APPM 4600 - HW #2 - Cambria C

1) (a) $(1+x)^n = 1 + nx + o(x)$ as $x \to 0$

$(1+x)^n - 1 - nx = o(x)$

$$\lim_{x\to 0} \frac{(1+x)^n - 1 - nx}{x} \overset{LH}{=} \lim_{x\to 0} \frac{n(1+x)^{n-1} - n}{1}$$

$$= n \cdot (1)^{n-1} - n = n - n = 0 \checkmark$$

(b) $x\sin\sqrt{x} = O(x^{3/2})$ as $x \to 0$

$$\lim_{x\to 0} \frac{x\sin(x^{1/2})}{x^{3/2}} = \lim_{x\to 0} \frac{\sin(x^{1/2})}{x^{1/2}}$$

$$\overset{L.H}{=} \lim_{x\to 0} \frac{\cos(x^{1/2})\left[\frac{1}{2}x^{-1/2}\right]}{\left[\frac{1}{2}x^{-1/2}\right]}$$

$$= \lim_{x\to 0} \cos(x^{1/2}) = 1 \checkmark$$

(c) $e^{-t} = o\left(\frac{1}{t^2}\right)$ as $t \to \infty$

$$\lim_{t\to\infty} \frac{e^{-t}}{1/t^2} = \lim_{t\to\infty} \frac{t^2}{\frac{1}{e^{-t}}}$$

$$\overset{L.H}{=} \lim_{t\to\infty} \frac{2t}{e^t} \overset{L.H}{=} \lim_{t\to\infty} \frac{2}{e^t} = 0 \checkmark$$

(d) $\int_0^{\varepsilon} e^{-x^2}\,dx = O(\varepsilon) \quad \varepsilon \to 0$

$$\lim_{\varepsilon \to 0} \frac{\int_0^{\varepsilon} e^{-x^2}\,dx}{\varepsilon} \overset{L.H}{=} \lim_{\varepsilon \to 0} e^{-\varepsilon^2} = 1 \checkmark$$

2) (a) $\Delta b = \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix}$ $x = A^{-1} b$) original

perturbed: $A\hat{x} = b + \Delta b$

$\hat{x} = A^{-1} b + A^{-1} \Delta b$

$$\Delta x = x - \hat{x} = A^{-1} b - A^{-1} b + A^{-1} \Delta b$$

$$\boxed{\Delta x = A^{-1} \Delta b}$$

(b) relative cond: $K = \lim_{\delta \to 0} \sup_{\|\delta x\| < \delta} \dfrac{\frac{\|\delta f\|}{\|f\|}}{\frac{\|\delta x\|}{\|x\|}}$

$\|\,\| = $ 2 norm ↳ largest sing value

$$\frac{\|\delta f\|}{\|f\|} = \frac{\|A^{-1} \Delta b\|}{\|x\|} \qquad \frac{\|\delta x\|}{\|x\|} = \frac{\|\Delta b\|}{\|b\|}$$

but cond # should be indep of type of norm

relative cond: $K = \dfrac{\|A^{-1} \Delta b\|}{\|x\|} \cdot \dfrac{\|b\|}{\|\Delta b\|}$

$$\leq \frac{\|A^{-1}\| \, \|b\|}{\|x\|} \leq \|A^{-1}\| \, \|A\| = \text{cond}(A)$$

in python

$= \boxed{2 \times 10^{10}}$

$\|A\| = \sup_{\|x\|=1} \|Ax\|$ } max of absolute column sums

(c) Δb_1 & $\Delta b_2 \approx 10^{-5}$ $\Delta b_1 = 1 \times 10^{-5}$ $\Delta b_2 = 2 \times 10^{-5}$

relative error: $\dfrac{|x - \hat{x}|}{|x|} = \dfrac{|A^{-1} \Delta b|}{|x|} = \dfrac{\|A^{-1} \Delta b\|}{\|x\|}$

$$\leq \frac{\|A^{-1}\| \, \|\Delta b\|}{\|x\|} = \frac{(2 \times 10^{10})(3 \times 10^{-5})}{2} = \frac{6 \times 10^5}{2}$$

$\boxed{3 \times 10^5}$ $\approx 10^5$

The relationship between relative error, condition number + perturbation is the following in the order of magnitude of the number.

relative error = condition number + perturbation

i.e. $10^5 = 10^{10} + 10^{-5}$ $\{$ $5 = 10-5$ in order of magnitude

It is probably more realistic to have different values of perturbation because each value has its own individual meaning and measure.

If the perturbations are the same, this behavior wouldn't change because the order of magnitudes of relative error, perturbation, & cond # are still the same.

3) $f(x) = e^x - 1$

(a) $K(f(x)) = ?$

Δx = perturbation in x $\quad y = f(x)$

$\Delta y = f(x + \Delta x) - f(x)$

$$K(x) = \frac{\frac{|\Delta y|}{|y|}}{\frac{|\Delta x|}{|x|}} = \frac{|\Delta y|}{|\Delta x|} \cdot \frac{|x|}{|y|} = \frac{|f'(c)|\,|x|}{|f(x)|}$$

$$\boxed{K(x) = \frac{|e^c|\,|x|}{|e^x - 1|}}$$

When $x=0$, it is ill-conditioned because you are dividing by 0.

divide by 0 or subtract to 0 for unstable

(b) this algorithm isn't stable because when $x=0$, you will be subtracting similar values, which gets us into floating point arithmetic trouble, making the algorithm unstable

(c) actual answer: 10^{-9} up to 16 decimal places
algorithm answer: 1.000000082740371e-09
↳ it gives 7 digits accurately

This is expected because the algorithm isn't stable & is affected by small changes in x.

(d) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$f(x) = e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

Taylor Series that is accurate to 16 digits:

$$\boxed{f(x) = x + \frac{x^2}{2}}$$

(e) Check code - it is verified - computer gets 1e-09

(f) You don't get any digits of precision because the "simpler" taylor series is just x.

(g) Both taylor & np.expm1 get the same value of 0.0.