Computational Finance
FIN-472
Homework 4 - Solutions

October 13, 2017

Exercise 1: Let X be a one-dimensional random variable with characteristic function ϕ. This variable could be a model for the log prices of a financial asset. Define $S = \exp(X)$. Show that for $\nu, \alpha \in \mathbb{R}$

$$|\phi(\nu - i\alpha)| \leq \mathbb{E}[S^\alpha].$$

In particular, $\phi(\nu - i\alpha)$ is well-defined if S has moments of order α.

Solution: We have

$$\begin{aligned}
|\phi(\nu - i\alpha)| &= |\mathbb{E}[\exp((i\nu + \alpha)X)]| \\
&\leq \mathbb{E}[\exp(\alpha X)] \\
&= \mathbb{E}[S^\alpha].
\end{aligned}$$

Exercise 2: The goal of this exercise is to show that the formula for the cumulative distribution function in terms of the Fourier Transform of the density (see (6) below) can be derived using a change in the contour of integration.

Let X be a random variable with characteristic function ϕ, cumulative distribution function F and probability density function f. Observe that

$$F(x) := Pr(X \leq x) = \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} g_\epsilon(y) f(y)\, dy, \tag{1}$$

where

$$g_\epsilon(y) = \exp(\epsilon(y - x)) 1_{\{y \leq x\}}, \tag{2}$$

with $1_{\{y \leq x\}}$ the indicator function. Assume that $\mathbb{E}[\exp(-\alpha X)] < \infty$ and $\alpha > 0$.

a) Derive a formula for the Fourier Transform of the function $g_\epsilon(y)$ in (2).

Solution: We have

$$\begin{aligned}
\widehat{g}_\epsilon(\nu) &= \int_{-\infty}^{x} \exp(\epsilon(y - x) + i\nu y)\, dy \\
&= \exp(-\epsilon x) \int_{-\infty}^{x} \exp((\epsilon + i\nu) y)\, dy \\
&= \frac{\exp(-\epsilon x + (\epsilon + i\nu) x)}{\epsilon + i\nu} \\
&= \frac{\exp(i\nu x)}{\epsilon + i\nu}.
\end{aligned}$$

b) Assume that the hypotheses of Plancherel's Theorem hold and rewrite (1) as

$$F(x) := Pr(X \leq x) = \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \overline{\widehat{g_\epsilon}(\nu)} \widehat{f}(\nu)\, d\nu. \tag{3}$$

Solution: Assuming that we can use Plancherel's formula, by (1)

$$\begin{aligned} F(x) &= \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} g_\epsilon(y) f(y)\, dy \\ &= \frac{1}{2\pi} \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \overline{\widehat{g_\epsilon}}(\nu) \widehat{f}(\nu)\, d\nu \\ &= \frac{1}{2\pi} \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \frac{\exp(-i\nu x)}{\epsilon - i\nu} \widehat{f}(\nu)\, d\nu. \end{aligned}$$

c) Change the contour of integration in (3) to deduce

$$F(x) := Pr(X \leq x) = \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \overline{\widehat{g_\epsilon}}(\nu + i\alpha) \widehat{f}(\nu + i\alpha)\, d\nu. \tag{4}$$

Why is the hypothesis $\mathbb{E}[\exp(-\alpha X)] < \infty$ important to justify this change in contour of integration?

Solution: Since $\mathbb{E}[\exp(-\alpha X)] < \infty$ the function $\widehat{f}$ is analytic on $\{\nu : 0 < Im(\nu) < \alpha\}$ and continuous on $\{\nu : 0 \leq Im(\nu) \leq \alpha\}$. The same properties hold for $\overline{\widehat{g_\epsilon}}$ for any $\epsilon > 0$. Cauchy's integration theorem implies that for all $R > 0$

$$0 = I_1(R) + I_2(R) + I_3(R) + I_4(R), \tag{5}$$

where

$$\begin{aligned} I_1(R) &= \int_{-R}^{R} \overline{\widehat{g_\epsilon}}(\nu) \widehat{f}(\nu)\, d\nu, \\ I_2(R) &= -\int_{-R}^{R} \overline{\widehat{g_\epsilon}}(\nu + i\alpha) \widehat{f}(\nu + i\alpha)\, d\nu, \\ I_3(R) &= \int_{0}^{\alpha} \overline{\widehat{g_\epsilon}}(R + i\theta) \widehat{f}(R + i\theta)\, d\theta, \\ I_4(R) &= -\int_{0}^{\alpha} \overline{\widehat{g_\epsilon}}(-R + i\theta) \widehat{f}(-R + i\theta)\, d\theta. \end{aligned}$$

