APPM 4600 - HW 5 - Cambria

1) $(x_0, y_0) = (1,1)$

$$f(x,y) = 3x^2 - y^2 = 0$$
$$g(x,y) = 3xy^2 - x^3 - 1 = 0$$

(a) $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} 1/6 & 1/18 \\ 0 & 1/6 \end{bmatrix} \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}$

check code but it converged to $\begin{bmatrix} 0.5 \\ 0.866025 \end{bmatrix}$ within 20 iterations.

(b) $\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 6x & -2y \\ 3y^2 - 3x^2 & 6xy \end{bmatrix}$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} =$$

$$= \frac{1}{36x^2y - 2y(3x^2 - 3y^2)} \begin{bmatrix} 6xy & 2y \\ 3x^2 - 3y^2 & 6x \end{bmatrix}$$

plug in $(x,y) = (1,1)$

$$= \frac{1}{36} \begin{bmatrix} 6 & 2 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1/6 & 1/18 \\ 0 & 1/6 \end{bmatrix}$$

This is the inverse Jacobian matrix evaluated at $x=1, y=1$ for all iterations.

(c) Running Newton's method with the Jacobian calculate in b, we get that the root is:

$$\begin{bmatrix} 0.5 \\ 0.8660254 \end{bmatrix}$$ w/ 11 iterations in 0.000415 seconds

(d) Exact soln: $x = 0.5$ $y = 0.8660254$

$$f(x,y) = 3\left(\frac{1}{4}\right) - (0.8660254)^2$$

$$= 6.5548 \times 10^{-9} \approx 0$$

$$g(x,y) = 3\left(\frac{1}{2}\right)(0.866)^2 - \left(\frac{1}{8}\right) - 1$$

$$= -9.832 \times 10^{-9} \approx 0$$

thus it is verified that these are the exact solutions.

2) $$x_{n+1} = \frac{1}{\sqrt{2}}\sqrt{1+(x_n+y_n)^2} - \frac{2}{3}$$

$$y_{n+1} = \frac{1}{\sqrt{2}}\sqrt{1+(x_n-y_n)^2} - \frac{2}{3}$$

$$G = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$ $$\{G(x^{(k-1)})\}_{k=1}^{\infty} \to p \in D$$

2) $\left|\frac{\partial g_i(x)}{\partial x_j}\right| \leq \frac{k}{n} \Rightarrow k=1$ because that's the largest value it can be

$n=2$ because we are in 2 dimensions

$$\left|\frac{\partial g(x)}{\partial x}\right| = \frac{1}{\sqrt{2}}\left[\frac{1}{2}\left[1+(x+y)^2\right]^{-1/2}(2(x+y))\right]$$

$$= \left|\frac{(x+y)}{\sqrt{2}\sqrt{1+(x+y)^2}}\right| \leq \frac{1}{2}$$

$$\left|\frac{\partial g(x)}{\partial y}\right| = \frac{1}{\sqrt{2}}\left[\frac{1}{2}\left[1+(x+y)^2\right]^{-1/2}(2(x+y))\right]$$

$$= \left|\frac{(x+y)}{\sqrt{2}\sqrt{1+(x+y)^2}}\right| \leq \frac{1}{2}$$

$$\left|\frac{\partial g(y)}{\partial x}\right| = \frac{1}{\sqrt{2}}\left[\frac{1}{2}\left(1+(x-y)^2\right)^{-1/2}(2(x-y))(-1)\right]$$

$$= \left|\frac{-(x-y)}{\sqrt{2}\sqrt{1+(x-y)^2}}\right| \leq \frac{1}{2}$$

$$\left|\frac{\partial g(y)}{\partial y}\right| = \frac{1}{\sqrt{2}}\left[\frac{1}{2}\left[1+(x-y)^2\right]^{-1/2}(2(x-y))\right]$$

$$= \left|\frac{x-y}{\sqrt{2}\sqrt{1+(x-y)^2}}\right| \leq \frac{1}{2}$$

$$\left|\frac{\partial g(x)}{\partial y}\right| = \left|\frac{\partial g(x)}{\partial x}\right| \le \frac{1}{2} \qquad \left|\frac{\partial g(y)}{\partial x}\right| = \left|\frac{\partial g(y)}{\partial x}\right| \le \frac{1}{2}$$

