A NOVEL METHOD FOR DETERMINING DOA FROM FAR-FIELD TDOA OR FDOA

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1. INTRODUCTION

Locating a radio-frequency transmitter, or *source localization*, is a vital step in many applications. Source localization is often performed using measurements of the transmitted signal obtained by several nearby receivers. Specifically, measurements of the transmitted signal at two distinct receivers allow one to compute the time difference of arrival (TDOA) and frequency difference of arrival (FDOA) between those receivers. With estimates of TDOA or FDOA measurements, one can compute various other quantities describing the location of the transmitter, including the angle of arrival (AOA) / direction of arrival (DOA), the range to the receiver, and thus the location of the transmitter in the global coordinate system (geolocation). If information about the source is known *a priori*, such as altitude (ALT), it is typically possible to estimate receiver location with the use of fewer measurements.

Source localization using TDOA measurements-only is a well-understood problem and many algorithms have been developed for its solution. Common approaches include linearization of the system or a multidimensional search [11]. Methods for managing data to deal with noise, including divide and conquer (DAC), the RANdom SAmpling Consensus method (RANSAC), and projection to the feasible set of TDOA measurements have been proposed [1, 3, 5, 8]. Geometrically, each TDOA measurement restricts the potential transmitter location to a hyperbola. Thus if several measurements are obtained, locating the emitter requires finding the intersection of several hyperbolae.

For a nearby source, simple geometric relationships between the TDOA measurements and the known receiver positions allow both the range to the transmitter and the DOA to be computed with a single array of sensors [2]. When the distance between the receivers and the transmitter is much greater than the distance between the receivers it is common to simplify the wave propagation model and assume that wave curvature is negligible in the region of the receivers. This assumption is commonly referred to as the far-field assumption [4]. This simplification modifies the geometric relationships between the TDOA and receiver positions so the DOA is uniquely determined. Range to the transmitter, however, can no longer be calculated in the far-field case without additional receivers or measurements. [2]

The equations relating the FDOA measurements to the receiver positions are more complicated than those used in the TDOA case. The FDOA model is nonlinear and depends on the receiver velocities, so source localization with FDOA measurements is more complicated than geolocation using TDOA measurements. While the FDOA measurements are often used as an additional constraint to the TDOA geolocation systems (TDOA/FDOA

localization) [6], only a few algorithms have been developed using FDOA alone [3, 7]. Some limitations of these algorithms are the high cost of computation that comes from nonlinear solver methods. There are some special cases where it is desirable to solve for the emitter location using FDOA only. For instance, in the case of a narrowband signal with a long pulse duration, Doppler resolution is finer than the range resolution so that it is difficult to measure the TDOA accurately [4, 7, 9].

While DOA estimation is typically performed using TDOA measurements, this paper introduces a novel approach to calculating the DOA using either TDOA or FDOA measurements, or both, by capitalizing on the simplified geometry of the source-localization problem under the far-field assumption. The main benefit of this method is its computational efficiency, as it simplifies the calculation of DOA to the solution of a linear system of equations. In section 2, we develop a far-field model for the FDOA measurements and discuss a technique for determining the signal direction of arrival. In section 3, we develop a similar far-field approximation for the TDOA model and present the analogous DOA technique. Finally, we summarize the results by presenting a source-localization algorithm in section 4 and a discussion of future directions for research in section 5.

2. DIRECTION OF ARRIVAL WITH FDOA MEASUREMENTS

Consider a stationary transmitter located at $\mathbf{x}$. Suppose we have N receivers located at $\mathbf{x}_1, ..., \mathbf{x}_N$ with velocities $\mathbf{v}_1, ..., \mathbf{v}_N$. The frequency shift of the signal between the emitter and the ith receiver is

$$d_i = \frac{f_0}{c} \left(\mathbf{v}_i^T \cdot \frac{\mathbf{x}_i - \mathbf{x}}{\|\mathbf{x}_i - \mathbf{x}\|} \right),$$
(1)

where f_0 is the emitted frequency and c is the speed of propagation. In our scenario, we know the receiver positions and velocities, and we would like to solve for the transmitter position. We cannot measure the frequency shift directly, but we can measure the difference in frequency shifts, $f_{i,j} = d_j - d_i$. Scaling by $\|\mathbf{x}_i - \mathbf{x}\|^{-1}$ means that each of these equations is nonlinear. Even so, by taking pairwise differences of the equations in (1), one can numerically solve the system by expressing it in terms of polynomials and using techniques like homotopy continuation [3]. While such an approach is able to find all solutions, it is computationally more expensive than a linear solve.

The nonlinearity of 1 makes it difficult to accurately solve for $\mathbf{x}$, so we simplify the model by deriving a far-field approximation for equation 1. Additionally, for simplicity we ignore the constant factor of f_0/c in equation 1. Thus d_i is now *proportional* to the frequency shift.

