Jacobi's Two-Square Theorem

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Abstract

This report explores Jacobi's Two-Square Theorem, which determines the number of ways a positive integer can be written as a sum of two squares. This builds upon Fermat's theorem, which determines if a positive integer can be written as a sum of two squares. This report includes a detailed example of Jacobi's Two-Square Theorem, followed by a proof based on Jacobi's triple product identity.

1 Introduction

Jacobi's Two-Square Theorem that counts the number of ways a given positive integer can be expressed as the sum of two squares. The theorem was proven in the 19th century by Carl Gustav Jacob Jacobi, and builds upon earlier work by Pierre de Fermat.

Fermat's Two-Square Theorem addresses the existence of such representations. The Theorem determines whether a given integer can be written as a sum of two squares. Jacobi extended this theorem by asking not just whether such representations exist, but how many ways a number can be represented in this form [3].

Jacobi's Two-Square Theorem 1 . *The number $r_2(n)$ of representations of the positive integer n as a sum of two squares is given by*

$$r_2(n) = 4(d_1(n) - d_3(n))$$

Where $d_i(n) =$ number of divisors of n that are congruent to i modulo 4.

2 Example Using Jacobi's Two-Square Theorem

Here is an example of how to use Jacobi's Two-Square Theorem: Suppose you want to find the number of ways 45 can be written as a sum of two squares. Jacobi's Theorem can help us determine this.

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2.1 Step 1: find the divisors of 45

The divisors of 45 would be: 1, 3, 5, 9, 15, 45.

2.2 Step 2: find $d_1(n)$

d_1 is the numbers of divisors of n that have a remainder of 1. For example 1 divided by 4, 4 fits into 1 zero times with a remainder of 1. After applying this logic to the rest of the divisors, we find d_1 to be 1, 5, 9, 45.

2.3 Step 3: find $d_3(n)$

d_3 is the numbers of divisors of n that have a remainder of 3. For example 3 divided by 4, 4 fits into 3 zero times with a remainder of 3. After applying this logic to the rest of the divisors, we find d_3 to be 3, 15.

2.4 Step 4: Use Jacobi's Two-Square Theorem

$$r_2(n) = 4(d_1(n) - d_3(n))$$

$4(d_1 - d_3) = 4(4 - 2) = 4 \cdot 2 = 8$

We find there are eight ways to write 45 as a sum of 2 squares, which are the following eight ways:

$$6^2 + 3^2, (-6)^2 + 3^2, 6^2 + (-3)^2, (-6)^2 + (-3)^2,$$
$$3^2 + 6^2 (-3)^2 + 6^2, 3^2 + (-6)^2, (-3)^2 + (-6)^2.$$

3 Proof of by Hirschhorn

The proof of Jacobi's Two-Square Theorem can be divided into three main parts. The first part is about using Jacobi's Triple Product Identity. The second part of the proof is to manipulate the power series. The third part is to understand the coefficients of x^n in the series and relate them to the number of representations of n as a sum of two squares [1].

3.1 Jacobi's Triple Product Identity

Jacobi's Triple Product Identity 1

$$\prod_{n\geq 1}(1 + ax^{2n-1})(1 + a^{-1}x^{2n-1})(1 - x^{2n}) = \sum_{n=-\infty}^{\infty} a^n x^{n^2}$$

and this holds for each pair of complex numbers a, x with $a \neq 0$ and $|x| < 1$.

3.2 Further changes to infinite products

Put $-a^2x$ for a, then x for x^2, multiply by a and we obtain the identity, invariant under $a \leftrightarrow -a^{-1}$

$$(a-a^{-1})\prod_{n\geq 1}(1-a^2x^n)(1-a^{-2}x^n)(1-x^n) = \sum_{n=-\infty}^{\infty}(-1)^n a^{2n+1}x^{(n^2+n)/2}.$$

