

# Summary: Introduction to Probability

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# Chapter 1

## Introduction: Part I

### 1.1 Sets

### 1.2 The principle of induction

### 1.3 Order structure of the real numbers

**Exercise 1.1** (1.11 TOOL). Let  $A$  be a set with  $n$  elements. Show that

1. the number of permutations of the elements from  $A$  is  $n!$  ;
2. for any  $0 \leq k \leq n$ , the number of subsets of  $A$  having  $k$  elements is given by

$$\frac{n!}{(n-k)!k!}.$$

## Chapter 2

# Introduction: Part II

### 2.1 Functions

**Proposition 2.1.** TODO prop 2.12

### 2.2 Cardinality of Sets

### 2.3 Euclidean distance

**Proposition 2.2.** let  $f : A \rightarrow B$  be a function. Let  $B_* \subset B$ . Then,

(a)  $f^{-1}(B_*^c) = f^{-1}(B_*)^c$ .

Let  $I$  and  $J$  be some sets and  $A_i \subset A, i \in I$ , and  $B_j \subset B, j \in J$ , be a collection of sets from  $A$  and  $B$ , respectively. Then,

(b) TODO

(c) TODO

(d) TODO

## Chapter 3

# Introduction: Part III

### 3.1 Real valued sequences

## Chapter 4

# Measurable sets: Part I

### 4.1 Measurable spaces

**Definition 4.1.** 4.2 TODO

**Definition 4.2** (Measurable space). let  $\Omega \neq \emptyset$  and  $\mathcal{F}$  be a  $\sigma$ -field on  $\Omega$ . The pair  $(\Omega, \mathcal{F})$  is referred to as a measurable space. if  $A \in \mathcal{F}$ , then  $A$  is said to be measurable. if  $\mathcal{A} \subset \mathcal{F}$  and  $\mathcal{A}$  is a  $\sigma$ -field on  $\Omega$ ,  $\mathcal{A}$  is referred to as a sub- $\sigma$ -field on  $\Omega$ .

## Chapter 5

# Measurable sets: Part II

5.1 Measure spaces

5.2 Semirings



## Chapter 6

# Measurable sets: Part III

### 6.1 Measure extensions

**Proposition 6.1.** Let  $(a, b]$ ,  $a < b \in \mathbb{R}$ , be any left-open interval. Let  $I$  be countable and  $(a_i, b_i]$ ,  $i \in I$ , be s.t.,  $(a, b] \subset \bigcup_{i \in I} (a_i, b_i]$ , then

$$b - a \leq \sum_{i \in I} (b_i - a_i). \quad (10)$$

**Proposition 6.2.** Let  $(a, b]$ ,  $a < b \in \mathbb{R}$ , be any left-open interval. let  $I$  be countable and  $\{(a_i, b_i] : i \in I\}$  be a disjoint collection of left-open intervals s.t.  $\bigcup_{i \in I} (a_i, b_i] \subset (a, b]$ . Then

$$\sum_{i \in I} (b_i - a_i) \leq b - a.$$

**Definition 6.1.** Let  $\Omega \neq \emptyset$  be a set and  $\mathcal{A}$  be a collection of subsets from  $\Omega$ . Let  $A \in \mathcal{P}(\Omega)$  be any subset of  $\Omega$ . A collection  $\{U_i : i \in I\}$  is said to be a covering of  $A$  by sets from  $\mathcal{A}$  if:

(i)  $\{U_i : i \in I\} \subset \mathcal{A}$  (Set membership condition)

NOTE that (i) means  $U_i \subset \mathcal{A} \forall i \in I$ , not  $\bigcup_{i \in I} U_i \subset \mathcal{A}$ .

(ii)  $A \subset \bigcup_{i \in I} U_i$  (Covering condition)

A covering  $\{\bigcup_i : i \in I\}$  of  $A$  by sets from  $\mathcal{A}$  is referred as countable (resp. finite) if  $I$  is countable (resp. finite). We write  $C_{\mathcal{A}}(A)$  for the set which contains all the countable covering of  $A$  by sets from  $\mathcal{A}$ , i.e.,

$$C_{\mathcal{A}}(A) = \{\xi : \xi \text{ is a countable covering of } A \text{ by sets from } \mathcal{A}\}.$$

**Why do we say  $A \in \mathcal{P}(\Omega)$  instead of  $A \in \Omega$ ?** When we use the notation  $A \in \mathcal{P}(\Omega)$ , it signifies that  $A$  is a subset of  $\Omega$ , not an element of  $\Omega$ . The power set  $\mathcal{P}(\Omega)$  represents all possible subsets of  $\Omega$ , including  $\Omega$  itself, any subset of it, or even an empty set. Using  $A \in \Omega$  would incorrectly imply that  $A$  is an individual element of  $\Omega$ , which does not align with the context of covering subsets with subsets.

**My Example 6.1** (Finite Covering). Let  $\Omega = \{1, 2, 3, 4, 5\}$ , and let  $\mathcal{A}$  be a collection of subsets of  $\Omega$ , such as  $\mathcal{A} = \{\{1\}, \{2, 3\}, \{3, 5\}\}$ , if we take  $A = \{1, 2, 3\}$ , a finite covering of  $A$  by sets from  $\mathcal{A}$  could be  $\{\{1\}, \{2, 3\}\}$ . This covering is finite, as  $I$  can be  $\{1, 2\}$ , which is finite. The 2 conditions both hold. Each  $U_i$  is a subset of  $\mathcal{A}$ , and  $A$  is covered by the union of  $U_i$ . In this case, the possible countable coverings of  $A$  that can be formed using subsets of  $\mathcal{A}$  are restricted to the one already provided. Therefore,  $C_{\mathcal{A}}(A) = \{\{1\}, \{2, 3\}\}$

**Important from Example 6.1 (Script)** Let  $\Omega = \mathbb{R}$  and  $\mathcal{R} = \{A : A = (a, b], a, b \in \mathbb{R}\} \cup \{\emptyset\}$ . We define the function  $\ell : \mathcal{R} \rightarrow [0, \infty)$  s.t.

$$\ell(U) = \begin{cases} b - a, & \text{if } U = (a, b], \\ 0, & \text{if } U = \emptyset. \end{cases}$$

Given  $A \in \mathcal{P}(\mathbb{R})$ , we also define the function  $v_\ell(\xi) : \mathcal{R} \rightarrow \mathbb{R}^+$ , where  $\xi \in C_{\mathcal{R}}(A)$  s.t.

$$v_\ell(\xi) = \sum_{U \in \xi} \ell(U).$$

We also show that

$$\inf\{v_\ell(\xi) : \xi \in C_{\mathcal{R}}((a, b])\} = \inf_{\xi \in C_{\mathcal{R}}((a, b])} v_\ell(\xi) = b - a, \quad (11)$$

i.e.,  $b - a$  is a lower bound for the values of  $v_\ell(\xi)$ ,  $\xi \in C_{\mathcal{R}}((a, b])$ . We also saw that there exists  $\xi \in C_{\mathcal{R}}((a, b])$  s.t.  $b - a = v_\ell(\xi)$ . Hence, the latter infimum is a minimum (Proposition 6.3).