By Exercise 1 the integrand in $I_3(R)$ can be bounded by

$$\mathbb{E}[\exp(-\theta(X - x))] \frac{1}{|\epsilon + \theta - iR|} = \mathbb{E}[\exp(-\theta(X - x))] \frac{1}{\sqrt{(\epsilon + \theta)^2 + R^2}}.$$

This quantity tends to 0 as R tends to infinity. We conclude

$$\lim_{R \to \infty} I_3(R) = 0.$$

A similar argument shows that

$$\lim_{R \to \infty} I_4(R) = 0.$$

	x=1		
α	L=1	L=10	L=50
0.1	0.4995	0.6338	0.6322
1	0.3618	0.6374	0.6322
10	56.8464	32.7307	0.3363

Table 1: Approximated probabilities using formula (6) for an exp(1) distribution. L denotes the truncation bound in the numerical integration. The exact value for $x = 1$ is 0.6321.

Hence by taking limit as $R \to \infty$ in (5) we obtain

$$\int_{-\infty}^{\infty} \overline{\widehat{g}_\epsilon}(\nu)\widehat{f}(\nu)\, d\nu = \int_{-\infty}^{\infty} \overline{\widehat{g}_\epsilon}(\nu + i\alpha)\widehat{f}(\nu + i\alpha)\, d\nu,$$

and (4) follows.

d) Take the limit in (4) and conclude that

$$F(x) = \frac{1}{\pi}\int_0^{\infty} Re\left(\frac{\widehat{f}(\nu + i\alpha)}{\alpha - i\nu}\exp(-ix(\nu + i\alpha))\right) d\nu. \tag{6}$$

Note: Here you have to recall the properties of the Fourier Transform for real valued functions.

Solution: We have

$$\begin{aligned} F(x) &= \frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \frac{\exp(-i(\nu + i\alpha)x)}{\epsilon - i(\nu + i\alpha)}\widehat{f}(\nu + i\alpha)\, d\nu \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\widehat{f}(\nu + i\alpha)}{\alpha - i\nu}\exp(-i(\nu + i\alpha)x))\, d\nu. \end{aligned}$$

After recalling that $F(x)$ is a real number, formula (6) follows from the following observation: the real part of the integrand above is an even function of ν. To see this one can use a similar argument as the one used to prove Proposition 1 in the lecture notes.

e) Test this formula when X has an exponential distribution with rate $\lambda = 1$ for $x = 1$, $\alpha = 0.1, 1, 10$ and different truncation limits $L = 1, 10, 50$. You can use the numerical integration routines provided in Matlab.

Solution: In this case we have

$$F(x) = \frac{1}{\pi}\int_0^{\infty} Re\left(\frac{1}{(1 + \alpha - i\nu)(\alpha - i\nu)}\exp(-ix(\nu + i\alpha))\right) d\nu.$$

Table 1 shows the results. For details see Matlab file hereunder.

```
function P=Tail_Probability(alpha,x,L)

% Uses formula (17) in Lecture notes to compute
% tail probabilities Pr(X\leq x) for a random variable X¬exp(1)
% alpha = damping factor
```

```
% L = Truncation bound in the integral

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Implementation of the integrand in (17)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
f = @(y)(1./(1-j*y));% Characteristic function
g = @(y)(real(f(y+j*alpha).*exp(-j*y.*x)./(alpha-j*y))); % Integrand ...
    in (17)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Integral evaluation
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
P=((exp(alpha*x)/pi)*integral(@(y)(g(y)),0,L));

end
```

Exercise 3: Using Heston model with the following parameters

$$S_0 = 100, \qquad V_0 = 0.0175, \qquad \rho = -0.6, \qquad r = 0,$$
$$\kappa = 1.5, \qquad \theta = 0.04, \qquad \sigma = 0.3$$

a) Compute the price of the European call options with maturities

$$T \in \{1/12, 1/6, 1/4, 1/2, 1, 2, 3, 4\}$$

and strikes

$$K \in \{50, 80, 90, 95, 100, 105, 110, 120, 150\},$$

using adequate numerical integration functions among the ones provided by Matlab.

