Summary: Introduction to Probability

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Contents

Chapter 1

Introduction: Part I

1.1 Sets

1.2 The principle of induction

1.3 Order structure of the real numbers

Exercise 1.1 (1.11 TOOL)**.** Let A be a set with n elements. Show that

1. the number of permutations of the elements from A is $n!$;
2. for any $0 \leq k \leq n$, the number of subsets of A having k elements if given by

$$\frac{n!}{(n-k)!k!}.$$

Chapter 2

Introduction: Part II

2.1 Functions

Proposition 2.1. TODO prop 2.12

2.2 Cardinality of Sets

2.3 Euclidean distance

Proposition 2.2. let $f : A \to B$ be a function. Let $B_* \subset B$. Then,

(a) $f^{-1}(B_c^*) = f^{-1}(B_*)^c$.

Let I and J be some sets and $A_i \subset A, i \in I$, and $B_j \subset B, j \in J$, be a collection of sets from A and B, respectively. Then,

(b) TODO

(c) TODO

(d) TODO

Chapter 3

Introduction: Part III

3.1 Real valued sequences

Chapter 4

Measurable sets: Part I

4.1 Measurable spaces

Definition 4.1 (σ-field)**.** Let Ω be a nonempty set. A family of subsets $\mathcal{F}$ of Ω is called a σ-field on Ω if the following three itmes are statisfied:

(i) $\Omega \in \mathcal{F}$;

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$;

(iii) if $\{A_i : i \in \mathbb{N}\}$ is a collection of sets s.t. $A_i \in \mathcal{F}$ for any $i \in \mathbb{N}$, then $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}$.

Definition 4.2. 4.2 TODO

Definition 4.3 (Measurable space)**.** let $\Omega \neq \emptyset$ and $\mathcal{F}$ be a σ-field on Ω. The pair $(\Omega, \mathcal{F}$ is referred to as a measurable space. if $A \in \mathcal{F}$, then A is said to be measurable. if $A \subset \mathcal{F}$ and $\mathcal{A}$ is a σ-field on Ω, $\mathcal{A}$ is referred to as a sub-σ-field on Ω.

Chapter 5

Measurable sets: Part II

5.1 Measure spaces

Definition 5.1 (Measure on $\mathcal{F}$)**.** TODO

5.2 Semirings

Chapter 6

Measurable sets: Part III

6.1 Measure extensions

Proposition 6.1. Let $(a, b]$, $a < b \in \mathbb{R}$, be any left-open interval. Let I be countable and $(a_i, b_i]$, $i \in I$, be s.t., $(a, b] \subset \bigcup_{i \in I}(a_i, b_i]$, then

$$b - a \leq \sum_{i \in I}(b_i - a_i). \tag{10}$$

Proposition 6.2. Let $(a, b]$, $a < b \in \mathbb{R}$, be any left-open interval. let I be countable and $\{(a_i, b_i] : i \in I\}$ be a disjoint collection of left-open intervals s.t. $\bigcup_{i \in I}(a_i, b_i] \subset (a, b]$. Then

$$\sum_{i \in I}(b_i - a_i) \leq b - a.$$

Definition 6.1. Let $\Omega \neq \emptyset$ be a set and $\mathcal{A}$ be a collection of subsets from Ω. Let $A \in \mathcal{P}(\Omega)$ be any subset of Ω. A collection $\{U_i : i \in I\}$ is said to be a covering of A by sets from $\mathcal{A}$ if:

(i) $\{U_i : i \in I\} \subset \mathcal{A}$ (Set membership condition)

NOTE that (i) means $U_i \subset \mathcal{A}\ \forall i \in I$, not $\bigcup_{i \in I} U_i \subset \mathcal{A}$.

(ii) $A \subset \bigcup_{i \in I} U_i$ (Covering condition)

A covering $\{\bigcup_i : i \in I\}$ of A by sets from $\mathcal{A}$ is referred as countable (resp. finite) if I is countable (resp. finite). We write $C_{\mathcal{A}}(A)$ for the set which contains all the countable covering of A by sets from $\mathcal{A}$, i.e.,

$$C_{\mathcal{A}}(A) = \{\xi : \xi \text{ is a countable covering of } A \text{ by sets from } \mathcal{A}\}.$$

Why do we say $A \in \mathcal{P}(\Omega)$ instead of $A \in \Omega$? When we use the notation $A \in \mathcal{P}(\Omega)$, it signifies that A is a subset of Ω, not an element of Ω. The power set $\mathcal{P}(\Omega)$ represents all possible subsets of Ω, including Ω itself, any subset of it, or even an empty set. Using $A \in \Omega$ would incorrectly imply that A is an individual element of Ω, which does not align with the context of covering subsets with subsets.

My Example 6.1 (Finite Covering)**.** Let $\Omega = \{1, 2, 3, 4, 5\}$, and let $\mathcal{A}$ be a collection of subests of Ω, such as $\mathcal{A} = \{\{1\}, \{2, 3\}, \{3, 5\}\}$, if we take $A = \{1, 2, 3\}$, a finite covering of A by sets from $\mathcal{A}$ could be $\{\{1\}, \{2, 3\}\}$. This covering is finite, as I can be $\{1, 2\}$, which is finite. The 2 conditions both hold. Each U_i is a subset of $\mathcal{A}$, and A is covered by the union of U_i. In this case, the possible countable coverings of A that can be formed using subsets of $\mathcal{A}$ are restricted to the one already provided. Therefore, $C_{\mathcal{A}}(A) = \{\{1\}, \{2, 3\}\}$

Important from Example 6.1 (Script) Let $\Omega = \mathbb{R}$ and $\mathcal{R} = \{A : A = (a, b], a, b \in \mathbb{R}\} \cup \{\emptyset\}$. We define the function $\ell : \mathcal{R} \to [0, \infty)$ s.t.

$$\ell(U) = \begin{cases} b - a, & \text{if } U = (a, b], \\ 0, & \text{if } U = \emptyset. \end{cases}$$

Given $A \in \mathcal{P}(\mathbb{R})$, we also define the function $v_\ell(\xi) : \mathcal{R} \to \mathbb{R}^+$, where $\xi \in C_\mathcal{R}(A)$ s.t.

$$v_\ell(\xi) = \sum_{U \in \xi} \ell(U).$$

We also show that

$$\inf\{v_\ell(\xi) : \xi \in C_\mathcal{R}((a, b])\} = \inf_{\xi \in C_\mathcal{R}((a,b])} v_\ell(\xi) = b - a, \tag{11}$$

i.e., $b - a$ is a lower bound for the values of $v_\ell(\xi)$, $\xi \in C_\mathcal{R}((a, b])$. We also saw that there exists $\xi \in C_\mathcal{R}((a, b])$ s.t. $b - a = v_\ell(\xi)$. Hence, the latter infimum is a minimum (Proposition 6.3).

Proposition 6.3. Given any left open interval $(a, b]$, $min_{\xi \in C_\mathcal{R}((a,b])} v_\ell(\xi) = b - a$

Define ℓ^* We build on the latter result and define the function

$$\ell^* = \inf_{\xi \in C_\mathcal{R}(A)} v_\ell(\xi), \quad A \in \mathcal{P}(\mathbb{R}).$$

Note, we know that if $A \in \mathcal{R}$, then $\ell^*(A) = b - a$

Chapter 7

Measurable functions

7.1 The concept of measurable functions

Definition 7.1 (Measurable function)**.** Let $(\Omega, \mathcal{F})$ and $(\Omega^*, \mathcal{F}^*)$ be two measurable spaces (cf. Definition 4.3). A function $f : \Omega \to \Omega^*$ is said to be measurable $\mathcal{F}/\mathcal{F}^*$ if for any $A^* \in \mathcal{F}^*, f^{-1}(A^*) \in \mathcal{F}$.

