APPM 4600 - #W4 - Cambria Chaney

1) $\frac{T(x,t) - T_S}{T_i - T_S} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$

T_S = constant temp = -15

T_i = initial soil temp = 20

α = thermal conductivity = 0.138×10^{-6}

$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-s^2)\,ds$

t = seconds x = m

(a) only freeze after 60 days = 5,184,000 s
root finding problem $f(x) = 0$
If $f(x_1, t) = 0$ then x_1 is the depth of the soil where temperature is 0, at time t

$$T(x, 60) = \operatorname{erf}\left(\frac{x}{2\sqrt{0.138 \times 10^{-6} \times 5{,}184{,}000\,s}}\right) \cdot (20 + 15) - 15$$

$$T(x, 60\text{ days}) = \operatorname{erf}\left(\frac{x}{1.6916}\right)(35) - 15$$

$$f(x) = \operatorname{erf}\left(\frac{x}{1.6916}\right)(35) - 15$$

$$f'(x) = 35\left[\frac{d}{dx} \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{1.6916}} e^{-s^2}\,ds\right]$$

$$f'(x) = 35\left[\frac{2}{\sqrt{\pi}} \left(e^{-\left(\frac{x}{1.6916}\right)^2}\right)\left(\frac{1}{1.6916}\right)\right]$$

$$f'(x) = 23.346\, e^{-\frac{1}{2.8615}x^2}$$

See plots attached

(b) root = 0.67695

(c) root = 0.6769 @ $x_0 = 0.01$
root = 0.6769 @ $x_0 = 1 = \bar{x}$ $f(1) > 0$

2) $f(x)$ - root α of multiplicity m

(a) A solution α of $f(x)=0$ is a zero/root of multiplicity m if for $x \neq \alpha$ we can write:

$$f(x) = (x-\alpha)^m q(x)$$

where $\lim_{x \to \alpha} q(x) \neq 0$

(b) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ show $|f'(p)| < 1$

$$x_{n+1} = x_n - \frac{(x-\alpha)^m q(x)}{(x-\alpha)^m q'(x) + q(x)(m)(x-\alpha)^{m-1}}$$

$$g(x) = x_{n+1} = x_n - \frac{(x_n-\alpha)q(x_n)}{(x_n-\alpha)q'(x_n) + mq(x_n)}$$

$$g'(x) = 1 - \frac{\left[[(x_n-\alpha)q'(x) + mq(x)][(x_n-\alpha)q'(x) + q(x)] - [(x_n-\alpha)q(x)][(x-\alpha)q''(x) + q'(x) + mq'(x)] \right]}{[(x_n-\alpha)q'(x_n) + mq(x)]^2}$$

$$g'(\alpha) = 1 - \frac{[mq(\alpha)][q(\alpha)] - 0}{m^2 q(\alpha)^2}$$

$$g'(\alpha) = 1 - \frac{1}{m}$$ Thus $|g'(\alpha)| < 1$ if $m > 1$ - thus newton converges linearly.

(c) $g(x) = x - m\frac{f(x)}{f'(x)}$ is 2nd order convergent

$$g'(x) = 1 - m\left[\frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2}\right]$$

plug α $\quad (f(x) = (x-\alpha)^m q(x))$

$$g'(\alpha) = 1 - m\left[\frac{1}{m}\right]$$

from part b

$$g'(\alpha) = 0$$

$$g''(x) = -m\left[\frac{[(x_n-\alpha)q''(x) + q'(x) + mq''(x)][(x_n-\alpha)q'(x) + q(x)] + [(x_n-\alpha)q''(x) + 2q'(x)][(x_n-\alpha)q'(x) + mq(x)] - [[(x-\alpha)q(x)][(x-\alpha)q'''(x) + q''(x) + q''(x) + mq''(x)] + [(x-\alpha)q''(x) + q'(x) + mq'(x)][(x-\alpha)q'(x) + q(x)]]}{[(x_n-\alpha)q'(x) + mq(x)]^2}\right.$$

$$\left. - \frac{[mq(\alpha)^2][2[(x_n-\alpha)q'(x) + mq(x)][(x_n-\alpha)q''(x) + q'(x) + mq'(x)]]}{low^2}\right]$$

