kappa > 0, \lambda > 0, \theta$ and a standard Brownian motion W. Moreover, denote by x_0 the starting value of the process.

a) *Solve* (2).

b) Using the solution found in a) *show* that

$$\lim_{t\to\infty} \mathbb{E}[X_t] = \theta.$$

c) In this exercise we compute the quantity $\mathbb{E}[X_T^2]$ in two different ways.

- *Compute* $\mathbb{E}[X_T^2]$ by using the solution found in a) and knowing that, if a function f is square-integrable on $[0,t]$, then $\int_0^t f(s)dW_s$ is normally distributed with mean 0 and variance $\int_0^t f^2(s)ds$.
- Define

$$v(t,x) = \mathbb{E}[(X_T^{t,x})^2]$$

where $X_s^{t,x}$ is the solution of

$$\begin{cases} dX_s = \kappa(\theta - X_s)\,ds + \lambda dW_s, & \text{for } t \le s \le T \\ X_t = x. \end{cases}$$

Derive the following partial differential equation (PDE) satisfied by v:

$$\begin{cases} v_t + \mathcal{G}v = 0 \\ v(T,x) = x^2, \end{cases} \tag{3}$$

where $\mathcal{G}$ denotes the generator of the process defined by

$$\mathcal{G}v = \kappa(\theta - x)v_x + \frac{\lambda^2}{2}v_{xx}.$$

Solve the PDE (3) and *derive* the value of $\mathbb{E}[X_T^2]$.

Hint: In order to solve (3)*, use the Ansatz* $v(t,x) = a(T-t) + b(T-t)x + c(T-t)x^2$*, derive the equations for* a, b *and* c *and solve them.*

Exercise 3: Heston model (8 points /40)
In the Heston model, the dynamics of the price process $(S_t)_{0\le t\le T}$ with respect to a risk-neutral measure $\mathbb{Q}$ are given by

$$dS_t = rS_tdt + S_t\rho\sqrt{V_t}dW_t^{(1)} + S_t\sqrt{1-\rho^2}\sqrt{V_t}dW_t^{(2)},$$

where the spot variance v_t (square of the volatility) is modeled by

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^{(1)}.$$

Here, $W^{(1)}$ and $W^{(2)}$ are independent standard Brownian motions, r is a risk-free interest rate, ρ is a correlation parameter and κ, θ, σ are additional model parameters.
Let $U(t,x,y)$ be the price at time t of a European option with payoff function Ψ, when $S_t = x$ and $V_t = y$, i.e. (by risk-neutral pricing rule)

$$U(t,x,y) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\Psi(S_T^{t,x,y})].$$

Derive the pricing PDE satisfied by $U(t,x,y)$.

Exercise 4: Up-and-Out option (12 points /40)
Let $(S_t)_{0\leq t\leq T}$ be a price process. The up-and-out call barrier option with discrete monitoring at two monitoring dates is a path-dependent option whose payoff function at maturity time T is given by

$$\Psi(S) := (S_T - K)^+ \ \mathbb{1}_{\{S_T<b\}}\mathbb{1}_{\{S_{T/2}<b\}} = \begin{cases} (S_T - K)^+ & \text{if } S_t < b, \text{ for } t = \frac{T}{2}, T, \\ 0 & \text{if } S_t \geq b, \text{ for } t = \frac{T}{2} \text{ or } t = T, \end{cases} \tag{4}$$

where K denotes the strike value and $b \in \mathbb{R}$ is the value of the barrier.

a) Consider a multi-period binomial model over the time interval $[0,T]$ with parameters r, d, u, N, with N even, and denote the initial stock price S_0 by s.
Write a Matlab function `BinomialpriceBarrierUODM.m` with input parameters

$$r, d, u, N, T, s, K, b$$

which computes the binomial price at time $t = 0$ of the up-and-out call barrier option with discrete monitoring at two monitoring dates defined in (4).

Hint: Adapt the iteration formula introduced in slide 31 (Week 2) in order to price this path-dependent option.

b) Consider now the Black Scholes model where the $\mathbb{Q}$-dynamics of the price process $(S_t)_{0\leq t\leq T}$ are given by

$$dS_t = rS_t\,dt + \sigma S_t dW_t, \tag{5}$$

for a risk-free interest rate r and volatility σ. Denote the initial price S_0 by s.
Let

$$\{t_k := \frac{Tk}{N_{time}} | k = 0, \cdots, N_{time}\}.$$

be a discretization of $[0,T]$ and let

$$\{(S^i_{t_k})_{k=0,\cdots,N_{time}} | i = 1, \cdots, N_{sim}\}$$

be a set of N_{sim} simulated paths generated by using the discretization scheme

$$S_{t_{k+1}} - S_{t_k} = rS_{t_k}\Delta t + \sigma S_{t_k}\sqrt{\Delta t}Z, \tag{6}$$

where $\Delta t = T/N_{time}$ and $Z \approx N(0,1)$. Then, the Monte Carlo (MC) price at time $t = 0$ of an option maturing at time T with payoff function Ψ is given by

$$P_{MC} := \frac{e^{-rT}}{N_{sim}} \sum_{i=1}^{N_{sim}} \Psi(S^i).$$

Write a Matlab function `MCpriceBarrierUODM.m` with input parameters

$$r, \sigma, N_{time}, N_{sim}, T, s, K, b$$

that computes the MC price P_{MC} of the up-and-out call barrier option with discrete monitoring at two monitoring dates.

c) *Compare* the binomial prices computed in part a) with the MC price computed in b), for $N \to \infty$. In particular, consider the following set of parameters

$$s = 1, \quad r = 0.1, \quad T = 0.5, \quad K = 0.9, \quad \sigma = 0.1, \quad b = 1.3$$

and *plot*

- The constant MC price P_{MC} computed in b) with $N_{time} = 100$ and $N_{sim} = 10^6$,
- The binomial price obtained for $N = 2, 4, \cdots, 200$.

against the values of N. What do you observe?

Remark: For this part, please submit on the moodle the Matlab script/function that generates the needed plot.