# Computational Finance
# FIN-472
# Homework 4

October 13, 2017

**Exercise 1:** Let $X$ be a one-dimensional random variable with characteristic function $\phi$. This variable could be a model for the log prices of a financial asset. Define $S = \exp(X)$. Show that for $\nu, \alpha \in \mathbb{R}$

$$|\phi(\nu - i\alpha)| \leq \mathbb{E}[S^\alpha].$$

In particular, $\phi(\nu - i\alpha)$ is well-defined if $S$ has moments of order $\alpha$.

**Exercise 2:** The goal of this exercise is to show that the formula for the cumulative distribution function in terms of the Fourier Transform of the density (see (5) below) can be derived using a change in the contour of integration.

Let $X$ be a random variable with characteristic function $\phi$, cumulative distribution function $F$ and probability density function $f$. Observe that

$$F(x) := Pr(X \leq x) = \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} g_\epsilon(y) f(y)\, dy, \tag{1}$$

where

$$g_\epsilon(y) = \exp(\epsilon(y - x)) 1_{\{y \leq x\}}, \tag{2}$$

with $1_{\{y \leq x\}}$ the indicator function. Assume that $\mathbb{E}[\exp(-\alpha X)] < \infty$ and $\alpha > 0$.

a) Derive a formula for the Fourier Transform of the function $g_\epsilon(y)$ in (2).

b) Assume that the hypotheses of Plancherel's Theorem hold and rewrite (1) as

$$F(x) := Pr(X \leq x) = \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \overline{\widehat{g_\epsilon}(\nu)} \widehat{f}(\nu)\, d\nu. \tag{3}$$

c) Change the contour of integration in (3) to deduce

$$F(x) := Pr(X \leq x) = \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \overline{\widehat{g_\epsilon}}(\nu + i\alpha) \widehat{f}(\nu + i\alpha)\, d\nu. \tag{4}$$

Why is the hypothesis $\mathbb{E}[\exp(-\alpha X)] < \infty$ important to justify this change in contour of integration?

d) Take the limit in (4) and conclude that

$$F(x) = \frac{1}{\pi}\int_0^\infty Re\left(\frac{\widehat{f}(\nu + i\alpha)}{\alpha - i\nu}\exp(-ix(\nu + i\alpha))\right)d\nu. \tag{5}$$

*Note:* Here you have to recall the properties of the Fourier Transform for real valued functions.

e) Test this formula when $X$ has an exponential distribution with rate $\lambda = 1$ for $x = 1$, $\alpha = 0.1, 1, 10$ and different truncation limits $L = 1, 10, 50$. You can use the numerical integration routines provided in Matlab.

**Exercise 3:** Using Heston model with the following parameters

$$S_0 = 100, \quad V_0 = 0.0175, \quad \rho = -0.6, \quad r = 0,$$
$$\kappa = 1.5, \quad \theta = 0.04, \quad \sigma = 0.3$$

a) Compute the price of the European call options with maturities

$$T \in \{1/12, 1/6, 1/4, 1/2, 1, 2, 3, 4\}$$

and strikes

$$K \in \{50, 80, 90, 95, 100, 105, 110, 120, 150\},$$

using adequate numerical integration functions among the ones provided by Matlab.

*Note:* You can use the formula for the characteristic function of the log price given in the lecture.

b) Do the same exercise, but this time replace the previous integrand function, say $\phi_H$, by $\phi_H - \phi_{BS}$, where $\phi_{BS}$ is the corresponding integrand function for the Black-Scholes model and then add the Black-Scholes price to the value of the integral. Does it change the results obtained in a)? *Note:* For the Black-Scholes formula take $\sigma = \sqrt{\theta}$.

c) Redo a) and b) with different damping factors $\alpha \in \{0.01, 0.5, 1, 1.5, 2, 5, 10\}$ and different truncation limits $L = 10, 25, 50, 100$ for the numerical integration. Are the results dependent on the value chosen for $\alpha$ or $L$?