Euler Angles

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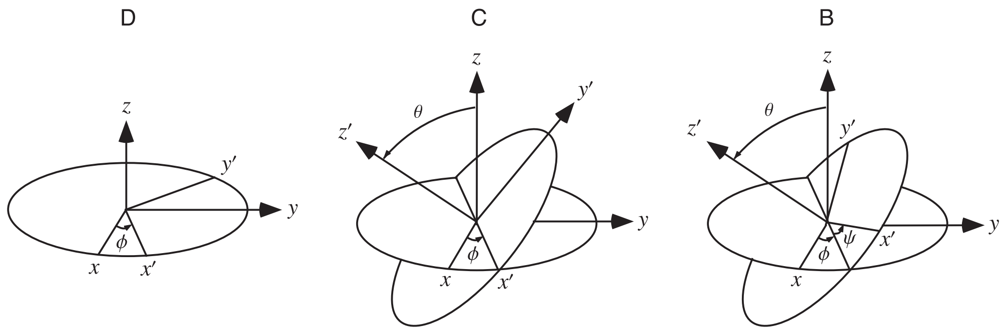

According to Euler's rotation theorem, any rotation may be described using three angles. If the rotations are written in terms of rotation matrices D, C, and B, then a general rotation A can be written as

$$\mathsf{A} = \mathsf{B}\,\mathsf{C}\,\mathsf{D}. \quad (1)$$

The three angles giving the three rotation matrices are called Euler angles. There are several conventions for Euler angles, depending on the axes about which the rotations are carried out. Write the matrix A as

$$\mathsf{A} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \quad (2)$$

The so-called "x-convention," illustrated above, is the most common definition. In this convention, the rotation given by Euler angles (ϕ, θ, ψ), where

1. the first rotation is by an angle ϕ about the z-axis using D,

2. the second rotation is by an angle $\theta \in [0, \pi]$ about the former x-axis (now x') using C, and

3. the third rotation is by an angle ψ about the former z-axis (now z') using B.

Note, however, that several notational conventions for the angles are in common use. Goldstein (1980, pp. 145-148) and Landau and Lifschitz (1976) use (ϕ, θ, ψ), Tuma (1974) says (ψ, θ, ϕ) is used in aeronautical engineering in the analysis of space vehicles (but claims that (ϕ, θ, ψ) is used in the analysis of gyroscopic motion), while Bate *et al.* (1971) use (Ω, i, ω). Goldstein remarks that continental authors usually use (ψ, θ, ϕ), and warns that left-handed coordinate systems are also in occasional use (Osgood 1937, Margenau and Murphy 1956-64). Varshalovich (1988, pp. 21-23) uses the notation (α, β, γ) or $(\alpha', \beta', \gamma')$ to denote the Euler angles, and gives three different angle conventions, none of which corresponds to the x-convention.

Here, the notation (ϕ, θ, ψ) is used, a convention that could be used in versions of the Wolfram Language prior to 6 as `RotationMatrix3D`[*phi*, *theta*, *psi*] (which could be run after loading ``Geometry`Rotations` ``) and `RotateShape`[*g*, *phi*, *theta*, *psi*] (which could be run after loading ``Geometry`Shapes` ``). In the x-convention, the component rotations are then given by

$$\mathsf{D} \equiv \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$\mathsf{C} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{4}$$

$$\mathsf{B} \equiv \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

so

$$a_{11} = \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi \tag{6}$$
$$a_{12} = \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi \tag{7}$$
$$a_{13} = \sin\psi\sin\theta \tag{8}$$
$$a_{21} = -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi \tag{9}$$
$$a_{22} = -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi \tag{10}$$
$$a_{23} = \cos\psi\sin\theta \tag{11}$$
$$a_{31} = \sin\theta\sin\phi \tag{12}$$
$$a_{32} = -\sin\theta\cos\phi \tag{13}$$
$$a_{33} = \cos\theta \tag{14}$$

To obtain the components of the angular velocity $\boldsymbol{\omega}$ in the body axes, note that for a matrix

$$\mathsf{A} \equiv [\,\mathbf{A}_1 \;\; \mathbf{A}_2 \;\; \mathbf{A}_3\,], \tag{15}$$

it is true that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} a_{11}\,\omega_x + a_{12}\,\omega_y + a_{13}\,\omega_z \\ a_{21}\,\omega_x + a_{22}\,\omega_y + a_{23}\,\omega_z \\ a_{31}\,\omega_x + a_{32}\,\omega_y + a_{33}\,\omega_z \end{bmatrix} \tag{16}$$

$$= \mathbf{A}_1\,\omega_x + \mathbf{A}_2\,\omega_y + \mathbf{A}_3\,\omega_z. \tag{17}$$

