Real Analysis HW 1

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1. Exercise 0.3.6

   (a) Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   Solution:

   Proof.

$$\begin{align} A \cap (B \cup C) &= \{z | z \in A \text{ and } (z \in B \text{ or } z \in C)\} \tag{1} \\ &= \{z | (z \in A \text{ and } z \in B) \text{ or } (z \in A \text{ and } z \in C)\} \tag{2} \\ &= \{z | (z \in A \cap B) \text{ or } (z \in (A \cap C)\} \tag{3} \\ &= (A \cap B) \cup (A \cap C) \tag{4} \end{align}$$

   $\square$

   (b) Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   Solution:

   Proof.

$$\begin{align} A \cup (B \cap C) &= \{z | z \in A \text{ or } (z \in B \text{ and } z \in C)\} \tag{5} \\ &= \{z | (z \in A \text{ or } z \in B) \text{ and } (z \in A \text{ or } z \in C)\} \tag{6} \\ &= \{z | (z \in A \cup B) \cap (z \in (A \cup C)\} \tag{7} \\ &= (A \cup B) \cap (A \cup C) \tag{8} \end{align}$$

   $\square$

2. Exercise 0.3.11

   (a) Prove by induction that $n < 2^n$ for all $n \in \mathbb{N}$

   Solution:

   Proof. The proof proceeds by induction

Base Case: Let P(n) be a statement depending on a natural number n

$$P(1) = 1 < 2^1 \tag{9}$$
$$= 1 < 2 \tag{10}$$
$$1 < 2 = \text{TRUE} \tag{11}$$

By plugging in n = 1, we see that P(1) is true because $1 < 2^1$

Inductive Step:

Suppose P(n) is true. That is, suppose $n < 2^n$ holds

Inductive Hypothesis:

$$P(n) = n < 2^n \tag{12}$$

Want to show:

$$n + 1 < 2^{n+1} \tag{13}$$

Take P(n) and multiply both sides by 2 to obtain:

$$2n < 2^{n+1} \tag{14}$$

□

(b) **Prove** $n + 1 < 2n \forall\ n \geq 2$

Proof.

Base Case for P(2):

$$2 + 1 < 2 * 2 \tag{15}$$
$$= 3 < 4 \tag{16}$$

Inductive Step

Assume P(n) $\rightarrow n + 1 < 2n$

WTS

$$(n + 1) + 1 < 2(n + 1) = P(n + 1) \tag{17}$$

Induction Hypothesis (IH):

$$n + 1 < 2n \tag{18}$$

Add 2 to both sides

$$n + 3 < 2n + 2 \tag{19}$$

$$[(n + 1) + 1 < 2(n + 1)] = P(n + 1)$$

Therefore,

$$n + 1 < 2n < 2^{n+1} \tag{20}$$

By transitivity

$$n + 1 < 2^{n+1} \tag{21}$$

□

3. Exercise 0.3.12

   (a) Show that for a finite set A of cardinality n, the cardinality of $\mathscr{P}(A)$ is 2^n

Solution:

Proof. The proof proceeds by induction

WTS P(1) is true and P(n+1) shows P(1) is true

$$P(1) = 2^n = 2^1 = 2 \tag{22}$$
$$P(0) = 2^n = 2^0 = 1 \tag{23}$$

□

4. Exercise 0.3.15 Prove that $n^3 + 5n$ is divisible by 6 for all $n \in N$

Solution:

Proof.

$$P(n) = \left[n^3 + 5n = 6z | z \in \mathbb{Z}\right] \tag{24}$$
$$P(1) = [\frac{1^3 + 5(1)}{6}] \tag{25}$$
$$= [\frac{1+5}{6}] \tag{26}$$
$$= [\frac{6}{6}] \tag{27}$$
$$= 1 \in \mathbb{Z} \tag{28}$$

Inductive Step Assume P(n) =$[\frac{n^3+5n}{6} \in \mathbb{Z}]$

$$P(n) = [n^3 + 5n = 6z | z \in \mathbb{Z}] \tag{29}$$
$$P(1) = [\frac{n^3 + 5n}{6}] \tag{30}$$