Note: You can use the formula for the characteristic function of the log price given in the lecture.

b) Do the same exercise, but this time replace the previous integrand function, say ϕ_H, by $\phi_H - \phi_{BS}$, where ϕ_{BS} is the corresponding integrand function for the Black-Scholes model and then add the Black-Scholes price to the value of the integral. Does it change the results obtained in a)? *Note:* For the Black-Scholes formula take $\sigma = \sqrt{\theta}$.

Solution for a) and b): Using $\alpha = 0.01$ and the truncation level $L = 50$ one obtains the results presented in Table 2. For the parameter σ in the BS model we take $\sqrt{\theta}$. A possible implementation of the option pricer can be found hereunder.

```
function [P Q] = ...
    Call_Price_Heston_mod(S,K,T,r,kappa,theta,sigma,rho,V,alpha,L)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% P= Price of a call with Maturity T and Strike K using the ...
    characteristic function of the
%  price and the Carr-Madan formula (see (22) in Lecture notes) - ...
    No FFT used
%  (Exercise 3 a)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Q = Price with alternative method subtracting Black Scholes prices
%  (Exercise 3 b)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% S= Initial price
% r= risk free rate
% kappa,theta,sigma,rho = parameters Heston
% V= initial vol in Heston model
% alpha = damping factor (alpha >0)
% L = truncation bound for the integral
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Heston characteristic function as in (23) in Lecture Notes
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

b=@(nu)(kappa-1i*rho*sigma.*nu);
gamma=@(nu)(sqrt(sigma^2*(nu.^2+1i.*nu)+b(nu).^2));
a=@(nu)(b(nu)./gamma(nu)).*sinh(T*0.5.*gamma(nu));
c=@(nu)(gamma(nu).*coth(0.5*T.*gamma(nu))+b(nu));
d=@(nu)(kappa*theta*T.*b(nu)/sigma^2);

f=@(nu)(1i*(log(S)+r*T).*nu+d(nu));
g=@(nu)(cosh(T* 0.5.*gamma(nu))+a(nu)).^(2*kappa*theta/sigma^2);
h=@(nu)(-(nu.^2+1i.*nu)*V./c(nu));

phi=@(nu)(exp(f(nu)).*exp(h(nu))./g(nu));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Black-Scholes characteristic function
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

sigmaBS=sqrt(theta); % Vol used in BS model

% charateristic funtion
phiBS=@(nu)(exp((1i*(log(S)+(r-0.5*sigmaBS^2)*T)*nu)-...
    0.5*(sigmaBS^2)*T*(nu.^2)));

%%%%%%%%%%%%%%%%%%%%%%%%%
%Integrands
%%%%%%%%%%%%%%%%%%%%%%%%%
integrand=@(nu)(real((phi(nu-1i*(alpha+1))./...
    (alpha^2+alpha-nu.^2+1i*(2*alpha+1)*nu)).*exp(-1i*log(K).*nu))); ...
       % Original integrand in (22)
integrandBS=@(nu)(real((phiBS(nu-1i*(alpha+1))./...
    (alpha^2+alpha-nu.^2+1i*(2*alpha+1)*nu)).*exp(-1i*log(K).*nu)));
integrandBS=@(nu)(integrand(nu)-integrandBS(nu)); % Integrand ...
   Subtracting BS integrand

%%%%%%%%%%%%%%%%%%%%%%%
% Pricing formula
%%%%%%%%%%%%%%%%%%%%%%%
P=(exp(-r*T-alpha*log(K))/pi)*integral(integrand,0, L);% Price with (22)

Diff=(exp(-r*T-alpha*log(K))/pi)*integral(integrandBS,0, L);
Call=blsprice(S, K, r, T, sigmaBS, 0);
Q=Call+Diff; % Price with alternative method
end
```

c) Redo a) and b) with different damping factors $\alpha \in \{0.01, 0.5, 1, 1.5, 2, 5, 10\}$ and different truncation limits $L = 10, 25, 50, 100$ for the numerical integration. Are the results

