A NOVEL METHOD FOR DETERMINING DOA FROM FAR-FIELD TDOA OR FDOA

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1. Introduction

Locating a radio-frequency transmitter is a vital step in many applications. Source localization is often performed using measurements of the transmitted signal obtained by several nearby receivers. Specific observable quantities include the time difference of arrival (TDOA) and frequency difference of arrival (FDOA) between receivers. When the distance between the receivers and the transmitter is much greater than the distance between the receivers (far-field case [3]), the angle of arrival (AOA) can be found using the geometric relationship with the TDOA and distance between receivers [1]. Additionally, the problem can be simplified if information about the transmitter is known *a priori*, such as altitude (ALT). This paper introduces a novel approach to calculating the AOA by capitalizing on the interesting geometry of the source-localization problem. This method allows for the calculation of AOA using either TDOA or FDOA measurements and simplifies the problem to the solution of a linear system of equations.

The FDOA equations are nonlinear and thus have a more complicated geometry than geolocation with TDOA measurements only. While the FDOA measurements are often used as an additional constraint to the TDOA geolocation systems (TDOA/FDOA localization) [5], only a few algorithms have been developed using FDOA alone ([2,6]). There are some special cases where it is desirable to solve for the emitter location using FDOA only. For instance in the case of a narrowband signal with a long pulse duration, the Doppler resolution is higher than the range resolution and it can be difficult to measure the TDOA accurately [3, 6, 7].

As mentioned above, the angle of arrival (AOA), sometimes referred to as the direction of arrival (DOA), is usually calculated using basic trigonometry and TDOA measurements. A nice overview of this relationship is covered in [1]. For a source in the near-field, both the ranges and AOA can be found using this method with a single array of sensors. In the far-field case, only the AOA can be calculated, so more than one set of measurements are needed for source localization [1]. In this paper, we present an alternative method for calculation of AOA using FDOA measurements in the far-field.

We derive the FDOA far-field approximation and its use for determining direction of arrival in section 2. A proposed source-localization algorithm is presented in section 4.

2. Far-field approximation for FDOA

Consider a stationary transmitter located at $\mathbf{x}$. Additionally, there are N receivers located at $\mathbf{x}_1, ..., \mathbf{x}_N$ with velocity $\mathbf{v}_1, ..., \mathbf{v}_N$. The frequency shift of the signal between the emitter and receiver i is,

(1) $$d_i = \mathbf{v}_i^T \cdot \frac{\mathbf{x}_i - \mathbf{x}}{\|\mathbf{x}_i - \mathbf{x}\|}.$$

We now derive a far-field approximation for equation 1.

Assume without loss of generality that the receivers are centered around the origin. We consider the far-field case, where the distance between receivers is much smaller than the distance to the emitter, i.e. $\|\mathbf{x}\| >> \|\mathbf{x}_i\|$, $\forall\ i$. The far-field approximation (as in [3]) for $1/\|\mathbf{x} - \mathbf{x_i}\|$ is:

$$\frac{1}{\|\mathbf{x} - \mathbf{x_i}\|} = \frac{1}{\|\mathbf{x}\|}\left(1 + \mathcal{O}\left(\frac{\|\mathbf{x}_i\|}{\|\mathbf{x}\|}\right)\right).$$

Truncating after the first term above allows for simplification of the factor (in eq. 1):

$$\frac{\mathbf{x}_i - \mathbf{x}}{\|\mathbf{x}_i - \mathbf{x}\|} = \frac{\mathbf{x}_i}{\|\mathbf{x}\|} - \frac{\mathbf{x}}{\|\mathbf{x}\|}.$$

Additionally, the far-field assumption implies that the first term will have small magnitude. Thus, $\dfrac{\mathbf{x_i} - \mathbf{x}}{\|\mathbf{x_i} - \mathbf{x}\|}$ is simplified to $\dfrac{-\mathbf{x}}{\|\mathbf{x}\|}$. Equation 1 becomes:

(2) $$d_i = -\mathbf{v}_i^T \cdot \hat{\mathbf{x}},$$

where $\hat{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$, is a unit vector from the centroid of the receivers. The entire system of frequency shifts can be written:

(3) $$\mathbf{d} = -\mathbf{V}\hat{\mathbf{x}},$$

where

$$\mathbf{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix} \qquad \mathbf{V} = \begin{pmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_N^T \end{pmatrix}.$$

In practice, the frequency shifts are not observable. Instead the frequency difference of arrival (FDOA) is measured between receivers. The FDOA is equivalent to the difference in frequency shifts,

(4) $$f_{i,j} = d_j - d_i.$$

A system equivalent to equation 3 can be constructed for the FDOA, with the use of a differencing matrix $\mathbf{P}$. The matrix $\mathbf{P}$ has entries of 0 and ± 1 corresponding to the differencing in equation 4. Thus, with the far-field simplification above, the vector of FDOA measurements, $\mathbf{f}$, is equivalent to,

(5) $$\mathbf{f} = -\mathbf{P}\mathbf{V}\hat{\mathbf{x}}.$$

The matrix $-\mathbf{PV}$ will be referred to as $\tilde{\mathbf{V}}$ for simplicity.