Proposition 7.1 (Measurable function)**.** let $(\Omega, \mathcal{F})$ and $(\Omega^*, \mathcal{F}^*)$ be two measurable spaces and $f : \Omega \to \Omega^*$ be a function. Suppose that $\mathcal{F}^* = \sigma(\mathcal{G})$ and for any $G \in \mathcal{G}$, $f^{-1}(G) \in \mathcal{F}$. Then, f is $\mathcal{F}/\mathcal{F}^*$ measurable.

Definition 7.2 (Borel function)**.** A function $f : \mathbb{R}^m \to \mathbb{R}^k$ is called Borel function if it is measurable $\mathfrak{B}(\mathbb{R}^m)/\mathfrak{B}(\mathbb{R}^k)$.

Proposition 7.2 (Continuous functions and Borel functions)**.** Any continuous function $f : \mathbb{R}^m \to \mathbb{R}^k$ is a Borel function.

Proposition 7.3 ($\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and $f : \Omega \to \mathbb{R}$ be a real-valued function. Suppose that $\{\omega \in \Omega : f(\omega) \leq x\} \in \mathcal{F}$ for any $x \in \mathbb{R}$, then f is $\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable. In other words: f is a measurable function if the pre-image of any interval $(-\infty, x]$ under f is a measurable set in $\mathcal{F}$, or $f^{-1}((-\infty, x]) \in \mathcal{F}$. since $\mathfrak{B}(\mathbb{R}) = \sigma(\{(-\infty, x] : x \in \mathbb{R}\})$, we also clearly see the proof (cf. Proposition 7.1).

Thinking about $f^{-1}((-\infty, x))$ If $B \in \mathfrak{B}(\mathbb{R})$, then, $f^{-1}(B) = \{\omega \in \Omega : f(\omega) \in B\}$ Is the same as saying, $f^{-1}((-\infty, x)) = \{\omega \in \Omega : f(\omega) \leq x\}$. $f^{-1}(B)$ will return ALL of the values $\omega \in \Omega$ for which $f(\omega) \in B$. See My Example 7.1 for further intuition.

Define $\mathbb{1}_A(\omega)$ TODO

Example 7.1 (Simple measurable function)**.** Let $\Omega = \{h, t\}$ and $\mathcal{F} = \mathcal{P}(\{h, t\}) = \{\emptyset, \{h\}, \{t\}, \{h, t\}\}$. Then, $\{h\} \in \mathcal{P}(\{h, t\})$. Thus

$$f(\omega) = \begin{cases} 1, & \text{if } \omega = h, \\ 0, & \text{if } \omega = t, \end{cases}$$

is $\mathcal{P}(\{h, t\})/\mathfrak{B}(\mathbb{R})$ measurable. In order for f to be $\mathcal{P}(\{h, t\})/\mathfrak{B}(\mathbb{R})$ measurable, the pre-image of every Borel set in $\mathbb{R}$ under f must be an element of $\mathcal{F}$. For any $x \in \mathbb{R}$, $f^{-1}((-\infty, x])$ will either be $\emptyset, \{h\}$, or $\{t\} \in \mathcal{F}$.

Proposition 7.4 ($\mathcal{F}/\mathfrak{B}(\mathbb{R}^k)$ measurable)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and $f : \Omega \to \mathbb{R}^k$, i.e.,

$$f(\omega) = (f_1(\omega), \ldots, f_k(\omega)).$$

Then, f is $\mathcal{F}/\mathfrak{B}(\mathbb{R}^k)$ measurable if and only if for any $i = 1, \ldots, k, f_i : \Omega \to \mathbb{R}$ is $\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable.

Proposition 7.5 (Composite measurable function)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and $f_i : \Omega \to \mathbb{R}, i = 1, \ldots, k,$ be $\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable. Suppose that $g : \mathbb{R}^k \to \mathbb{R}$ is $\mathfrak{B}(\mathbb{R}^k)/\mathfrak{B}(\mathbb{R})$ measurable. Then,

$$w \mapsto g((f_1(\omega), \ldots, f_k(\omega))) = g(f_1(\omega), \ldots, f_k(\omega)).$$

is $\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable. (Composite function usually written without double brackets)

Proposition 7.6 (Continuity preserves measurability in function composition)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and $f_i : \Omega \to \mathbb{R}, i = 1, \ldots, k,$ be $\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable. Then, if $g : \mathbb{R}^k \to \mathbb{R}$ is continuous,

$$w \mapsto g(f_1(\omega), \ldots, f_k(\omega)).$$

is $\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable.

Example 7.2 (Continuity preserves measurability)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and $f_i : \Omega \to \mathbb{R}, i = 1, \ldots, k,$ be $\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable. Then, $\sum_{i=1}^k f_i$ is $\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable (cf. Proposition 2.1).

Example 7.3 (Continuity preserves measurability)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and $f_i : \Omega \to \mathbb{R}, i = 1, \ldots, k,$ be $\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable. Then, $\prod_{i=1}^k f_i$ is $\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable (cf. Proposition 2.1).

Definition 7.3 (Simple functions)**.** A function $f : \Omega \to \mathbb{R}$ is called simple if there exists $n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and sets $A_1, \ldots, A_n \subset \Omega$ s.t.

$$f(\omega) = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}(\omega) \quad \omega \in \Omega.$$

That is, a simple function is a finite linear combination of indicator functions.

Example 7.4 (Simple function)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and f be a simple function on Ω, i.e., $f(\omega) = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}(\omega)$. Then, if $A_i \in \mathcal{F}$ for any $i = 1, \ldots, n, f$ is $\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable.

My Example 7.1 (Simple function)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and $f : \Omega \to \mathbb{R}$ be the function defined in 7.3. For this simplified setting, suppose $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{F} = \{\emptyset, \{1, 2\}, \{3, 4\}, \Omega\}$. Moreover, we define our function with $n = 2$, where $\alpha_1 = 3, \alpha_2 = 5, A_1 = \{1, 2\}$ and $A_2 = \{3, 4\}$. Then,

$$f(\omega) = 3 \cdot \mathbb{1}_{\{1,2\}}(\omega) + 5 \cdot \mathbb{1}_{\{3,4\}}(\omega).$$

Now, let's consider two preimages of this function, $f^{-1}(\{3\})$ and $f^{-1}(\{12\})$. Note that both of these sets are Borel sets in $\mathbb{R}$. Also note that, if $B \in \mathfrak{B}(\mathbb{R})$, then,

$$f^{-1}(B) = \{\omega \in \Omega : f(\omega) \in B\}.$$

As seen in Thinking about 7.1. Since f takes the value 3 for $\omega \in \{1, 2\}$, $f^{-1}(\{3\}) = \{1, 2\} \in \mathcal{F}$. And, as f doesn't take any value for values $\notin \{\{1, 2\}, \{3, 4\}\}$, $f^{-1}(\{12\}) = \emptyset \in \mathcal{F}$. So indeed, f is $\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable.

Definition 7.4 (Simple functions in standard form)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and $f : \Omega \to \mathbb{R}$ be a simple function, as defined in Definition 7.3. f is called standard if $\cup_{i=1}^n A_i = \Omega$ and $\{A_1, \ldots, A_n\} \subset \mathcal{F}$ is disjoint. if f is standard, we say that it is a simple function in standard form.