**Proposition 6.3.** Given any left open interval  $(a, b]$ ,  $\min_{\xi \in C_{\mathcal{R}}((a, b])} v_\ell(\xi) = b - a$

**Define  $\ell^*$**  We build on the latter result and define the function

$$\ell^* = \inf_{\xi \in C_{\mathcal{R}}(A)} v_\ell(\xi), \quad A \in \mathcal{P}(\mathbb{R}).$$

Note, we know that if  $A \in \mathcal{R}$ , then  $\ell^*(A) = b - a$

# Chapter 7

## Measurable functions

### 7.1 The concept of measurable functions

**Definition 7.1** (Measurable function). Let  $(\Omega, \mathcal{F})$  and  $(\Omega^*, \mathcal{F}^*)$  be two measurable spaces (cf. Definition 4.2). A function  $f : \Omega \rightarrow \Omega^*$  is said to be measurable  $\mathcal{F}/\mathcal{F}^*$  if for any  $A^* \in \mathcal{F}^*$ ,  $f^{-1}(A^*) \in \mathcal{F}$ .

**Proposition 7.1** (Measurable function). Let  $(\Omega, \mathcal{F})$  and  $(\Omega^*, \mathcal{F}^*)$  be two measurable spaces and  $f : \Omega \rightarrow \Omega^*$  be a function. Suppose that  $\mathcal{F}^* = \sigma(\mathcal{G})$  and for any  $G \in \mathcal{G}$ ,  $f^{-1}(G) \in \mathcal{F}$ . Then,  $f$  is  $\mathcal{F}/\mathcal{F}^*$  measurable.

**Definition 7.2** (Borel function). A function  $f : \mathbb{R}^m \rightarrow \mathbb{R}^k$  is called Borel function if it is measurable  $\mathfrak{B}(\mathbb{R}^m)/\mathfrak{B}(\mathbb{R}^k)$ .

**Proposition 7.2** (Continuous functions and Borel functions). Any continuous function  $f : \mathbb{R}^m \rightarrow \mathbb{R}^k$  is a Borel function.

**Proposition 7.3** ( $\mathcal{F}/\mathfrak{B}(\mathbb{R})$  measurable). Let  $(\Omega, \mathcal{F})$  be a measurable space and  $f : \Omega \rightarrow \mathbb{R}$  be a real-valued function. Suppose that  $\{\omega \in \Omega : f(\omega) \leq x\} \in \mathcal{F}$  for any  $x \in \mathbb{R}$ , then  $f$  is  $\mathcal{F}/\mathfrak{B}(\mathbb{R})$  measurable. In other words:  $f$  is a measurable function if the pre-image of any interval  $(-\infty, x]$  under  $f$  is a measurable set in  $\mathcal{F}$ , or  $f^{-1}((-\infty, x]) \in \mathcal{F}$ . since  $\mathfrak{B}(\mathbb{R}) = \sigma(\{(-\infty, x] : x \in \mathbb{R}\})$ , we also clearly see the proof (cf. Proposition 7.1).

**Thinking about**  $f^{-1}((-\infty, x])$  If  $B \in \mathfrak{B}(\mathbb{R})$ , then,  $f^{-1}(B) = \{\omega \in \Omega : f(\omega) \in B\}$  Is the same as saying,  $f^{-1}((-\infty, x]) = \{\omega \in \Omega : f(\omega) \leq x\}$ .  $f^{-1}(B)$  will return ALL of the values  $\omega \in \Omega$  for which  $f(\omega) \in B$ . See My Example 7.1 for further intuition.

**Define**  $\mathbb{1}_A(\omega)$  TODO

**Example 7.1** (Simple measurable function). Let  $\Omega = \{h, t\}$  and  $\mathcal{F} = \mathcal{P}(\{h, t\}) = \{\emptyset, \{h\}, \{t\}, \{h, t\}\}$ . Then,  $\{h\} \in \mathcal{P}(\{h, t\})$ . Thus

$$f(\omega) = \begin{cases} 1, & \text{if } \omega = h, \\ 0, & \text{if } \omega = t, \end{cases}$$

is  $\mathcal{P}(\{h, t\})/\mathfrak{B}(\mathbb{R})$  measurable. In order for  $f$  to be  $\mathcal{P}(\{h, t\})/\mathfrak{B}(\mathbb{R})$  measurable, the pre-image of every Borel set in  $\mathbb{R}$  under  $f$  must be an element of  $\mathcal{F}$ . For any  $x \in \mathbb{R}$ ,  $f^{-1}((-\infty, x])$  will either be  $\emptyset$ ,  $\{h\}$ , or  $\{t\} \in \mathcal{F}$ .

**Proposition 7.4** ( $\mathcal{F}/\mathfrak{B}(\mathbb{R}^k)$  measurable). Let  $(\Omega, \mathcal{F})$  be a measurable space and  $f : \Omega \rightarrow \mathbb{R}^k$ , i.e.,

$$f(\omega) = (f_1(\omega), \dots, f_k(\omega)).$$

Then,  $f$  is  $\mathcal{F}/\mathfrak{B}(\mathbb{R}^k)$  measurable if and only if for any  $i = 1, \dots, k$ ,  $f_i : \Omega \rightarrow \mathbb{R}$  is  $\mathcal{F}/\mathfrak{B}(\mathbb{R})$  measurable.

