A NOVEL METHOD FOR DETERMINING DOA FROM FAR-FIELD TDOA OR FDOA

KARLEIGH J. CAMERON AND SAMUEL J. PINE

1. Introduction

Locating a radio-frequency transmitter, or *source localization*, is a vital step in many applications. Source localization is often performed using measurements of the transmitted signal obtained by several nearby receivers. Specifically, measurements of the transmitted signal at two distinct receivers allow one to compute the time difference of arrival (TDOA) and frequency difference of arrival (FDOA) between those receivers. With estimates of TDOA or FDOA measurements, one can compute various other quantities describing the location of the receiver, including the angle of arrival (AOA) / direction of arrival (DOA), the range to the receiver, and thus the location of the receiver in the global coordinate system (geolocation).

For a nearby source, simple geometric relationships between the TDOA measurements and the known receiver positions allow both the range to the transmitter and the AOA to be computed. When the distance between the receivers and the transmitter is much greater than the distance between the receivers it is common to simplify the wave propagation model and assume that wave curvature is negligible in the region of the receivers. This assumption is commonly referred to as the far-field assumption [3]. This simplification modifies the geometric relationships between the TDOA and receiver positions so that one can compute only the AOA in the case where a single array of sensors is available [1]. If information about the transmitter is known *a priori*, such as altitude (ALT), it is typically possible to remedy the situation and also solve for the range to the transmitter.

The equations relating the FDOA measurements to the receiver positions are much more complicated than those used in the TDOA case. The FDOA model is nonlinear and depends on the receiver velocities, and so source localization with FDOA measurements is more complicated than geolocation using TDOA measurements. While the FDOA measurements are often used as an additional constraint to the TDOA geolocation systems (TDOA/FDOA localization) [5], only a few algorithms have been developed using FDOA alone [2, 6]. There are some special cases where it is desirable to solve for the emitter location using FDOA only. For instance, in the case of a narrowband signal with a long pulse duration, Doppler resolution is finer than the range resolution so that it is difficult to measure the TDOA accurately [3, 6, 7].

This paper introduces a novel approach to calculating the AOA/DOA using either TDOA or FDOA measurements, or both, by capitalizing on the simplified geometry of the source-localization problem under the far-field assumption. The main benefit of this method is its computational efficiency, as it simplifies the calculation of AOA to the solution of a linear system of equations. In section 2, we develop a far-field model for the FDOA measurements and discuss a technique for determining the signal direction of arrival. In section 3, we develop a similar far-field approximation for the TDOA model and present the analogous AOA technique. Finally, we summarize the results by presenting a source-localization algorithm in section 4 and a discussion of future directions for research in section 5.

2. Angle of Arrival with FDOA Measurements

Consider a stationary transmitter located at $\mathbf{x}$. Additionally, there are N receivers located at $\mathbf{x}_1, ..., \mathbf{x}_N$ with velocity $\mathbf{v}_1, ..., \mathbf{v}_N$. The frequency shift of the signal between the emitter and receiver i is,

$$d_i = \mathbf{v}_i^T \cdot \frac{\mathbf{x}_i - \mathbf{x}}{\|\mathbf{x}_i - \mathbf{x}\|}. \tag{1}$$

2.1. **Far-field approximation for FDOA.** We now derive a far-field approximation for equation 1.

Assume without loss of generality that the receivers are centered around the origin. We consider the far-field case, where the distance between receivers is much smaller than the distance to the emitter, i.e. $\|\mathbf{x}\| >> \|\mathbf{x}_i\|$, $\forall\ i$. The far-field approximation (as in [3]) for $1/\|\mathbf{x} - \mathbf{x_i}\|$ is:

$$\frac{1}{\|\mathbf{x} - \mathbf{x_i}\|} = \frac{1}{\|\mathbf{x}\|}\left(1 + \mathcal{O}\left(\frac{\|\mathbf{x}_i\|}{\|\mathbf{x}\|}\right)\right).$$

