APPM 4600 - HW 10 - Cambria C.

1) $f(x) = \sin(x)$

(a) $P_3^3(x) = \dfrac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2 + b_3 x^3}$

$T_6(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + 0x^2 + 0x^4 + 0x^6 + 0$

for sin(x)

$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = (1 + b_1 x + b_2 x^2 + b_3 x^3)\left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}\right)$

constant	$a_0 = 0$	$b_3 = \dfrac{b_1}{3!}$
x	$a_1 = 1$	
x^2	$a_2 = b_1$	$b_2 = 6/120$
x^3	$a_3 = -\dfrac{1}{3!} + b_2$	$\dfrac{b_3}{3!} = \dfrac{b_1}{5!}$
x^4	$a_4 = 0 = b_3 + b_1/3!$	$5!\,b_3 = 3!\,b_1$
x^5	$a_5 = 0 = \dfrac{b_2}{3!} + \dfrac{1}{120}$	$5!\,\dfrac{b_1}{3!} = 3!\,b_1$
x^6	$a_6 = 0 = \dfrac{b_1}{5!} - \dfrac{b_3}{3!}$	$b_1 = 0$
x^7	$a_7 = b_2/5! = 0$	$b_3 = 0$
x^8	$a_8 = b_3/5! = 0$	$a_2 = 0$
		$a_1 = 1$
		$a_3 = -\dfrac{1}{6} + \dfrac{6}{120} = -7/60$

only look at, up to 6 →

$p(x) = \dfrac{-\frac{7}{60}x^3 + x}{1 + \frac{1}{20}x^2}$

(b) $P_2^4(x) = \dfrac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4} = T_6(x)$

$a_0 + a_1 x + a_2 x^2 = (1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)$

constant	$a_0 = 0$	$b_2 = \frac{1}{6}$
x	$a_1 = 1$	$b_3 = b_1/6$
x^2	$a_2 = b_1$	$b_4 = \frac{1}{36} - \frac{1}{120}$
x^3	$a_3 = 0 = b_2 - \frac{1}{3!}$	$= 7/360$
x^4	$a_4 = 0 = -\frac{b_1}{3!} + b_3$	$\frac{b_3}{3!} = \frac{b_1}{5!}$
x^5	$a_5 = 0 = b_4 + \frac{1}{5!} - \frac{b_2}{3!}$	$\frac{b_1}{36} = \frac{b_1}{120}$ $b_1 = 0 = b_3$
x^6	$a_6 = 0 = -\frac{b_3}{3!} + \frac{b_1}{5!}$	$a_2 = 0$ $a_1 = 1$ $a_0 = 0$
x^7	$a_7 = 0 = \frac{b_2}{5!} - \frac{b_4}{3!}$	
x^8	$a_8 = 0 = \frac{b_3}{5!}$	$b_3 = 0$
x^9	$a_9 = 0 = \frac{b_4}{5!}$	$b_4 = 0$

only look at up to 6

$$P(x) = \frac{x}{1 + \frac{1}{6}x^2 + \frac{7}{360}x^4}$$

(c) $P_4^2(x) = \dfrac{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4}{1 + b_1x + b_2x^2}$

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 = (1 + b_1x + b_2x^2)\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)$$

constant	$a_0 = 0$	$b_2 = 1/20$
x	$a_1 = 1$	$b_1 = 0$
x^2	$a_2 = b_1$	$a_2 = 0$
x^3	$a_3 = -\frac{1}{6} + b_2$	$a_1 = 1$
x^4	$a_4 = -b_1/6$	$a_0 = 0$
x^5	$a_5 = 0 = 1/120 - b_2/6$	$a_3 = -7/60$
x^6	$a_6 = 0 = b_1/120$	$a_4 = 0$
x^7	$a_7 = 0 = b_2/120$	

only consider up to x^6

$$p(x) = \frac{x - \frac{7}{60}x^3}{1 + \frac{1}{20}x^2}$$

2) $\int_0^1 f(x)\,dx = \frac{1}{2}f(x_0) + c_1 f(x_1)$

By trapezoidal rule $E(h) = \frac{(b-a)^3}{12} f''(u)$

Since there are 3 unknowns, we would expect the rule to be exact for at least $1, x, x^2$. This yields