Proposition 7.7 (7.7)**.** TODO

Proposition 7.8 (7.8)**.** TODO

7.2 Functions taking values in the extended real numbers

Definition 7.5 (Measurable functions in $\overline{\mathbb{R}}$)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and $f : \Omega \to \overline{\mathbb{R}}$. We say that f is $\mathcal{F}$ measurable if for any $A \in \mathfrak{B}(\mathbb{R})$, $\{\omega \in \Omega : f(\omega) \in A\} \in \mathcal{F}$ and $\{\omega \in \Omega : f(\omega) = -\infty\} \in \mathcal{F}$ and $\{\omega \in \Omega : f(\omega) = \infty\} \in \mathcal{F}$. Or, in other words, $f^{-1}(A), f^{-1}(-\infty), f^{-1}(\infty) \in \mathcal{F}$.

Remark 7.2 As seen in the script, as, if $f : \Omega \to \mathbb{R}, f^{-1}(-\infty), f^{-1}(\infty) = \emptyset$, any results on $\mathcal{F}$ meeasurable functions $f : \Omega \to \overline{\mathbb{R}}$ also apply to $\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable functions $f : \Omega \to \mathbb{R}$.

Remark 7.3 TODO, but important for notation, read it from the script.

Proposition 7.9 (7.9)**.** TODO

Proposition 7.10 (7.10)**.** TODO

Definition 7.6 (Positive and negative parts of a function)**.** TODO

Proposition 7.11. This proposition states that any $\mathcal{F}$-measurable function f can be approximated by a sequence of $\mathcal{F}$-measurable simple functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

My Example 7.2. Consider $\Omega = [0, 1]$ and $\mathcal{F}$ be the Borel σ-field on $[0, 1]$. Let $f(x) = x$. Define the sequence of simple functions $f_n(x) = \frac{\lfloor nx \rfloor}{n}$. Each f_n is $\mathcal{F}$-measurable and $f_n(x) \to x$ as $n \to \infty$.

Proposition 7.12. This proposition extends 7.11 by specifying that if f is non-negative, the convergence of the simple functions can be made monotone, i.e., $f_n(\omega)$ increases with n and converges to $f(\omega)$.

My Example 7.3. Using the same function $f(x) = x$ on $\Omega = [0, 1]$, define $f_n(x) = \frac{\lfloor nx \rfloor}{n}$. Note that $f_n(x) \leq f_{n+1}(x)$ for all $x \in [0, 1]$ and $n \in \mathbb{N}$, ensuring that $f_n(x) \uparrow f(x)$ as $n \to \infty$.

7.3 Sequence of measurable functions

Chapter 8

Integration: Part I

8.1 The integral for non-negative functions

If $f : \Omega \to \overline{\mathbb{R}}$ is s.t. $f(\omega) \geq 0$ for any $\omega \in \Omega$, f is said to be nonnegative.

Definition 8.1 (Finite partitions)**.** Let Ω be a set. A partiton of Ω is a disjoint collection $\{A : A \in P\}, P \subset \mathcal{P}(\Omega)$, s.t. $\cup_{A\in P} A = \Omega$. That is, a partition of Ω is a disjoint collection of subets of Ω whose union is Ω. If ξ is a partition of Ω, a set $A \in \xi$ is referred to as an atom of ξ. A partition ξ of Ω is said to be finite, if it contains a finite number of atoms.

Example 8.1 (Finite partition)**.** Let $\Omega = \{0, 1, \ldots, N\}, N \in \mathbb{N}$. Then, $\xi = \{\{\omega\} : w \in \Omega\}$ is a finite partition of Ω. (Partition contains $N + 1$ elements).

Definition 8.2 ($Z_0^{\mathcal{F}}$)**.** Let $(\Omega, \mathcal{F})$ be a measurable space. We use the notation $Z_0^{\mathcal{F}}(\Omega) = Z_0^{\mathcal{F}}$ for the set which contains all the finite partitions of Ω with atoms from $\mathcal{F}$. That is,

$$Z_0^{\mathcal{F}} = \{\xi : \xi \text{ is finite partition of } \Omega \text{ s.t. for any A } \in \xi, A \in \mathcal{F}\}.$$

Definition 8.3 (Integral for a nonnegative standard simple function)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f : \Omega \to \overline{\mathbb{R}}$ be nonnegative and $\mathcal{F}$ measurable. Then, we define

$$S_\mu^f(\xi) = \sum_{A\in\xi} (\inf_{\omega\in A} f(\omega))\mu(A), \quad \xi \in Z_0^{\mathcal{F}},$$

Essentially, $S_\mu^f(\xi)$ approximates the integral of f by considering the smallest value f takes on each piece of the partition and multiplying this by the measure of the piece. And

$$\int_\Omega f(\omega)\mu(d\omega) = \sup_{\xi\in Z_0^{\mathcal{F}}} S_\mu^f(\xi).$$

The integral of f over Ω with respect to μ, is the supremum of $S_\mu^f(\xi)$ over all possible partitions ξ of Ω in $Z_0^{\mathcal{F}}$. This definition captures the idea of the integral as the limit of finer and finer approximations of f by simple functions. Upon the latter definition, we deduce the integral for a (nonnegative) standard simple function (cf. Definition 7.4).

Proposition 8.1. TODO

My Example 8.1 (Integral of a nonnegative standard simple function)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with $\Omega = \{a, b, c, d\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, and μ is the counting measure, i.e., $\mu(A)$ is the number of elements in A. Let $f : \Omega \to \overline{\mathbb{R}}$,

$$f(\omega) = \begin{cases} 1 & \text{if } \omega = a, \\ 2 & \text{if } \omega = b, \\ 3 & \text{if } \omega = c, \\ 0 & \text{if } \omega = d \end{cases}$$

Consider the partition $\xi = \{\{a\}, \{b\}, \{c\}, \{d\}\}$. $\inf_{\omega \in \{a\}} f(\omega) = 1$, $\inf_{\omega \in \{b\}} f(\omega) = 2$, $\inf_{\omega \in \{c\}} f(\omega) = 3$, $\inf_{\omega \in \{d\}} f(\omega) = 4$. Since each singleton set in ξ as measure of 1 under μ,

$$S_\mu^f(\xi) = (1 \times 1) + (2 \times 1) + (3 \times 1) + (0 \times 1) = 6$$

if $\sup_{\xi \in Z_0^{\mathcal{F}}} S_\mu^f = 6$, which I think it should be, then $\int_\Omega f(\omega)\mu(d\omega) = 6$.

Example 8.2. Example 8.2 interesting and clear, TODO.

