APPM 4600 - #W4 - Cambria Chaney

1) $\frac{T(x,t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$

T_s = constant temp = -15
T_i = initial soil temp = 20
α = thermal conductivity = 0.138×10^{-6}

$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-s^2)\,ds$

t = seconds x = m

(a) only freeze after 60 days = 5,184,000 s
root finding problem $f(x) = 0$
If $f(x_1, t) = 0$ then x_1 is the depth of the soil where temperature is 0, at time t

$$T(x, 60) = \text{erf}\left(\frac{x}{2\sqrt{0.138 \times 10^{-6} \times 5{,}184{,}000\,s}}\right) \cdot (20 + 15) - 15$$

$$T(x, 60\text{ days}) = \text{erf}\left(\frac{x}{1.6916}\right)(35) - 15$$

$$f(x) = \text{erf}\left(\frac{x}{1.6916}\right)(35) - 15$$

$$f'(x) = 35\left[\frac{d}{dx}\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{1.6916}} e^{-s^2}\,ds\right]$$

$$f'(x) = 35\left[\frac{2}{\sqrt{\pi}}\left(e^{-\left(\frac{x}{1.6916}\right)^2}\right)\left(\frac{1}{1.6916}\right)\right]$$

$$f'(x) = 23.346\, e^{-\frac{1}{2.8615}x^2}$$

See plots attached

(b) root = 0.67695

(c) root = 0.6769 @ $x_0 = 0.01$
root = 0.6769 @ $x_0 = 1$ $= \bar{x}$ $f(1) > 0$

2) $f(x)$ - root α of multiplicity m

(a) A solution α of $f(x)=0$ is a zero/root of multiplicity m if for $x \neq \alpha$ we can write:

$$f(x) = (x-\alpha)^m q(x)$$

where $\lim_{x \to \alpha} q(x) \neq 0$

(b) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ show $|f'(p)| < 1$

$$x_{n+1} = x_n - \frac{(x-\alpha)^m q(x)}{(x-\alpha)^m q'(x) + q(x)(m)(x-\alpha)^{m-1}}$$

$$g(x) = x_{n+1} = x_n - \frac{(x_n-\alpha) q(x_n)}{(x_n-\alpha) q'(x_n) + m q(x_n)}$$

$$g'(x) = 1 - \frac{\begin{bmatrix} [(x_n-\alpha)q'(x) + mq(x)] \\ [(x_n-\alpha)q'(x) + q(x)] - \\ [(x_n-\alpha)q(x)][(x-\alpha)q''(x) + \\ q'(x) + mq'(x)] \end{bmatrix}}{[(x_n-\alpha)q'(x_n) + mq(x)]^2}$$

$$q'(\alpha) = 1 - \frac{[mq(\alpha)][q(\alpha)] - 0}{m^2 q(\alpha)^2}$$

$q'(\alpha) = 1 - \frac{1}{m}$ Thus $|g'(\alpha)| < 1$ if $m > 1$ - thus newton converges linearly.

(c) $g(x) = x - m\frac{f(x)}{f'(x)}$ is 2nd order convergent

$g'(x) = 1 - m\left[\frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2}\right]$ $\quad (f(x) = (x-\alpha)^m q(x))$

plug α

$g'(\alpha) = 1 - m\left[\frac{1}{m}\right]$ from part b

$g'(\alpha) = 0$

$$g''(x) = -m\left[\frac{[(x_n-\alpha)q''(x) + q'(x) + mq''(x)][(x_n-\alpha)q'(x) + q(x)] + [(x_n-\alpha)q''(x) + 2q'(x)][(x_n-\alpha)q'(x) + mq(x)] - [[(x-\alpha)q(x)][(x-\alpha)q'''(x) + q''(x) + q''(x) + mq''(x)] + [(x-\alpha)q''(x) + q'(x) + mq'(x)][(x-\alpha)q'(x) + q(x)]]}{[(x_n-\alpha)q'(x) + mq(x)]^2} - \frac{[mq(\alpha)^2][2[(x_n-\alpha)q'(x) + mq(x)][(x_n-\alpha)q''(x) + q'(x) + mq'(x)]]}{low^2}\right]$$