**Proposition 7.5** (Composite measurable function). Let  $(\Omega, \mathcal{F})$  be a measurable space and  $f_i : \Omega \rightarrow \mathbb{R}, i = 1, \dots, k$ , be  $\mathcal{F}/\mathfrak{B}(\mathbb{R})$  measurable. Suppose that  $g : \mathbb{R}^k \rightarrow \mathbb{R}$  is  $\mathfrak{B}(\mathbb{R}^k)/\mathfrak{B}(\mathbb{R})$  measurable. Then,

$$w \mapsto g((f_1(\omega), \dots, f_k(\omega))) = g(f_1(\omega), \dots, f_k(\omega)).$$

is  $\mathcal{F}/\mathfrak{B}(\mathbb{R})$  measurable. (Composite function usually written without double brackets)

**Proposition 7.6** (Continuity preserves measurability in function composition). Let  $(\Omega, \mathcal{F})$  be a measurable space and  $f_i : \Omega \rightarrow \mathbb{R}, i = 1, \dots, k$ , be  $\mathcal{F}/\mathfrak{B}(\mathbb{R})$  measurable. Then, if  $g : \mathbb{R}^k \rightarrow \mathbb{R}$  is continuous,

$$w \mapsto g(f_1(\omega), \dots, f_k(\omega)).$$

is  $\mathcal{F}/\mathfrak{B}(\mathbb{R})$  measurable.

**Example 7.2** (Continuity preserves measurability). Let  $(\Omega, \mathcal{F})$  be a measurable space and  $f_i : \Omega \rightarrow \mathbb{R}, i = 1, \dots, k$ , be  $\mathcal{F}/\mathfrak{B}(\mathbb{R})$  measurable. Then,  $\sum_{i=1}^k f_i$  is  $\mathcal{F}/\mathfrak{B}(\mathbb{R})$  measurable (cf. Proposition 2.1).

**Example 7.3** (Continuity preserves measurability). Let  $(\Omega, \mathcal{F})$  be a measurable space and  $f_i : \Omega \rightarrow \mathbb{R}, i = 1, \dots, k$ , be  $\mathcal{F}/\mathfrak{B}(\mathbb{R})$  measurable. Then,  $\prod_{i=1}^k f_i$  is  $\mathcal{F}/\mathfrak{B}(\mathbb{R})$  measurable (cf. Proposition 2.1).

**Definition 7.3** (Simple functions). A function  $f : \Omega \rightarrow \mathbb{R}$  is called simple if there exists  $n \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in \mathbb{R}$  and sets  $A_1, \dots, A_n \subset \Omega$  s.t.

$$f(\omega) = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}(\omega) \quad \omega \in \Omega.$$

That is, a simple function is a finite linear combination of indicator functions.

**Example 7.4** (Simple function). Let  $(\Omega, \mathcal{F})$  be a measurable space and  $f$  be a simple function on  $\Omega$ , i.e.,  $f(\omega) = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}(\omega)$ . Then, if  $A_i \in \mathcal{F}$  for any  $i = 1, \dots, n$ ,  $f$  is  $\mathcal{F}/\mathfrak{B}(\mathbb{R})$  measurable.

**My Example 7.1** (Simple function). Let  $(\Omega, \mathcal{F})$  be a measurable space and  $f : \Omega \rightarrow \mathbb{R}$  be the function defined in 7.3. For this simplified setting, suppose  $\Omega = \{1, 2, 3, 4\}$  and  $\mathcal{F} = \{\emptyset, \{1, 2\}, \{3, 4\}, \Omega\}$ . Moreover, we define our function with  $n = 2$ , where  $\alpha_1 = 3, \alpha_2 = 5, A_1 = \{1, 2\}$  and  $A_2 = \{3, 4\}$ . Then,

$$f(\omega) = 3 \cdot \mathbb{1}_{\{1,2\}}(\omega) + 5 \cdot \mathbb{1}_{\{3,4\}}(\omega).$$

Now, let's consider two preimages of this function,  $f^{-1}(\{3\})$  and  $f^{-1}(\{12\})$ . Note that both of these sets are Borel sets in  $\mathbb{R}$ . Also note that, if  $B \in \mathfrak{B}(\mathbb{R})$ , then,

$$f^{-1}(B) = \{\omega \in \Omega : f(\omega) \in B\}.$$

As seen in Thinking about 7.1. Since  $f$  takes the value 3 for  $\omega \in \{1, 2\}$ ,  $f^{-1}(\{3\}) = \{1, 2\} \in \mathcal{F}$ . And, as  $f$  doesn't take any value for values  $\notin \{\{1, 2\}, \{3, 4\}\}$ ,  $f^{-1}(\{12\}) = \emptyset \in \mathcal{F}$ . So indeed,  $f$  is  $\mathcal{F}/\mathfrak{B}(\mathbb{R})$  measurable.

**Definition 7.4** (Simple functions in standard form). Let  $(\Omega, \mathcal{F})$  be a measurable space and  $f : \Omega \rightarrow \mathbb{R}$  be a simple function, as defined in Definition 7.3.  $f$  is called standard if  $\cup_{i=1}^n A_i = \Omega$  and  $\{A_1, \dots, A_n\} \subset \mathcal{F}$  is disjoint. if  $f$  is standard, we say that it is a simple function in standard form.

**Proposition 7.7** (7.7). TODO

**Proposition 7.8** (7.8). TODO

## 7.2 Functions taking values in the extended real numbers

**Definition 7.5** (Measurable functions in  $\overline{\mathbb{R}}$ ). Let  $(\Omega, \mathcal{F})$  be a measurable space and  $f : \Omega \rightarrow \overline{\mathbb{R}}$ . We say that  $f$  is  $\mathcal{F}$  measurable if for any  $A \in \mathfrak{B}(\mathbb{R})$ ,  $\{\omega \in \Omega : f(\omega) \in A\} \in \mathcal{F}$  and  $\{\omega \in \Omega : f(\omega) = -\infty\} \in \mathcal{F}$  and  $\{\omega \in \Omega : f(\omega) = \infty\} \in \mathcal{F}$ . Or, in other words,  $f^{-1}(A), f^{-1}(-\infty), f^{-1}(\infty) \in \mathcal{F}$ .

**Remark 7.2** As seen in the script, as, if  $f : \Omega \rightarrow \mathbb{R}, f^{-1}(-\infty), f^{-1}(\infty) = \emptyset$ , any results on  $\mathcal{F}$  measurable functions  $f : \Omega \rightarrow \overline{\mathbb{R}}$  also apply to  $\mathcal{F}/\mathfrak{B}(\mathbb{R})$  measurable functions  $f : \Omega \rightarrow \mathbb{R}$ .