$f(x)=1$ $\quad \int_0^1 1\,dx = 1 = \frac{1}{2} + c_1 \Rightarrow c_1 = \frac{1}{2}$

$f(x)=x$ $\quad \int_0^1 x\,dx = 1/2 = \frac{1}{2}x_0 + \frac{1}{2}x_1$

$f(x)=x^2$ $\quad \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3} = \frac{1}{2}x_0^2 + \frac{1}{2}x_1^2$

Thus we have the system

$1 = x_0 + x_1$

$\frac{2}{3} = x_0^2 + x_1^2$

$\frac{2}{3} = x_0^2 + (1-x_0)^2 = x_0^2 + 1 - 2x_0 + x_0^2$

$\frac{2}{3} = 2x_0^2 - 2x_0 + 1$

$0 = 2x_0^2 - 2x_0 + \frac{1}{3}$

in interval [0,1] $\quad \frac{2 \pm \sqrt{4-(4)(2/3)}}{4} = 0.788, 0.21132$

$\frac{1}{2} \pm \frac{\sqrt{3}}{6}$ $\quad \boxed{x_0 = 0.788 \quad x_1 = 0.21132 \quad c_1 = \frac{1}{2}}$

$\int_0^1 x^3 dx = \frac{1}{4} = \frac{1}{2}x_0^3 + c_1 x_1^3$

plugging in x_0, x_1 & c_1, we get an exact equality, so the degree of precision is 3 or greater

3) (a) see attached code in git

(b) Trap error $= \frac{(b-a)\left(\frac{b-a}{n}\right)^2}{12} f''(u)$

$b-a = 5-(-5) = 10 \qquad [-5,5]$

$f = \frac{1}{1+s^2} \quad f' = -(1+s^2)^{-2}(2s)$

$f'' = (2s)(2(1+s^2)^{-3}(2s)) + (-(1+s^2)^{-2}(2))$

$f'' = \frac{8s^2}{(1+s^2)^3} - \frac{2}{(1+s^2)^2}$ } want this to be maximized in absolute value

$\left| \frac{1000}{12n^2} f''(u) \right| < 10^{-4}$

this happens at $u = 0 \in [-5,5]$

$$|f''(0)| = \left|\frac{-2}{1}\right| = 2$$

thus $\left|\frac{1000}{12n^2} \cdot 2\right| < 10^{-4}$

$$10^4 < \frac{12n^2}{2000}$$

$$n > 1290.99 \qquad n = 1291$$

Simpson's error $= \frac{b-a}{180} h^4 f^{(4)}(u)$

$$\left|\frac{10^5}{180 n^4} f^{(4)}(u)\right| < 10^{-4}$$

$$f^{(3)} = \frac{(1+s^2)^3(16s) - 8s^2(3(1+s^2)^2\overbrace{(2s)}^{48s^3(1+s^2)^2})}{(1+s^2)^6} + \underbrace{4(1+s^2)^{-3}(2s)}_{\frac{8s}{(1+s^2)^3}}$$

$$f^{(4)} = \frac{(1+s^2)^6\left[(1+s^2)^3(16) + (16s)(3(1+s^2)^2(2s)) - 48s^3(1+s^2)(2)(2s) - (1+s^2)^2(144s^2)\right] - \left[(1+s^2)^3(16s) - 48s^3(1+s^2)^2\right]\left[\underbrace{6(1+s^2)^5(2s)}_{12s(1+s^2)^5}\right]}{(1+s^2)^{12}}$$

$$+ \frac{(1+s^2)^{\cancel{3}}(8) - 8s(3\cancel{(1+s^2)^2}(2s))}{(1+s^2)^{\cancel{6}\,4}}$$

$$= \frac{16(1+s^2)^{\cancel{9}\,2} + 96s^2(1+s^2)^{\cancel{8}} - 192s^4\cancel{(1+s^2)^7} - 144s^2(1+s^2)^{\cancel{8}} - 192s^2(1+s^2)^{\cancel{8}} + 576s^4\cancel{(1+s^2)^7}}{(1+s^2)^{\cancel{12}\,5}}$$

$$= \frac{16(1+s^2)^2 - 240s^2(1+s^2) + 384s^4}{(1+s^2)^5} + \frac{8(1+s^2) - 48s^2}{(1+s^2)^4}$$