K	Methods	T							
		1/12	1/6	1/4	1/2	1	2	3	4
50	ϕ_H	50.0018	50.0012	50.0006	50.0003	50.0183	50.1814	50.4634	50.8092
	$\phi_H - \phi_{BS}$	50.0018	50.0012	50.0006	50.0003	50.0183	50.1814	50.4634	50.8092
80	ϕ_H	20.0086	20.0066	20.0325	20.2687	21.0713	22.871	24.5422	26.0552
	$\phi_H - \phi_{BS}$	20.008	20.0066	20.0325	20.2687	21.0713	22.871	24.5422	26.0552
90	ϕ_H	10.0358	10.174	10.4011	11.2087	12.8022	15.5172	17.7414	19.6444
	$\phi_H - \phi_{BS}$	10.0351	10.174	10.4011	11.2087	12.8022	15.5172	17.7414	19.6444
95	ϕ_H	5.2363	5.70314	6.14385	7.32165	9.28805	12.3693	14.8017	16.8504
	$\phi_H - \phi_{BS}$	5.23715	5.70314	6.14385	7.32165	9.28805	12.3693	14.8017	16.8504
100	ϕ_H	1.57131	2.24418	2.80208	4.17165	6.33063	9.62058	12.186	14.3349
	$\phi_H - \phi_{BS}$	1.57041	2.24417	2.80208	4.17165	6.33063	9.62058	12.186	14.3349
105	ϕ_H	0.121837	0.477942	0.854297	1.97869	4.00982	7.28874	9.89654	12.0944
	$\phi_H - \phi_{BS}$	0.122568	0.477948	0.854297	1.97869	4.00982	7.28874	9.89654	12.0944
110	ϕ_H	0.00399647	0.0391579	0.154807	0.761317	2.34397	5.37316	7.92703	10.1205
	$\phi_H - \phi_{BS}$	0.00371532	0.0391572	0.154807	0.761317	2.34397	5.37316	7.92703	10.1205
120	ϕ_H	-0.013686	-0.00354475	0.0023677	0.0718768	0.640098	2.69031	4.88156	6.91829
	$\phi_H - \phi_{BS}$	-0.0129997	-0.00353755	0.00236777	0.0718768	0.640098	2.69031	4.88156	6.91829
150	ϕ_H	-0.00356424	0.000714532	0.00107198	0.000167906	0.00699855	0.209023	0.862132	1.86686
	$\phi_H - \phi_{BS}$	-0.00324074	0.000717027	0.001072	0.000167906	0.00699855	0.209023	0.862132	1.86686

Table 2: European call option prices for $\alpha = 0.01$ and $L = 50$.

dependent on the value chosen for α or L?

Solution: The prices corresponding to the various values of α and $L = 25$ are presented in Tables 3, 4, 5, 6, 7, 8 and 9.

K	Methods	T							
		1/12	1/6	1/4	1/2	1	2	3	4
50	ϕ_H	50.0452	50.0284	50.0181	50.0043	50.0174	50.1814	50.4634	50.8092
	$\phi_H - \phi_{BS}$	50.02	50.0198	50.0152	50.0042	50.0174	50.1814	50.4634	50.8092
80	ϕ_H	20.1037	20.0602	20.0608	20.2689	21.0686	22.871	24.5422	26.0552
	$\phi_H - \phi_{BS}$	20.0413	20.0414	20.0554	20.2688	21.0686	22.871	24.5422	26.0552
90	ϕ_H	9.84778	10.0874	10.3594	11.2085	12.8055	15.5172	17.7414	19.6444
	$\phi_H - \phi_{BS}$	9.93891	10.1101	10.3656	11.2087	12.8055	15.5172	17.7414	19.6444
95	ϕ_H	5.15363	5.62597	6.07866	7.29414	9.28667	12.3694	14.8017	16.8504
	$\phi_H - \phi_{BS}$	5.16194	5.62665	6.07883	7.29415	9.28667	12.3694	14.8017	16.8504
100	ϕ_H	1.84091	2.36966	2.85979	4.16787	6.32621	9.62066	12.186	14.3349
	$\phi_H - \phi_{BS}$	1.73507	2.34564	2.8535	4.16772	6.32621	9.62066	12.186	14.3349
105	ϕ_H	0.181017	0.565654	0.935245	2.01139	4.0098	7.28866	9.89654	12.0944
	$\phi_H - \phi_{BS}$	0.163992	0.55838	0.932788	2.0113	4.0098	7.28866	9.89654	12.0944
110	ϕ_H	-0.23232	-0.0725335	0.108904	0.7745	2.34888	5.37303	7.92703	10.1205
	$\phi_H - \phi_{BS}$	-0.143508	-0.0529642	0.113726	0.774598	2.34888	5.37303	7.92703	10.1205
120	ϕ_H	0.104143	0.0383804	0.00338739	0.0473246	0.636857	2.69044	4.88156	6.91829
	$\phi_H - \phi_{BS}$	0.0630475	0.0321074	0.00260733	0.0473527	0.636857	2.69044	4.88156	6.91829
150	ϕ_H	0.0228252	0.0013249	-0.0100293	-0.0148092	0.00489141	0.209139	0.862128	1.86686
	$\phi_H - \phi_{BS}$	0.0128183	0.00101929	-0.00930125	-0.0147334	0.00489155	0.209139	0.862128	1.86686