$$\frac{x+y}{\sqrt{1+(x+y)^2}} < \frac{\sqrt{2}}{2} \qquad \frac{x-y}{\sqrt{1+(x+y)^2}} \le \frac{\sqrt{2}}{2}$$

$$2(x+y) < \sqrt{2+2(x+y)^2} \qquad 2(x-y) \le \sqrt{2+2(x-y)^2}$$

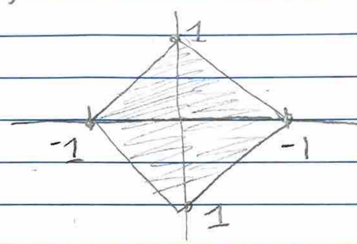
$$4(x+y)^2 \le 2+2(x+y)^2 \qquad 4(x-y)^2 < 2+2(x-y)^2$$

$$2(x+y)^2 \le 2 \qquad 2(x-y)^2 < 2$$

$$(x+y)^2 \le 1 \qquad (x-y)^2 \le 1$$

$$-1 \le x+y \le 1 \qquad -1 < x-y < 1$$

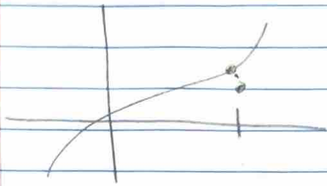
Plotting the Intersection we get



Which is our basin of convergence where the fixed point iteration is guaranteed to converge to a unique solution for any starting point $(x_0, y_0) \in D$.

3) $f(x, y) = 0$

$$\Delta x = -df_x$$
$$\Delta y = -df_y$$



(a) $\nabla f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$ evaluated at (x_n, y_n)

$f(x_n, y_n) = 0$

equation of a line in the direction of a gradient $\quad g(x, y) = \dfrac{x - x_n}{f_x} - \dfrac{y - y_n}{f_y} = 0$

$x_n, y_n, f_x, f_y = \text{const}$

$$\frac{\partial g}{\partial x} = \frac{1}{f_x} \qquad \frac{\partial g}{\partial y} = -\frac{1}{f_y}$$

$$J = \begin{bmatrix} f_x & f_y \\ \frac{1}{f_x} & -\frac{1}{f_y} \end{bmatrix} \qquad J^{-1} = \begin{bmatrix} -\frac{1}{f_y} & -f_y \\ -\frac{1}{f_x} & f_x \end{bmatrix} \frac{1}{-\frac{f_x}{f_y} - \frac{f_y}{f_x}}$$

$$J^{-1} = \begin{bmatrix} \frac{1}{f_y} & f_y \\ \frac{1}{f_y} & -f_x \end{bmatrix} \frac{f_x f_y}{f_x^2 + f_y^2}$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - J^{-1} \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) = 0 \end{bmatrix}$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} f/f_y \\ f/f_x \end{bmatrix} \begin{bmatrix} \frac{f_x f_y}{f_x^2 + f_y^2} \end{bmatrix}$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} \dfrac{f \cdot f_x}{f_x^2 + f_y^2} \\ \dfrac{f \cdot f_y}{f_x^2 + f_y^2} \end{bmatrix}$$

where $d = \dfrac{f}{f_x^2 + f_y^2}$

$x_{n+1} = x_n - d f_x$

$y_{n+1} = y_n - d f_y$

(b) $x^2 + 4y^2 + 4z^2 = 16$ $(1,1,1) = \bar{x}_0$

$= 0$

$f_x = 2x$ $f_y = 8y$ $f_z = 8z$

$z_{n+1} = z_n - d f_z$ $d = \dfrac{f}{f_x^2 + f_y^2 + f_z^2}$

$$\boxed{(x, y, z) = (1.09364, 1.36033, 1.36033)}$$

Using the order of convergence code in lab for the x, y, & z values separately, the order of convergence is 2 for each x, y, z, when there is a sufficient number of iterations