

## Simulation Methods in Physics I

### Worksheet 3: Molecular Dynamics 2 and Observables

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## 1 Command line parameters

For the simulation different command line options were defined. In the following code blocks you can see them. They will be explained during the report.

Code block 1: Commands for ljsim.py

script: src/ljsim.py

```
14 # command line arguments
15 parser = argparse.ArgumentParser()
16 parser.add_argument("--cont", type=double, help="continue
    calculation with for cont further time")
17 parser.add_argument("--time", type=double, help="How long
    do you want to run the simulation? | default time=10")
18 parser.add_argument("--tstat", type=double, help="Uses a
    thermostat with a given temperature")
19 parser.add_argument("--warm", type=double, help="Use the
    warming up | pass force")
20 parser.add_argument("--ctstat", type=double, help="
    continue for the simulation with tstat")
21 parser.add_argument("--cwarm", type=double, help="continue
    for the simulation with warm")
22
23 args = parser.parse_args()
```

---

Code block 2: Commands for ljanalyze.py

script: src/ljanalyze.py

```
8 # command line arguments
9 parser = argparse.ArgumentParser()
10 parser.add_argument("--M", type=int, default=10, help="
    window size | default: M=10")
11 parser.add_argument("--datafile", type=str, default='../
    dat/ljsim.dat', help="datafilename | default: '../dat/
    ljsim.dat'")
12 parser.add_argument("--teq", type=double, help="
    equilibration time | calculates averages after
    equilibration time")
13 parser.add_argument("--tlim", type=double, nargs='+', help
    ="type in the time limits you want to plot")
14
15 args = parser.parse_args()
```

---

## 2 Restart the program where left it

First we want to change the program in a way, that it is able to restart the simulation where it ended last time. The first thing that we have to do is to store the necessary information about the end state of the system, the position  $x$  and velocity  $v$  of each particle, into the file `ljsim.dat`. This happens in code block 3. Furthermore there are the new variables  $T_s$  and  $P_s$ . They represent the temperature and pressure of the system over time and will be explained in the next chapter.

Code block 3: Storing data

script: `src/ljsim.py`

```
221 # write out simulation data
222 print("Writing simulation data to {}".format(datafilename
    ))
223 datafile = open(datafilename, 'w')
224 pickle.dump([ts, Es, Ts, Ps, x, v, hs], datafile)
225 datafile.close()
```

Now this data must be read by `ljsim.py`. Therefore we define a command line option `--cont` which accepts as argument the time for which the simulation should be continued (code block 1). If the parameter is not used the simulation will start new. Furthermore there is another argument `--time` which takes the simulation time for a new simulation. The code in code block 4 does exactly this.

Code block 4: Continue simulation

script: `src/ljsim.py`

```
74 # Import previous data
75 if args.cont:
76     # open datafile
77     datafile = open(datafilename, 'r')
78     ts, Es, Ts, Ps, x, v, hs = pickle.load(datafile)
79     datafile.close()
80
81     # length of run
82     t = ts[-1]
83     tmax = t+args.cont
84     step = 0
85
86 else:
87     # length of run
88     if args.time:
89         tmax = args.time
90     else:
91         tmax = 10.0
```

### 3 Simple Observables

#### 3.1 Expanding Energy Calculation

First the energy calculation should be changed in order to also get the kinetic and potential energy. This is done by changing the `compute_energy()` function call and the parameter `Es` in `ljsim.py` (code block 5).

Code block 5:  $E_{\text{kin}} + E_{\text{pot}}$

script: `src/ljsim.py`

```

189     if step % measurement_stride == 0:
190         E_pot, E_kin, E_tot = compute_energy(x, v)
191         T = 2*E_kin/(3*N)
192         P = compute_pressure(E_kin, x)
193         print("t={}: \n\tE_pot={}\n\tE_kin={}\n\tE_tot={}\n\tT={}\n\tP={}"
              .format(t, E_pot, E_kin, E_tot, T, P))
194         compute_distances(x, r)
195         h = compute_histogram(r)
196
197         ts.append(t)
198         Es.append([E_pot, E_kin, E_tot])
199         Ts.append(T)
200         Ps.append(P)
201         hs.append(h)

```

---

#### 3.2 Calculating the Temperature

The temperature of the system can be calculated from the kinetic Energy  $E_{\text{kin}}$  like in code block 5 shown. This is a single scalar calculation and therefore can be done in python.