Proposition 8.2 (Monotone convergence theorem)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f_n : \Omega \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$, be a sequence of nonnegative $\mathcal{F}$ measurable functions s.t. for any $\omega \in \Omega, f_n(\omega) \uparrow f(\omega)$ for some $f : \Omega \to \overline{\mathbb{R}}$. Then,

$$\int_\Omega f_n(\omega)\mu(d\omega) \uparrow \int_\Omega f(\omega)\mu(d\omega).$$

Proposition 8.3 (The integral of nonnegative functions is linear)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space, $f, g : \Omega \to \overline{\mathbb{R}}$ be two nonnegative and $\mathcal{F}$ measurable functions. Given $\alpha, \beta \in [0, \infty)$ we have that

$$\int_\Omega (\alpha f + \beta g)(\omega)\mu(d\omega) = \alpha \int_\Omega f(\omega)\mu(d\omega) + \beta \int_\Omega g(\omega)\mu(d\omega).$$

As a consequence of the latter two proposition we have the following result:

Proposition 8.4. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f_i : \Omega \to \overline{\mathbb{R}}, i \in \mathbb{N}$, be a sequence of nonnegative $\mathcal{F}$ measurable functions, then

$$\int_\Omega \left(\sum_{i \in \mathbb{N}} f_i \right)(\omega)\mu(d\omega) = \sum_{i \in \mathbb{N}} \left(\int_\Omega f_i(\omega)\mu(d\omega) \right).$$

Definition 8.4 (True almost everywhere $(a.e.)$)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Suppose that for any $\omega \in \Omega, S(\omega)$ is a statment on Ω. We say S is true μ almost everywhere $(a.e.)$ if $\mu(\{\omega : S(\omega) \text{ is false}\}) = 0$.

Example 8.3 $(\mu(a.e.))$**.** Interesting and clear. TODO.

Proposition 8.5. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Assume that $f, g : \Omega \to \overline{\mathbb{R}}$ be two nonnegatibe and $\mathcal{F}$ measurable functions.

(i) If $\mu(\{\omega : f(\omega) > 0\}) > 0$, then $\int_\Omega f(\omega)\mu(d\omega) > 0$;

(ii) If $\int_\Omega f(\omega)\mu(d\omega) < \infty$, then $f < \infty$ μ $a.e.$;

(iii) If $f \leq g$ μ $a.e.$, then $\int_\Omega f(\omega)\mu(d\omega) \leq \int_\Omega g(\omega)\mu(d\omega)$;

(iv) If $f = g$ μ $a.e.$, then $\int_\Omega f(\omega)\mu(d\omega) = \int_\Omega g(\omega)\mu(d\omega)$.

8.2 Integrable functions

We recall the definiton of the positive (f^+) and negative (f^-) parts of a function (cf. Definition 7.6). Pay attention, f^- is basically the negative part of the function, but reflected by the x-axis. The result is positive. Also see 7.2

Definition 8.5 (Integral of an integrable function)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f : \Omega \to \overline{\mathbb{R}}$ be a $\mathcal{F}$ measurable function. The integral of f is defined by:

$$\int_\Omega f(\omega)\mu(d\omega) = \int_\Omega f^+(\omega)\mu(d\omega) - \int_\Omega f^-(\omega)\mu(d\omega),$$

unless $\int_\Omega f^+(\omega)\mu(d\omega) = \int_\Omega f^-(\omega)\mu(d\omega) = \infty$, in which case $\int_\Omega f(\omega)\mu(d\omega)$ is not defined.
If both $\int_\Omega f^+(\omega)\mu(d\omega) < \infty$ and $\int_\Omega f^-(\omega)\mu(d\omega) < \infty$, f is said to be integrable.

(NOTE) This assumption is definied upon the measure μ, i.e., if one wants to further refer to the measure of integration one specifies that f is integrable with respect to μ.

Proposition 8.6 (Generalisation of the condition for f to be integrable)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f : \Omega \to \overline{\mathbb{R}}$ be $\mathcal{F}$ measurable. Then, f is integrable if and only if $\int_\Omega |f(\omega)|\mu(d\omega) < \infty$.

Proposition 8.7 (Extension (cf. (iii) Proposition 8.5))**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f, g : \Omega \to \overline{\mathbb{R}}$ be $\mathcal{F}$ measurable. If f and g are integrable and $f \leq g$ *a.e.*, then, $\int_\Omega f(\omega)\mu(d\omega) \leq \int_\Omega g(\omega)\mu(d\omega)$.

Proposition 8.8 (Extension (c.f. Proposition 8.3))**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space, $f, g : \Omega \to \overline{\mathbb{R}}$ be two integrable and $\mathcal{F}$ measurable functions. Then, for any $\alpha, \beta \in \mathbb{R}$ we have that $\alpha f + \beta g$ is integrable and

$$\int_\Omega (\alpha f + \beta g)(\omega)\mu(d\omega) = \alpha \int_\Omega f(\omega)\mu(d\omega) + \beta \int_\Omega g(\omega)\mu(d\omega).$$

8.3 Fatou's lemma and Lebesgue's dominated convergence theorem

Proposition 8.9 (Fatou's lemma)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f_n : \Omega \to \overline{\mathbb{R}}, n \in \mathbb{N}$, be a sequence of nonnegative and $\mathcal{F}$ measurable function. Then,

$$\int_\Omega \liminf_{n\to\infty} f_n(\omega)\mu(d\omega) \leq \liminf_{n\to\infty} \int_\Omega f_n(\omega)\mu(d\omega).$$

8.4 Integration over measurable sets

Tool 8.1 (Integration over $\bigcup_{i\in I} A_i$)**.** (From Ex. 8.9). Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f : \Omega \to \overline{\mathbb{R}}$ be a $\mathcal{F}$ measurable function. Suppose that either f is nonnegative of integrable and let $\{A_i : i \in I\} \subset \mathcal{F}$ be disjoint, where $I \subset \mathbb{N}$. Then

$$\int_{\bigcup_{i\in I} A_i} f(\omega)\mu(d\omega) = \sum_{i\in I} \int_{A_i} f(\omega)\mu(d\omega).$$

Chapter 9

Integration: Part II

9.1 Pushforward measure

Definition 9.1 (Pushforward function)**.** Let $(\Omega, \mathcal{F})$ and $(\Omega^*, \mathcal{F}^*)$ be two measurable spaces and $g : \Omega \to \Omega^*$ be $\mathcal{F}/\mathcal{F}^*$ measurable. Let μ be a measure on $\mathcal{F}$. Define the function

$$\mu g^{-1}(A^*) = \mu(g^{-1}(A^*)) = \mu(\{\omega \in \Omega : g(\omega \in A^*)\}), \quad A^* \in \mathcal{F}^*.$$

The measure μg^{-1} is referred to as the pushforward measure of μ. This means that μg^{-1} measures, in terms of μ, the pre-image of each set A^* under g. Hence, μ is a valid measure on $(\Omega^*, \mathcal{F}^*)$!! It provides a way to "transfer" the measure from $(\Omega, \mathcal{F})$ to $(\Omega^*, \mathcal{F}^*)$ via the function g.

Proposition 9.1. TODO

9.2 Densities

Proposition 9.2 (ν is a measure on $\mathcal{F}$)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\phi : \Omega \to \overline{\mathbb{R}}$ be a nonnegative and $\mathcal{F}$ measurable function. Then, ν defined by

$$\nu(A) = \int_A \phi(\omega)\mu(d\omega), \quad A \in \mathcal{F},$$

is a measure on $\mathcal{F}$

Definition 9.2 (ϕ, density of ν in respect to μ)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and ν be a measure on $\mathcal{F}$. A nonnegative and $\mathcal{F}$ measurable funtion $\phi : \Omega \to \overline{\mathbb{R}}$ is said to be a density of ν with respect to μ if for any $A \in \mathcal{F}, \nu(A) = \int_A \phi(\omega)\mu(d\omega)$.

Proposition 9.3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Suppose that ν is a measure on $\mathcal{F}$ with density ϕ with respect to μ. Then

(i) for any nonnegative and $\mathcal{F}$ measurable function f,

$$\int_A f(\omega)\nu(d\omega) = \int_A f(\omega)\phi(w)\nu(d\omega), \quad A \in \mathcal{F};$$

(ii) f is integrable with respect to ν if and only if $f\phi$ (the product of the two functions) is integrable with respect to μ. This is clear in (i).

