Chapter 16.2

2.1. Free Profinite Monoids
2.2. Lattices
2.3. More Than Lattices
2.4. Varieties of Languages
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2.1. Free Profinite Monoids

Def. A finite monoid M separates two words $u, v \in A^*$ if $\exists$ morphism $\varphi : A^* \to M$ s.t. $\varphi(u) \neq \varphi(v)$.

Example 1. $\underbrace{M}_{\mathbb{Z}/2\mathbb{Z}}$

Let $\varphi : A^* \to \{0,1\}$ s.t. $w \mapsto |w| \pmod 2$

$\varphi(aba) = 1$
$\varphi(ab) = 0$ #

Let $r(u,v)$ denote the minimum cardinality of a monoid separating u and v.

$r(u,v) = 0$ iff $u = v$

Let $d(u,v) = \frac{1}{2^{r(u,v)}}$

Prop.1 d is ultrametric. Thus $\forall u,v,w \in A^*$

(1) $d(u,v) = 0$ iff $u = v$
(2) $d(u,v) = d(v,u)$
(3) $d(u,w) \leq \max(d(u,v), d(v,w))$

Also, $d(\overset{wu}{uw}, \overset{wv}{vw}) \leq d(u,v)$

Prop 2 (A^*, d) is discrete

WTS every subset is clopen

$u \in A^*$ $\quad n$ = size of the syntactic monoid of $\{u\}$

WTS $\{u\}$ is open $\quad n = |\text{synt}(\{u\})|$

Let $v \in A^*$ be arbitrary st. $d(u,v) < 2^{-n}$
$\therefore r(u,v) > n$
$\therefore M$ does not separate u and v
$\therefore u = v$

$B(u, 2^{-n}) = \{u\}$
$\therefore$ open

Every singleton is open

Let $L \subseteq A^*$
$L = \bigcup_{u \in L} \{u\}$ $\therefore L$ is open

L^c is open

Prop 3
$\to$ Every open ball of (A^*, d) is regular
Every regular language is a finite union of open balls.

Def. $\hat{A}^*$ is the completion of (A^*, d)
We construct $\hat{A}^*$ as follows:

$C(A^*)$
$x \sim y$ iff $x_1 y_1 x_2 y_2 \ldots$ is Cauchy
$x, y \in \hat{A}^*$
$d(x,y) = \lim_{n \to \infty} d(x_n, y_n)$

Prop 2.2.
1. Multiplication in A^* is uniformly continuous
2. Every morphism $\varphi : A^* \to B^*$ and $\psi : A^* \to M$ is uniformly continuous
3. $\hat{A}^*$ is a compact space

Extend multiplication of A^* to $\hat{A}^*$

$\varphi : A^* \to B^* \longrightarrow \hat{\varphi} : \hat{A}^* \to \hat{B}^*$ $\quad \psi : \hat{A}^* \to M$

Prop 2.3.1
L is regular iff $\bar{L}$ is clopen

2.2. Lattices

Intro to Families of Regular Languages

Def. Profinite equation
$u, v \in \hat{A}^*$
(u,v)
$u \to v$

$L \subseteq A^*$ satisfies $u \to v$ if $u \in \bar{L} \Rightarrow v \in \bar{L}$
$L \vdash u \to v$

Def. Lattice of languages over an alphabet A is a set of languages $\mathcal{L}$ s.t.
$A^* \in \mathcal{L}$
$\emptyset \in \mathcal{L}$
$\mathcal{L}$ is closed under finite union and intersection

$\mathcal{L}$ is a lattice iff exists some set E such $\mathcal{L} = \mathcal{L}(E)$

$\mathcal{L}(E)$ = set of reg languages in A^* which satisfy all relations in E

$A^* \in \mathcal{L}(E)$
$\emptyset \in \mathcal{L}(E)$

$E_L = \{(u,v) \in \hat{A}^* \times \hat{A}^* \mid L \vdash u \to v\}$

If L is regular, then E_L is clopen

$\mathcal{L}$ be a lattice

$E_{\mathcal{L}} = \bigcap_{L \in \mathcal{L}} E_L$

Let L be a regular language $\in \mathcal{L}(E_{\mathcal{L}})$

$\exists$ finite subset K of $\mathcal{L}$ s.t. $L \in \mathcal{L}(E_K)$

$xuy \to xvy \;\; \forall x,y \in \hat{A}^*$

$\to L \vdash v \leq u$ if $\forall x,y \in \hat{A}^*$ $L \vdash xuy \to xvy$

$\to L \vdash u = v$ if $L \vdash u \leq v$ and $L \vdash v \leq u$

$\mathcal{V}$
$A^* \mathcal{V}$