**Remark 7.3** TODO, but important for notation, read it from the script.

**Proposition 7.9** (7.9). TODO

**Proposition 7.10** (7.10). TODO

**Definition 7.6** (Positive and negative parts of a function). TODO

**Proposition 7.11.** This proposition states that any  $\mathcal{F}$ -measurable function  $f$  can be approximated by a sequence of  $\mathcal{F}$ -measurable simple functions  $(f_n)_{n \in \mathbb{N}}$  such that  $f_n(\omega) \rightarrow f(\omega)$  for all  $\omega \in \Omega$ .

**My Example 7.2.** Consider  $\Omega = [0, 1]$  and  $\mathcal{F}$  be the Borel  $\sigma$ -field on  $[0, 1]$ . Let  $f(x) = x$ . Define the sequence of simple functions  $f_n(x) = \frac{\lfloor nx \rfloor}{n}$ . Each  $f_n$  is  $\mathcal{F}$ -measurable and  $f_n(x) \rightarrow x$  as  $n \rightarrow \infty$ .

**Proposition 7.12.** This proposition extends 7.11 by specifying that if  $f$  is non-negative, the convergence of the simple functions can be made monotone, i.e.,  $f_n(\omega)$  increases with  $n$  and converges to  $f(\omega)$ .

**My Example 7.3.** Using the same function  $f(x) = x$  on  $\Omega = [0, 1]$ , define  $f_n(x) = \frac{\lfloor nx \rfloor}{n}$ . Note that  $f_n(x) \leq f_{n+1}(x)$  for all  $x \in [0, 1]$  and  $n \in \mathbb{N}$ , ensuring that  $f_n(x) \uparrow f(x)$  as  $n \rightarrow \infty$ .

### 7.3 Sequence of measurable functions

## Chapter 8

# Integration: Part I

### 8.1 The integral for non-negative functions

If  $f : \Omega \rightarrow \overline{\mathbb{R}}$  is s.t.  $f(\omega) \geq 0$  for any  $\omega \in \Omega$ ,  $f$  is said to be nonnegative.

**Definition 8.1** (Finite partitions). Let  $\Omega$  be a set. A partition of  $\Omega$  is a disjoint collection  $\{A : A \in P\}$ ,  $P \subset \mathcal{P}(\Omega)$ , s.t.  $\cup_{A \in P} A = \Omega$ . That is, a partition of  $\Omega$  is a disjoint collection of subsets of  $\Omega$  whose union is  $\Omega$ . If  $\xi$  is a partition of  $\Omega$ , a set  $A \in \xi$  is referred to as an atom of  $\xi$ . A partition  $\xi$  of  $\Omega$  is said to be finite, if it contains a finite number of atoms.

**Example 8.1** (Finite partition). Let  $\Omega = \{0, 1, \dots, N\}$ ,  $N \in \mathbb{N}$ . Then,  $\xi = \{\{\omega\} : \omega \in \Omega\}$  is a finite partition of  $\Omega$ . (Partition contains  $N + 1$  elements).

**Definition 8.2** ( $Z_0^{\mathcal{F}}$ ). Let  $(\Omega, \mathcal{F})$  be a measurable space. We use the notation  $Z_0^{\mathcal{F}}(\Omega) = Z_0^{\mathcal{F}}$  for the set which contains all the finite partitions of  $\Omega$  with atoms from  $\mathcal{F}$ . That is,

$$Z_0^{\mathcal{F}} = \{\xi : \xi \text{ is finite partition of } \Omega \text{ s.t. for any } A \in \xi, A \in \mathcal{F}\}.$$

**Definition 8.3** (Integral for a nonnegative standard simple function). Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space and  $f : \Omega \rightarrow \overline{\mathbb{R}}$  be nonnegative and  $\mathcal{F}$  measurable. Then, we define

$$S_{\mu}^f(\xi) = \sum_{A \in \xi} \left( \inf_{\omega \in A} f(\omega) \right) \mu(A), \quad \xi \in Z_0^{\mathcal{F}},$$

Essentially,  $S_{\mu}^f(\xi)$  approximates the integral of  $f$  by considering the smallest value  $f$  takes on each piece of the partition and multiplying this by the measure of the piece. And

$$\int_{\Omega} f(\omega) \mu(d\omega) = \sup_{\xi \in Z_0^{\mathcal{F}}} S_{\mu}^f(\xi).$$

The integral of  $f$  over  $\Omega$  with respect to  $\mu$ , is the supremum of  $S_{\mu}^f(\xi)$  over all possible partitions  $\xi$  of  $\Omega$  in  $Z_0^{\mathcal{F}}$ . This definition captures the idea of the integral as the limit of finer and finer approximations of  $f$  by simple functions. Upon the latter definition, we deduce the integral for a (nonnegative) standard simple function (cf. Definition 7.4).

**Proposition 8.1.** TODO

**My Example 8.1** (Integral of a nonnegative standard simple function). Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space with  $\Omega = \{a, b, c, d\}$ ,  $\mathcal{F} = \mathcal{P}(\Omega)$ , and  $\mu$  is the counting measure, i.e.,  $\mu(A)$  is the number of elements in  $A$ . Let  $f : \Omega \rightarrow \overline{\mathbb{R}}$ ,

$$f(\omega) = \begin{cases} 1 & \text{if } \omega = a, \\ 2 & \text{if } \omega = b, \\ 3 & \text{if } \omega = c, \\ 0 & \text{if } \omega = d \end{cases}$$

Consider the partition  $\xi = \{\{a\}, \{b\}, \{c\}, \{d\}\}$ .  $\inf_{\omega \in \{a\}} f(\omega) = 1$ ,  $\inf_{\omega \in \{b\}} f(\omega) = 2$ ,  $\inf_{\omega \in \{c\}} f(\omega) = 3$ ,  $\inf_{\omega \in \{d\}} f(\omega) = 4$ . Since each singleton set in  $\xi$  as measure of 1 under  $\mu$ ,

$$S_\mu^f(\xi) = (1 \times 1) + (2 \times 1) + (3 \times 1) + (4 \times 1) = 6$$

if  $\sup_{\xi \in \mathcal{Z}_0^f} S_\mu^f = 6$ , which I think it should be, then  $\int_\Omega f(\omega) \mu(d\omega) = 6$ .

