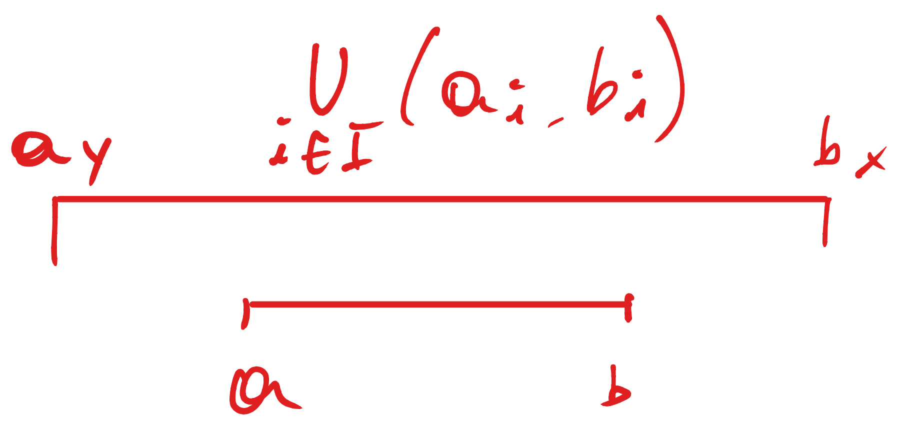
I had mixed feeling about what I wanted to do. At the end, I worked as usual on the script (as seen in the first 2 pages). I solved 2 exercises and then decided to learn how to use latex, in order to write a summary for the course. I got to the point seen in the last 2 pages.



6 Measurable sets: Part III

6.1 Measure extensions

conceptually ok

Proposition 6.1. *Let $(a,b]$, $a<b\in\mathbb{R}$, be any left-open interval. Let I be countable and $(a_i,b_i]$, $i\in I$, be s.t., $(a,b]\subset\cup_{i\in I}(a_i,b_i]$, then*

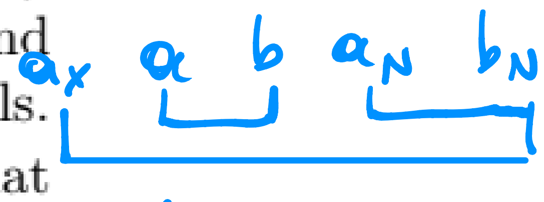
$$b-a\le\sum_{i\in I}(b_i-a_i). \tag{10}$$

$i=1$

finite case

Proof. The finite case, i.e., there exists $N\in\mathbb{N}$ s.t. $\cup_{i\in I}(a_i,b_i]=\cup_{i=1}^{N}(a_i,b_i]$ follows by induction. The base step of the induction is clear, if $(a,b]\subset(c,d]$, then $c\le a<b\le d$ and hence $b-a\le d-c$. For the induction step assume that (10) holds for $N-1$ intervals. Let $(a,b]\subset\cup_{i=1}^{N}(a_i,b_i]$. We want to show that $b-a\le\sum_{i=1}^{N}(b_i-a_i)$. Notice first that we can always assume that $b_1\le b_2\le\cdots\le b_N$. If not, we can just consider a relabeling and the union would remain unchanged. Assume first that $b\notin(a_N,b_N]$. Then, $b\le a_N$ since $b>b_N$ is not possible. To see this, assume by contradiction that $b>b_N$. Then, since $b_1\le b_2\le\cdots\le b_N$, $b\notin(a_i,b_i]$ for any $i=1,\dots,N$. Since $b\in(a,b]\subset\cup_{i=1}^{N}(a_i,b_i]$, this is not possible. Hence, $b\notin(a_N,b_N]\Rightarrow b\le a_N$. Hence, $(a,b]\subset\cup_{i=1}^{N-1}(a_i,b_i]$ since if $y\in(a,b]$, $y\le b\le a_N$ and hence $y\notin(a_N,b_N]$. By the induction hypothesis, the result follows. Thus, in the remaining we assume that $b\in(a_N,b_N]$. If $a_N\le a$ then $a_N\le a<b\le b_N$ and the result follows. Hence, assume that $a<a_N$. Then, $(a,a_N]\subset\cup_{i=1}^{N-1}(a_i,b_i]$. This is because $y\in(a,a_N]$ implies that $y\notin(a_N,b_N]$. Further $y\in(a,a_N]$ implies that $a<y\le a_N<b$ ($b\in(a_N,b_N]$) and hence $(a,a_N]\subset(a,b]$. Since $(a,b]$ is a subset of $\cup_{i=1}^{N}(a_i,b_i]$ it follows that $y\in(a_i,b_i]$ for some $i\ne N$, i.e., $(a,a_N]\subset\cup_{i=1}^{N-1}(a_i,b_i]$. By the induction hypothesis, $\sum_{i=1}^{N-1}(b_i-a_i)\ge a_N-a$. Therefore, $\sum_{i=1}^{N}(b_i-a_i)\ge a_N-a+b_N-a_N\ge a_N-a+b-a_N=b-a$.