This far-field simplification reduces the FDOA equations to a linear system. This suggests that feasible FDOA measurements in the far-field case lie on the image of the unit circle transformed by the matrix $\tilde{\mathbf{V}}$. This image is an ellipse with rotation and scaling determined by the singular value decomposition of $\tilde{\mathbf{V}}$. Indeed, this can be confirmed by computing the singular value decomposition of generated far-field FDOA measurements and confirming they lie on the same subspace as $\tilde{\mathbf{V}}$. This relationship can also be demonstrated visually with a plot of generated FDOA measurements (fig. 1).

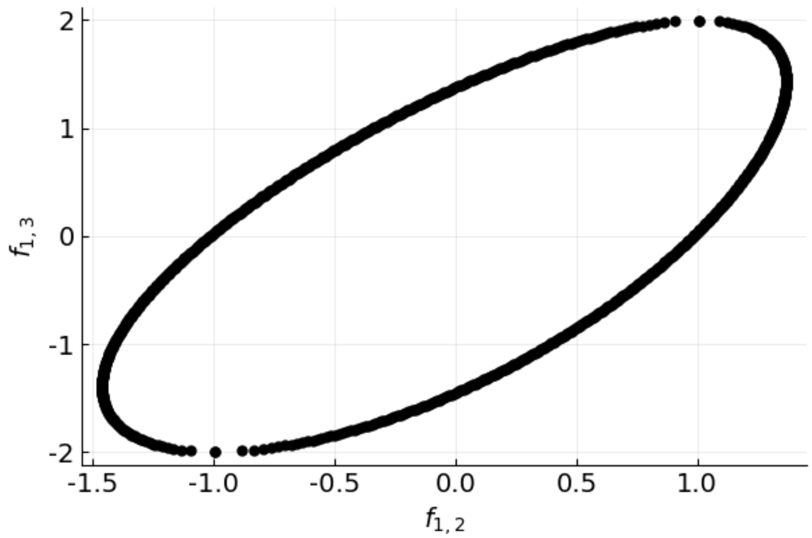

FIGURE 1. Plot of far-field $f_{1,2}$ vs. $f_{1,3}$ for a system of three receivers centered around the origin. Note the image is an ellipse with scaling in the direction of the left-singular vectors of $\tilde{\mathbf{V}}$.

One method for denoising in TDOA-based geolocation is the projection of noisy measurements onto the range of the differencing matrix $\mathbf{P}$ [4, 8]. This ensures that the TDOA measurements are physically realizable and consistent between receivers. One benefit of the method for DOA calculation proposed above is that denoising is automatically performed since projection onto the range of $-\mathbf{PV}$ is equivalent to projection onto the range of $\mathbf{P}$.

2.1. **Calculating direction of arrival (DOA).** The far-field approximated form of the FDOA equations is linear with variable $\hat{\mathbf{x}}$, representing the direction of arrival (DOA) of the signal. Thus, the DOA can be found with a linear solve of equation 5. If there are more FDOA measurements than direction components, we can find the least squares solution to the problem, which is also the pseudo-inverse solution:

$$\hat{\mathbf{x}} = (\tilde{\mathbf{V}}^T\tilde{\mathbf{V}})^{-1}\tilde{\mathbf{V}}^T\mathbf{f}. \tag{6}$$

If the receivers move and repeat this process, the intersection of the lines generated by equation 6 will provide an estimate for the location of the emitter. This is the idea behind our source localization algorithm (section 4).

3. Far-field approximation for TDOA

Although the time difference of arrival (TDOA) is simpler than the FDOA case, we include its far-field approximation for completeness.

Using the same problem setup as above, the time it takes for the signal to travel between the emitter and receiver i is:

$$\tau_i = \frac{1}{c}\|\mathbf{x_i} - \mathbf{x}\|,$$

from here the scalar $\frac{1}{c}$ will be left out for simplicity. The far-field approximation for $\|\mathbf{x_i} - \mathbf{x}\|$ is given [3],

$$\|\mathbf{x_i} - \mathbf{x}\| = \|\mathbf{x}\| \left(1 - \frac{\mathbf{x}_i \cdot \hat{\mathbf{x}}}{\|\mathbf{x}\|} + \mathcal{O}\left(\frac{\|\mathbf{x}_i\|}{\|\mathbf{x}\|}\right)\right).$$

Thus, τ_i becomes,

$$\tau_i = \|\mathbf{x}\| - \mathbf{x}_i \cdot \hat{\mathbf{x}}.$$

As in the FDOA case, τ_i is not observable. Instead we look to the time difference of arrival (TDOA) between receivers i and j,

$$\begin{aligned}\tau_{i,j} &= (\|\mathbf{x}\| - \mathbf{x}_j \cdot \hat{\mathbf{x}}) - (\|\mathbf{x}\| - \mathbf{x}_i \cdot \hat{\mathbf{x}}) \\ &= \mathbf{x}_i \cdot \hat{\mathbf{x}} - \mathbf{x}_j \cdot \hat{\mathbf{x}} \\ &= (\mathbf{x}_i - \mathbf{x}_j) \cdot \hat{\mathbf{x}}.\end{aligned}$$