**Example 8.2.** Example 8.2 interesting and clear, TODO.

**Proposition 8.2** (Monotone convergence theorem). Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space and  $f_n : \Omega \rightarrow \overline{\mathbb{R}}$ ,  $n \in \mathbb{N}$ , be a sequence of nonnegative  $\mathcal{F}$  measurable functions s.t. for any  $\omega \in \Omega$ ,  $f_n(\omega) \uparrow f(\omega)$  for some  $f : \Omega \rightarrow \overline{\mathbb{R}}$ . Then,

$$\int_\Omega f_n(\omega) \mu(d\omega) \uparrow \int_\Omega f(\omega) \mu(d\omega).$$

**Proposition 8.3** (The integral of nonnegative functions is linear). Let  $(\Omega, \mathcal{F}, \mu)$  be a measurable space,  $f, g : \Omega \rightarrow \overline{\mathbb{R}}$  be two nonnegative and  $\mathcal{F}$  measurable functions. Given  $\alpha, \beta \in [0, \infty)$  we have that

$$\int_\Omega (\alpha f + \beta g)(\omega) \mu(d\omega) = \alpha \int_\Omega f(\omega) \mu(d\omega) + \beta \int_\Omega g(\omega) \mu(d\omega).$$

As a consequence of the latter two proposition we have the following result:

**Proposition 8.4.** Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space and  $f_i : \Omega \rightarrow \overline{\mathbb{R}}$ ,  $i \in \mathbb{N}$ , be a sequence of nonnegative  $\mathcal{F}$  measurable functions, then

$$\int_\Omega \left( \sum_{i \in \mathbb{N}} f_i \right) (\omega) \mu(d\omega) = \sum_{i \in \mathbb{N}} \left( \int_\Omega f_i(\omega) \mu(d\omega) \right).$$

**Definition 8.4** (True almost everywhere (*a.e.*)). Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space. Suppose that for any  $\omega \in \Omega$ ,  $S(\omega)$  is a statment on  $\Omega$ . We say  $S$  is true  $\mu$  almost everywhere (*a.e.*) if  $\mu(\{\omega : S(\omega) \text{ is false}\}) = 0$ .

**Example 8.3** ( $\mu(a.e.)$ ). Interesting and clear. TODO.

**Proposition 8.5.** Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space. Assume that  $f, g : \Omega \rightarrow \overline{\mathbb{R}}$  be two nonnegatibe and  $\mathcal{F}$  measurable functions.

- (i) If  $\mu(\{\omega : f(\omega) > 0\}) > 0$ , then  $\int_\Omega f(\omega) \mu(d\omega) > 0$ ;
- (ii) If  $\int_\Omega f(\omega) \mu(d\omega) < \infty$ , then  $f < \infty$   $\mu$  *a.e.*;
- (iii) If  $f \leq g$   $\mu$  *a.e.*, then  $\int_\Omega f(\omega) \mu(d\omega) \leq \int_\Omega g(\omega) \mu(d\omega)$ ;
- (iv) If  $f = g$   $\mu$  *a.e.*, then  $\int_\Omega f(\omega) \mu(d\omega) = \int_\Omega g(\omega) \mu(d\omega)$ .

## 8.2 Integrable functions

We recall the definiton of the positive ( $f^+$ ) and negative ( $f^-$ ) parts of a function (cf. Definition 7.6). Pay attention,  $f^-$  is basically the negative part of the function, but reflected by the x-axis. The result is positive. Also see 7.2

**Definition 8.5** (Integral of an integrable function). Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space and  $f : \Omega \rightarrow \overline{\mathbb{R}}$  be a  $\mathcal{F}$  measurable function. The integral of  $f$  is defined by:

$$\int_\Omega f(\omega) \mu(d\omega) = \int_\Omega f^+(\omega) \mu(d\omega) - \int_\Omega f^-(\omega) \mu(d\omega),$$

unless  $\int_\Omega f^+(\omega) \mu(d\omega) = \int_\Omega f^-(\omega) \mu(d\omega) = \infty$ , in which case  $\int_\Omega f(\omega) \mu(d\omega)$  is not defined. If both  $\int_\Omega f^+(\omega) \mu(d\omega) < \infty$  and  $\int_\Omega f^-(\omega) \mu(d\omega) < \infty$ ,  $f$  is said to be integrable.

(NOTE) This assumption is defined upon the measure  $\mu$ , i.e., if one wants to further refer to the measure of integration one specifies that  $f$  is integrable with respect to  $\mu$ .

**Proposition 8.6** (Generalisation of the condition for  $f$  to be integrable). Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space and  $f : \Omega \rightarrow \overline{\mathbb{R}}$  be  $\mathcal{F}$  measurable. Then,  $f$  is integrable if and only if  $\int_{\Omega} |f(\omega)| \mu(d\omega) < \infty$ .

**Proposition 8.7** (Extension (cf. (iii) Proposition 8.5)). Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space and  $f, g : \Omega \rightarrow \overline{\mathbb{R}}$  be  $\mathcal{F}$  measurable. If  $f$  and  $g$  are integrable and  $f \leq g$  a.e., then,  $\int_{\Omega} f(\omega) \mu(d\omega) \leq \int_{\Omega} g(\omega) \mu(d\omega)$ .

**Proposition 8.8** (Extension (c.f. Proposition 8.3)). Let  $(\Omega, \mathcal{F}, \mu)$  be a measurable space,  $f, g : \Omega \rightarrow \overline{\mathbb{R}}$  be two integrable and  $\mathcal{F}$  measurable functions. Then, for any  $\alpha, \beta \in \mathbb{R}$  we have that  $\alpha f + \beta g$  is integrable and

$$\int_{\Omega} (\alpha f + \beta g)(\omega) \mu(d\omega) = \alpha \int_{\Omega} f(\omega) \mu(d\omega) + \beta \int_{\Omega} g(\omega) \mu(d\omega).$$

### 8.3 Fatou's lemma and Lebesgue's dominated convergence theorem

**Proposition 8.9** (Fatou's lemma). Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space and  $f_n : \Omega \rightarrow \overline{\mathbb{R}}, n \in \mathbb{N}$ , be a sequence of nonnegative and  $\mathcal{F}$  measurable function. Then,

$$\int_{\Omega} \liminf_{n \rightarrow \infty} f_n(\omega) \mu(d\omega) \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n(\omega) \mu(d\omega).$$