base step: N-1



① $a_x\ a_N\ a\ b\ b_N$ or $a_x\ a\ a_N\ b\ b_N$ ②

infinite case

We use the Heine-Borel theorem for intervals (cf. Proposition 2.9) to prove the infinite case, i.e., $\cup_{i\in I}(a_i,b_i]=\cup_{i=1}^{\infty}(a_i,b_i]$. Suppose that $(a,b]\subset\cup_{i=1}^{\infty}(a_i,b_i]$. Let $\varepsilon>0$ be s.t. $b-a>\varepsilon$. This is possible since $b\ne a$. Clearly, the family of intervals $(a_i,b_i+\varepsilon2^{-i})$, $i\in\mathbb{N}$, are s.t.

$$[a+\varepsilon,b]\subset\bigcup_{i\in\mathbb{N}}(a_i,b_i+\varepsilon2^{-i}).$$

By Proposition 2.9 it follows that there exists $i_1,\dots,i_N$, s.t.

TO UNDERSTAND

$$[a+\varepsilon,b]\subset\bigcup_{k=1}^{N}(a_{i_k},b_{i_k}+\varepsilon2^{-i_k}).$$

Hence, by the finite case,

$$\begin{aligned}b-a+\varepsilon&\le\sum_{k=1}^{N}(b_{i_k}-a_{i_k}+\varepsilon2^{-i_k})=\sum_{k=1}^{N}(b_{i_k}-a_{i_k})+\varepsilon\sum_{k=1}^{N}2^{-i_k}\\&\le\sum_{i=1}^{\infty}(b_i-a_i)+\varepsilon\sum_{i=1}^{\infty}2^{-i}=\sum_{i=1}^{\infty}(b_i-a_i)+\frac{\varepsilon}{2}\sum_{i=0}^{\infty}2^{-i}.\end{aligned}$$

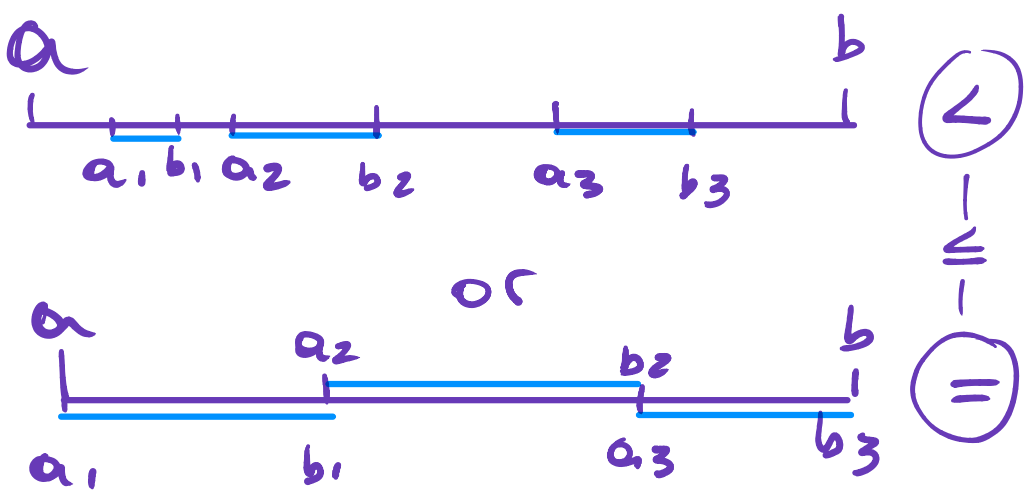
By Exercise 3.15, we obtain that $b-a+\varepsilon\le\sum_{i=1}^{\infty}(b_i-a_i)+\varepsilon$. This completes the argument. □

If the collection $\{(a_i,b_i]\colon i\in I\}$ is disjoint we also have the following result (Exercise 6.10).