max value is at 0 w/ value of 24

$$\left|\frac{10^5}{180n^4}\cdot 24\right| < 10^{-4}$$

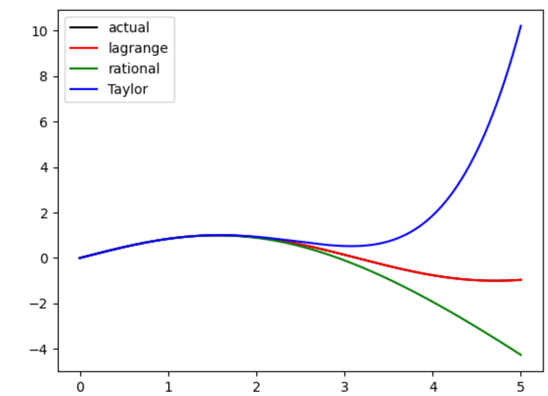
$$10^4 < \frac{180n^4}{24\times 10^5}$$

$107.456 < n$ $\quad n = 108$

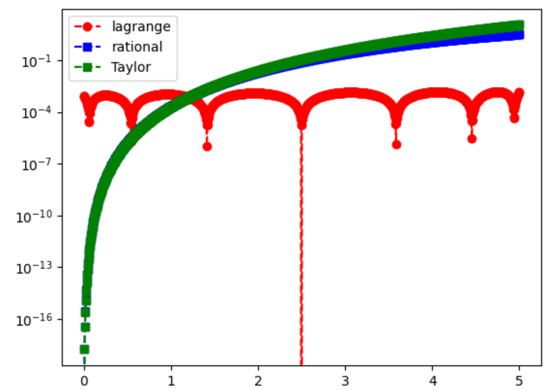
(c) Scipy - 2.746801 53 (no tolerance specified)
Trap - 2.74680138
Simp - 2.74680152
Scipy - 10^{-6} - 2.7468015 3389 w/ neval = 147
Scipy - 10^{-4} - 2.7468015 3390 w/ neval = 63

The values of n for scipy are significantly less than what I calculated for n in part b

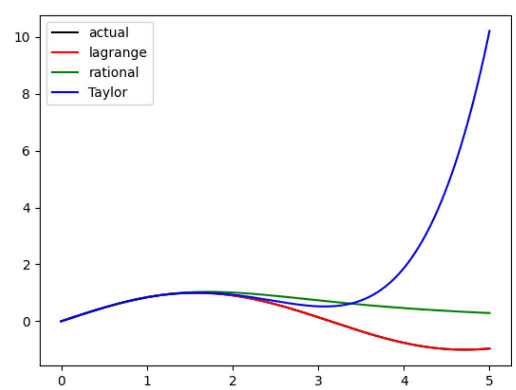
1a) Pade Plot



Pade Error



1b) Pade Plot



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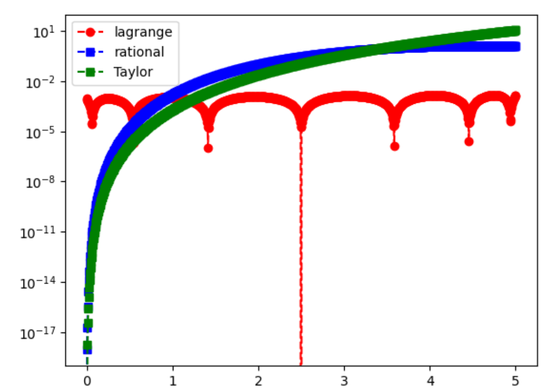
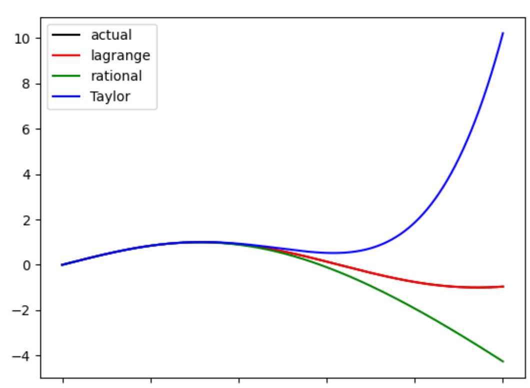


1c) Pade Plot



Pade Error