## Chapter 9

# Integration: Part II

### 9.1 Pushforward measure

**Definition 9.1** (Pushforward function). Let  $(\Omega, \mathcal{F})$  and  $(\Omega^*, \mathcal{F}^*)$  be two measurable spaces and  $g : \Omega \rightarrow \Omega^*$  be  $\mathcal{F}/\mathcal{F}^*$  measurable. Let  $\mu$  be a measure on  $\mathcal{F}$ . Define the function

$$\mu g^{-1}(A^*) = \mu(g^{-1}(A^*)) = \mu(\{\omega \in \Omega : g(\omega) \in A^*\}), \quad A^* \in \mathcal{F}^*.$$

The measure  $\mu g^{-1}$  is referred to as the pushforward measure of  $\mu$ . This means that  $\mu g^{-1}$  measures, in terms of  $\mu$ , the pre-image of each set  $A^*$  under  $g$ . Hence,  $\mu$  is a valid measure on  $(\Omega^*, \mathcal{F}^*)$ !! It provides a way to "transfer" the measure from  $(\Omega, \mathcal{F})$  to  $(\Omega^*, \mathcal{F}^*)$  via the function  $g$ .

**Proposition 9.1.** TODO

### 9.2 Densities

**Proposition 9.2** ( $\nu$  is a measure on  $\mathcal{F}$ ). Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space and  $\phi : \Omega \rightarrow \overline{\mathbb{R}}$  be a nonnegative and  $\mathcal{F}$  measurable function. Then,  $\nu$  defined by

$$\nu(A) = \int_A \phi(\omega) \mu(d\omega), \quad A \in \mathcal{F},$$

is a measure on  $\mathcal{F}$

**Definition 9.2** ( $\phi$ , density of  $\nu$  in respect to  $\mu$ ). Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space and  $\nu$  be a measure on  $\mathcal{F}$ . A nonnegative and  $\mathcal{F}$  measurable function  $\phi : \Omega \rightarrow \overline{\mathbb{R}}$  is said to be a density of  $\nu$  with respect to  $\mu$  if for any  $A \in \mathcal{F}$ ,  $\nu(A) = \int_A \phi(\omega) \mu(d\omega)$ .

**Proposition 9.3.** Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space. Suppose that  $\nu$  is a measure on  $\mathcal{F}$  with density  $\phi$  with respect to  $\mu$ . Then

(i) for any nonnegative and  $\mathcal{F}$  measurable function  $f$ ,

$$\int_A f(\omega) \nu(d\omega) = \int_A f(\omega) \phi(\omega) \mu(d\omega), \quad A \in \mathcal{F};$$

(ii)  $f$  is integrable with respect to  $\nu$  if and only if  $f\phi$  (the product of the two functions) is integrable with respect to  $\mu$ . This is clear in (i).

(iii) if  $f\phi$  is integrable with respect to  $\mu$ , then (i) holds.

### 9.3 Integration with respect to the Lebesgue measure on the real line

**Definition 9.3.** Consider the measure space  $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \lambda)$ , where  $\lambda$  is the Lebesgue measure on the Borel  $\sigma$ -field  $\mathfrak{B}(\mathbb{R})$ . In accordance with Definition 8.5, a  $\mathfrak{B}(\mathbb{R})$  measurable function  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  is Lebesgue integrable if  $\int_{\mathbb{R}} |f(x)| \lambda(dx) < \infty$ . The integral of  $f$  with respect to  $\lambda$  is denoted with  $\int_{\mathbb{R}} f(x) dx$ , i.e.,  $\int_{\mathbb{R}} f(x) dx = \int_{\mathbb{R}} f(x) \lambda(dx)$ . If  $E \subset \mathbb{R}$  and  $\lambda|_E$  is the restriction of  $\lambda$  to  $\mathfrak{B}(E)$  (cf. Definition 4.1), then a  $\mathfrak{B}(E)$  measurable function  $f : E \rightarrow \overline{\mathbb{R}}$  is referred to as Lebesgue integrable if  $\int_E |f(x)| \lambda|_E(dx) < \infty$ . Also in this case we write  $\int_E |f(x)| \lambda|_E(dx) = \int_E f(x) dx$ .

In accordance with the fact that the Lebesgue measure of a single point is zero, we adapt the following definition.

**Definition 9.4.** TODO. Interesting but easy and well known.

We review the definition of a Riemann integrable function:

**Definition 9.5** (title).

## 9.4 Lecture

Partial integration and substitution TODO.

**Exercise 9.1** (9.6).  $\nu$  is a measure with density  $\phi$  with respect to  $\mu$ .  $f$  nonnegative and  $\mathcal{F}$  measurable. Prove:

$$(i) \int_A f(\omega) \nu(d\omega) = \int_A f(\omega) \phi(\omega) \mu(d\omega)$$

NOTE  $\nu(d\omega) = \phi(\omega) \mu(d\omega)$  short notation for  $\nu$  has density  $\phi$ :

1. Definition of  $\nu$  having a density  $\phi$  with respect to  $\mu$ : When we say that  $\nu$  has a density  $\phi$  with respect to  $\mu$ , it means that for any measurable set  $A \in \mathcal{F}$ , the measure  $\nu$  of  $A$  can be computed as:

$$\nu(A) = \int_A \phi(\omega) \mu(d\omega).$$

This is the integral of the function  $\phi$  over the set  $A$ , with respect to the measure  $\mu$ .

2. Notation  $\nu(d\omega) = \phi(\omega) \mu(d\omega)$ : This notation is shorthand and is used to express how  $\nu$  acts on infinitesimal elements in a manner analogous to how  $\mu$  acts, but scaled by the function  $\phi$ . It is essentially saying that for a small element  $d\omega$ , the measure  $\nu(d\omega)$  is given by  $\phi(\omega) \mu(d\omega)$ .

3. Clarification on  $\int_{d\omega} \phi(\omega) \mu(d\omega)$ : The correct notation or expression should not involve integrating over an "infinitesimal element"  $d\omega$ . The differential notation  $\nu(d\omega) = \phi(\omega) \mu(d\omega)$  is symbolic and used to express the relationship between  $\nu$  and  $\mu$  at a small scale, rather than an actual operation.