2.1. **Far-field Approximation for FDOA.** Assume without loss of generality that the centroid of the receiver positions is the origin. We consider the far-field case, where the distance between receivers is much smaller than the distance to the emitter, i.e. $\|\mathbf{x}\| \gg \|\mathbf{x}_i\|$, $\forall i$. The far-field approximation (as in [4]) for $1/\|\mathbf{x} - \mathbf{x_i}\|$ is:

$$\frac{1}{\|\mathbf{x} - \mathbf{x_i}\|} = \frac{1}{\|\mathbf{x}\|} \left(1 + \mathcal{O} \left(\frac{\|\mathbf{x}_i\|}{\|\mathbf{x}\|} \right) \right).$$

Truncating after the first term above allows for simplification of the factor (in eq. 1):

$$\frac{\mathbf{x}_i - \mathbf{x}}{\|\mathbf{x}_i - \mathbf{x}\|} \approx \frac{\mathbf{x}_i}{\|\mathbf{x}\|} - \frac{\mathbf{x}}{\|\mathbf{x}\|}.$$

Additionally, the far-field assumption implies that the first term will have small magnitude. Thus, $\frac{\mathbf{x_i} - \mathbf{x}}{\|\mathbf{x_i} - \mathbf{x}\|}$ is simplified to $\frac{-\mathbf{x}}{\|\mathbf{x}\|}$. Equation 1 becomes:

$$(2) \qquad\qquad d_i = -\mathbf{v}_i^T \cdot \hat{\mathbf{x}},$$

where $\hat{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$, is the unit vector in the direction of $\mathbf{x}$. The entire system of frequency shifts can be written:

$$(3) \qquad\qquad \mathbf{d} = -\mathbf{V}\hat{\mathbf{x}},$$

where

$$\mathbf{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix} \qquad \mathbf{V} = \begin{pmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_N^T \end{pmatrix}.$$

In practice, the frequency shifts are not observable. Instead the frequency difference of arrival (FDOA) is measured between receivers. The FDOA is equivalent to the difference in frequency shifts,

$$(4) \qquad\qquad f_{i,j} = d_j - d_i.$$

A system equivalent to equation 3 can be constructed for the FDOA, with the use of a differencing matrix $\mathbf{P}$. The matrix $\mathbf{P}$ has entries of 0 and ±1 corresponding to the differencing in equation 4. Thus, with the far-field simplification above, the vector of FDOA measurements, $\mathbf{f}$, is equivalent to,

$$(5) \qquad\qquad \mathbf{f} = -\mathbf{P}\mathbf{V}\hat{\mathbf{x}}.$$

The matrix $-\mathbf{P}\mathbf{V}$ will be referred to as $\tilde{\mathbf{V}}$ for simplicity.

This far-field simplification reduces the FDOA equations to a linear system. This suggests that feasible FDOA measurements in the far-field case lie on the image of the unit circle transformed by the matrix $\tilde{\mathbf{V}}$. This image is an ellipse with rotation and scaling determined by the singular value decomposition of $\tilde{\mathbf{V}}$. Indeed, this can be confirmed by computing the singular value decomposition of generated far-field FDOA measurements and confirming they lie on the same subspace as $\tilde{\mathbf{V}}$. This relationship can be demonstrated visually with a plot of generated FDOA measurements (fig. 1).

One method for denoising in TDOA-based geolocation is the projection of noisy measurements onto the range of the differencing matrix $\mathbf{P}$ [5, 10]. This ensures that the TDOA measurements are physically realizable and consistent between receivers. One benefit of the method for DOA calculation proposed above is that denoising is automatically performed since projection onto the range of $-\mathbf{P}\mathbf{V}$ is equivalent to projection onto the range of $\mathbf{P}$.
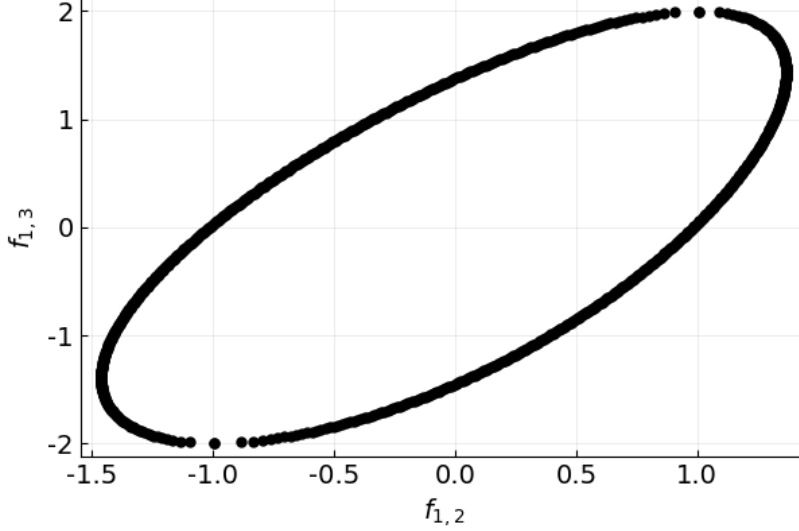
FIGURE 1. Plot of far-field $f_{1,2}$ vs. $f_{1,3}$ for a system of three receivers centered around the origin. Note the image is an ellipse with scaling in the direction of the left-singular vectors of $\tilde{\mathbf{V}}$.