WTS P(n + 1) = $[\frac{(n+1)^3+5(n+1)}{6} \in \mathbb{Z}]$

$$
\begin{align}
&= \frac{(n+1)^3 + 5n + 5}{6} \tag{31}\\
&= \frac{n^3 + 3n^2 + 3n + 1 + 5n + 5}{6} \tag{32}\\
&= \frac{n^3 + 5n}{6} + \frac{3n^2 + 3n + 1 + 5}{6} \tag{33}\\
&= \frac{n^3 + 5n}{6} + \frac{3n^2 + 3n + 6}{6} \tag{34}\\
&= \frac{n^3 + 5n}{6} + \frac{n^2 + n}{2} + \frac{6}{6} \tag{35}\\
&= \frac{n^3 + 5n}{6} + \frac{n^2 + n}{2} + \frac{6}{6} \tag{36}
\end{align}
$$

□

Prove $\frac{n^2+n}{2} \in \mathbb{Z}$

Case 1
n is even
$n + n^2 =$ sum of two even numbers which is even

Case 2
n is odd
n^2 is odd
n^2 + n = odd + odd = even

Proof.

$$\frac{n^2 + n}{2} = \frac{even}{2} \in \mathbb{Z}$$

□

5. Exercise 0.3.19

(a) Give an example of a countably infinite collection of finite sets A1,A2,..., whose union is not a finite set. Prove that is is countably infinite

Solution:

Proof.

$$
\begin{align}
\{1\} \cup \{2\} \cup \{3\} \tag{37}\\
\bigcup_{i=1}^{\infty} i = \{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup ... \tag{38}\\
\bigcup_{i=1}^{s} i = 1 + 2 + 3 + 4... \tag{39}\\
\tag{40}
\end{align}
$$

$\square$

6. Exercise 1.1.1

   (a) Prove part (iii) of Proposition 1.1.8. That is, let F be an ordered field and $x, y, z \in F$.

   Prove $x < 0$and $y < z,$ then $xy > xz$.

Solution:

Proof. A field F is an ordered field if F is also an ordered set s.t.

$$(i) \text{ For } x, y, z \in F, x < y \text{ implies } x + z < y + z \tag{41}$$
$$(ii) \text{ For } x, y \in F, x > 0 \text{ and } y > 0 \text{ implies } xy > 0. \tag{42}$$

By taking (i)

Assume $x < y$ implies $x + z < y + z$ but also assume $x < y < z$ (By transitivity)

Since $x < 0$ and $y < z$

$$y < z$$

Therefore, $y - z < 0 \rightarrow z - y¿0$

$$\text{and } -x > 0$$

Now apply (ii)

$$(z - y)(-x) > 0$$
$$-zx + xy > 0$$
$$xy > zx$$

$\square$

7. Exercise 1.1.2

   (a) Let S be an ordered set. Let $A \subset S$ be a nonempty finite subset. Then A is bounded. Furthermore, inf A exists and is in A and sup A exists and is in A. **Hint:** Use induction.

Solution:
If A is a finite set then ∃ k ∈ ℕ s.t. ∃ a bijection f: ℕ ≤ k → A

Proof. The proof proceeds by induction
Inductive Step

P(n) := S ordered set

A $\subseteq$ S nonempty subset s.t. $\|A\| = n$ then:

1. A is bounded
2. inf(A) and sup(A) exist
3. inf(A), sup(A) $\in$ A

WTS $P(n+1)$

Let A $\subset$ S be nonempty with $\|A\| = n+1$

$$A := \{a_1, a_2, .., a_n\} \cup \{a_n + 1\} \tag{43}$$
$$A = S \cup T \rightarrow \|S\| = n, \|T\| = 1 \tag{44}$$

Inductive Hypothesis
$\|S\| = n \Rightarrow$ S is bounded

$\exists\ m_s, M_s$ s.t.

$$m_s \leq s \leq M_s \forall s \in S \tag{45}$$
$$\tag{46}$$

Base Case

$|T| = 1 \Rightarrow$ T is bounded

$\exists\ m_T, M_T$ s.t.

$$m_T \leq t \leq M_T\ \forall t \in T \tag{47}$$
$$m := min\{m_s, m_T\} \tag{48}$$
$$M := max\{M_s, M_T\} \tag{49}$$
$$\tag{50}$$