$$g''(\alpha) = -m\left[\frac{[q'(\alpha) + mq''(\alpha)][q(\alpha)] + [2q'(\alpha)][mq(\alpha)] - [0 + (q'(\alpha) + mq'(\alpha))(q(\alpha))]}{[m^2 q(\alpha)^2]}\right.$$

$$\left. - \frac{[mq(\alpha)^2][2[mq(\alpha)][q'(\alpha) + mq'(\alpha)]]}{m^4 q(\alpha)^4}\right]$$

$$= -m\left[\cancel{q(\alpha)q'(\alpha)} + mq''(\alpha)q(\alpha) + 2mq'(\alpha)q(\alpha) - \cancel{q'(\alpha)q(\alpha)} + mq'(\alpha)q(\alpha)\right]$$

$$= -m\left[3m^3 q'(\alpha)q(\alpha)^3 + m^3 q''(\alpha)q(\alpha)^3 - 2m^2 q(\alpha)^3 q'(\alpha) - 2m^3 q(\alpha)^3 q'(\alpha)\right]$$

num $= -m\left[m^3 q(\alpha)^3 q'(\alpha) + m^3 q''(\alpha)q(\alpha)^3 + 2m^2 q(\alpha)^3 q'(\alpha)\right]$

$$g''(\alpha) = -\left\{\frac{m^4 q(\alpha)^3 q'(\alpha) + m^4 q''(\alpha)\, q(\alpha)^3 + 2m^3 q(\alpha)^3 q'(\alpha)}{m^4\, q(\alpha)^4}\right\}$$

$$q''(\alpha) = -\left[\frac{m q'(\alpha) + m q''(\alpha) + 2q'(\alpha)}{m q(\alpha)}\right] \neq 0$$

thus second order convergent

(d) If $m > 1$, Newton's method still converges just second order, if you add the multiplicity constant m to the iteration

3) $\{x_k\}_{k=1}^{\infty}$ converges to α

$$\lim_{n \to \infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^p} = \lambda \quad \text{for positive } \lambda \,\&\, \alpha$$

$$\lim_{n \to \infty} \log(|x_{n+1} - \alpha|) - \log(|x_n - \alpha|^p) = \log(\lambda)$$

$$\lim_{n \to \infty} \log(|x_{n+1} - \alpha|) - p\log(|x_n - \alpha|) = \log(\lambda)$$

$$\log(|x_{n+1} - \alpha|) = p\log(|x_n - \alpha|) + \log(\lambda)$$

let $y = \log(|x_{n+1} - \alpha|)$ & $x = \log(|x_n - \alpha|)$ and $b = \log(\lambda)$.

Thus $y = px + b$. $\log(|x_{n+1} - \alpha|)$ & $\log(|x_n - \alpha|)$ have a linear relationship where the order p is the slope of this linear relationship.

$f(x) = (e^x - 3x^2)^3$

4) $f(x) = e^{3x} - 27x^6 + 27x^4e^x - 9x^2e^{2x}$ $\in(3,5)$

$f'(x) = 3e^{3x} - 27\cdot 6x^5 + 27\cdot 4x^3e^x + 27x^4e^x - 9x^2(2e^{2x}) - 18xe^{2x}$

Newton's method: $p_0 = 4$

root: 3.733, 52 iterations

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ order of convergence: 2 → not exactly 1 due to the 50 iterations

$g'(x) = \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}$

① Modified Newtons: $g(x) = \frac{f(x)}{f'(x)}$

$\left(\frac{f}{f'}, \left(\frac{f}{f'}\right)', N_{max}, tol\right)$ root: 3.733, 5 iterations

order of convergence: 2

② Modified Newton's: $g(x) = x - m\frac{f(x)}{f'(x)}$ ↳ becomes iteration

zero has multiplicity 3

root: 3.733 4 iterations

order of convergence: 2 ✓

I prefer the original newtons method because it is less calculation heavy than the ① modified newton and also doesn't require the multiplicity like the ② modified Newton. It is hard to know the multiplicity of a complicated function so that would be my last choice

5) Newton: root = 1.1347, $x_0 = 2$
8 iterations

Secant: root = 1.1347, 9 iterations
$x_0 = 2$ $x_1 = 1$

Slope of secant log graph: 1.6237

slope of newton log graph: 1.9922

This relates to the order because Newton's method has an order of convergence 2 and secant has an order of convergence between 1 and 2.

See plots below.