APPM 4600 - Homework 9 - cambria c.

1) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\quad M^T M \bar{a} = M^T b$

overdetermined system

$$M^T M \bar{a} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ 2v \end{bmatrix}$$

$$M^T \bar{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u \\ 2v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{matrix} u = 1 \\ 2v = 1 \end{matrix} \quad \boxed{\begin{matrix} u = 1 \\ v = 1/2 \end{matrix}}$$

2) $\begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 + 1 \\ b_2 + 2 \\ b_3 + 3 \\ b_4 + 4 \end{bmatrix}$

$$\min E^2 = b_1^2 + 4b_2^2 + 25b_3^2 + 9b_4^2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 12 & -2 \\ 20 & 0 \\ 6 & 21 \end{bmatrix}$$

set $b_1 = b_2 = b_3 = b_4 = 0$ to get into the form $Ax = b$ where

$$b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 15 \\ 12 \end{bmatrix}$$

our system: $\begin{bmatrix} 1 & 3 \\ 12 & -2 \\ 20 & 0 \\ 6 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 15 \\ 12 \end{bmatrix}$

$M^T M \bar{a} = M^T b$

$$M^T M \bar{a} = \begin{bmatrix} 1 & 12 & 20 & 6 \\ 3 & -2 & 0 & 21 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 12 & -2 \\ 20 & 0 \\ 6 & 21 \end{bmatrix} = \begin{bmatrix} 581 & 105 \\ 105 & 454 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$M^T \bar{b} = \begin{bmatrix} 1 & 12 & 20 & 6 \\ 3 & -2 & 0 & 21 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 15 \\ 12 \end{bmatrix} = \begin{bmatrix} 1177 \\ 247 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -454/252749 & -15/36107 \\ -15/36107 & 83/36107 \end{bmatrix} \begin{bmatrix} 1177 \\ 247 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2.011 \\ 0.0788 \end{bmatrix} = \begin{bmatrix} 508423/252749 \\ 2846/36107 \end{bmatrix}$$

3) If $f=0$ in $[a,b]$ then $f^{(n)}=0$ in $[a,b]$ $\forall n$

(a) $\{1, x, x^2, \dots, x^n\}$ are linearly independent

$$c_0 \cdot 1 + c_1 x + c_2 x^2 + \dots + c_n x^n = 0$$

$f_0' = 0$ $f_1' = 1$ $f_2' = 2x$ ... $f_n' = nx^{n-1}$

$f_0'' = 0$ $f_0'' = 0$ $f_2'' = 2$ ... $f_n'' = n(n-1)x^{n-2}$

$f_n^{(m)} = n!\, x^{n-m}$

Wronskian:

$$\begin{bmatrix} 1 & x & x^2 & \cdots & x^n \\ 0 & 1 & 2x & & nx^{n-1} \\ \vdots & 0 & 2 & \ddots & \vdots \\ 0 & \vdots & 0 & \ddots & \vdots \\ 0 & & 0 & 0 & n! \end{bmatrix}$$

determinant of an upper diagonal matrix is the product of the diagonal entries

All diagonal entries of this Wronskian are non-zero, so its determinant is non-zero, meaning the set of functions is linearly independent

2n+1 functions → 2n deriv.

(b) $\{1, \cos(x), \cos(2x), \ldots \cos(nx), \sin(x), \ldots \sin(nx)\}$

$\cos(x)$:
$f' = -\sin(nx)\cdot n$
$f''(x) = -\cos nx \cdot n^2$
$f'''(x) = \sin nx \cdot n^3$
$f^{(4)}(x) = \cos nx \cdot n^4$

$\sin(x)$:
$f' = \cos(nx)\cdot n$
$f'' = -\sin(nx)\cdot n^2$
$f''' = -\cos(nx)\cdot n^3$
$f^{(4)} = \sin(nx)\cdot n^4$

want to show they are orthogonal, which means they are linearly independent

$$\int_0^{2\pi} \sin(mt)\sin(nt)\,dt = \begin{cases} 0, & n \neq m \\ \pi, & n = m \end{cases}$$

from Fourier

$$\int_0^{2\pi} \sin(mt)\cos(nt)\,dt = 0$$

$$\int_0^{2\pi} \cos(mt)\cos(nt)\,dt = \begin{cases} 0, & n \neq m \\ \pi, & n = m \end{cases}$$

$$\int_0^{2\pi} 1\cdot\cos(mt) = \int_0^{2\pi} 1\cdot\sin(nt) = 0$$

Thus every combination of functions on the periodic interval $[0, 2\pi]$ is orthogonal. Thus, the set of functions is linearly independent

4) $\phi_K(x) = (x - b_K)\,\phi_{K-1}(x) - c_K\,\phi_{K-2}(x)$

where $b_K = \dfrac{\langle x\phi_{K-1}, \phi_{K-1}\rangle}{\langle \phi_{K-1}, \phi_{K-1}\rangle}$