In summary,  $\nu(d\omega) = \phi(\omega) \mu(d\omega)$  is a concise way to denote that  $\nu$  is derived by weighting  $\mu$  by the density  $\phi$ , and this relationship is used to transform integrals with respect to  $\nu$  into integrals with respect to  $\mu$  weighted by  $\phi$ .

(ii)  $f$  integrable w.r.t.  $\nu \iff f\phi, (f(\omega)\phi(\omega))$ , integrable w.r.t.  $\mu$ .

(iii) if either of the two statements in (ii) holds, then (i) holds.

Proof:

(i). Let  $f$  be a standard simple function,  $f = \sum_{n=1}^N \alpha_i \mathbb{1}_{A_i}$ , then

$$\begin{aligned} \int_A f(\omega) \nu(d\omega) &= \int_A \left( \sum_{n=1}^N \alpha_i \mathbb{1}_{A_i}(\omega) \right) \nu(d\omega) = \sum_{n=1}^N \alpha_i \int_A \mathbb{1}_{A_i}(\omega) \nu(d\omega) = \sum_{n=1}^N \alpha_i \int_{\Omega} \mathbb{1}_A(\omega) \mathbb{1}_{A_i}(\omega) \nu(d\omega) \\ &= \sum_{n=1}^N \alpha_i \int_{\Omega} \mathbb{1}_{A \cap A_i}(\omega) \nu(d\omega) = \sum_{n=1}^N \alpha_i \nu(A \cap A_i) = \sum_{n=1}^N \alpha_i \int_{A \cap A_i} \phi(\omega) \mu(d\omega) = \sum_{n=1}^N \alpha_i \int_A \mathbb{1}_{A_i}(\omega) \phi(\omega) \mu(d\omega) \\ &= \int_A \sum_{n=1}^N \alpha_i \mathbb{1}_{A_i}(\omega) \phi(\omega) \mu(d\omega) = \int_A f(\omega) \phi(\omega) \mu(d\omega). \end{aligned}$$

Hence we have verified (i) if  $f$  is standard and simple.

In order to verify it for nonnegative functions:

(IMPORTANT; TOOL, TO ADD) Recall (chapter 7): Any  $f$  nonnegative and  $\mathcal{F}$  measurable can be approximated by a standard simple function, i.e.,  $\exists (f_n)_{n \in \mathbb{N}}$  s.t.  $f_n(\omega) \uparrow f(\omega)$ . By the monotone convergence theorem,

$$\int_{\Omega} f(\omega) \nu(d\omega) = \lim_{n \rightarrow \infty} \int_{\Omega} f_n(\omega) \nu(d\omega) = \lim_{n \rightarrow \infty} \int_{\Omega} f_n(\omega) \phi(\omega) \mu(d\omega)$$

$f_n$  converges to  $f$

$$\stackrel{\text{(again monotone convergence)}}{=} \int_{\Omega} f(\omega) \phi(\omega) \mu(d\omega).$$

This proves (i).

(ii).  $\int_A |f(\omega)|\nu(d\omega) < \infty$  (definition of integrability),  $= \int_A |f(\omega)\phi(\omega)|\mu(d\omega)$ , and we know that the equality holds by (i). This shows (ii).

(iii). Recall  $f^+ = \max(f, 0)$ ,  $f^- = \max(-f, 0)$ . Positive and negative parts of  $f$ . Cuts out all negative points. We know,

$$f(\omega) = f^+ - f^-(\omega).$$

$f$  integrable w.r.t.  $\nu$  implies that,

$$\int_{\Omega} f(\omega)\nu(d\omega) = \int_{\Omega} f^+\nu(d\omega) - \int_{\Omega} f^-(\omega)\nu(d\omega).$$

By (i) applied to  $f^+$  and  $f^-$ ,

$$= \int_{\Omega} f^{(+)}(\omega)\phi(\omega)\mu(d\omega) - \int_{\Omega} f^-(\omega)\phi(\omega)\mu(d\omega) = \int_{\Omega} f(\omega)\phi(\omega)\mu(d\omega).$$

**Exercise 9.2** (9.7). b) TODO

$\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-(\frac{x^2+y^2}{2})} d(x, y)$ , continuous as composition of continuous functions, and nonnegative. Fobini - Tonelli Theorem:

$$\begin{aligned} &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\frac{x^2}{2}} \left( \int_{\mathbb{R}} e^{-\frac{y^2}{2}} dy \right) dx. \\ &= \frac{1}{2\pi} \left( \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx \right)^2. \end{aligned}$$

$u = \frac{x}{\sqrt{2}}$  substitute

$$\begin{aligned} &= \frac{1}{2\pi} \left( \int_{\mathbb{R}} e^{-u^2} \sqrt{2} du \right)^2. \\ &= \frac{1}{\pi} \left( \int_{\mathbb{R}} e^{-u^2} du \right)^2 = \frac{\pi}{\pi}. \end{aligned}$$

Remember Gaussian integral:

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}.$$

# Chapter 10

## General notions in Probability

### 10.1 Probability spaces

**Definition 10.1.** Let  $(\Omega, \mathcal{F})$  be a measurable space. A probability  $\mathbb{P}$  on  $\mathcal{F}$  is a measure on  $\mathcal{F}$  s.t.  $\mathbb{P}(\Omega) = 1$ . The triple  $(\Omega, \mathcal{F}, \mathbb{P})$  is referred to as a probability space.

**Example 10.1.** Let  $\Omega$  be a finite and nonempty set. Define

$$\mathbb{P}(A) = \frac{\#A}{\#\Omega}, \quad A \in \mathcal{P}(\Omega),$$

Where  $\mathcal{P}(\Omega)$  is the power set on  $\Omega$ . Then,  $\mathbb{P}$  is a probability on  $\mathcal{P}(\Omega)$ .

**Example 10.2.** Let  $C$  be a set s.t.  $\#C = 52$ . Suppose that

$$C = S_1 \cup S_2 \cup S_3 \cup S_4,$$

with  $\{S_1, S_2, S_3, S_4\}$  disjoint and s.t.  $\#S_i = 13$  for all  $i = 1, 2, 3, 4$ . We remain in the setting of the previous example with

$$\Omega = \{A \subset C : \#A = 5\},$$

and  $\mathbb{P}$  on  $\mathcal{P}(\Omega)$  defined as in exercise 10.1. Upon exercise 1.1, we already know that  $\#\Omega = \binom{52}{5}$ . Let

$$A_i = \{A \subset S_i : \#A = 5\}, \quad i = 1, 2, 3, 4,$$

TODO

### 10.2 Random variables and random vectors

**Definition 10.2** (Random variable). Let  $(\Omega, \mathcal{F})$  be a measurable space. A map  $X : \Omega \rightarrow \mathbb{R}$  is referred to as a random variable on  $(\Omega, \mathcal{F})$  if it is  $\mathcal{F}/\mathfrak{B}(\mathbb{R})$  measurable.