#### 3.3 Calcating Pressure

##### Derivation of the pressure

The virial of a system is defined as:

$$G = \sum_i^N \mathbf{p}_i \cdot \mathbf{r}_i \quad (1)$$

In the case

$$0 = \left\langle \frac{dG}{dt} \right\rangle = \left\langle \sum_i^N \frac{\mathbf{p}_i^2}{m_i} \right\rangle + \sum_i^N \langle \mathbf{F}_i \cdot \mathbf{r}_i \rangle \quad (2)$$

the following equation can be derived:

$$-\sum_i^N \langle \mathbf{F}_i \cdot \mathbf{r}_i \rangle = \left\langle \sum_i^N \frac{\mathbf{p}_i^2}{m_i} \right\rangle = 2 \langle E_{\text{kin}} \rangle = 3Nk_B T \stackrel{\text{id.gas}}{=} 3PV \quad (3)$$

For a lennard jones pair interaction (like in the simulation) let the forces  $\mathbf{f}_{ij}$  and the distance vectors  $\mathbf{r}_{ij}$  be defined as follows. Thereby  $f^{\text{lj}}(|\mathbf{r}_{ij}|)$  is the scalar lennard jones force for the distance  $|\mathbf{r}_{ij}|$ , where a positive value means repulsion.

$$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i \quad (4)$$

$$\mathbf{f}_{ij} = f^{\text{lj}}(|\mathbf{r}_{ij}|) \cdot \frac{\mathbf{r}_{ij}}{|\mathbf{r}_{ij}|} \quad (5)$$

$$\mathbf{f}_{ij} = -\mathbf{f}_{ji} \quad \mathbf{r}_{ij} = -\mathbf{r}_{ji} \quad (6)$$

With this knowledge we can derive the pressure of the system by splitting up the force  $\mathbf{F}$  in equation (3) into an ideal gas part  $\mathbf{F}^{\text{id.gas}}$  and a lennard jones part  $\mathbf{F}^{\text{lj}}$ .

$$-\sum_i^N \langle \mathbf{F}_i \cdot \mathbf{r}_i \rangle = -\sum_i^N \langle \mathbf{F}_i^{\text{id.gas}} \cdot \mathbf{r}_i \rangle - \sum_i^N \langle \mathbf{F}_i^{\text{lj}} \cdot \mathbf{r}_i \rangle \quad (7)$$

$$\left\langle \sum_i^N \frac{\mathbf{p}_i^2}{m_i} \right\rangle = 3PV - \sum_i^N \langle \mathbf{F}_i^{\text{lj}} \cdot \mathbf{r}_i \rangle \quad (8)$$

$$P = \frac{Nk_B T}{V} + \frac{1}{3V} \sum_i^N \langle \mathbf{F}_{i,\text{interaction}} \cdot \mathbf{r}_i \rangle \quad (9)$$

$$= \frac{1}{3V} \left[ \left\langle \sum_i^N \frac{\mathbf{p}_i^2}{m_i} \right\rangle + \left\langle \sum_{i,j \neq i}^N -\mathbf{f}_{ij} \cdot \mathbf{r}_i \right\rangle \right] \quad (10)$$

$$= \frac{1}{3V} \left[ \left\langle \sum_i^N \frac{\mathbf{p}_i^2}{m_i} \right\rangle + \left\langle \sum_{i,j > i}^N \mathbf{f}_{ij} \cdot \mathbf{r}_j - \mathbf{f}_{ij} \cdot \mathbf{r}_i \right\rangle \right] \quad (11)$$

$$= \frac{1}{3V} \left[ \left\langle \sum_i^N \frac{\mathbf{p}_i^2}{m_i} \right\rangle + \left\langle \sum_{i,j > i}^N \mathbf{f}_{ij} \cdot \mathbf{r}_{ij} \right\rangle \right] \quad (12)$$

$$= \frac{1}{3V} \left[ 2 \langle E_{\text{kin}} \rangle + \left\langle \sum_{i,j > i}^N \mathbf{f}_{ij} \cdot \mathbf{r}_{ij} \right\rangle \right] \quad (13)$$

This works also for a non lennard jones pair interaction with analogous definition. In the programm we will use equation (13) but we use the actual state instead of calculating the expectation values.

## Pressure in Cython

Because of the vectorial calculations for each particle pair the calculation has to be done in C if it should be fast. Therefor the function `c_compute_pressure()` is written in `c_lj.cpp`.

