APPM 4600 - HW1 - Cambria Chaney

1) (i) code
(ii) code
(iii) The difference is that one graph has noise while the other does not. What's causing this is the floating point arithmetic difference's in the two equations. The coefficient one has a lot more floats than the $(x-2)^9$ one, which is causing the noise. The $(x-2)^9$ one is correct because it is simpler and avoids cancellation.

2) (i) $\sqrt{x+1} - 1 \quad x \simeq 0$

$$\sqrt{x+1} - 1 \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \frac{x+1-1}{\sqrt{x+1}+1} = \frac{x}{\sqrt{x+1}+1}$$

now when evaluated you get 0

(ii) $\sin(x) - \sin(y)$ for $x \simeq y$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

identity: $\sin(x) - \sin(y) = \underbrace{2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)}$

this is better because you are doing the subtraction inside the sin instead of outside. this evaluates to 0 when $x \simeq y$

(iii) $\dfrac{1-\cos(x)}{\sin(x)}$ for $x \simeq 0$

$$\frac{(1-\cos(x))(1+\cos(x))}{\sin(x)+\sin(x)\cos(x)} \qquad \underbrace{1-\cos^2(x)}_{=}$$

$$= \frac{\sin^2(x)}{\sin(x)+\sin(x)\cos(x)} = \frac{\sin(x)}{1+\cos(x)}$$

now it evaluates to 0

3) $P_2(x)$ for $(1+x+x^3)\cos(x)$ about $x_0=0$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n \qquad f^{(0)}(0)=1$$

$$f^{(1)}(0) = \cos(x)(1+3x^2) - (1+x+x^3)\sin(x)$$
$$= \qquad 1 - 0 = 1$$
$$\Rightarrow x$$

$$f^{(2)}(0) = \cos(x)(6x) - (1+3x^2)\sin(x)$$
$$- [\sin(x)(1+3x^2) + (1+x+x^3)\cos(x)]$$
$$= 0 - 0 - 0 - 1 = -1$$
$$\Rightarrow -\frac{1}{2}x^2$$

$$\boxed{P_2(x) = 1 + x - \frac{1}{2}x^2}$$

(a) $P_2(1/2) = \dfrac{3}{2} - \dfrac{1}{2}\left(\dfrac{1}{4}\right) = \dfrac{12}{8} - \dfrac{1}{8} = \boxed{\dfrac{11}{8}}$

error formula $= R_2(x) = \dfrac{f^{(3)}(0)}{3!}x^3$

$$f^3(0) = \cos(x)(6) - 6x(\sin x) - 2(1+3x^2)\cos(x) - 2\sin(x)(6x) - \cos(x)(1+3x^2) + (1+x+x^3)\sin(x)$$

$$= 6 - 0 - 2 - 0 - 1 + 0 = 3$$

$$R_2(x) = \frac{3}{6}x^3 = \frac{1}{2}x^3$$

upper bound of error at 0.5 = $\boxed{\frac{1}{16}}$

Actual error = $\left(1+\frac{1}{2}+\frac{1}{8}\right)\cos\left(\frac{1}{2}\right) - \frac{11}{8}$

$$= \left|\frac{13}{8}\left(\cos\left(\frac{1}{2}\right)\right) - \frac{11}{8}\right| = 0.05107$$

according to python

less than upper error bound

(b) $|f(x) - P_2(x)| = \frac{f'''(x)}{3!}x^3$

$= [3\cos(x) - 17x\sin(x) - 9x^2\cos x + \sin(x) + x^3\sin(x)]\ x^3/6$

$$\boxed{= \frac{1}{2}x^3\cos(x) - \frac{17}{6}x^4\sin(x) - \frac{3}{2}x^5\cos(x) + \frac{x^3}{6}\sin x + \frac{x^6}{6}\sin x}$$

(c) $\int_0^1 f(x)\,dx \approx \int_0^1 P_2(x)\,dx$

$$= \int_0^1 1 + x - \frac{1}{2}x^2\,dx = x + \frac{1}{2}x^2 - \frac{1}{6}x^3\Big|_0^1$$

$$= 1 + \frac{1}{2} - \frac{1}{6} - 0 = \frac{6}{6} + \frac{3}{6} - \frac{1}{6} = \frac{8}{6} = \boxed{\frac{4}{3}}$$

bound max

(d) using part b $= \int_0^1 \frac{1}{12}x^3\cos x - \int_0^1 \frac{17}{6}x^4\sin x$
$\int_0^1 \frac{3}{2}x^5\cos x + \int_0^1 \frac{x^3}{6}\sin x + \int_0^1 \frac{x^6}{6}\sin x \; dx$