$c_K = \dfrac{\langle x\phi_{K-1}, \phi_{K-2}\rangle}{\langle \phi_{K-2}, \phi_{K-2}\rangle}$

$\phi_K = x^K +$ (lower order terms)

$$\phi_K(x) = (x - b_K)\phi_{K-1}(x) - c_K\,\phi_{K-2}(x) - \{a_{K-3}\,\phi_{K-3}(x) + a_{K-4}\,\phi_{K-4}(x) + \ldots + a_0\phi_0(x)\}$$

To make the original expression true, we'd need to show that $a_j = 0$ $j \leq K-3$
To do this, form the scalar product
$\langle \phi_j(x), \phi_K(x)\rangle = 0$ for $j \neq K$

$$\langle (x - b_K)\,\phi_{K-1} - c_K\,\phi_{K-2}, \phi_j\rangle$$
$$= \langle (x - b_K)\phi_{K-1}, \phi_j\rangle - c_K\,\langle \phi_{K-2}, \phi_j\rangle$$

when $j \neq K-1$ and $j \neq K-2$, we get that the inner product is 0 since $\{\phi_j\}_{j=0}^{n}$ is an orthogonal set of functions.

Thus, all a_j for $j \leq K-3$ are 0

If $j = k-1$ we get

from orthogonality

$$0 = \langle (x-b_k)\phi_{k-1}, \phi_{k-1}\rangle - \cancelto{0}{\langle c_k \phi_{k-2}, \phi_{k-1}\rangle}$$

$$0 = \langle x\phi_{k-1} - b_k\phi_{k-1}, \phi_{k-1}\rangle$$

$$0 = \langle x\phi_{k-1}, \phi_{k-1}\rangle - b_k\langle \phi_{k-1}, \phi_{k-1}\rangle$$

$$b_k = \frac{\langle x\phi_{k-1}, \phi_{k-1}\rangle}{\langle \phi_{k-1}, \phi_{k-1}\rangle}$$

If $j = k-2$ we get

$$0 = \langle (x-b_k)\phi_{k-1}, \phi_{k-2}\rangle - \langle c_k\phi_{k-2}, \phi_{k-2}\rangle$$

orthogonality

$$= \langle x\phi_{k-1}, \phi_{k-2}\rangle - \cancelto{0}{b_k\langle \phi_{k-1}, \phi_{k-2}\rangle} - c_k\langle \phi_{k-2}, \phi_{k-2}\rangle$$

$$0 = \langle x\phi_{k-1}, \phi_{k-2}\rangle - c_k\langle \phi_{k-2}, \phi_{k-2}\rangle$$

$$c_k = \frac{\langle x\phi_{k-1}, \phi_{k-2}\rangle}{\langle \phi_{k-2}, \phi_{k-2}\rangle}$$

5)

$T_n(x) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$

$x = \frac{1}{2}\left(z + \frac{1}{z}\right)$

normal way: $T_n(x) = \cos(n \cos^{-1}(x))$

3 term recursion: $T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$

$\hookrightarrow T_0 = 1 \quad T_1 = x$

$T_0(x) = \frac{1}{2}\left(z^0 + \frac{1}{z^0}\right) = \frac{1}{2}(1+1) = 1$

$T_1(x) = \frac{1}{2}\left(z + \frac{1}{z}\right) = x$ by definition

$$\frac{1}{2}\left(z^{n+1} + \frac{1}{z^{n+1}}\right) = \cancel{2}x\left(\frac{1}{\cancel{2}}\left(z^n + \frac{1}{z^n}\right)\right) - \frac{1}{2}\left(z^{n-1} + \frac{1}{z^{n-1}}\right)$$

$$= \frac{1}{2}\left(z + \frac{1}{z}\right)\left(z^n + \frac{1}{z^n}\right) - \frac{1}{2}\left(z^{n-1} + \frac{1}{z^{n-1}}\right)$$

$$= \frac{1}{2}\left[z^{n+1} + \frac{1}{z^{n-1}} + z^{n-1} + \frac{1}{z^{n+1}}\right] - \frac{1}{2}\left[z^{n-1} + \frac{1}{z^{n-1}}\right]$$

$$= \frac{1}{2}\left[z^{n+1} + \frac{1}{z^{n+1}}\right] + \frac{1}{2}\left[z^{n-1} + \frac{1}{z^{n-1}}\right] - \frac{1}{2}\left[z^{n-1} + \frac{1}{z^{n-1}}\right]$$

$$\frac{1}{2}\left(z^{n+1} + \frac{1}{z^{n+1}}\right) = \frac{1}{2}\left[z^{n+1} + \frac{1}{z^{n+1}}\right]$$

Thus this equation generates the same polynomials as the standard definition