Truncating after the first term above allows for simplification of the factor (in eq. 1):

$$\frac{\mathbf{x}_i - \mathbf{x}}{\|\mathbf{x}_i - \mathbf{x}\|} = \frac{\mathbf{x}_i}{\|\mathbf{x}\|} - \frac{\mathbf{x}}{\|\mathbf{x}\|}.$$

Additionally, the far-field assumption implies that the first term will have small magnitude. Thus, $\dfrac{\mathbf{x_i} - \mathbf{x}}{\|\mathbf{x_i} - \mathbf{x}\|}$ is simplified to $\dfrac{-\mathbf{x}}{\|\mathbf{x}\|}$. Equation 1 becomes:

$$d_i = -\mathbf{v}_i^T \cdot \hat{\mathbf{x}}, \tag{2}$$

where $\hat{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$, is a unit vector from the centroid of the receivers. The entire system of frequency shifts can be written:

$$\mathbf{d} = -\mathbf{V}\hat{\mathbf{x}}, \tag{3}$$

where

$$\mathbf{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix} \qquad \mathbf{V} = \begin{pmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_N^T \end{pmatrix}.$$

In practice, the frequency shifts are not observable. Instead the frequency difference of arrival (FDOA) is measured between receivers. The FDOA is equivalent to the difference in frequency shifts,

$$f_{i,j} = d_j - d_i. \tag{4}$$

A system equivalent to equation 3 can be constructed for the FDOA, with the use of a differencing matrix $\mathbf{P}$. The matrix $\mathbf{P}$ has entries of 0 and ± 1 corresponding to the differencing in equation 4. Thus, with the far-field simplification above, the vector of FDOA measurements, $\mathbf{f}$, is equivalent to,

$$\mathbf{f} = -\mathbf{P}\mathbf{V}\hat{\mathbf{x}}. \tag{5}$$

The matrix $-\mathbf{P}\mathbf{V}$ will be referred to as $\tilde{\mathbf{V}}$ for simplicity.

This far-field simplification reduces the FDOA equations to a linear system. This suggests that feasible FDOA measurements in the far-field case lie on the image of the unit circle transformed by the matrix $\tilde{\mathbf{V}}$. This image is an ellipse with rotation and scaling determined by the singular value decomposition of $\tilde{\mathbf{V}}$. Indeed, this can be confirmed by computing the singular value decomposition of generated far-field FDOA measurements and confirming they lie on the same subspace as $\tilde{\mathbf{V}}$. This relationship can also be demonstrated visually with a plot of generated FDOA measurements (fig. 1).

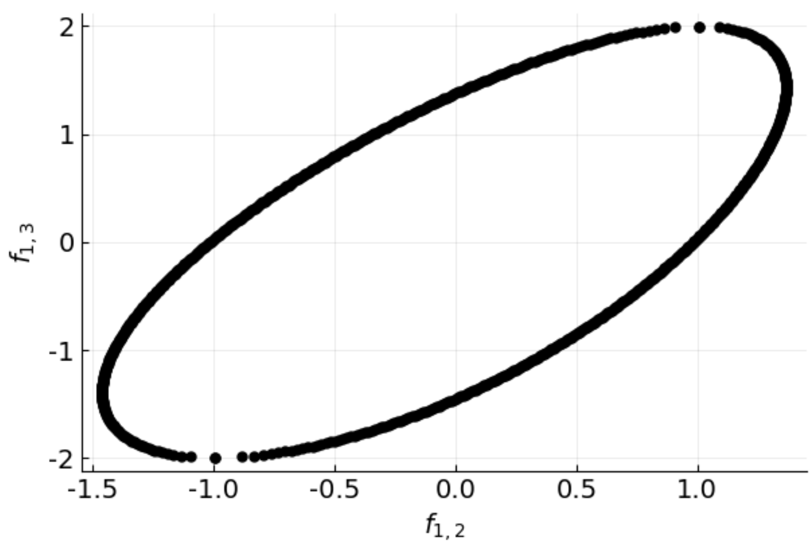

FIGURE 1. Plot of far-field $f_{1,2}$ vs. $f_{1,3}$ for a system of three receivers centered around the origin. Note the image is an ellipse with scaling in the direction of the left-singular vectors of $\tilde{\mathbf{V}}$.