Table 3: European call option prices for $\alpha = 0.01$ and $L = 25$.

K	Methods	T							
		1/12	1/6	1/4	1/2	1	2	3	4
50	ϕ_H	50.0637	50.0397	50.0251	50.0051	50.0167	50.1814	50.4634	50.8092
	$\phi_H - \phi_{BS}$	50.028	50.0276	50.0209	50.005	50.0167	50.1814	50.4634	50.8092
80	ϕ_H	20.1123	20.0637	20.0615	20.267	21.0679	22.8711	24.5422	26.0552
	$\phi_H - \phi_{BS}$	20.0447	20.0433	20.0555	20.2668	21.0679	22.8711	24.5422	26.0552
90	ϕ_H	9.84239	10.086	10.3603	11.2108	12.8062	15.5171	17.7414	19.6444
	$\phi_H - \phi_{BS}$	9.9369	10.1098	10.3668	11.211	12.8062	15.5171	17.7414	19.6444
95	ϕ_H	5.1378	5.61542	6.07156	7.29243	9.28709	12.3694	14.8017	16.8504
	$\phi_H - \phi_{BS}$	5.1549	5.61924	6.07285	7.29249	9.28709	12.3694	14.8017	16.8504
100	ϕ_H	1.83829	2.36578	2.85547	4.16478	6.3258	9.62069	12.186	14.3349
	$\phi_H - \phi_{BS}$	1.73353	2.34234	2.84945	4.16465	6.3258	9.62069	12.186	14.3349
105	ϕ_H	0.194524	0.57334	0.939359	2.01112	4.00914	7.28867	9.89654	12.0944
	$\phi_H - \phi_{BS}$	0.169769	0.563432	0.936009	2.011	4.00914	7.28867	9.89654	12.0944
110	ϕ_H	-0.220039	-0.063272	0.116011	0.777255	2.34879	5.37301	7.92704	10.1205
	$\phi_H - \phi_{BS}$	-0.137855	-0.0461582	0.119929	0.777308	2.34879	5.37301	7.92704	10.1205
120	ϕ_H	0.0898789	0.0297897	-0.00165935	0.0469813	0.637442	2.69044	4.88155	6.91829
	$\phi_H - \phi_{BS}$	0.0568634	0.026312	-0.00147672	0.0470476	0.637442	2.69044	4.88155	6.91829
150	ϕ_H	0.0154478	-0.00176134	-0.0106303	-0.0131979	0.00549715	0.209114	0.862128	1.86686
	$\phi_H - \phi_{BS}$	0.009861	-0.000654947	-0.00945818	-0.0131102	0.00549729	0.209114	0.862128	1.86686

Table 4: European call option prices for $\alpha = 0.5$ and $L = 25$.

