

# APPM 4100 - HW 10 - Cambria C.

1)  $f(x) = \sin(x)$

$$(a) P_3^3(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2 + b_3 x^3}$$

$$T_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + 0x^2 + 0x^4 + 0x^6 + 0$$

for  $\sin(x) \in$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = (1 + b_1 x + b_2 x^2 + b_3 x^3) \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} \right)$$

constant	$a_0 = 0$	$b_3 = \frac{b_1}{3!}$
$x^2$	$q_1 = 1$	
$x^3$	$q_2 = b_1$	$b_2 = \frac{b_1}{120}$
	$q_3 = -\frac{1}{3!} + b_2$	$b_3 = \frac{-b_1}{5!}$
$x^4$	$a_4 = 0 = b_3 + \frac{b_1}{3!}$	$5! b_3 = 3! b_1$
$x^5$	$a_5 = 0 = \frac{b_2}{3!} + \frac{1}{120}$	$5! \frac{b_1}{3!} = 3! b_1$

only 100% off up to 6	$x^6$	$q_6 = 0 = \frac{b_1}{5!} - \frac{b_3}{3!}$	$b_1 = 0$
	$x^7$	$a_7 = \frac{b_2}{5!} = 0$	$b_2 = 0$
	$x^8$	$a_8 = \frac{b_3}{5!} = 0$	$b_3 = -\frac{1}{6} + \frac{6}{120}$

$$P(x) = \frac{-\frac{7}{60}x^3 + x}{1 + \frac{1}{20}x^2}$$

$$(b) P_2^4(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4} = T_0(x)$$

$$a_0 + a_1 x + a_2 x^2 = (1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4) \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} \right)$$

constant

$$a_0 = 0$$

$$b_2 = \frac{1}{6}$$

$$x^2$$

$$a_1 = 1$$

$$b_3 = b_1 / 6$$

$$x^3$$

$$a_2 = b_1$$

$$b_4 = \frac{1}{36} - \frac{1}{120}$$

$$x^4$$

$$a_3 = 0 = b_2 - \frac{1}{3!}$$

$$= \frac{1}{720}$$

$$x^5$$

$$a_4 = 0 = -\frac{b_1}{3!} + b_3$$

$$\frac{b_4}{3!} = \frac{b_1}{5!}$$

$$x^6$$

$$a_5 = 0 = b_4 + \frac{1}{5!} - \frac{b_2}{3!}$$

$$\frac{b_1}{3!} = \frac{b_1}{120}$$

$$x^7$$

$$a_6 = 0 = -\frac{b_3}{3!} + b_1$$

$$a_2 = 0$$

$$x^8$$

$$a_7 = 0 = \frac{b_2}{5!} - \frac{b_4}{3!}$$

$$a_1 = 1 \quad a_0 = 0$$

$$x^9$$

$$a_8 = 0 = \frac{b_3}{5!}$$

$$b_3 = 0$$

$$x^{10}$$

$$a_9 = 0 = \frac{b_4}{5!}$$

$$b_4 = 0$$

only look  
at up to  
 $x^0$

$$P(x) = \frac{x}{1 + \frac{1}{6}x^2 + \frac{7}{360}x^4}$$

$$(C) P_4^2(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}{1 + b_1 x + b_2 x^2}$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 = (1 + b_1 x + b_2 x^2)(x - \frac{x^3}{6} + \frac{x^5}{120})$$

<u>constant</u>	$a_0 = 0$	$b_2 = 1/120$
$x$	$a_1 = 1$	$b_1 = 0$
$x^2$	$a_2 = b_1$	$a_2 = 0$
$x^3$	$a_3 = -\frac{1}{6} + b_2$	$a_3 = 1$
$x^4$	$a_4 = -b_1/6$	$a_0 = 0$
$x^5$	$a_5 = 0 = 1/120 - b_2/6$	$a_3 = -7/120$
$x^6$	$a_6 = 0 = b_1/120$	$a_4 = 0$
$x^7$	$a_7 = 0 = b_2/120$	

only consider up to  $x^6$

$$P(x) = x - \frac{\frac{7}{60}x^3}{1 + \frac{1}{20}x^2}$$

$$2) \int_0^1 f(x) dx = \frac{1}{2} f(x_0) + c_1 f(x_1)$$

$$\text{By trapezoidal rule } E(h) = -\frac{(b-a)^3}{12} f''(u)$$

since there are 3 unknowns, we would expect the rule to be exact for at least  $1, x, x^2$ . This yields

$$f(x) = \int_0^1 1 dx = 1 = \frac{1}{2} + c_1 \Rightarrow c_1 = \frac{1}{2}$$

$$f(x) = x \quad \int_0^1 x dx = \frac{1}{2} = \frac{1}{2} x_0 + \frac{1}{2} x_1$$