2.2. Calculating direction of arrival (DOA).

The far-field approximated form of the FDOA equations is linear in $\hat{\mathbf{x}}$, the direction of arrival (DOA) of the signal. As long as the matrix $\tilde{\mathbf{V}}$) is of full rank, we can compute the solution to equation 5 and use it as an estimate of the DOA. In general, all 3 receivers will have different velocities and If there are more FDOA measurements than direction components, we can find the least-squares solution to the problem,

$$(6) \qquad \hat{\mathbf{x}} = (\tilde{\mathbf{V}}^T \tilde{\mathbf{V}})^{-1} \tilde{\mathbf{V}}^T \mathbf{f}.$$

If the receivers move and repeat this process, the intersection of the lines generated by equation 6 will provide an estimate for the location of the emitter. This is the idea behind our source localization algorithm in section 4.

3. Direction of Arrival with TDOA Measurements

3.1. Far-field approximation for TDOA.

Although the time difference of arrival (TDOA) is simpler than the FDOA case, we include its far-field approximation for completeness.

Using the same problem setup as above, the time it takes for the signal to travel between the emitter and receiver i is:

$$\tau_i = \frac{1}{c} \|\mathbf{x_i} - \mathbf{x}\|,$$

from here the scalar $\frac{1}{c}$ will be left out for simplicity. The far-field approximation for $\|\mathbf{x_i} - \mathbf{x}\|$ is given [4],

$$\|\mathbf{x_i} - \mathbf{x}\| = \|\mathbf{x}\| \left(1 - \frac{\mathbf{x}_i \cdot \hat{\mathbf{x}}}{\|\mathbf{x}\|} + \mathcal{O}\left(\frac{\|\mathbf{x}_i\|}{\|\mathbf{x}\|} \right) \right).$$

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Thus, τ_i becomes,

$$\tau_i = \|\mathbf{x}\| - \mathbf{x}_i \cdot \hat{\mathbf{x}}.$$

As in the FDOA case, τ_i is not observable. Instead we look to the time difference of arrival (TDOA) between receivers i and j,

$$\begin{aligned} \tau_{i,j} &= (\|\mathbf{x}\| - \mathbf{x}_j \cdot \hat{\mathbf{x}}) - (\|\mathbf{x}\| - \mathbf{x}_i \cdot \hat{\mathbf{x}}) \\ &= \mathbf{x}_i \cdot \hat{\mathbf{x}} - \mathbf{x}_j \cdot \hat{\mathbf{x}} \\ &= (\mathbf{x}_i - \mathbf{x}_j) \cdot \hat{\mathbf{x}}. \end{aligned}$$

The system of TDOA measurements are equivalent to:

$$(7) \qquad\qquad \boldsymbol{\tau} = -\mathbf{PX}\hat{\mathbf{x}},$$

where $\mathbf{X}$ is the matrix of receiver locations and $\mathbf{P}$ is a differencing matrix as before. This suggests that feasible far-field TDOA measurements lie in the image of the unit circle under transformation of $-\mathbf{PX}$.

3.2. Calculating direction of arrival (DOA).
As in the FDOA case, the least-squares estimate of direction of arrival can be calculated using the pseudoinverse:

$$(8) \qquad\qquad \hat{\mathbf{x}} = -((\mathbf{PX})^T \mathbf{PX})^{-1} (\mathbf{PX})^T \mathbf{f}.$$

4. Source-localization algorithm

Formulas for calculating the direction of arrival (DOA) of a signal from FDOA or TDOA measurements were given in equations 6 and 8, respectively. If the receivers move, measurements are recorded for a second time step, and the DOA is recalculated, the intersection of these lines will provide an estimate for the emitter location. This simple algorithm (in 2D) is described more precisely below. A schematic of an example run is shown in figure 2.

Although the receivers appear to be traveling together in figure 2, the receivers must have different velocities when the measurements are taken for all FDOA values to be nonzero. This could be achieved by having the receivers fly together in a group between time steps, then rotate or move apart from each other for FDOA measurement calculation.