(iii) if $f\phi$ is integrable with respect to μ, then (i) holds.

9.3 Integration with respect to the Lebesgue measure on the real line

Definition 9.3. Consider the measure space $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \lambda)$, where λ is the Lebesgue measure on the Borel σ-field $\mathfrak{B}(\mathbb{R})$. In accordance with Definition 8.5, a $\mathfrak{B}(\mathbb{R})$ measurable function $f : \mathbb{R} \to \overline{\mathbb{R}}$ is Lebesgue integrable if $\int_{\mathbb{R}} |f(x)|\lambda(dx) < \infty$. The integral of f with respect to λ is denoted with $\int_{\mathbb{R}} f(x)dx$, i.e., $\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} f(x)\lambda(dx)$. If $E \subset \mathbb{R}$ and $\lambda|_E$ is the restriction of λ to $\mathfrak{B}(E)$ (cf. Definiton 4.2), then a $\mathfrak{B}(E)$ measurable function $f : E \to \overline{\mathbb{R}}$ is referred to as Lebesgue integrable if $\int_E |f(x)|\lambda|_E(dx) < \infty$. Also in this case we write $\int_E |f(x)|\lambda|_E(dx) = \int_E f(x)dx$.

In accordance with the fact that the Lebesgue measure of a single point is zero, we adapt the following definition.

Definition 9.4. TODO. Interesting but easy and well known.

We review the definition of a Riemann integrable function:

Definition 9.5 (title)**.**

9.4 Lecture

Partial integration and substitution TODO.

Exercise 9.1 (9.6)**.** ν is a measure with density ϕ with respect to μ. f nonnegative and $\mathcal{F}$ measurable. Prove:

(i) $\int_A f(\omega)\nu(d\omega) = \int_A f(\omega)\phi(w)\mu(d\omega)$

NOTE $\nu(d\omega) = \phi(\omega)\mu(d\omega)$ short notation for ν has density ϕ:

1. Definition of ν having a density ϕ with respect to μ: When we say that ν has a density ϕ with respect to μ, it means that for any measurable set $A \in \mathcal{F}$, the measure ν of A can be computed as:

$$\nu(A) = \int_A \phi(\omega)\mu(d\omega).$$

This is the integral of the function ϕ over the set A, with respect to the measure μ.

2. Notation $\nu(d\omega) = \phi(\omega)\mu(d\omega)$: This notation is shorthand and is used to express how ν acts on infinitesimal elements in a manner analogous to how μ acts, but scaled by the function ϕ. It is essentially saying that for a small element $d\omega$, the measure $\nu(d\omega)$ is given by $\phi(\omega)\mu(d\omega)$.

3. Clarification on $\int_{d\omega} \phi(\omega)\mu(d\omega)$: The correct notation or expression should not involve integrating over an "infinitesimal element" $d\omega$. The differential notation $\nu(d\omega) = \phi(\omega)\mu(d\omega)$ is symbolic and used to express the relationship between ν and μ at a small scale, rather than an actual operation.

In summary, $\nu(d\omega) = \phi(\omega)\mu(d\omega)$ is a concise way to denote that ν is derived by weighting μ by the density ϕ, and this relationship is used to transform integrals with respect to ν into integrals with respect to μ weighted by ϕ.

(ii) f integrable w.r.t. $\nu \iff f\phi$, $(f(\omega)\phi(\omega))$, integrable w.r.t. μ.

(iii) if either of the two statments in (ii) holds, then (i) holds.

Proof:
(i). Let f be a standard simple function, $f = \sum_{n=1}^{\mathbb{N}} \alpha_i \mathbb{1}_{Ai}$, then

$$\int_A f(\omega)\nu(d\omega) = \int_A (\sum_{n=1}^{\mathbb{N}} \alpha_i \mathbb{1}_{Ai}(\omega))\nu(d\omega) = \sum_{n=1}^{\mathbb{N}} \alpha_i \int_A \mathbb{1}_{Ai}(\omega)\nu(d\omega) = \sum_{n=1}^{\mathbb{N}} \alpha_i \int_\Omega \mathbb{1}_A(\omega)\mathbb{1}_{Ai}(\omega)\nu(d\omega)$$

$$= \sum_{n=1}^{\mathbb{N}} \alpha_i \int_\Omega \mathbb{1}_{A\cap A_i}(\omega)\nu(d\omega) = \sum_{n=1}^{\mathbb{N}} \alpha_i \nu(A \cap A_i) = \sum_{n=1}^{\mathbb{N}} \alpha_i \int_{A\cap A_i} \phi(\omega)\mu(d\omega) = \sum_{n=1}^{\mathbb{N}} \alpha_i \int_A \mathbb{1}_{A_i}(\omega)\phi(\omega)\mu(d\omega)$$

$$= \int_A \sum_{n=1}^{\mathbb{N}} \alpha_i \mathbb{1}_{A_i}(\omega)\phi(\omega)\mu(d\omega) = \int_A f(\omega)\phi(w)\mu(d\omega).$$

Hence we have verified (i) if f is standard and simple.
In order to verify it for nonnegative functions:
(IMPORTANT; TOOL, TO ADD) Recall (chapter 7): Any f nonnegative and $\mathcal{F}$ measurable can be approximated by a standard simple function, i.e., $\exists (f_n)_{n\in\mathbb{N}}$ s.t. $f_n(\omega) \uparrow f(\omega)$. By the monotone convergence theorem,

$$\int_\Omega f(\omega)\nu(d\omega) = \lim_{n\to\infty} \int_\Omega f_n(\omega)\nu(d\omega) = \lim_{n\to\infty} \int_\Omega f_n(\omega)\phi(\omega)\mu(d\omega)$$

f_n converges to f

$$\overset{\text{(again monotone convergence)}}{=} \int_\Omega f(\omega)\phi(\omega)\mu(d\omega).$$

This proves (i).

(ii). $\int_A |f(\omega)|\nu(d\omega) < \infty$ (definition of integrability), $= \int_A |f(\omega)\phi(\omega)|\mu(d\omega)$, and we know that the equality holds by (i). This shows (ii).

(iii). Recall $f^+ = max(f, 0)$, $f^- = max(-f, 0)$. Positive and negative parts of f. Cuts out all negative points. We know,

$$f(\omega) = f^+ - f^-(\omega).$$

f integrable w.r.t. ν implies that,

$$\int_\Omega f(\omega)\nu(d\omega) = \int_\Omega f^+\nu(d\omega) - \int_\Omega f^-(\omega)\nu(d\omega).$$

By (i) applied to f^+ and f^-,

$$= \int_\Omega f^{(+)}(\omega)\phi(\omega)\mu(d\omega) - \int_\Omega f^-(\omega)\phi(\omega)\mu(d\omega) = \int_\Omega f(\omega)\phi(\omega)\mu(d\omega).$$

Exercise 9.2 (9.7)**.** b) TODO
$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-(\frac{x^2+y^2}{2})} d(x, y)$, continuoius as composition of continuous functions, and nonnegative. Fobini - Tonelli Theorem:

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{x^2}{2}}(\int_{\mathbb{R}} e^{-\frac{y^2}{2}} dy)dx.$$

$$= \frac{1}{2\pi}(\int_{\mathbb{R}} e^{\frac{-x^2}{2}} dx)^2.$$

u $= \frac{x}{\sqrt{2}}$ substitute

$$= \frac{1}{2\pi}(\int_{\mathbb{R}} e^{-u^2}\sqrt{2}du)^2.$$

$$= \frac{1}{\pi}(\int_{\mathbb{R}} e^{-u^2} du)^2 = \frac{\pi}{\pi}.$$

Remember Gaussian integral:

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}.$$

Chapter 10

General notions in Probability

10.1 Probability spaces

Definition 10.1. Let $(\Omega, \mathcal{F})$ be a measurable space. A probability $\mathbb{P}$ on $\mathcal{F}$ is a measure on $\mathcal{F}$ s.t. $\mathbb{P}(\Omega) = 1$. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is referred to as a probability space.