One method for denoising in TDOA-based geolocation is the projection of noisy measurements onto the range of the differencing matrix $\mathbf{P}$ [4, 8]. This ensures that the TDOA measurements are physically realizable and consistent between receivers. One benefit of the method for DOA calculation proposed above is that denoising is automatically performed since projection onto the range of $-\mathbf{P}\mathbf{V}$ is equivalent to projection onto the range of $\mathbf{P}$.

2.2. **Calculating direction of arrival (DOA).** The far-field approximated form of the FDOA equations is linear with variable $\hat{\mathbf{x}}$, representing the direction of arrival (DOA) of the signal. Thus, the DOA can be found with a linear solve of equation 5. If there are more FDOA measurements than direction components, we can find the least squares solution to the problem, which is also the pseudo-inverse solution:

$$\hat{\mathbf{x}} = (\tilde{\mathbf{V}}^T\tilde{\mathbf{V}})^{-1}\tilde{\mathbf{V}}^T\mathbf{f}. \tag{6}$$

If the receivers move and repeat this process, the intersection of the lines generated by equation 6 will provide an estimate for the location of the emitter. This is the idea behind our source localization algorithm (section 4).

3. Angle of Arrival with TDOA Measurements

3.1. **Far-field approximation for TDOA.** Although the time difference of arrival (TDOA) is simpler than the FDOA case, we include its far-field approximation for completeness.

Using the same problem setup as above, the time it takes for the signal to travel between the emitter and receiver i is:

$$\tau_i = \frac{1}{c}\|\mathbf{x_i} - \mathbf{x}\|,$$

from here the scalar $\frac{1}{c}$ will be left out for simplicity. The far-field approximation for $\|\mathbf{x_i} - \mathbf{x}\|$ is given [3],

$$\|\mathbf{x_i} - \mathbf{x}\| = \|\mathbf{x}\| \left(1 - \frac{\mathbf{x}_i \cdot \hat{\mathbf{x}}}{\|\mathbf{x}\|} + \mathcal{O}\left(\frac{\|\mathbf{x}_i\|}{\|\mathbf{x}\|}\right)\right).$$

Thus, τ_i becomes,

$$\tau_i = \|\mathbf{x}\| - \mathbf{x}_i \cdot \hat{\mathbf{x}}.$$

As in the FDOA case, τ_i is not observable. Instead we look to the time difference of arrival (TDOA) between receivers i and j,

$$\begin{aligned}\tau_{i,j} &= (\|\mathbf{x}\| - \mathbf{x}_j \cdot \hat{\mathbf{x}}) - (\|\mathbf{x}\| - \mathbf{x}_i \cdot \hat{\mathbf{x}}) \\ &= \mathbf{x}_i \cdot \hat{\mathbf{x}} - \mathbf{x}_j \cdot \hat{\mathbf{x}} \\ &= (\mathbf{x}_i - \mathbf{x}_j) \cdot \hat{\mathbf{x}}.\end{aligned}$$

The system of TDOA measurements are equivalent to:

$$\boldsymbol{\tau} = -\mathbf{P}\mathbf{X}\hat{\mathbf{x}}, \tag{7}$$

where $\mathbf{X}$ is the matrix of receiver locations and $\mathbf{P}$ is a differencing matrix as before. This suggests that feasible far-field TDOA measurements lie in the image of the unit circle under transformation of $-\mathbf{PX}$.