Which then leads to

$$\begin{aligned} a\prod_{n\geq 1}((1+a^4x^{4n-1})(1+a^{-4}x^{4n-3})(1-x^{4n})- \\ -a^{-1}\prod_{n\geq 1}(1+a^4x^{4n-3})(1+a^{-4}x^{4n-1})(1-x^{4n}) = \\ = \sum_{n=-\infty}^{\infty}(a^{4n+1}x^{2n^2+n}) - \sum_{n=-\infty}^{\infty}(a^{4n-1}x^{2n^2-n}), \end{aligned}$$

3.3 Eliminating parameter a

Differentiate with respect to a, put $a=1$, divide by 2, and we find

$$\begin{aligned} &\prod_{n\geq 1}(1-x^n)^3 = \\ &= \prod_{n\geq 1}(1+x^{4n-3})(1+x^{4n-1})(1-x^{4n})\cdot(1-4\sum_{n\geq 1}(\frac{x^{4n-3}}{1+x^{4n-3}}-\frac{x^{4n-1}}{1+x^{4n-1}})). \end{aligned}$$

Divide by

$$\prod_{n\geq 1}(1+x^n)^2(1-x^n) = \prod_{n\geq 1}(1+x^n)(1+x^n)(1-x^n)$$

$$= \prod_{n\geq 1}(1+x^n)(1-x^{2n}).$$

Because the following equation is a product you can expand

$$\prod_{n\geq 1}(1+x^n) = (1+x)(1+x^2)(1+x^3)(1+x^4)\dots$$

which can also be written as

$$(1+x)(1+x^2)(1+x^3)(1+x^4)\dots = \prod_{n\geq 1}(1+x^{2n-1})(1+x^{2n}).$$

substitute this formula back into this equation

$$= \prod_{n\geq 1}(1+x^n)(1-x^{2n})$$

and we get

$$= \prod_{n\geq 1}(1+x^{2n-1})(1+x^{2n})(1-x^{2n}) = \prod_{n\geq 1}(1+x^{2n-1})(1-x^{4n}).$$

Through a similar process we can write $\prod_{n\geq 1}(1+x^{2n-1})$ as

$$\prod_{n\geq 1}(1+x^{4n-3})(1+x^{4n-1}) = \prod_{n\geq 1}(1+x^{4n-3})(1+x^{4n-1})(1-x^{4n}).$$

After dividing the left side of the equation at the beginning of 3.3 by

$$\prod_{n\geq 1}(1+x^n)^2(1-x^n)$$

and the right side of the equation at the beginning of 3.3 by

$$\prod_{n\geq 1}(1+x^{4n-3})(1+x^{4n-1})(1-x^{4n}),$$

We get

$$(\prod_{n\geq 1}\frac{1-x^n}{1+x^n})^2 = 1-4\sum_{n\geq 1}(\frac{x^{4n-3}}{1+x^{4n-3}} - \frac{x^{4n-1}}{1+x^{4n-1}}).$$

3.4 Simplifying the left-hand side

Now, let us focus on

$$\prod_{n\geq 1}\frac{1-x^n}{1+x^n}.$$

You can rewrite $(1-x^n)$ using the method shown in the previous step as

$$\prod_{n\geq 1}\frac{(1-x^{2n-1})(1-x^{2n})}{(1+x^n)}.$$

You can further simplify by dividing $(1-x^{2n})$ with $(1+x^n)$

$$= \prod_{n\geq 1}(1-x^{2n-1})(1-x^n).$$

As shown earlier you can rewrite $(1-x^n)$ which equals

$$\prod_{n\geq 1}(1-x^{2n-1})(1-x^{2n-1})(1-x^{2n}).$$

We can once again use Jacobi's Triple Product Identity where $a = 1$ to obtain the following formula for this product

$$= \sum_{n=-\infty}^{\infty}(-1)^n x^{n^2}.$$

3.5 The final relation between two series

Substituting the last step back into the equation,

$$(\sum_{n=-\infty}^{\infty}(-1)^n x^{n^2})^2 = 1 - 4\sum_{n\geq 1}(\frac{x^{4n-3}}{1+x^{4n-3}} - \frac{x^{4n-1}}{1+x^{4n-1}})$$