Now, ω_z corresponds to rotation about the ϕ axis, so look at the ω_z component of $\mathsf{A}\ \boldsymbol{\omega}$,

$$\boldsymbol{\omega}_\phi = \mathbf{A}_3\ \omega_z \tag{18}$$

$$= \begin{bmatrix} \sin\psi \sin\theta \\ \cos\psi \sin\theta \\ \cos\theta \end{bmatrix} \dot{\phi}. \tag{19}$$

The line of nodes corresponds to a rotation by θ about the ξ-axis, so look at the ω_ξ component of $\mathsf{B}\ \boldsymbol{\omega}$,

$$\boldsymbol{\omega}_\theta = \mathbf{B}_1\ \omega_\xi \tag{20}$$

$$= \mathbf{B}_1\ \dot{\theta} \tag{21}$$

$$= \begin{bmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{bmatrix} \dot{\theta}. \tag{22}$$

Similarly, to find rotation by ψ about the remaining axis, look at the ω_ψ component of $\mathsf{B}\ \boldsymbol{\omega}$,

$$\boldsymbol{\omega}_\psi = \mathbf{B}_3\ \omega_\psi \tag{23}$$

$$= \mathbf{B}_3\ \dot{\psi} \tag{24}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\psi}. \tag{25}$$

Combining the pieces gives

$$\boldsymbol{\omega} = \begin{bmatrix} \sin\psi \sin\theta\, \dot{\phi} + \cos\psi\, \dot{\theta} \\ \cos\psi \sin\theta\, \dot{\phi} - \sin\psi\, \dot{\theta} \\ \cos\theta\, \dot{\phi} + \dot{\psi}. \end{bmatrix} \tag{26}$$

For more details, see Goldstein (1980, p. 176) and Landau and Lifschitz (1976, p. 111).

The x-convention Euler angles are given in terms of the Cayley-Klein parameters by

$$\phi = -2\, i \ln\left[\pm \frac{\alpha^{1/2}\, \gamma^{1/4}}{\beta^{1/4}\, (1+\beta\gamma)^{1/4}}\right],\ -2\, i \ln\left[\pm \frac{i\, \alpha^{1/2}\, \gamma^{1/4}}{\beta^{1/4}\, (1+\beta\gamma)^{1/4}}\right] \tag{27}$$

$$\psi = -2\, i \ln\left[\pm \frac{\alpha^{1/2}\, \beta^{1/4}}{\gamma^{1/4}\, (1+\beta\gamma)^{1/4}}\right],\ -2\, i \ln\left[\pm \frac{i\, \alpha^{1/2}\, \beta^{1/4}}{\gamma^{1/4}\, (1+\beta\gamma)^{1/4}}\right] \tag{28}$$

$$\theta = \pm 2 \cos^{-1}\left(\pm\sqrt{1+\beta\gamma}\right). \tag{29}$$

In the "y-convention,"

$$\phi_x \equiv \phi_y + \tfrac{1}{2}\,\pi \tag{30}$$

$$\psi_x \equiv \psi_y - \tfrac{1}{2}\,\pi. \tag{31}$$

Therefore,

$$\sin\phi_x = \cos\phi_y \tag{32}$$

$$\cos\phi_x = -\sin\phi_y \tag{33}$$

$$\sin\psi_x = -\cos\psi_y \tag{34}$$

$$\cos\psi_x = \sin\psi_y, \tag{35}$$

giving rotation matrices

$$\mathsf{D} \equiv \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{36}$$

$$\mathsf{C} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{37}$$

$$\mathsf{B} \equiv \begin{bmatrix} \sin\psi & -\cos\psi & 0 \\ \cos\psi & \sin\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{38}$$

and A is given by

$$a_{11} = -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi \tag{39}$$

$$a_{12} = \sin\psi\cos\phi + \cos\theta\sin\phi\cos\psi \tag{40}$$

$$a_{13} = -\cos\psi\sin\theta \tag{41}$$

$$a_{21} = -\cos\psi\sin\phi - \cos\theta\cos\phi\sin\psi \tag{42}$$

$$a_{22} = \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi \tag{43}$$

$$a_{23} = \sin\psi\sin\theta \tag{44}$$

$$a_{31} = \sin\theta\cos\phi \tag{45}$$

$$a_{32} = \sin\theta\sin\phi \tag{46}$$

$$a_{33} = \cos\theta. \tag{47}$$

In the "$x\ y\ z$ (pitch-roll-yaw) convention," θ is pitch, ψ is roll, and ϕ is yaw.

$$\mathsf{D} \equiv \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{48}$$

$$\mathsf{C} \equiv \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{49}$$

$$\mathsf{B} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \tag{50}$$

and A is given by

$$a_{11} = \cos\theta\cos\phi \tag{51}$$
$$a_{12} = \cos\theta\sin\phi \tag{52}$$
$$a_{13} = -\sin\theta \tag{53}$$
$$a_{21} = \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi \tag{54}$$
$$a_{22} = \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi \tag{55}$$
$$a_{23} = \cos\theta\sin\psi \tag{56}$$
$$a_{31} = \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi \tag{57}$$
$$a_{32} = \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi \tag{58}$$
$$a_{33} = \cos\theta\cos\psi. \tag{59}$$