3.2. **Calculating direction of arrival (DOA).** As in the FDOA case, the direction of arrival can be calculated using the pseudoinverse:

$$\hat{\mathbf{x}} = -((\mathbf{PX})^T\mathbf{PX})^{-1}(\mathbf{PX})^T\mathbf{f}. \tag{8}$$

4. Source-localization algorithm

Formulas for calculating the direction of arrival (DOA) of a signal from FDOA or TDOA measurements were given in equations 6 and 8, respectively. If the receivers move, measurements are recorded for a second time step, and the DOA is recalculated, the intersection of these lines will provide an estimate for the emitter location. This simple algorithm (in 2D) is described more precisely below. A schematic of an example run is shown in figure 2.

Input:

- Receiver data: Location and velocity for two time steps. Without loss of generality, assume the receivers are centered at the origin in time step 1, centered at (c_x, c_y) at timestep 2.
- FDOA measurements between all pairs of receivers of the same time step.

Output:

- Estimate for emitter location, $\tilde{\mathbf{x}} = [\tilde{x}, \tilde{y}]^T$.

(1) Calculate DOA for time step 1 and time step 2 ($\hat{\mathbf{x}}^{(1)} = [\hat{x}^{(1)}, \hat{y}^{(1)}]^T, \hat{\mathbf{x}}^{(2)} = [\hat{x}^{(2)}, \hat{y}^{(2)}]^T$) using equation 6.

(2) Find the intersection of the lines generated by (1):

$$\begin{aligned}\tilde{x} =& \frac{c_y - \hat{y}^{(2)} c_x / \hat{x}^{(2)}}{\hat{y}^{(1)}/\hat{x}^{(1)} - \hat{y}^{(2)}/\hat{x}^{(2)}} \\ \tilde{y} =& \frac{\hat{y}^{(1)}}{\hat{x}^{(1)}} \cdot \tilde{x}.\end{aligned}$$

(3) Return $\tilde{\mathbf{x}}$.

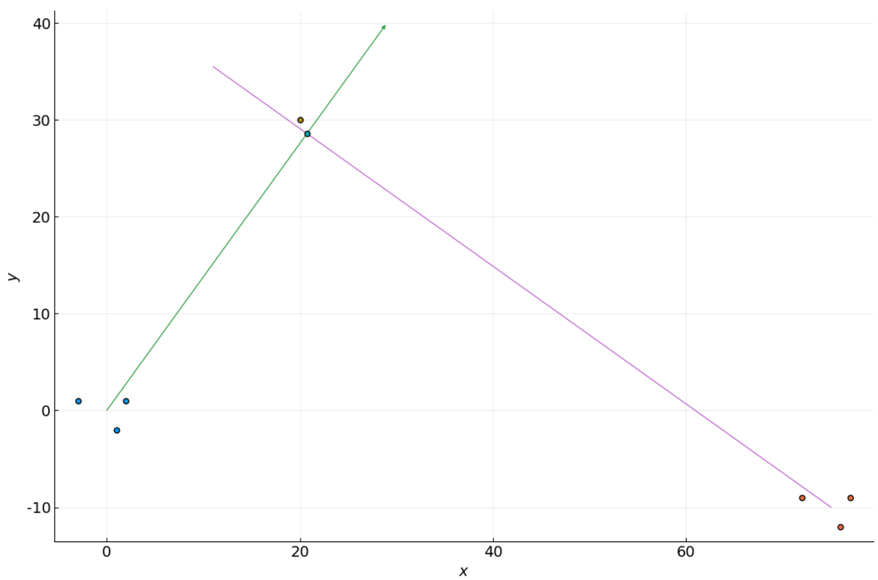

Figure 2. Proof of concept example for the proposed source localization algorithm in 2D with three receivers. In this example, the estimated location is $\tilde{\mathbf{x}} = (20.7318, 28.5771)$ and the correct location is $\mathbf{x} = (20, 30)$. The DOA calculations used FDOA measurements.

5. Conclusion

Considering far-field FDOA-based geolocation naturally leads to a simple method for determining direction of arrival. This calculation requires only a linear solve which makes the corresponding source-localization technique very efficient. Additionally, since FDOA measurement data is projected onto the range of the differencing matrix, the solution is naturally de-noised in a method consistent with [4, 8].

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