The system of TDOA measurements are equivalent to:

$$\boldsymbol{\tau} = -\mathbf{P}\mathbf{X}\hat{\mathbf{x}}, \tag{7}$$

where $\mathbf{X}$ is the matrix of receiver locations and $\mathbf{P}$ is a differencing matrix as before. This suggests that feasible far-field TDOA measurements lie in the image of the unit circle under transformation of $-\mathbf{PX}$.

3.1. **Calculating direction of arrival (DOA).** As in the FDOA case, the direction of arrival can be calculated using the pseudoinverse:

$$\hat{\mathbf{x}} = -((\mathbf{PX})^T\mathbf{PX})^{-1}(\mathbf{PX})^T\mathbf{f}. \tag{8}$$

4. Source-localization algorithm

Formulas for calculating the direction of arrival (DOA) of a signal from FDOA or TDOA measurements were given in equations 6 and 8, respectively. If the receivers move, measurements are recorded for a second time step, and the DOA is recalculated, the intersection of these lines will provide an estimate for the emitter location. This simple algorithm (in 2D) is described more precisely below. A schematic of an example run is shown in figure 2.

Input:

- Receiver data: Location and velocity for two time steps. Without loss of generality, assume the receivers are centered at the origin in time step 1, centered at (c_x, c_y) at timestep 2.
- FDOA measurements between all pairs of receivers of the same time step.

Output:

- Estimate for emitter location, $\tilde{\mathbf{x}} = [\tilde{x}, \tilde{y}]^T$.

(1) Calculate DOA for time step 1 and time step 2 ($\hat{\mathbf{x}}^{(1)} = [\hat{x}^{(1)}, \hat{y}^{(1)}]^T, \hat{\mathbf{x}}^{(2)} = [\hat{x}^{(2)}, \hat{y}^{(2)}]^T$) using equation 6.

(2) Find the intersection of the lines generated by (1):

$$\tilde{x} = \frac{c_y - \hat{y}^{(2)} c_x / \hat{x}^{(2)}}{\hat{y}^{(1)}/\hat{x}^{(1)} - \hat{y}^{(2)}/\hat{x}^{(2)}}$$

$$\tilde{y} = \frac{\hat{y}^{(1)}}{\hat{x}^{(1)}} \cdot \tilde{x}.$$

(3) Return $\tilde{\mathbf{x}}$.

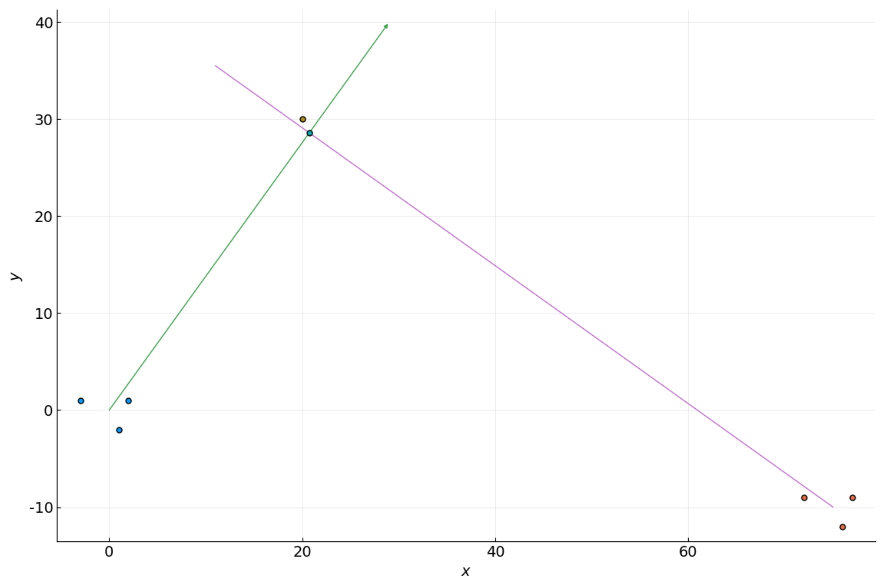

FIGURE 2. Proof of concept example for the proposed source localization algorithm in 2D with three receivers. In this example, the estimated location is $\tilde{\mathbf{x}} = (20.7318, 28.5771)$ and the correct location is $\mathbf{x} = (20, 30)$. The DOA calculations used FDOA measurements.

Conclusion

Considering far-field FDOA-based geolocation naturally leads to a simple method for determining direction of arrival. This calculation requires only a linear solve which makes the corresponding source-localization technique very efficient. Additionally, since FDOA measurement data is projected onto the range of the differencing matrix, the solution is naturally de-noised in a method consistent with [4, 8].

References

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