A set of parameters sometimes used instead of angles are the Euler parameters e_0, e_1, e_2 and e_3, defined by

$$e_0 \equiv \cos\left(\frac{\phi}{2}\right) \tag{60}$$
$$\mathbf{e} \equiv \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \hat{\mathbf{n}}\sin\left(\frac{\phi}{2}\right). \tag{61}$$

Using Euler parameters (which are quaternions), an arbitrary rotation matrix can be described by

$$a_{11} = e_0^2 + e_1^2 - e_2^2 - e_3^2 \tag{62}$$
$$a_{12} = 2\,(e_1\,e_2 + e_0\,e_3) \tag{63}$$
$$a_{13} = 2\,(e_1\,e_3 - e_0\,e_2) \tag{64}$$
$$a_{21} = 2\,(e_1\,e_2 - e_0\,e_3) \tag{65}$$
$$a_{22} = e_0^2 - e_1^2 + e_2^2 - e_3^2 \tag{66}$$
$$a_{23} = 2\,(e_2\,e_3 + e_0\,e_1) \tag{67}$$
$$a_{31} = 2\,(e_1\,e_3 + e_0\,e_2) \tag{68}$$
$$a_{32} = 2\,(e_2\,e_3 - e_0\,e_1) \tag{69}$$
$$a_{33} = e_0^2 - e_1^2 - e_2^2 + e_3^2 \tag{70}$$

(Goldstein 1980, p. 153).

If the coordinates of two sets of n points $\mathbf{x}_i$ and $\mathbf{x}_i'$ are known, one rotated with respect to the other, then the Euler rotation matrix can be obtained in a straightforward manner using least squares fitting. Write the points as arrays of vectors, so

$$[\,\mathbf{x}_1{}' \;\cdots\; \mathbf{x}_n{}'\,] = \mathsf{A}\,[\,\mathbf{x}_1 \;\cdots\; \mathbf{x}_n\,]. \tag{71}$$

Writing the arrays of vectors as matrices gives

$$\mathsf{X}' = \mathsf{A}\,\mathsf{X} \tag{72}$$

$$\mathsf{X}'\,\mathsf{X}^{\mathrm{T}} = \mathsf{A}\,\mathsf{X}\,\mathsf{X}^{\mathrm{T}}, \tag{73}$$

and solving for A gives

$$\mathsf{A} = \mathsf{X}'\,\mathsf{X}^{\mathrm{T}}\left(\mathsf{X}\,\mathsf{X}^{\mathrm{T}}\right)^{-1}. \tag{74}$$

However, we want the angles θ, ϕ, and ψ, not their combinations contained in the matrix A. Therefore, write the 3×3 matrix

$$\mathsf{A} = \begin{bmatrix} f_1(\theta, \phi, \psi) & f_2(\theta, \phi, \psi) & f_3(\theta, \phi, \psi) \\ f_4(\theta, \phi, \psi) & f_5(\theta, \phi, \psi) & f_6(\theta, \phi, \psi) \\ f_7(\theta, \phi, \psi) & f_7(\theta, \phi, \psi) & f_9(\theta, \phi, \psi) \end{bmatrix} \tag{75}$$

as a 1×9 vector

$$\mathbf{f} = \begin{bmatrix} f_1(\theta, \phi, \psi) \\ \vdots \\ f_9(\theta, \phi, \psi) \end{bmatrix}. \tag{76}$$

Now set up the matrices

$$\begin{bmatrix} \left.\frac{\partial f_1}{\partial \theta}\right|_{\theta_i,\phi_i,\psi_i} & \left.\frac{\partial f_1}{\partial \phi}\right|_{\theta_i,\phi_i,\psi_i} & \left.\frac{\partial f_1}{\partial \psi}\right|_{\theta_i,\phi_i,\psi_i} \\ \vdots & \vdots & \vdots \\ \left.\frac{\partial f_9}{\partial \theta}\right|_{\theta_i,\phi_i,\psi_i} & \left.\frac{\partial f_9}{\partial \phi}\right|_{\theta_i,\phi_i,\psi_i} & \left.\frac{\partial f_9}{\partial \psi}\right|_{\theta_i,\phi_i,\psi_i} \end{bmatrix} \begin{bmatrix} d\,\theta \\ d\,\phi \\ d\,\psi \end{bmatrix} = d\,\mathbf{f}. \tag{77}$$

Using nonlinear least squares fitting then gives solutions which converge to (θ, ϕ, ψ).

SEE ALSO

Cayley-Klein Parameters, Euler Parameters, Euler's Rotation Theorem, Infinitesimal Rotation, Quaternion, Rotation, Rotation Formula, Rotation Matrix

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