K	Methods	T							
		1/12	1/6	1/4	1/2	1	2	3	4
50	ϕ_H	50.0899	50.0558	50.0346	50.0058	50.0156	50.1814	50.4634	50.8092
	$\phi_H - \phi_{BS}$	50.0396	50.0386	50.0288	50.0056	50.0156	50.1814	50.4634	50.8092
80	ϕ_H	20.1211	20.067	20.0615	20.2644	21.0671	22.8711	24.5422	26.0552
	$\phi_H - \phi_{BS}$	20.0485	20.0451	20.0551	20.2642	21.0671	22.8711	24.5422	26.0552
90	ϕ_H	9.83751	10.0853	10.3619	11.2137	12.8069	15.5171	17.7414	19.6444
	$\phi_H - \phi_{BS}$	9.93492	10.1098	10.3685	11.2139	12.8069	15.5171	17.7414	19.6444
95	ϕ_H	5.12108	5.60434	6.0642	7.29086	9.28765	12.3694	14.8017	16.8504
	$\phi_H - \phi_{BS}$	5.14748	5.61145	6.06664	7.29097	9.28765	12.3694	14.8017	16.8504
100	ϕ_H	1.8347	2.36109	2.85048	4.16143	6.32545	9.62072	12.186	14.3349
	$\phi_H - \phi_{BS}$	1.73177	2.33861	2.84491	4.16133	6.32545	9.62072	12.186	14.3349
105	ϕ_H	0.207567	0.580596	0.943057	2.01053	4.00843	7.2887	9.89654	12.0944
	$\phi_H - \phi_{BS}$	0.175456	0.568269	0.938925	2.01039	4.00843	7.2887	9.89654	12.0944
110	ϕ_H	-0.20755	-0.0538691	0.123164	0.779889	2.34861	5.37299	7.92704	10.1205
	$\phi_H - \phi_{BS}$	-0.132225	-0.0393424	0.126121	0.779897	2.34861	5.37299	7.92704	10.1205
120	ϕ_H	0.0768063	0.0220263	-0.00608792	0.0469325	0.638061	2.69042	4.88155	6.91829
	$\phi_H - \phi_{BS}$	0.0511272	0.0210277	-0.00507895	0.0470264	0.638062	2.69042	4.88155	6.91829
150	ϕ_H	0.00985449	-0.00384364	-0.0107011	-0.0116042	0.00601747	0.209088	0.862129	1.86686
	$\phi_H - \phi_{BS}$	0.00750168	-0.00181163	-0.00928228	-0.0115167	0.00601756	0.209088	0.862129	1.86686

Table 5: European call option prices for $\alpha = 1$ and $L = 25$.

K	Methods	T							
		1/12	1/6	1/4	1/2	1	2	3	4
50	ϕ_H	50.1264	50.0778	50.0473	50.006	50.0137	50.1815	50.4634	50.8092
	$\phi_H - \phi_{BS}$	50.0558	50.0538	50.0393	50.0057	50.0137	50.1815	50.4634	50.8092
80	ϕ_H	20.1298	20.0696	20.0607	20.2611	21.0661	22.8712	24.5422	26.0552
	$\phi_H - \phi_{BS}$	20.0524	20.0466	20.0542	20.2609	21.0661	22.8712	24.5422	26.0552
90	ϕ_H	9.83338	10.0855	10.3644	11.2171	12.8076	15.517	17.7414	19.6444
	$\phi_H - \phi_{BS}$	9.93304	10.1102	10.3708	11.2172	12.8076	15.517	17.7414	19.6444
95	ϕ_H	5.10384	5.59298	6.05677	7.28953	9.28833	12.3694	14.8017	16.8504
	$\phi_H - \phi_{BS}$	5.13981	5.60345	6.06036	7.28968	9.28833	12.3694	14.8017	16.8504
100	ϕ_H	1.83021	2.35567	2.84493	4.1579	6.32516	9.62076	12.186	14.3349
	$\phi_H - \phi_{BS}$	1.72981	2.33452	2.83998	4.15784	6.32516	9.62076	12.186	14.3349
105	ϕ_H	0.219841	0.587237	0.946232	2.00962	4.00767	7.28873	9.89654	12.0944
	$\phi_H - \phi_{BS}$	0.180937	0.572787	0.941464	2.00947	4.00767	7.28873	9.89654	12.0944
110	ϕ_H	-0.19517	-0.0445714	0.130173	0.782322	2.34833	5.37299	7.92704	10.1205
	$\phi_H - \phi_{BS}$	-0.126735	-0.0326668	0.132156	0.782286	2.34833	5.37299	7.92704	10.1205
120	ϕ_H	0.065127	0.0152097	-0.00983697	0.0471584	0.638691	2.6904	4.88155	6.91829
	$\phi_H - \phi_{BS}$	0.0459196	0.0163241	-0.00815907	0.047268	0.638691	2.6904	4.88155	6.91829
150	ϕ_H	0.00577308	-0.00511846	-0.0103893	-0.010097	0.00644559	0.209063	0.86213	1.86686
	$\phi_H - \phi_{BS}$	0.00566725	-0.00255122	-0.00887735	-0.0100184	0.00644563	0.209063	0.86213	1.86686

Table 6: European call option prices for $\alpha = 1.5$ and $L = 25$.