Example 10.1. Let Ω be a finite and nonempety set. Define

$$\mathbb{P}(A) = \frac{\#A}{\#\Omega}, \quad A \in \mathcal{P}(\Omega), .$$

Where $\mathcal{P}(\Omega)$ is the power set on Ω. Then, $\mathbb{P}$ is a probability on $\mathcal{P}(\Omega)$.

Example 10.2. Let C be a set s.t. $\#C = 52$. Suppose that

$$C = S_1 \cup S_2 \cup S_3 \cup S_4,$$

with $\{S_1, S_2, S_3, S_4\}$ disjoint and s.t. $\#S_i = 13$ for all $i = 1, 2, 3, 4$. We remain in the setting of the previous example with

$$\Omega = \{A \subset C : \#A = 5\},$$

and $\mathbb{P}$ on $\mathcal{P}(\Omega)$ defined as in exercise 10.1. Upon exercise 1.1, we already know that $\#\Omega = \binom{52}{5}$. Let

$$A_i = \{A \subset S_i : \#A = 5\}, \quad i = 1, 2, 3, 4,$$

TODO

10.2 Random variables and random vectors

Definition 10.2 (Random variable)**.** Let $(\Omega, \mathcal{F})$ be a measurable space. A map $X : \Omega \to \mathbb{R}$ is referred to as a random variable on $(\Omega, \mathcal{F})$ if it if $\mathcal{F}/\mathfrak{B}(\mathbb{R})$ measurable.

Definition 10.3 (Random vector)**.** Let $(\Omega, \mathcal{F})$ be a measurable space. A map $X : \Omega \to \mathbb{R}^k$ is referred to as a random vector on $(\Omega, \mathcal{F})$ if it is $\mathcal{F}/\mathfrak{B}(\mathbb{R}^k)$ measurable.

Proposition 10.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X be a random vector on $(\Omega, \mathcal{F})$. A random variable Y on $(\Omega, \mathcal{F})$ is $\sigma(X)$ measurable if and only if there exists a function $f : \mathbb{R}^k \to \mathbb{R}$ which is $\mathfrak{B}(\mathbb{R}^k)$ measurable s.t. $Y = f(X)$.

Definition 10.4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The distribution or law of a random vector on $(\Omega, \mathcal{F})$ is the pushforward measure $P_X = \mathbb{P}X^{-1}$ on $\mathfrak{B}(\mathbb{R}^k)$ (cf. Definition 9.1). In particular, for any $B \in \mathfrak{B}(\mathbb{R}^k)$, we use the simplified notation

$$\{\omega \in \Omega : X(\omega) \in B\} = \{X \in B\},$$

and hence

$$P_X(B) = \mathbb{P}(X^{-1}(B)) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in B\}) = \mathbb{P}(X \in B).$$

For now, unless mentioned otherwise, if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, any random vector X is a random vector on $(\Omega, \mathcal{F})$, i.e., a $\mathcal{F}$ measurable function with values in $\mathbb{R}^k$.

10.3 Discrete laws

Definition 10.5 (Discrete random vector)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random vector is referred to as discrete if there exists a countable set $E = E_1 \times \ldots \times E_k \subset \mathbb{R}^k$ s.t. $P_X(E) = 1$. That is to say that the law of X has a countable support.

Proposition 10.2 (Discrete random vector)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random vector X is discrete if and only if

$$P_X = \sum_{x \in E} p_x \delta_x, \quad p_x = \mathbb{P}(X = x),$$

for some countable set $E = E_1 \times \ldots \times E_k \subset \mathbb{R}^k$. In particular, for any $B \in \mathfrak{B}(\mathbb{R}^k)$, $P_X(B) = \sum_{x \in B \cap E} p_x$.

Proof of Proposition 10.2. Suppose that X is discrete. Let $B \in \mathfrak{B}(\mathbb{R}^k)$. We have that

$$P_X(B) = P_X(B \cap E) = \mathbb{P}(X \in B \cap E) = \mathbb{P}(\bigcup_{x \in B \cap E} \{X = x\})) = \sum_{x \in B \cap E} p_x = \sum_{x \in E} p_x \delta_x(B).$$

with respect to the other direction, if P_X is given as in Prop. 10.2, then

$$1 = P_X(\mathbb{R}^k) = \sum_{x \in E} p_x \delta_x(\mathbb{R}^k) = \sum_{x \in E} p_x = \mathbb{P}(X \in E) = P_x(E),$$

i.e., X is a discrete random vector according to Def. 10.5.

Example 10.3 (Tail, head)**.** Let $\Omega = \{t, h\}$ and

$$X(\omega) = \begin{cases} 0, & \text{if } w = t, \\ 1, & \text{if } w = h. \end{cases}$$

Then, X is a random variable on $(\Omega, \mathcal{P}(\Omega))$.

Explanation: for the cases $X^{-1}(0) = t$ and $X^{-1}(1) = h$ it is clear. Consider then $X^{-1}(\omega) = \emptyset \in \mathcal{P}(\Omega)$ for any $\omega \notin \{1, 2\}$.

Suppose that $\mathbb{P}$ is a probability on $\mathcal{P}(\Omega)$ s.t. $\mathbb{P}(X^{-1}(0)) = \mathbb{P}(X = 0) = 1 - p$ and $\mathbb{P}(X = 1) = p$. Clearly, $\mathbb{P}(X \in \{0, 1\}) = P_X(\{0, 1\}) = 1$. By Prop. 10.2, we deduce that the law of X is given by

$$P_X = (1 - p)\delta_0 + p\delta_1.$$

That is, for any $B \in \mathfrak{B}(\mathbb{R})$,

$$P_X(B) = (1 - p)\delta_0(B) + p\delta_1(B) = \begin{cases} 0, & \text{if } 0 \notin B \text{ and } 1 \notin B \\ 1 - p, & \text{if } 0 \in B \text{ and } 1 \notin B \\ p, & \text{if } 0 \notin B \text{ and } 1 \in B \\ 1, & \text{if } 0 \in B \text{ and } 1 \in B \end{cases}.$$

For example, for $B = (2, 4]$, $X^{-1}(B) = \emptyset \in \mathcal{P}(\Omega)$, and $P_X(B) = P_X(\emptyset) = 0$. Also note that, for example, $\mathbb{P}(X = 0) = \mathbb{P}(X^{-1}(0)) = \mathbb{P}(t) = 1 - p$.