Proposition 6.2. *Let $(a,b]$, $a<b\in\mathbb{R}$, be any left-open interval. Let I be countable and $\{(a_i,b_i]\colon i\in I\}$ be a disjoint collection of left-open intervals s.t. $\cup_{i\in I}(a_i,b_i]\subset(a,b]$. Then*

OK

$$\sum_{i\in I}(b_i-a_i)\le b-a.$$



Hence, $\mathcal{F}$ is a σ-field. The given σ-field $\mathcal{F}$ is referred to as the power set of Ω and denoted with $\mathcal{P}(\Omega)$ (or 2^Ω). It is the largest possible σ-field on Ω.

$\neq$

Definition 6.1. *Let $\Omega \neq \emptyset$ be a set and $\mathcal{A}$ be a collection of subsets from Ω. Let $A \in \mathcal{P}(\Omega)$ be any subset of Ω. A collection $\{U_i : i \in I\}$ is said to be a covering of A by sets from $\mathcal{A}$ if $\{U_i : i \in I\} \subset \mathcal{A}$ and $A \subset \cup_{i \in I} U_i$. A covering $\{U_i : i \in I\}$ of A by sets from $\mathcal{A}$ is referred to as countable (resp. finite) if I is countable (resp. finite). We write $C_{\mathcal{A}}(A)$ for the set which contains all the countable coverings of A by sets from $\mathcal{A}$, i.e.,*

$\xi = \{U_i : i \in I\}$

Example 4.6.

$$C_{\mathcal{A}}(A) = \{\xi : \xi \text{ is a countable covering of } A \text{ by sets from } \mathcal{A}\}.$$

Example 6.1. *Consider the setting of Example 4.6 and let $\Omega = \mathbb{R}$ and $\mathcal{R}$ be the family of left-open intervals with the empty set adjoined:*

$$\mathcal{R} = \{A : A = (a, b],\ a, b \in \mathbb{R}\} \cup \{\emptyset\}.$$

$\underset{x-r}{a} \quad x \quad \underset{x+r}{b}$

Let $B_r(x)$ be any open ball with center $x \in \mathbb{R}$ and radius $r > 0$. That is, $B_r(x) = (x - r, x + r)$ is an open interval with endpoints $a = x - r$ and $b = x + r$. Consider the set $\xi_1 = \{(a, r], (r, b]\}$. Then, $\xi_1 \in C_{\mathcal{R}}((a, b))$. As another example, let for $n \in \mathbb{N}$,

$(a,b) \subset (a,r] \cup (r,b]$

$$U_i^n = \left(a + \frac{2ri}{2^n}, a + \frac{2r(i+1)}{2^n}\right], \quad i = 0, \dots 2^n - 1.$$

$\left(a + \frac{2ri}{2^n}\right.$ $\quad i = 0 \Rightarrow a$

$a + \frac{2ri}{2^n} + \frac{2r}{2^n}\Big]$ $\quad i = 2^n - 1$

$a + \frac{2r(2^n-1)}{2^n} + \frac{2r}{2^n}$

$= a + \frac{2r2^n}{2^n} - \cancel{\frac{2r}{2^n}} + \cancel{\frac{2r}{2^n}}$

$= a + 2r = b$

Then, $\xi_n^2 = \{U_i^n : i = 0, \dots 2^n - 1\} \in C_{\mathcal{R}}((a, b))$ for any $n \in \mathbb{N}$. As a final example, let $\varepsilon > 0$ and define

a

$K \to \varepsilon \Rightarrow K \to 0$

$\left(\frac{a}{2^0}, \frac{b}{2^0}\right]$

$$U_k^\varepsilon = \left(\frac{a}{2^k}, \frac{b}{2^k}\right], \quad k \in \mathbb{N} \cup \{0\}.$$

b

Then, $\xi_\varepsilon^3 = \{U_k^\varepsilon : k \in \mathbb{N} \cup \{0\}\} \in C_{\mathcal{R}}((a, b))$ for any $\varepsilon > 0$. Each of the coverings ξ, ξ_ε^2 and ξ_ε^3 of (a, b) by sets from $\mathcal{R}$ offers an approach to quantify the length of (a, b) by summing up the respective lengths of the sets from $\mathcal{R}$. Given $A \in \mathcal{P}(\mathbb{R})$, we define the function $v_\ell(\xi) = \sum_{U \in \xi} \ell(U)$, $\xi \in C_{\mathcal{R}}(A)$ where $\ell : \mathcal{R} \to [0, \infty)$ is s.t.