K	Methods	T							
		1/12	1/6	1/4	1/2	1	2	3	4
50	ϕ_H	50.1771	50.1078	50.0641	50.005	50.0108	50.1816	50.4634	50.8092
	$\phi_H - \phi_{BS}$	50.0788	50.075	50.0535	50.0047	50.0108	50.1816	50.4634	50.8092
80	ϕ_H	20.1384	20.0715	20.0589	20.2568	21.0651	22.8712	24.5422	26.0552
	$\phi_H - \phi_{BS}$	20.0566	20.0478	20.0524	20.2567	21.0651	22.8712	24.5422	26.0552
90	ϕ_H	9.83009	10.0866	10.3677	11.221	12.8083	15.517	17.7414	19.6444
	$\phi_H - \phi_{BS}$	9.93127	10.1111	10.3738	11.2211	12.8083	15.517	17.7414	19.6444
95	ϕ_H	5.08612	5.58141	6.04933	7.28847	9.28916	12.3694	14.8017	16.8504
	$\phi_H - \phi_{BS}$	5.13188	5.59524	6.05404	7.28865	9.28916	12.3694	14.8017	16.8504
100	ϕ_H	1.82484	2.34953	2.83884	4.15422	6.32497	9.62079	12.186	14.3349
	$\phi_H - \phi_{BS}$	1.72765	2.33006	2.83465	4.15421	6.32497	9.62079	12.186	14.3349
105	ϕ_H	0.231331	0.593246	0.948868	2.00839	4.00688	7.28877	9.89654	12.0944
	$\phi_H - \phi_{BS}$	0.186217	0.576985	0.943621	2.00824	4.00688	7.28877	9.89654	12.0944
110	ϕ_H	-0.182959	-0.0354315	0.136995	0.784534	2.34796	5.37299	7.92704	10.1205
	$\phi_H - \phi_{BS}$	-0.121384	-0.0261402	0.138019	0.78446	2.34796	5.37299	7.92704	10.1205
120	ϕ_H	0.0547376	0.00927193	-0.0129574	0.0476286	0.639318	2.69037	4.88156	6.91829
	$\phi_H - \phi_{BS}$	0.0411937	0.0121501	-0.0107633	0.0477423	0.639318	2.69037	4.88156	6.91829
150	ϕ_H	0.00284998	-0.00579994	-0.00982363	-0.00869945	0.00678959	0.209039	0.862131	1.86686
	$\phi_H - \phi_{BS}$	0.0042457	-0.00298294	-0.00832697	-0.00863482	0.00678958	0.209039	0.862131	1.86686

Table 7: European call option prices for $\alpha = 2$ and $L = 25$.