Put $-x$ for x, and we obtain

$$(\sum_{n=-\infty}^{\infty} x^{n^2})^2 = 1 + 4\sum_{n\geq 1}(\frac{x^{4n-3}}{1-x^{4n-3}} - \frac{x^{4n-1}}{1-x^{4n-1}}).$$

3.6 Relation to Jacobi's Two-Square Theorem

The above equation relates to Jacobi's Two-Square Theorem by the following:

The left-hand side of the equation is $(\sum_{n=-\infty}^{\infty} x^{n^2})^2$. One can write this as:

$$\sum_{n=-\infty}^{\infty} x^{n^2} \cdot \sum_{n=-\infty}^{\infty} x^{n^2}$$

Which can be further expanded to

$$(\ldots+x^{3^2}+x^{2^2}+x^{1^2}+1+x^{1^2}+x^{2^2}+\ldots)\cdot(\ldots+x^{3^2}+x^{2^2}+x^{1^2}+1+x^{1^2}+x^{2^2}\ldots)$$

With the above equation, one can see that $x^n = x^{a^2} \cdot x^{b^2}$, which means that the coefficient of x^n equals the number of ways that an integer can be as a sum of two squares. $(n = a^2 + b^2)$

The right-hand side of the equation $1 + 4\sum_{n\geq 1}(\frac{x^{4n-3}}{1-x^{4n-3}} - \frac{x^{4n-1}}{1-x^{4n-1}})$ relates to Jacobi's Two-Square Theorem by the following:

We consider power series and consider the coefficients of x,

$$\sum_{n\geq 1}\frac{x^{4n-3}}{1-x^{4n-3}} = \frac{x}{1-x} + \frac{x^5}{1-x^5} + \ldots$$

With the first expansion, $\frac{x}{1-x} = x + x^2 + x^3 + x^4 + \ldots$, in this x^n appears here only if 1 divides n.

With the second expansion, $\frac{x^5}{1-x^5} = x^5 + x^{10} + x^{15} + x^{20} + \dots$, in this x^n appears here only if 5 divides n. And so on.

The overall coefficient of x^n in expansion of $\sum_{n\geq 1} \frac{x^{4n-3}}{1-x^{4n-3}}$ as power series is $d_1(n)$.

A similar process will be used for $\sum_{n\geq 1} \frac{x^{4n-1}}{1-x^{4n-1}}$, where we consider power series and consider the coefficients of x,

$$\sum_{n\geq 1} \frac{x^{4n-1}}{1-x^{4n-1}} = \frac{x^3}{1-x^3} + \frac{x^7}{1-x^7} + \dots$$

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With the first expansion, $\sum_{n\geq 1} \frac{x^3}{1-x^3} = x^3 + x^6 + x^9 + x^{12} + \dots$, in this x^n appears here only if 3 divides n.

With the second expansion, $\sum_{n\geq 1} \frac{x^7}{1-x^7} = x^7 + x^{14} + x^{21} + x^{28} + \dots$, in this x^n appears here only if 7 divides n.

The overall coefficient of x^n in expansion of $\sum_{n\geq 1} \frac{x^{4n-3}}{1-x^{4n-3}}$ as power series is $d_3(n)$.

4 Conclusion

In this report, we explore Jacobi's Two-Square Theorem, which determines the number of ways a positive integer can be written as a sum of two squares. The report illustrated an example of how to use Jacobi's Two-Square Theorem. Furthermore, the report examined the proof of the theorem, using Jacobi's Triple-Product Identity. Future work involves Jacobi's Four-Square Theorem, which involves similar methods. Jacobi's Four-Square Theorem determines the number of representations of a positive integer as a sum of four squares. More details about Jacobi's Four-Square Theorem can be found in the following article [2].

References

[1] Michael D. Hirschhorn. A simple proof of jacobi's two-square theorem. *The American Mathematical Monthly*, 92(8):579–580, 1985.

[2] Michael D. Hirschhorn. A simple proof of jacobi's four-square theorem. *Proceedings of the American Mathematical Society*, 101(3):436–438, 1987.

[3] Elemér Kiss. A short proof of fermat's two-square theorem given by jános bolyai. *Mathematica Pannonica*, 8(2):293–295, 1997.