$$g''(\alpha) = -m\left[\frac{[q'(\alpha) + mq''(\alpha)][q(\alpha)] + [2q'(\alpha)][mq(\alpha)] - [0 + (q'(\alpha) + mq'(\alpha))(q(\alpha))]}{[m^2 q(\alpha)^2]} - \frac{[mq(\alpha)^2][2[mq(\alpha)][q'(\alpha) + mq'(\alpha)]]}{m^4 q(\alpha)^4}\right]$$

$$= -m\left[\cancel{q(\alpha)q'(\alpha)} + mq''(\alpha)q(\alpha) + 2mq'(\alpha)q(\alpha) - \cancel{q'(\alpha)q(\alpha)} + mq'(\alpha)q(\alpha)\right]$$

$$= -m\left[3m^3 q'(\alpha)q(\alpha)^3 + m^3 q''(\alpha)q(\alpha)^3 - 2m^2 q(\alpha)^3 q'(\alpha) - 2m^3 q(\alpha)^3 q'(\alpha)\right]$$

num $= -m\left[m^3 q(\alpha)^3 q'(\alpha) + m^3 q''(\alpha)q(\alpha)^3 + 2m^2 q(\alpha)^3 q'(\alpha)\right]$

$$g''(\alpha) = -\left\{\frac{m^4 q(\alpha)^3 q'(\alpha) + m^4 q''(\alpha) q(\alpha)^3 + 2m^3 q(\alpha)^3 q'(\alpha)}{m^4 q(\alpha)^4}\right\}$$

$$g''(\alpha) = -\left[\frac{mq'(\alpha) + mq''(\alpha) + 2q'(\alpha)}{mq(\alpha)}\right] \neq 0$$

thus second order convergent

(d) If $m > 1$, Newton's method still converges just second order, if you add the multiplicity constant m to the iteration

3) $\{x_k\}_{k=1}^{\infty}$ converges to α

$$\lim_{n\to\infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^p} = \lambda \quad \text{for positive } \lambda \,\&\, \alpha$$

$$\lim_{n\to\infty} \log(|x_{n+1} - \alpha|) - \log(|x_n - \alpha|^p) = \log(\lambda)$$

$$\lim_{n\to\infty} \log(|x_{n+1} - \alpha|) - p\log(|x_n - \alpha|) = \log(\lambda)$$

$$\log(|x_{n+1} - \alpha|) = p\log(|x_n - \alpha|) + \log(\lambda)$$

let $y = \log(|x_{n+1} - \alpha|)$ & $x = \log(|x_n - \alpha|)$ and $b = \log(\lambda)$.

Thus $y = px + b$. $\log(|x_{n+1} - \alpha|)$ & $\log(|x_n - \alpha|)$ have a linear relationship where the order p is the slope of this linear relationship.

$f(x) = (e^x - 3x^2)^3$

4) $f(x) = e^{3x} - 27x^6 + 27x^4e^x - 9x^2e^{2x}$ $\in(3,5)$

$f'(x) = 3e^{3x} - 27\cdot 6x^5 + 27\cdot 4x^3e^x + 27x^4e^x - 9x^2 12e^{2x} - 18xe^{2x}$

Newton's method: $p_0 = 4$

root: 3.733, 52 iterations

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ order of convergence: 2 → not exactly 1 due to the 50 iterations

$g'(x) = \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}$

① Modified Newtons: $g(x) = \frac{f(x)}{f'(x)}$

$\left(\frac{f}{f'}, \left(\frac{f}{f'}\right)', Nmax, tol\right)$ root: 3.733, 5 iterations

order of convergence: 2

② Modified Newton's: $g(x) = x - m\frac{f(x)}{f'(x)}$

zero has multiplicity 3 ↳ becomes iteration

root: 3.733 4 iterations

order of convergence: 2 ✓

I prefer the original newtons method because it is less calculation heavy than the ① modified newton and also doesn't require the multiplicity like the ② modified Newton. It is hard to know the multiplicity of a complicated function so that would be my last choice