Input:
- Receiver data: Location and velocity for two time steps. Without loss of generality, assume the receivers are centered at the origin in time step 1, and centered at (c_x, c_y) at timestep 2.
- FDOA measurements between all pairs of receivers of the same time step.

Output:
- Estimate for emitter location, $\tilde{\mathbf{x}} = [\tilde{x}, \tilde{y}]^T$.

(1) Calculate DOA for time step 1 and time step 2 ($\hat{\mathbf{x}}^{(1)} = [\hat{x}^{(1)}, \hat{y}^{(1)}]^T, \hat{\mathbf{x}}^{(2)} = [\hat{x}^{(2)}, \hat{y}^{(2)}]^T$) using equation 6.

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(2) Find the intersection of the lines generated by (1):

$$\tilde{x} = \frac{c_y - \hat{y}^{(2)} c_x / \hat{x}^{(2)}}{\hat{y}^{(1)} / \hat{x}^{(1)} - \hat{y}^{(2)} / \hat{x}^{(2)}}$$

$$\tilde{y} = \frac{\hat{y}^{(1)}}{\hat{x}^{(1)}} \cdot \tilde{x}.$$
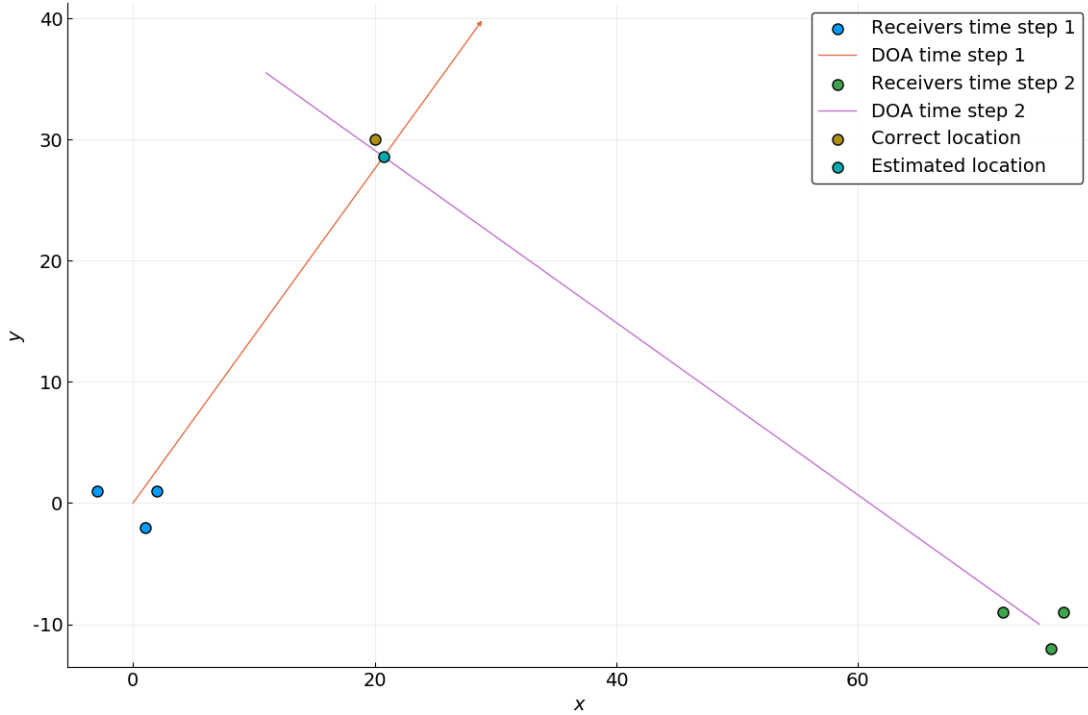
(3) Return $\tilde{\mathbf{x}}$.

FIGURE 2. Proof of concept example for the proposed source localization algorithm in 2D with three receivers. In this example, the estimated location is $\tilde{\mathbf{x}} = (20.7318, 28.5771)$ and the correct location is $\mathbf{x} = (20, 30)$. The DOA calculations used FDOA measurements.

5. CONCLUSION

Considering far-field FDOA-based geolocation naturally leads to a simple method for determining direction of arrival. This calculation requires only a linear solve which makes the corresponding source-localization technique very efficient. Additionally, since FDOA measurement data is projected onto the range of the differencing matrix, the solution is naturally de-noised in a method consistent with [5, 10]. Another benefit of this method is the generality that allows DOA to be calculated with either TDOA or FDOA measurements. This allows for accurate source localization using a variety of waveforms. Extensive numerical

tests have not yet been performed, but the algorithm appears to do quite well in the proof of concept tests we have run so far. More thorough testing will be performed in the future.

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