K	Methods	T							
		1/12	1/6	1/4	1/2	1	2	3	4
50	ϕ_H	51.2307	50.6577	50.2882	49.8356	49.9265	50.1881	50.4629	50.8092
	$\phi_H - \phi_{BS}$	50.6191	50.5197	50.274	49.8392	49.9265	50.1881	50.4629	50.8092
80	ϕ_H	20.1711	20.0494	20.0092	20.2033	21.0594	22.8718	24.5422	26.0552
	$\phi_H - \phi_{BS}$	20.0861	20.0393	20.0122	20.2038	21.0594	22.8718	24.5422	26.0552
90	ϕ_H	9.83512	10.1189	10.4118	11.2572	12.8103	15.5167	17.7414	19.6444
	$\phi_H - \phi_{BS}$	9.92428	10.1301	10.4106	11.2568	12.8104	15.5167	17.7414	19.6444
95	ϕ_H	4.97322	5.511	6.00828	7.29103	9.29692	12.3689	14.8017	16.8504
	$\phi_H - \phi_{BS}$	5.0787	5.54227	6.01708	7.29104	9.29692	12.3689	14.8017	16.8504
100	ϕ_H	1.77529	2.29902	2.79239	4.13094	6.32666	9.62078	12.1859	14.3349
	$\phi_H - \phi_{BS}$	1.71045	2.29551	2.79481	4.13114	6.32666	9.62078	12.1859	14.3349
105	ϕ_H	0.283542	0.61565	0.953162	1.9947	4.00243	7.28911	9.89652	12.0944
	$\phi_H - \phi_{BS}$	0.213896	0.595512	0.948441	1.99477	4.00243	7.28911	9.89652	12.0944
110	ϕ_H	-0.115872	0.0138251	0.172196	0.792432	2.34409	5.3732	7.92703	10.1205
	$\phi_H - \phi_{BS}$	-0.092125	0.00954388	0.168901	0.792341	2.34409	5.3732	7.92703	10.1205
120	ϕ_H	0.0140732	-0.0116295	-0.0213355	0.0539023	0.642459	2.69014	4.88157	6.91829
	$\phi_H - \phi_{BS}$	0.0207948	-0.00417888	-0.0186406	0.0538975	0.642459	2.69014	4.88157	6.91829
150	ϕ_H	-0.00237647	-0.0044827	-0.00510195	-0.00292517	0.00759229	0.208964	0.862137	1.86686
	$\phi_H - \phi_{BS}$	0.000523212	-0.0026443	-0.00445978	-0.00293568	0.00759231	0.208964	0.862137	1.86686

Table 8: European call option prices for $\alpha = 5$ and $L = 25$.

K	Methods	T							
		1/12	1/6	1/4	1/2	1	2	3	4
50	ϕ_H	70.3536	51.567	40.0187	33.0441	50.6423	49.6875	50.5598	50.797
	$\phi_H - \phi_{BS}$	68.5692	58.3118	44.0861	32.8682	50.6427	49.6875	50.5598	50.797
80	ϕ_H	19.9028	19.607	19.5212	19.9271	21.1322	22.8602	24.5435	26.0551
	$\phi_H - \phi_{BS}$	20.12	19.7856	19.5876	19.924	21.1322	22.8602	24.5435	26.0551
90	ϕ_H	9.9984	10.3262	10.6185	11.3545	12.7786	15.5211	17.7409	19.6444
	$\phi_H - \phi_{BS}$	9.9443	10.2619	10.5945	11.3555	12.7786	15.5211	17.7409	19.6444
95	ϕ_H	4.79114	5.42486	5.98935	7.35171	9.309	12.3689	14.8015	16.8504
	$\phi_H - \phi_{BS}$	4.96554	5.44952	5.98425	7.3516	9.30899	12.3689	14.8015	16.8504
100	ϕ_H	1.64434	2.1804	2.69593	4.10749	6.34381	9.61898	12.1861	14.3349
	$\phi_H - \phi_{BS}$	1.66436	2.20795	2.70423	4.10709	6.34381	9.61898	12.1861	14.3349
105	ϕ_H	0.312979	0.606111	0.922088	1.95753	4.00373	7.28844	9.89664	12.0943
	$\phi_H - \phi_{BS}$	0.246378	0.601384	0.925973	1.95745	4.00373	7.28844	9.89664	12.0943
110	ϕ_H	-0.0376031	0.0661567	0.202944	0.785264	2.33663	5.37371	7.92702	10.1205
	$\phi_H - \phi_{BS}$	-0.0533161	0.0544609	0.20092	0.785391	2.33663	5.37371	7.92702	10.1205
120	ϕ_H	-0.0059066	-0.0153854	-0.0152385	0.0669266	0.643172	2.69025	4.88154	6.91829
	$\phi_H - \phi_{BS}$	0.00522198	-0.0118154	-0.0154735	0.0669211	0.643171	2.69025	4.88154	6.91829
150	ϕ_H	-0.000864149	-0.00102618	-0.00094049	-0.000119254	0.00722352	0.209022	0.862129	1.86686
	$\phi_H - \phi_{BS}$	-0.000131116	-0.000818587	-0.000986606	-0.000117368	0.00722351	0.209022	0.862129	1.86686

Table 9: European call option prices for $\alpha = 10$ and $L = 25$.