Classical examples of discrete probability distributions

My Example 10.1 (Examples of discrete probability distributions)**.**
Discrete uniform: $E \subset \mathbb{R}$ is a finite set s.t. $\#E = n$, and $p_x = \frac{1}{n}$ for any $x \in E$.
Bernoulli: $E = \{0, 1\}$ and $p_0 = 1 - p$ and $p_1 = p$, $p \in [0, 1]$.
Binomial: $E = \{0, 1, \ldots, n\}$, $n \in \mathbb{N}$ and $p_x = \binom{n}{x} p^x (1 - p)^{n - x}$, $p \in [0, 1]$.
Geometric: $E = \mathbb{N}$ and $p_x = (1 - p)^{x - 1} p$, $p \in (0, 1)$.
Poisson: $E = \mathbb{N} \cup \{0\}$ and $p_x = (\frac{\lambda}{x!}) e^{-\lambda}$, $\lambda > 0$.
Multinomial: TODO: Write multinomial discrete probability distribution.

10.4 Continuous laws

10.5 Expectation

Definition 10.6 (Expectation of X)**.** TODO: Write definition

Proposition 10.3 (Expectation of $f(X)$)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X be a random vector. Then, for any nonnegative and $\mathfrak{B}(\mathbb{R}^k)$ measurable map $f : \mathbb{R}^k \to \overline{\mathbb{R}}$,

$$\mathbb{E}[f(X)] = \int_{\mathbb{R}^k} f(x) P_X(dx).$$

In addition, if f is not necessarily nonnegative, this proposition holds if $\mathbb{E}[|f(X)|] < \infty$.

Chapter 11

Collection of random vectors

11.1 Independence

Definition 11.1 (Independent sub-σ-fields)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be n sub-σ-fields on Ω. $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are said to be independent if for any $A_1 \in \mathcal{A}_1, \ldots, A_n \in \mathcal{A}_n$,

$$\mathbb{P}(A_1 \cap \ldots \cap A_n) = \mathbb{P}(A_1) \times \ldots \times \mathbb{P}(A_n).$$

11.2 Sums of independent random vectors

11.3 Gauss vectors

Definition 11.2 (Gauss vector)**.** A random vector $X = (X_1, \ldots, X_k)$ is said to be a Gauss vector if and only if for any $v \in \mathbb{R}^k$, the random variable

$$v^t X = v_1 X_1 + \ldots + v_k X_k,$$

is Gaussian.

Remark 11.1. TODO

11.4 Lecture

Exercise 11.1. $X_1, \dots, X_n$ independent discrete uniform, on $\{1, \dots, p\}$, $p \in \mathbb{N}$. Meaning that each X_i can take any value in $\{1, \dots, p\}$ with equal probability $\frac{1}{p}$. Find the law of $M = max\{X_1, \dots, X_n\}$.

Note, Let X be discrete uniform on $\{1, \dots, p\}$, then the support of X is $\{1, \dots, p\}$. (The set of all values s.t. $\mathbb{P}(X = x) > 0$)

The support of M also is $\{1, \dots, p\}$. Why?

$$\mathbb{P}(M \notin \{1, \dots, p\}) \leq \mathbb{P}(\bigcup_{i=1}^{n} \{X_i \notin \{1, \dots, p\}\}) \leq \sum_{i=1}^{n} \mathbb{P}(X_i \notin \{1, \dots, p\}) = 0.$$

$$\Rightarrow \mathbb{P}(M \notin \{1, \dots, p\}) = 0 \Rightarrow \mathbb{P}(M \in \{1, \dots, p\}) = 1.$$

Let $t \in \mathbb{R}$,

$$\mathbb{P}(M \leq t) = F_M(t) = \mathbb{P}(max\{X_1, \dots, X_n\} \leq t) = \mathbb{P}(\bigcap_{1=1}^{n} \{x_i \leq t\}).$$

$$\overset{\text{independence}}{=} \prod_{i=1}^{n} \mathbb{P}(X_i \leq t) = \prod_{i=1}^{n} P_{X_i}((-\infty, t]) = \mathbb{P}(X \leq t)^n.$$

Then, the law of X is,

$$\mathbb{P}(X \leq t) = \begin{cases} 0, & t < 1 \\ \frac{\#\{k : k \leq t\}}{p} & 1 \leq t \leq p \\ 1, & t \geq p \end{cases}.$$

And the law of M,

$$F_M(t) = \mathbb{P}(X \leq t)^n = \begin{cases} 0, & t < 1 \\ (\frac{\#\{k : k \leq t\}}{p})^n, & 1 \leq t \leq p \\ 1, & t > p \end{cases}.$$

Also note, as it is discrete,

$$F_M(i) - F_M(i+1) = \sum_{k=1}^{i} \mathbb{P}(M = k) - \sum_{k=1}^{i-1} \mathbb{P}(M = k) = \mathbb{P}(M = i).$$

Exercise 11.2. X_1, X_2 Poisson with parameters λ and μ respectively. What is the law of $X_1 + X_2$?

Note in general, for $X_1 + X_2 = z$:

Discrete case, $E_1 + E_2 = E_{sum}$, get support of X_1, X_2, and then, $\forall z \in Z_{SUM}$,

$$P_Z(\{z\}) = \sum_{X_2 \in E_2} P_{X_1}(\{z - x_2\}) P_{X_2}(\{x_2\}).$$

Continuous case, (densities ϕ_1, ϕ_2), density of Z

$$\phi(z) = \int_{\mathbb{R}} \phi_1(z - x_2) \phi_2(x_2) dx_2.$$

Let $E_1 = \mathbb{N} \cup \{0\}$, $E_2 = \mathbb{N} \cup \{0\}$, the support is

$$E_1 + E_2 = \mathbb{N} \cup \{0\} \overset{\text{by definition}}{=} \{x_1 + x_2 : x_1 \in E_1, x_2 \in E_2\}.$$

Knowing the support helps us, we know where we can sum. Here we know that for $k > z$, $P_{X_1} = 0$. We then apply the formula for the discrete case:

$$P_Z(\{z\}) = \sum_{k \in \mathbb{N} \cup \{0\}} P_{X_1}(\{z - k\}) P_{X_2}(\{k\}) \overset{\text{for } k > z,\ P_{X_1} = 0}{=} \sum_{k=0}^{z} P_{X_1}(\{z - k\}) P_{X_2}(\{k\})$$

$$\overset{\text{plug in distribution}}{=} \sum_{k=0}^{z} e^{-\lambda} \frac{\lambda^{(z-k)}}{(z-k)!} e^{-\mu} \frac{\mu^k}{k!} = e^{-(\lambda+\mu)} \sum_{k=0}^{z} \frac{1}{(z-k)!k!} \mu^k \lambda^{(z-k)}$$

Use binomial theorem, $(\mu+\lambda)^z = \sum_{k=0}^{z} \binom{z}{k} \mu^k \lambda^{(z-k)}$. Multiply by $z!$ inside and divide outside of the sum.

$$= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{k=0}^{z} \frac{1 \times z!}{(z-k)!k!} \mu^k \lambda^{(z-k)} = e^{-(\lambda+\mu)} \frac{(\mu+\lambda)^z}{z!}.$$

Hence,

$$P_Z(\{z\}) = e^{-(\lambda+\mu)} \frac{(\mu+\lambda)^z}{z!}.$$

Note, we see that the law is the same as before, with the parameters added.

Chapter 12

Mock exam 1

Solve with the pdf of the mock exam on the side.

Notation: We recall some of the terminology:

- Given a nonempty set Ω, $\mathcal{P}(\Omega)$ is the power set on Ω;
- $\mathfrak{B}(\mathbb{R}^k)$ denotes the Borel σ-field on $\mathbb{R}^k$, $k \geq 1$;
- The measure
$$\mu(A) = \begin{cases} \#A, & \text{if } A \text{ is finite} \\ \infty, & \text{otherwise,} \end{cases} \qquad A \in \mathcal{P}(\Omega),$$
is referred to as the counting measure on $\mathcal{P}(\Omega)$;
- Given a measurable space $(\Omega, \mathcal{F})$ and $x \in \Omega$, we write δ_x for the measure
$$\mathcal{F} \ni A \mapsto \delta_x(A) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 12.1.