Code block 6: Pressure calculation in C

script: src/c\_lj.cpp

```

253  double c_compute_pressure(double E_kin, double* x){
254      double rij[3];
255      double fij[3];
256      double interaction = 0.0;
257
258      // for (int i = 1; i < N; i++)
259      //     for (int j = 0; j < i; j++) {
260
261          // add up fij*rij
262      vector<int>::iterator it = verlet_list.begin();
263      vector<int>::iterator end = verlet_list.end();
264      while (it != end) {
265          int i = *it;
266          ++it;
267          int j = *it;
268          ++it;
269
270          if (i!=j){
271              minimum_image(x, i, j, rij);
272              compute_lj_force(rij,fij);
273
274              for (int k = 0; k < 3; k++)
275                  interaction += fij[k]*rij[k];
276          }
277      }
278      return (2.*E_kin+interaction)/(3*L*L*L);
279  }

```

---

As you can see the first part is the same like in `c_compute_forces()`, but the last part differs. In order to save run time the function takes the kinetic energy, which is already calculated, as an argument.

## 4 Molecular Dynamics at a Desired Temperature

### Programming

For the *velocity rescaling* thermostat we can derive the rescaling-factor  $f_{\text{re}}$  from equation (14).

$$\frac{3}{2}k_B T_0 = \frac{E_{\text{kin},0}}{N} \quad (14)$$

$$= \frac{1}{N} \sum_{i=1}^N \frac{(f_{\text{re}} \cdot \mathbf{v}^{(i)})^2}{2m} \quad (15)$$

$$= f_{\text{re}}^2 \frac{E_{\text{kin}}}{N} \quad (16)$$

$$= f_{\text{re}}^2 \frac{3}{2} k_B T \quad (17)$$

$$f_{\text{re}} = \sqrt{\frac{T_0}{T}} \quad (18)$$

This rescaling is implemented in C. In fact the implementation in C is not much faster than in python, but the thermostat in python did some crazy stuff. The function `c_velocity_rescaling` can be seen in code block 7.

Code block 7: Velocity rescaling

script: `src/c_lj.cpp`

```

281 void c_velocity_rescaling(double T0, double T, double* v
    ){
282     double f = sqrt(T0/T);
283     for (int k = 0; k < 3*N; k++) {
284         v[k] *= f;
285     }
286 }
```

This function is used in the main loop in `ljsim.py` every time after measuring the observables.

To start the thermostat you can use the command line option `--tstat` which accepts the desired temperature as argument (see code block 1). The option `--ctstat` is there for continuing the simulation for the interrupted simulation with temperature `T` thermostat, but now with deactivated thermostat. This is necessary because the program saves the results of the simulation with different names for different temperatures.

### Measurement

Now we can test the simulation for desired temperatures of  $T \in \{0.3, 1.0, 2.0\}$ . Therefore we must type into the command line:

```
>> python ljsim.py --tstat T0 --time 250.
```

Figure 1 shows the results respectively.

We can easily see the system reach the desired temperature and the energies reach equilibrium. The pressure is a bit harder to see as the fluctuations are so large and take large amounts of time.

As you can see the equilibration time seems to be highest for  $T_O = 0.3$  and lowest for  $T_O = 2.0$ . This is because the temperature  $T_0 = 0.3$  is much lower than the initial temperature. Therefore the total energy of the system must decrease. But the positive kinetic energy is much lower, than the necessary energy difference. If the temperature should be  $T = 0.3$  the potential energy of the system must decrease, but it is not directly effected by the velocity rescaling. By contrast for  $T_0 = 2.0$  the necessary energy can simply be reached by increasing the kinetic energy because it is much higher than the initial energy. For  $T_0 = 1.0$  the necessary energy is not much different from the initial one and therefore the other effects on equilibration have a higher influence.