Claim
m $\leq$ a $\leq$ M, $\forall$ a $\in A = S \cup T$

Let us begin with **m** $\leq$ **a** for a $\in$ S $\cup$ T
Case 1
a $\in$ S

$$m = min\{m_s, m_T\} \leq m_s \leq a \tag{51}$$

Case 2
$a \in T$

$$m = min\{m_s, m_T\} \leq m_T \leq a \tag{52}$$

M $\geq$ a for a $\in$ S $\cup$ T
Case 1
$a \in S$

$$M = max\{M_S, M_T\} \geq M_S \geq a \tag{53}$$
$$\tag{54}$$

Case 2
$a \in T$

$$M = max\{M_S, M_T\} \geq M_T \geq a \tag{55}$$
$$\tag{56}$$

□

8. Exercise 1.1.5

   (a) Let S be an ordered set. Let $A \subset S$ and suppose b is an upper bound for A. Suppose $b \in A$. Show that $b = supA$.

Solution:
Assume b is an upper bound for A.

$b \geq supA$

Prove $b \geq supA$

Proof. Proof by definition of supremum
A supremum is $\leq$ b $\forall$ upper bounds of A, and b is an upper bound of A.

Now we want to prove that b $\leq$ supA.

Definiton of Upper Bound M is an upper bound for A if M $\geq$ a $\forall$ a $\in$ A.

SupA is an upper bound for A, therefore, SupA $\geq$ a $\forall$ a $\in$ A.

Therefore, since b $\in$ A, b $\leq$ SupA.

SupA $\leq$ b and SupA $\geq$ a therefore SupA = b.

□

9. Exercise 1.1.6

   (a) Let S be an ordered set. Let A $\subset$ S be nonempty and bounded above.
   Suppose sup A exists and sup A $\notin$ A. Show that A contains a countably infinite subset.

Solution:
We want to prove that SupA exists and that SupA $\notin$ A.

By definition If there exists a $b \in S$ such that $x \leq x \; \forall \; x \in A$, then we say A is bounded above and b is an upper bound of A.

Also if there exists an upper bound b_0 of A, s.t. whenever b is an upper bound for A we have:

$$x_0 \leq b$$

then x_0 is called the least upper bound or the supremum of A.

An ordered set S has the least-upper-bound property if every nonempty subset

A $\subset$ S that is bounded above has a least upper bound, that is sup A exists in S.

10. (a) In this exercise you will prove that

$$|\{q \in \mathbb{Q} : q > 0\}| = |\mathbb{N}|$$

In what follows, we will use the following theorem without proof:

Theorem Let q $\in \mathbb{Q}$ with $q > 0$. Then

i. If $q \in \mathbb{N}$ and $q \neq 1$, then there exists unique prime numbers $p_1 < p_2 ... < p_n$ and unique exponents $r_1, ..., r_n \in \mathbb{N}$ s.t.

$$q = p_1^{r_1} p_2^{r_2} ... p_N^{r_N}$$

ii. If $q \notin \mathbb{N}$, then there exist unique prime numbers $p_1 < p_2 < ... < p_N, q_1 < q_2 < ... < q_M$ with $p_i \neq q_j \forall i \in \{1, ..., N\}, j \in \{1, ..., M\}$, and unique exponents $r_1, ..., r_n, s_1, ..., s_M \in \mathbb{N}$ s.t.

$$q = \frac{p_1^{r_1} p_2^{r_2} ... p_N^{r_N}}{q_1^{s_1} q_2^{s_2} ... q_M^{s_M}}$$

Define $f : \{q \in \mathbb{Q} : q > 0\} \to \mathbb{N}$ as follows : $f(1) = 1$, if $q \in \mathbb{N} \backslash \{1\}$ is given by (†), then

$$f(q) = p_1^{2r_1}, ..., p_N^{2r_N}$$

and if $q \in \mathbb{Q} \backslash \mathbb{N}$ is given by (‡) then

$$\text{f(q)} = p_1^{2r_1}, ..., p_N^{2r_N} q_1^{2s_1 - 1} ... q_M^{2s_M - 1}$$

A. Compute f(4/15). Find q s.t. f(q)= 108.

Solution:

Proof. **Compute f(4/15)**

$q = 4/15$

According to the theorem q $\in \mathbb{Q}$ and $q \notin \mathbb{N}$

We compute the prime factorization of $\frac{4}{15}$ and plug the numbers into the equation $q = \frac{p_1^{r_1} p_2^{r_2} ... p_N^{r_N}}{q_1^{s_1} q_2^{s_2} ... q_M^{s_M}}$ to get $\frac{4}{15}$

$$(q) = \frac{p_1^{r_1} p_2^{r_2} ... p_N^{r_N}}{q_1^{s_1} q_2^{s_2} ... q_M^{s_M}} \tag{57}$$

$$= \frac{2^2}{3^1 5^1} \tag{58}$$

$$= [\frac{4}{15}] \tag{59}$$

Since $q \in \mathbb{Q}$ and $\notin \mathbb{N}$ Therefore, we use the equation

$$f(q) = p_1^{2r_1}, ..., p_N^{2r_N} q_1^{2s_1 - 1} ... q_M^{2s_M - 1}$$

And plug in the prime factors from above into the same equation

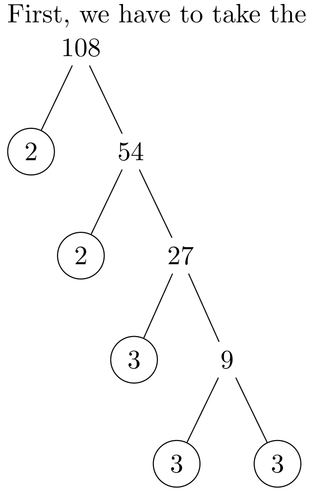
$$f(4/15) = f(q) = 2^{2(2)} * 3^{2(1)-1} * 5^{2(1)-1} \tag{60}$$

$$= 16 * 3 * 5 \tag{61}$$

$$= 240 \tag{62}$$

Find q s.t. f(q)= 108

First, we have to take the prime factorization



We get 2^2 and 3^3

We work backwards from $\mathbb{Q} \to \mathbb{N}$ and since it is $\notin \mathbb{N}$ from part (i) of the theorem by taking $f(q) = p_1^{2r_1}, ..., p_N^{2r_N} q_1^{2s_1 - 1} ... q_M^{2s_M - 1}$
and plugging it into it's inverse $(q) = \frac{p_1^{r_1} p_2^{r_2} ... p_N^{r_N}}{q_1^{s_1} q_2^{s_2} ... q_M^{s_M}}$

$$p_1^{2r_1} = 2^2 \quad (63)$$
$$q_1^{2s_1-1} = 3^3 \quad (64)$$
$$(65)$$

$p_1 = 2$, $r_1 = 1$, $q_1 = 3$, $s_1 = 2$

$$\frac{p_1^{r_1}}{q_1^{s_1}} = \frac{2^1}{3^2} \quad (66)$$
$$= \frac{2}{9} \quad (67)$$

Now let's double check that what we wrote is equal to 108 and plug p_1, r_1, q_1, s_1 into $f(q) = p_1^{2r_1}, ..., p_N^{2r_N} q_1^{2s_1-1} ... q_M^{2s_M-1}$ from (‡) since q $\notin \mathbb{N}$

$$f(2/9) = \frac{2^1}{3^2} \quad (68)$$

$p_1 = 2, r_1 = 1, q_1 = 3, s_1 = 2$

$$f(2/9) = p_1^{r_1*2} q_1^{2s_1-1} \quad (69)$$
$$= 2^{1*2} * 3^{2*2-1} \quad (70)$$
$$= 2^2 * 3^3 \quad (71)$$
$$= 4 * 27 \quad (72)$$
$$= 2 * 54 \quad (73)$$
$$= 108 \quad (74)$$

□

B. Use the **Theorem** to prove that f is a bijection.

Solution:

Proof. **Definition**: Bijection
By definition of Bijection we need to show that f is injective (one-to-one) and surjective (onto).

f is injective if $f(x) = f(y) \rightarrow x = y$

By taking $\mathbb{N} \rightarrow \mathbb{Q}$ or $f^{-1}(q)$

We need to compute the prime factorization of an arbitrary number q*.

$$q* = p_1^{r_1} p_2^{r_2} ... p_N^{r_N} \quad (75)$$

for $k = 1 \rightarrow$ Q :

if r_N is even: $r_i = \frac{r_N}{2}$
if r_N is odd $r_i = \frac{r_N+1}{2}$

This ends up looking like:

$$f^{-1}(q) = \frac{p_1^{r_1} p_2^{r_2} ... p_N^{r_N}}{q_1^{s_1} q_2^{s_2} ... q_M^{s_M}} \tag{76}$$

$\square$