$\max|\cos x| = \max|\sin x| = 1$ thus

$$\int_0^1 \frac{1}{2}x^3 - \frac{17}{6}x^4 - \frac{3}{2}x^5 + \frac{x^3}{6} + \frac{x^6}{6}\,dx =$$

$$\frac{x^4}{8} - \frac{17}{30}x^5 - \frac{3}{12}x^6 + \frac{x^4}{24} + \frac{x^7}{42}\Big|_0^1$$

$$= \frac{1}{8} - \frac{17}{30} - \frac{3}{12} + \frac{1}{24} + \frac{1}{42} - 0 = \boxed{-0.62619 = -\frac{263}{420}}$$

4) $ax^2 + bx + c = 0$ w/ $a=1, b=-56, c=1$

(a) $\dfrac{56 \pm \sqrt{56^2 - 4(1)(1)}}{2}$

$= \dfrac{56 \pm 55.964}{2}$

$= 0.018, 55.982$

relative error $= \dfrac{|x - \tilde{x}|}{|x|}$

find values

$$= \frac{\left|\underbrace{\frac{56 - \sqrt{3132}}{2}}^{0.0178628} - 0.018\right|}{\left|\frac{56 - \sqrt{3132}}{2}\right|} = -0.007678$$

$$= \frac{\left|\underbrace{\frac{56 + \sqrt{3132}}{2}}^{55.98213716} - 55.982\right|}{\left|\frac{56 + \sqrt{3132}}{2}\right|} = 2.45 \times 10^{-6}$$

(b) $(x-r_1)(x-r_2)=0$

$$x^2 - xr_2 - xr_1 + r_1r_2 = 0$$

$$x^2 - (r_2+r_1)x + r_1r_2 = 0$$

$a=1$ $b=-56=-(r_2+r_1)$ $c=1=r_1r_2$

system of equations

$r_1 = \frac{1}{r_2}$

$56 = r_2 + r_1$ $\quad 1 = r_1 * r_2$

$r_2 = 56 - r_1$ $\quad 1 = r_1(56-r_1)$

Vietta's Relations: $r_1+r_2 = -b/a$ $\quad r_1r_2 = c/a$

$1 = 56r_1 - r_1^2 \Rightarrow 0 = r_1^2 - 56r_1 + 1$

we're back at original problem so it didn't work.

5) (a) $|\Delta y| = |\Delta x_1 - \Delta x_2| \leq |\Delta x_1| + |\Delta x_2|$

$$\frac{|\Delta y|}{|y|} = \frac{|\Delta x_1 - \Delta x_2|}{|x_1 - x_2|} \leq \frac{|\Delta x_1| + |\Delta x_2|}{|x_1 - x_2|}$$

Relative error is large when $x_1 - x_2$ is small, meaning x_1 is similar to x_2

5) (b) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\cos(A+B) - \cos(A-B) = -2\sin A \sin B$

$\cos(x+\delta) - \cos(x)$

$A+B = x+\delta$ $\quad$ $x+2B = x+\delta$ $\quad$ $B = \frac{\delta}{2}$

$A-B = x$ $\quad$ $A = x+B$, $A = x+\frac{\delta}{2}$

$\cos(x+\frac{\delta}{2}+\frac{\delta}{2}) - \cos(x+\frac{\delta}{2}-\frac{\delta}{2})$
$= -2\sin(x+\delta/2)\sin(\delta/2)$

$\cos(x+\delta) - \cos(x) = -2\sin(x+\delta/2)\sin(\delta/2)$

This expression doesn't divide by 0 and doesn't subtract values to get close to 0, making it more stable than the original equation. In the original equation if δ is small, then we are subtracting 2 similar values making it unstable.

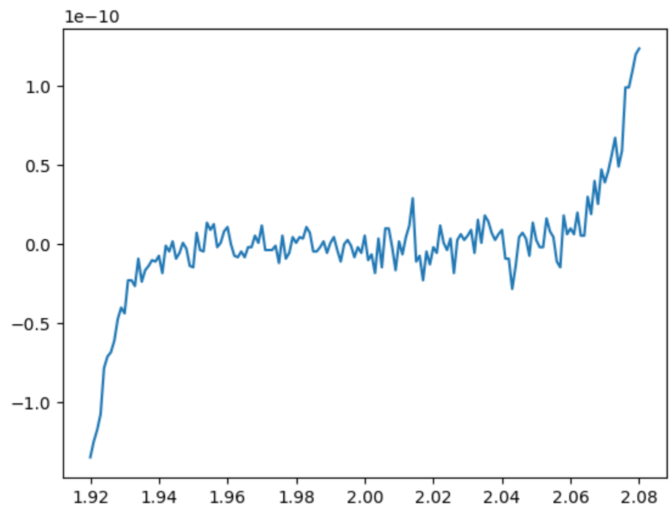
5) (c) $f(x+\delta) - f(x) = \delta f'(x) + \frac{\delta^2}{2!} f''(\varepsilon)$ $\quad \varepsilon \in [x, x+\varepsilon]$