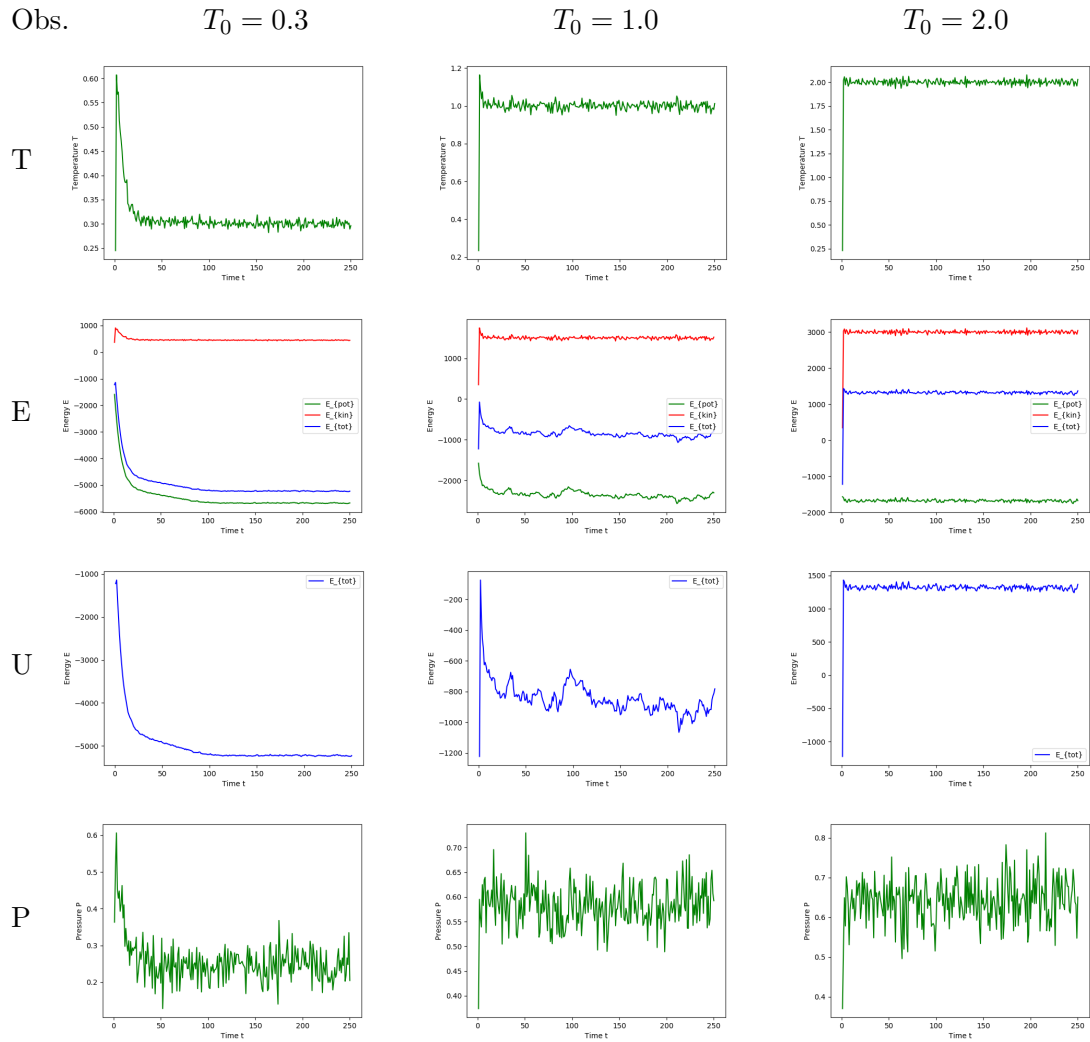


Fig. 1: Measured Observables (Obs.) for a thermostat temperature  $T_0$ .

T is the real temperature of the system, E are the energies (green: potential, red: kinetic, blue: total).

U is the total energy of the system alone.

P is the pressure of the system.

## 5 Warming up the System

When warming up the system you want to state a initial maximum force. Therefor the command line option `--warm` can be used (code block 1). The option `--cwarm` works analougus to `--ctstat`.

The first thing needed is the function `c_force_capping()` which is implemented in C. The reason for this is because you have to do many if requests and calculations for each particle.

Code block 8: Force capping

script: src/c\_lj.cpp

```

288  double c_force_capping(double* f, double fcap){
289      double fcap2 = fcap*fcap;
290      double test_cap = true; // if true: no force is capped
          any more
291      for (int k = 0; k < N; k++) {
292          double fabs2 = f[k]*f[k] + f[k+1]*f[k+1] + f[k+2]*f[
              k+2];
293          if (fabs2 > fcap2) {
294              test_cap = false;
295              double ff = sqrt(fcap2/fabs2);
296              f[k] *= ff;
297              f[k+N] *= ff;
298              f[k+2*N] *= ff;
299          }
300      }
301      if (test_cap){
302          fcap = 0;
303      }
304      return fcap;
305  }

```

The function in code block 8 limits the forces to a given maximum force `fcap`, but conserves the direction if the forces. Although this process is not physically correct. Furthermore the function tests if no force is capped any more. When this is the case it sets `fcap` to zero. The function is called in the `vv_step()` function in `ljsim.py`.