5) Newton: root = 1.1347 , $x_0 = 2$
8 iterations

secant: root = 1.1347 , 9 iterations
$x_0 = 2$ $x_1 = 1$

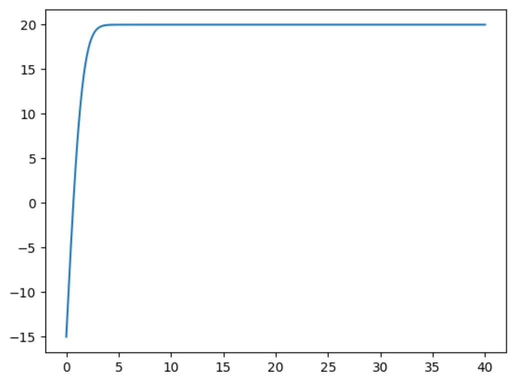
Slope of secant log graph: 1.6237

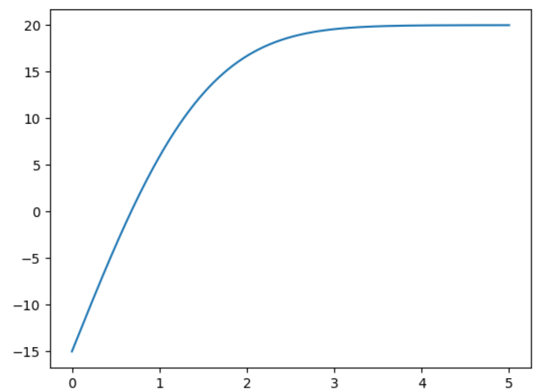
slope of newton log graph: 1.9922

This relates to the order because Newton's method has an order of convergence 2 and secant has an order of convergence between 1 and 2.

See plots below.

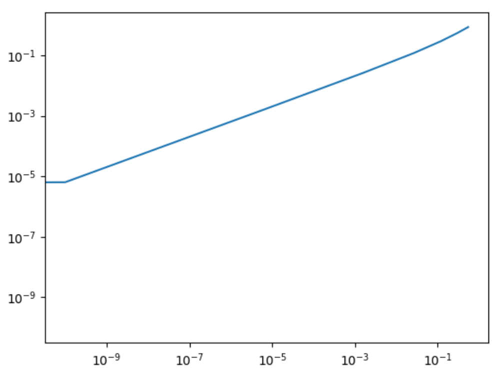
Problem 1: (xbar = 5 in the plot on the right, but in the computation xbar = 1)





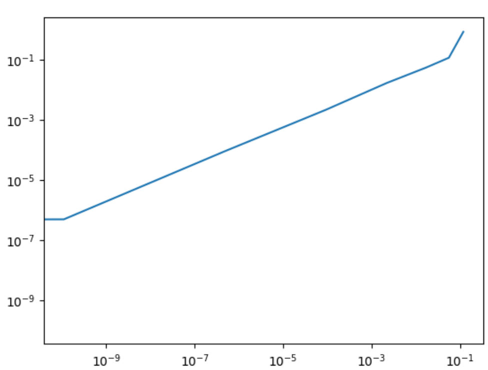
Problem 5:

Newton's Method:



Iteration	Error
0	0.865276
1	0.545904
2	0.296015
3	0.120247
4	0.0268143
5	0.00162914
6	6.38994e-06
7	9.87017e-11
8	0

Secant Method:



Iteration	Error
0	0.865276
1	-0.118595
2	0.0558536
3	-0.0170683
4	-0.00219259
5	9.26696e-05
6	-4.92453e-07
7	-1.10304e-10
8	0

The errors in the tables decreases, just as you would expect, for both newton's method and the secant method. The order of convergence is the same as the slope of the log graphs above for each method.