$f(x) = \cos(x)$ $\quad f'(x) = -\sin(x)$ $\quad f''(\varepsilon) = -\cos(\varepsilon)$

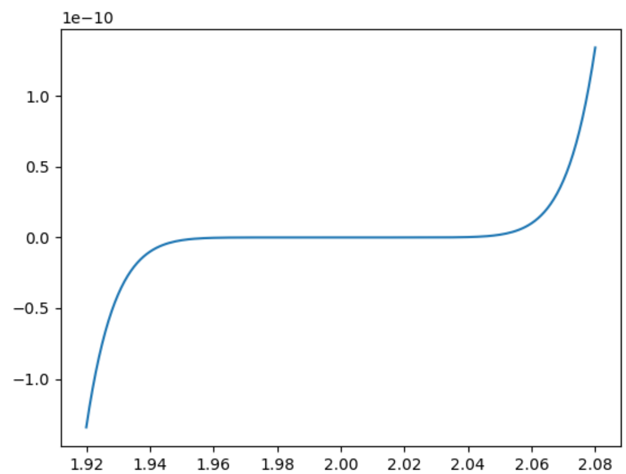
$\cos(x+\delta) - \cos(x) = -\delta\sin(x) - \frac{\delta^2}{2}\cos(\varepsilon)$

The algorithm I am going to use is using $-\delta\sin(x)$ as an approximation of $\cos(x+\delta) - \cos(x)$ since δ^2 in the second term will be close to 0.

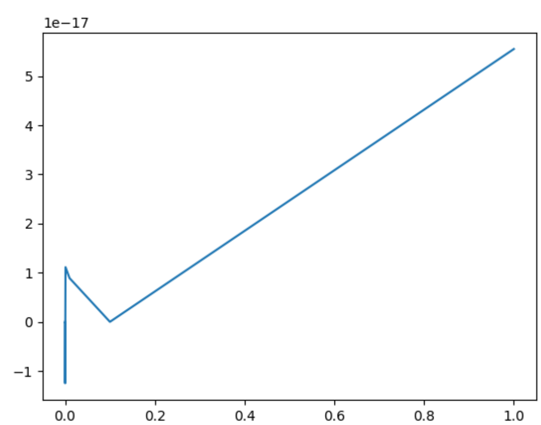
1.) (i)



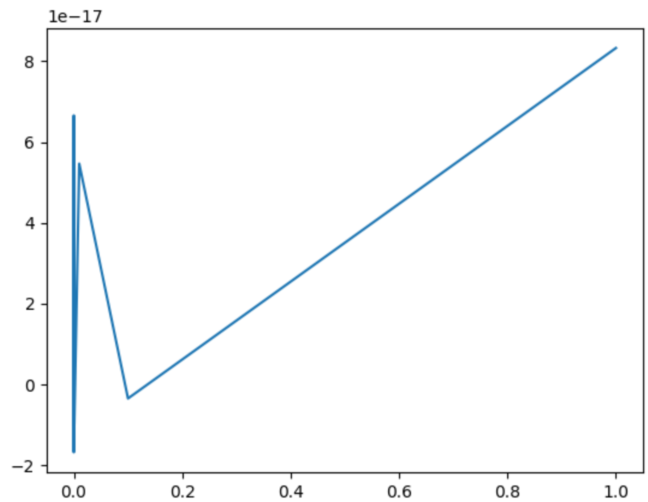
1.) (ii)



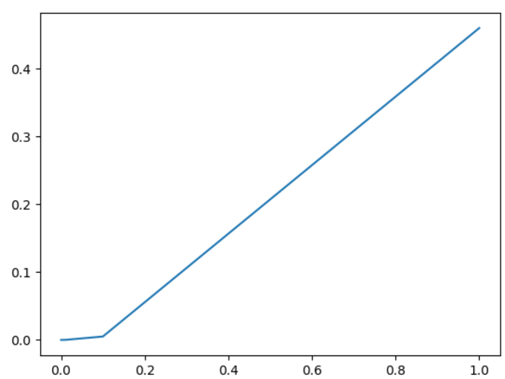
5.) (b) x = pi – plot of error



5.) (b) x = 10**6 – plot of error



5.) (c) x = pi – plot of error with my algorithm



5.) (c) x = 10**6 – plot of error with my algorithm