$$\ell(U) = \begin{cases} b - a, & \text{if } U = (a, b], \\ 0, & \text{if } U = \emptyset. \end{cases}$$

! $\ell : b - a$

$v_\ell : \sum b - a$

As an example, we have that $v_\ell(\xi_1) = r - a + b - r = b - a$. Notice also, that

$\{(a,r],(r,b]\} \Rightarrow (r-a)+(b-r) = b-a$

$\cancel{a} + \frac{2r(i+1)}{2^n}$

$-\cancel{a} - \frac{2ri}{2^n}$

$$\begin{aligned} v_\ell(\xi_n^2) &= \sum_{i=0}^{2^n-1} \frac{2r(i+1) - i}{2^n} \\ &= \frac{2r}{2^n} + \frac{4r}{2^n} - \frac{2r}{2^n} + \frac{6r}{2^n} - \frac{4r}{2^n} + \dots + \frac{2r(2^n-1)}{2^n} + 2r - \frac{2r(2^n-1)}{2^n} \\ &= 2r = b - a. \end{aligned}$$

Exercise 6.1. *Verify that $v_\ell(\xi_\varepsilon^3) = 2(b - a)$.*

In the following we show that

$$\inf\{v_\ell(\xi) : \xi \in C_{\mathcal{R}}((a, b])\} = \inf_{\xi \in C_{\mathcal{R}}((a,b])} v_\ell(\xi) = b - a, \tag{11}$$

i.e., $b - a$ is a lower bound for the values of $v_\ell(\xi)$, $\xi \in C_{\mathcal{R}}((a, b])$.

Exercise 6.2. *Verify that $\inf_{\xi \in C_{\mathcal{R}}((a,b])} v_\ell(\xi) \leq b - a$.*

Upon the later exercise, it remains to show that $b - a \leq \inf_{\xi \in C_{\mathcal{R}}((a,b])} v_\ell(\xi)$. Let ξ be any countable covering of $(a, b]$ by sets from $\mathcal{R}$. That is, $\xi = \{U_i : i \in I\}$, with $U_i = (a_i, b_i]$ or $U_i = \emptyset$, $i \in I$, where I is countable. Since $\ell(\emptyset) = 0$, we assume without loss of generality that $U_i = (a_i, b_i]$ for any $i \in I$. Therefore, we have that $(a, b] \subset \cup_{i \in I}(a_i, b_i]$ and $v_\ell(\xi) = $

Exercise 6.1. *Verify that* $v_\ell(\xi^3_\varepsilon) = 2(b-a)$.

$$\xi^3_\varepsilon = \bigcup_K^{\varepsilon} \left(\frac{a}{2^K}, \frac{b}{2^K}\right]$$

$$v_e(\xi^3_\varepsilon) = \sum_{K=0}^{\varepsilon=\infty} \frac{b-a}{2^K} = (b-a) + \frac{(b-a)}{2} + \frac{(b-a)}{4}$$

$$= (b-a)\sum_{K=0}^{\varepsilon=\infty} \frac{1}{2^K} = (b-a)^2$$

for any

$K \Rightarrow$ lowest is 0

$\varepsilon \Rightarrow$ highest is $+\infty$

Exercise 6.2. *Verify that* $\inf_{\xi \in C_{\mathcal{R}}((a,b])} v_\ell(\xi) \leq b-a$.

as $A \subset \xi$, the smallest ξ possible is $(a, b]$, and $v_e(\xi)$ is $b-a$.

Summary: Introduction to Probability

Daniele Cambria

2024

Chapter 1

Measurable sets: Part III

1.1 Measure extensions

Proposition 1.1. *Let $(a, b]$, $a < b \in \mathbb{R}$, be any left-open interval. Let I be countable and $(a_i, b_i]$, $i \in I$, be s.t., $(a, b] \subset \bigcup_{i \in I}(a_i, b_i]$, then*

$$b - a \leq \sum_{i \in I}(b_i - a_i). \tag{10}$$

Proposition 1.2. *Let $(a, b]$, $a < b \in \mathbb{R}$, be any left-open interval. let I be countable and $\{(a_i, b_i] : i \in I\}$ be a disjoint collection of left-open intervals s.t. $\bigcup_{i \in I}(a_i, b_i] \subset (a, b]$. Then*

$$\sum_{i \in I}(b_i - a_i) \leq b - a.$$

Definition 1.1. Let $\Omega \neq \emptyset$ be a set and $\mathcal{A}$ be a collection of subsets from Ω. Let $A \in \mathcal{P}(\Omega)$ be any subset of Ω. A collection $\{U_i : i \in I\}$ is said to be a covering of A by sets from $\mathcal{A}$ if:

(i) $\{U_i : i \in I\} \subset \mathcal{A}$ (Set membership condition)

NOTE that (i) means $U_i \subset \mathcal{A}$ $\forall i \in I$, not $\bigcup_{i \in I} U_i \subset \mathcal{A}$.

(ii) $A \subset \bigcup_{i \in I} U_i$ (Covering condition)

A covering $\{\bigcup_i : i \in I\}$ of A by sets from $\mathcal{A}$ is referred as countable (resp. finite) if I is countable (resp. finite). We write $C_{\mathcal{A}}(A)$ for the set which contains all the countable covering of A by sets from $\mathcal{A}$, i.e.,

$$C_{\mathcal{A}}(A) = \{\xi : \xi \text{ is a countable covering of } A \text{ by sets from } \mathcal{A}\}.$$

Why do we say $A \in \mathcal{P}(\Omega)$ instead of $A \in \Omega$? When we use the notation $A \in \mathcal{P}(\Omega)$, it signifies that A is a subset of Ω, not an element of Ω. The power set $\mathcal{P}(\Omega)$ represents all possible subsets of Ω, including Ω itself, any subset of it, or even an empty set. Using $A \in \Omega$ would incorrectly imply that A is an individual element of Ω, which does not align with the context of covering subsets with subsets.

My Example 1.1 (Finite Covering)**.** Let $\Omega = \{1, 2, 3, 4, 5\}$, and let $\mathcal{A}$ be a collection of subests of Ω, such as $\mathcal{A} = \{\{1\}, \{2, 3\}, \{3, 5\}\}$, if we take $A = \{1, 2, 3\}$, a finite covering of A by sets from $\mathcal{A}$ could be $\{\{1\}, \{2, 3\}\}$. This covering is finite, as I can be $\{1, 2\}$, which is finite. The 2 conditions both hold. Each U_i is a subset of $\mathcal{A}$, and A is covered by the union of U_i. In this case, the possible countable coverings of A that can be formed using subsets of $\mathcal{A}$ are restricted to the one already provided. Therefore, $C_{\mathcal{A}}(A) = \{\{1\}, \{2, 3\}\}$

Important from Example 6.1 (Script) Let $\Omega = \mathbb{R}$ and $\mathcal{R} = \{A : A = (a, b], a, b \in \mathbb{R}\} \cup \{\emptyset\}$. We define the function $\ell : \mathcal{R} \to [0, \infty)$ s.t.

$$\ell(U) = \begin{cases} b - a, & \text{if } U = (a, b], \\ 0, & \text{if } U = \emptyset. \end{cases}$$

Given $A \in \mathcal{P}(\mathbb{R})$, we also define the function $v_\ell(\xi) : \mathcal{R} \to [0, \infty)$, where $\xi \in C_{\mathcal{R}}(A)$ s.t.

$$v_\ell(\xi) = \sum_{U \in \xi} \ell(U).$$

We also show that

$$\inf\{v_\ell(\xi) : \xi \in C_{\mathcal{R}}((a, b])\} = \inf_{\xi \in C_{\mathcal{R}}((a,b])} v_\ell(\xi) = b - a, \tag{11}$$

i.e., $b - a$ is a lower bound for the values of $v_\ell(\xi)$, $\xi \in C_{\mathcal{R}}((a, b])$. We also saw that there exists $\xi \in C_{\mathcal{R}}((a, b])$ s.t. $b - a = v_\ell(\xi)$. Hence, the latter infimum is a minimum (Proposition 1.3).

Proposition 1.3. *Given any left open interval $(a, b]$, $min_{\xi \in C_{\mathcal{R}}((a,b])} v_\ell(\xi) = b - a$*

Define ℓ^* We build on the latter result and define the function

$$\ell^* = \inf_{\xi \in C_{\mathcal{R}}(A)} v_\ell(\xi), \quad A \in \mathcal{P}(\mathbb{R}).$$

Note, we know that if $A \in \mathcal{R}$, then $\ell^*(A) = b - a$.