(a) Refer to Def. 4.1.

(b) Measure on $\mathcal{F}$ (cf. Def 5.1).

- (i) $\mu_1(\emptyset) = C\mu(\emptyset) = 0$;

  (ii) We know that item ii holds for the counting measure by definition. For our redefined counting measure,
$$\mu_1(\bigcup_{i\in\mathbb{N}} A_i) = C\mu(\bigcup_{i\in\mathbb{N}} A_i) = C\sum_{i\in\mathbb{N}} \mu(A_i) = \sum_{i\in\mathbb{N}} C\mu(A_i) = \sum_{i\in\mathbb{N}} \mu_1(A_i).$$
- (i) $\mu_2(\emptyset) = \int_\emptyset f(\omega)\mu(d\omega) = 0$;

  (ii)
$$\mu_2(\bigcup_{i\in\mathbb{N}} A_i) = \int_{\bigcup_{i\in\mathbb{N}} A_i} f(\omega)\mu(d\omega) \overset{\text{Tool } 8.1}{=} \sum_{i\in\mathbb{N}} \int_{A_i} f(\omega)\mu(d\omega) = \sum_{i\in N} \mu_2(A_i).$$
- (i) $\mu_3(\emptyset) = \frac{1}{2} + \lambda(\emptyset) = \frac{1}{2}$.

We see that μ_3 is clearly not a measure on $\mathcal{F}$.

(c) Probability measure cf. Def. 10.1.

- $$P_1(\mathbb{R}) = \int_{\mathbb{R}} \mathbb{1}_{[0,\infty)}(x)e^{-x}dx = \int_{[0,\infty)} e^{-x}dx = (-e^{-x})|_0^\infty = (0-(-1)) = 1.$$

- $$P_2(\mathbb{N}) = \int_{\mathbb{N}} \mathbb{1}_{\{0,1\}}(x)x^2\mu(dx) = \int_{\{0,1\}} x^2\mu(dx) = 0^2 \cdot \mu(\{0\}) + 1^2 \cdot \mu(\{1\}) = 0 \cdot 1 + 1 \cdot 1 = 1.$$

- **Tool 12.1** (Integral with respect to a dirac measure)**.**

  $$\int_\Omega f(x)\delta_\omega(dx) = f(\omega).$$

  $$P_3(\mathbb{R}) = \int_{\mathbb{R}} x^2\mu(dx) = \int_{\mathbb{R}} x^2(\delta_{-1}(dx) + \delta_1(dx)) = \int_{\mathbb{R}} x^2\delta_{-1}(dx) + \int_{\mathbb{R}} x^2\delta_1(dx)$$
  $$= (-1)^2 + 1^2 = 2.$$

We see that P_3 is not a probability measure on $\mathcal{B}$.

(d) Calculate:

1. λ Lebesgue measure on $\mathfrak{B}(\mathbb{R})$.

$$\int_{\mathbb{R}} \mathbb{1}_{[-1,1]}(x)\lambda(dx) = \int_{[-1,1]} 1\lambda(dx) = 1 \cdot \lambda([-1,1]) = 1 \cdot 2 = 2.$$

2. $P(A) = (1-p)\delta_0(A) + p\delta_1(A)$, $A \in \mathfrak{B}(\mathbb{R})$, $p \in (0,1)$.

$$\int_{\mathbb{R}} (x-p)^2P(dx) = \int_{\mathbb{R}} (x-p)^2((1-p)\delta_0(dx)) + \int_{\mathbb{R}} (x-p)^2(p\delta_1(dx)) = (0-p)^2(1-p) + (1-p)^2p$$
$$= p^2 - p^3 + p + p^3 - 2p^2 = p - p^2.$$

3. λ Lebesgue measure on $\mathfrak{B}(\mathbb{R})$. As the Lebesgue measure of a singleton is equal to 0

$$\int_{\mathbb{N}} \log(x)\lambda(dx) = 0.$$

(e) Refer to Def. 10.5, Prop. 10.2.

1. The support is $E = \{0,1\}$, countable.

$$P_1(E) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1.$$

2. As F_X is continuous, $\mathbb{P}(X = x) = 0$, $\forall x \in \mathbb{R}$. This means that there exists no countable set E s.t. $P_X(E) = 1$.
3. The support is $E = \{0,1\}$, countable.

$$P_3(A) = \mathbb{P}(X = 1) \cdot \delta_1(A) + \mathbb{P}(X = 0) \cdot \delta_0(A).$$

Where $\mathbb{P}(X = 1) = \mathbb{P}(X^{-1}(1)) = \mathbb{P}(h)$.

P_2 is not a discrete law.

(f) TODO: Understand and complete. Cf. Sec. 11.3.

Exercise 12.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X be a discrete random variable on Ω with support $\{-1, 1\}$ and law

$$P_X(A) = \frac{1}{2}\delta_{-1}(A) + \frac{1}{2}\delta_1(A).$$

(a) $\mathbb{P}(X = -1) = P_X(\{-1\}) = \frac{1}{2}$, $\mathbb{P}(X = 1) = \frac{1}{2}$

(b) We have that $f(X) = |X|^2$, cf. Prop. 10.3

$$\mathbb{E}(|X|^2) = \int_{\{-1,1\}} |x|^2 P_X(dx) = |-1|^2 \cdot P_X(\{-1\}) + |1|^2 \cdot P_X(\{1\}) = 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1.$$

(c) $\mathbb{E}[X] = -\frac{1}{2} + \frac{1}{2} = 0$. We than know that

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 1 - 0 = 1.$$

(d) We can find the support of $\frac{X+1}{2}$.

$$\mathbb{P}(\frac{X+1}{2} = \omega) \neq 0 \Rightarrow X = 2\omega - 1 = \{-1, 1\}.$$

For $2\omega - 1 = -1$, $\omega = 0$, and for $2\omega - 1 = 1$, $\omega = 1$. The support of $\frac{X+1}{2}$ is $\{0, 1\}$. TODO: Finish explaining

(e)

Exercise 12.3.

Exercise 12.4.

Exercise 12.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X be a discrete random variable on Ω with support $\{1, \dots, N\}$, where $N \geq 2$ and N is even. Suppose that X has law defined upon:

$$\mathbb{P}(X = k) = C_N max\{k, N - k\}, \quad k = 1, \dots, N,$$

Where $C_N \in \mathbb{R}$. Find C_N.

As N is even, we can find a middle point $m = \frac{N}{2}$. I will use $N = 2m$.

$$\begin{aligned}
\sum_{k=1}^{N} \mathbb{P}(X = k) &= C_N(\sum_{k=1}^{m}(2m - k) + \sum_{k=m+1}^{2m} k) \\
&= C_N(\sum_{k=1}^{m}(2m - k) + \sum_{j=1}^{m}(m + j)) \\
&= C_N(2m \cdot m - \sum_{k=1}^{m} k + m \cdot m + \sum_{j=1}^{m} j) \\
&= C_N(3m^2) \\
&= C_N(3(\frac{N}{2})^2)
\end{aligned}$$

By definition

$$C_N(\frac{3N^2}{4}) = 1 \Rightarrow C_N = \frac{4}{3N^2}.$$