$$f(x) = x^2 \quad \int_0^1 x^2 dx = \frac{1}{3} x^3 \Big|_0^1 = \frac{1}{3} = \frac{1}{2} x_0^2 + \frac{1}{2} x_1^2$$

thus we have the system

$$1 = x_0 + x_1$$

$$\frac{2}{3} = x_0^2 + x_1^2$$

$$\frac{2}{3} = x_0^2 + (1-x_0)^2 = x_0^2 + 1 - 2x_0 + x_0^2$$

$$\frac{2}{3} = 2x_0^2 - 2x_0 + 1$$

$$0 = 2x_0^2 - 2x_0 + \frac{1}{3}$$

in interval  $[0, 1]$   $2 \pm \frac{\sqrt{4 - (4)(\frac{1}{3})}}{4} = 0.788, 0.21132$

$$\frac{1}{2} + \frac{\sqrt{3}}{6}$$

$$x_0 = 0.788 \quad x_1 = 0.21132 \quad c_1 = \frac{1}{2}$$

$$\int_0^1 x^3 dx = \frac{1}{4} = \frac{1}{2}x_0^3 + c_1 x_1^3$$

plugging in  $x_0, x_1 + c_1$ , we get an exact equality, so the degree of precision is 3 or greater

3) (a) see attached code in git

$$(b) \text{Trap error} = \frac{(b-a)}{n} \left| \frac{f''(u)}{12} \right|$$

$$b-a = 5 - (-5) = 10 \quad [-5, 5]$$

$$f = \frac{1}{1+s^2} \quad f' = -\frac{(1+s^2)^{-2}}{2s} (2s)$$

$$f'' = \frac{(2s)(2(1+s^2)^{-3}(2s)) + (-1+s^2)^{-2}(2)}{(1+s^2)^3}$$

$$f''' = \frac{8s^2}{(1+s^2)^3} - \frac{2}{(1+s^2)^2}$$

$$\left| \frac{1000}{12n^2} f'''(u) \right| < 10^{-4}$$

maximized  
in absolute  
value

this happens at  $u=0 \in [-5, 5]$

$$|f''(0)| = \left| -\frac{2}{1} \right| = 2$$

thus  $\left| \frac{1000}{12n^2} \cdot 2 \right| < 10^{-4}$

$$10^4 < \frac{12n^2}{2000}$$

$$n > 1290.99 \quad n = 1291$$

Simpson's error =  $\frac{b-a}{180} h^4 f^{(4)}(u)$

$$\left| \frac{10^5}{180n^4} f^{(4)}(u) \right| < 10^{-4}$$

$$f^{(3)} = \frac{(1+s^2)^3 (16s) - 8s^2 (3(1+s^2)^2 (2s))}{(1+s^2)^6} + \frac{4(1+s^2)(2s)}{(1+s^2)^3}$$

$$f^{(4)} = \frac{(1+s^2)^6 \left[ (1+s^2)^3 (16) + (16s) (3(1+s^2)^2 (2s)) - 48s^3 (1+s^2)^2 (2) (2s) - (1+s^2)^2 (144s^2) \right]}{(1+s^2)^12} - \frac{\left[ (1+s^2)^3 (16s) - 48s^3 (1+s^2)^2 \right] [6(1+s^2)^5 (2s)]}{(1+s^2)^12}$$

$$+ \frac{(1+s^2)^6 (8) - 8s (3(1+s^2)^2 (2s))}{(1+s^2)^16}$$

$$= \frac{16(1+s^2)^9 + 96s^2(1+s^2)^8 - 192s^4(1+s^2)^7 - 144s^2(1+s^2)^6 - 192s^2(1+s^2)^8 + 576s^4(1+s^2)^7}{(1+s^2)^{125}}$$

$$= \frac{16(1+s^2)^2 - 240s^2(1+s^2) + 384s^4 + 8(1+s^2) - 48s^2}{(1+s^2)^5}$$

max value is at 0 w/ value of 24

$$\left| \frac{10^5}{180n^4} \cdot 24 \right| < 10^{-4}$$

$$10^4 < \frac{180n^4}{24 \times 10^5}$$

$$107,456 < n \quad n = 108$$

(c)	Scipy	-	2.74680153	(no tolerance specified)
	TRAP	-	2.74680138	
	SIMP	-	2.74680152	
	SCIPY - $10^{-6}$	-	2.74680153389	w/neval = 147
	SCIPY - $10^{-4}$	-	2.74680153390	w/neval = 163

The values of n for SCIPY are significantly less than what calculated for n in part b