**Definition 10.3** (Random vector). Let  $(\Omega, \mathcal{F})$  be a measurable space. A map  $X : \Omega \rightarrow \mathbb{R}^k$  is referred to as a random vector on  $(\Omega, \mathcal{F})$  if it is  $\mathcal{F}/\mathfrak{B}(\mathbb{R}^k)$  measurable.

**Proposition 10.1.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $X$  be a random vector on  $(\Omega, \mathcal{F})$ . A random variable  $Y$  on  $(\Omega, \mathcal{F})$  is  $\sigma(X)$  measurable if and only if there exists a function  $f : \mathbb{R}^k \rightarrow \mathbb{R}$  which is  $\mathfrak{B}(\mathbb{R}^k)$  measurable s.t.  $Y = f(X)$ .

**Definition 10.4.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. The distribution or law of a random vector on  $(\Omega, \mathcal{F})$  is the pushforward measure  $P_X = \mathbb{P}X^{-1}$  on  $\mathfrak{B}(\mathbb{R}^k)$  (cf. Definition 9.1). In particular, for any  $B \in \mathfrak{B}(\mathbb{R}^k)$ , we use the simplified notation

$$\{\omega \in \Omega : X(\omega) \in B\} = \{X \in B\},$$

and hence

$$P_X(B) = \mathbb{P}(X^{-1}(B)) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in B\}) = \mathbb{P}(X \in B).$$

For now, unless mentioned otherwise, if  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space, any random vector  $X$  is a random vector on  $(\Omega, \mathcal{F})$ , i.e., a  $\mathcal{F}$  measurable function with values in  $\mathbb{R}^k$ .

### 10.3 Discrete laws

**Definition 10.5** (Discrete random vector). Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. A random vector is referred to as discrete if there exists a countable set  $E = E_1 \times \dots \times E_k \subset \mathbb{R}^k$  s.t.  $P_X(E) = 1$ . That is to say that the law of  $X$  has a countable support.

**Proposition 10.2** (Discrete random vector). Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. A random vector  $X$  is discrete if and only if

$$P_X = \sum_{x \in E} p_x \delta_x, \quad p_x = \mathbb{P}(X = x),$$

for some countable set  $E = E_1 \times \dots \times E_k \subset \mathbb{R}^k$ . In particular, for any  $B \in \mathfrak{B}(\mathbb{R}^k)$ ,  $P_X(B) = \sum_{x \in B \cap E} p_x$ .

Proof of Proposition 10.2. Suppose that  $X$  is discrete. Let  $B \in \mathfrak{B}(\mathbb{R}^k)$ . We have that

$$P_X(B) = P_X(B \cap E) = \mathbb{P}(X \in B \cap E) = \mathbb{P}\left(\bigcup_{x \in B \cap E} \{X = x\}\right) = \sum_{x \in B \cap E} p_x = \sum_{x \in E} p_x \delta_x(B).$$

with respect to the other direction, if  $P_X$  is given as in Prop. 10.2, then

$$1 = P_X(\mathbb{R}^k) = \sum_{x \in E} p_x \delta_x(\mathbb{R}^k) = \sum_{x \in E} p_x = \mathbb{P}(X \in E) = P_X(E),$$

i.e.,  $X$  is a discrete random vector according to Def. 10.5.

**Example 10.3** (Tail, head). Let  $\Omega = \{t, h\}$  and

$$X(\omega) = \begin{cases} 0, & \text{if } \omega = t, \\ 1, & \text{if } \omega = h. \end{cases}$$

Then,  $X$  is a random variable on  $(\Omega, \mathcal{P}(\Omega))$ .

Explanation: for the cases  $X^{-1}(0) = t$  and  $X^{-1}(1) = h$  it is clear. Consider then  $X^{-1}(\omega) = \emptyset \in \mathcal{P}(\Omega)$  for any  $\omega \notin \{1, 2\}$ .

Suppose that  $\mathbb{P}$  is a probability on  $\mathcal{P}(\Omega)$  s.t.  $\mathbb{P}(X^{-1}(0)) = \mathbb{P}(X = 0) = 1 - p$  and  $\mathbb{P}(X = 1) = p$ . Clearly,  $\mathbb{P}(X \in \{0, 1\}) = P_X(\{0, 1\}) = 1$ . By Prop. 10.2, we deduce that the law of  $X$  is given by

$$P_X = (1 - p)\delta_0 + p\delta_1.$$

That is, for any  $B \in \mathfrak{B}(\mathbb{R})$ ,

$$P_X(B) = (1 - p)\delta_0(B) + p\delta_1(B) = \begin{cases} 0, & \text{if } 0 \notin B \text{ and } 1 \notin B \\ 1 - p, & \text{if } 0 \in B \text{ and } 1 \notin B \\ p, & \text{if } 0 \notin B \text{ and } 1 \in B \\ 1, & \text{if } 0 \in B \text{ and } 1 \in B \end{cases}.$$

For example, for  $B = (2, 4]$ ,  $X^{-1}(B) = \emptyset \in \mathcal{P}(\Omega)$ , and  $P_X(B) = P_X(\emptyset) = 0$ . Also note that, for example,  $\mathbb{P}(X = 0) = \mathbb{P}(X^{-1}(0)) = \mathbb{P}(t) = 1 - p$ .

## Chapter 11

# Collection of random vectors

### 11.1 Independence

**Definition 11.1** (Independent sub- $\sigma$ -fields). Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $\mathcal{A}_1, \dots, \mathcal{A}_n$  be  $n$  sub- $\sigma$ -fields on  $\Omega$ .  $\mathcal{A}_1, \dots, \mathcal{A}_n$  are said to be independent if for any  $A_1 \in \mathcal{A}_1, \dots, A_n \in \mathcal{A}_n$ ,

$$\mathbb{P}(A_1 \cap \dots \cap A_n) = \mathbb{P}(A_1) \times \dots \times \mathbb{P}(A_n).$$