Code block 9: Limiting forces

script: src/ljsim.py

```

164      # compute new forces
165      # we assume that m=1 for all particles
166      f = compute_forces(x)
167      if args.warm:
168          force_capping(f,args.warm)

```

As you can see, the function will not be activated any more if `fcap = args.warm` is set to zero.

At least the `fcap` should be increased by ten percent every time the measurements are done. This is done in code block 10.

Code block 10: Increase maximum force

script: `src/ljsim.py`

```
213         # rescale fcap
214         if args.warm:
215             args.warm *= 1.1
```

But all the capping is useless without random initial conditions. The velocity is already random, so only the initial position has to be changed. You can do this with a similar command like in 11.

Code block 11: Random initial position

script: `src/ljsim.py`

```
98         # Initialize particle position
99         if args.warm:
100             # The warming up random positions
101             x = L*random.random((3,N))
```

## 6 Radial Distribution Function

An important observable for our simulation is the radial distribution function. It is the probability to find a particle at a given radial distance away from another particle compared to an ideal gas. Mathematically it is written as

$$g(r) = \frac{1}{\rho 4\pi r^2 dr} \sum_{i,j} \langle \delta(r - |\mathbf{r}_{ij}|) \rangle \quad (19)$$

For an ideal gas we expect  $g(r) = 1$  if we normalize correctly.

In our simulation we implement the `rdf` using both C and python. First we calculate all the pair distances for our particles in C.

Code block 12: Radial Distances

script: `src/c_lj.cpp`

```
307 void c_compute_distances(double* x, double* r){
308     double rij[3];
309     int k = 0;
310     for ( int i = 0; i < N; i++){
311         for ( int j = i+1; j < N; j++){
312             minimum_image(x, i, j, rij);
313             r[k] = sqrt(rij[0]*rij[0]+rij[1]*rij[1]+rij
                        [2]*rij[2]);
```

```

314         k++;
315     }
316 }
317 }

```

We then send this to an array in python and compute a histogram using numpy and normalize it in the correct way

Code block 13: Numpy Histogram

script: src/ljsim.py

```

10 def compute_histogram(r):
11     h, bins = histogram(r, bins=100, range=(0.8, 5.),
12                        density=True)
13     return h*vol/(4*pi*bins[:-1]*bins[:-1])

```

Finally we implement an average RDF after equilibrium in `ljanalyze.py`. We compute this using the following function

Code block 14: Compute Mean RDF

script: src/ljanalyze.py

```

51 def compute_mean_rdf(O, keq):
52     N = O.shape[0]
53     Om = zeros(O.shape[1])
54     for k in xrange(keq, N):
55         Om += O[k, :]
56     return Om/(N-keq)

```

and plot it.

## 7 Measuring Equilibrium Mean Values of the Observables

We now run the simulation for the same three temperatures 0.3, 1.0, 2.0 including random particle positions and the previously described warm up. We run the simulations for 1200 time steps. Using our command line arguments an example looks like:

```
python ljsim.py --warm 20 --tstat 0.3 --time 120
```

This sets an initial force cap of 20. The equilibration of the observables is plotted in the following figures with the first 100 timesteps omitted to preserve plot scaling. The command to produce these plots with our command line arguments looks like:

```
python ljanalyze.py --datafile ../dat/ljsim_T2d0_F20d0.dat --tlim 100 1199
--M 100
```

Here we used a window of  $M = 100$ .

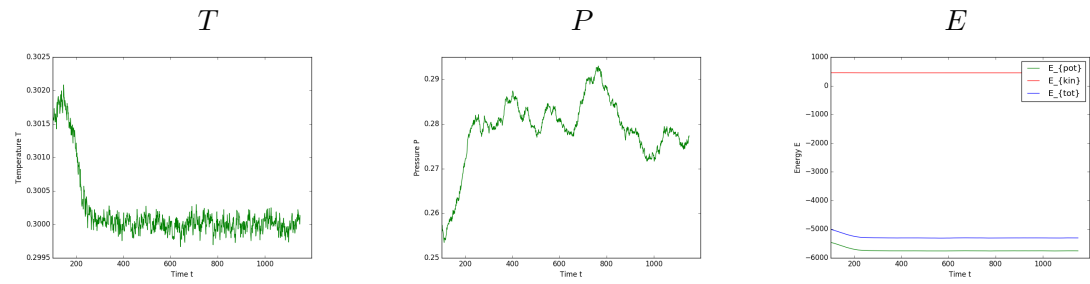


Fig. 2: Desired Temperature:  $T_0 = 0.3$ . Average values of temperature, pressure and energies from the left to the right.

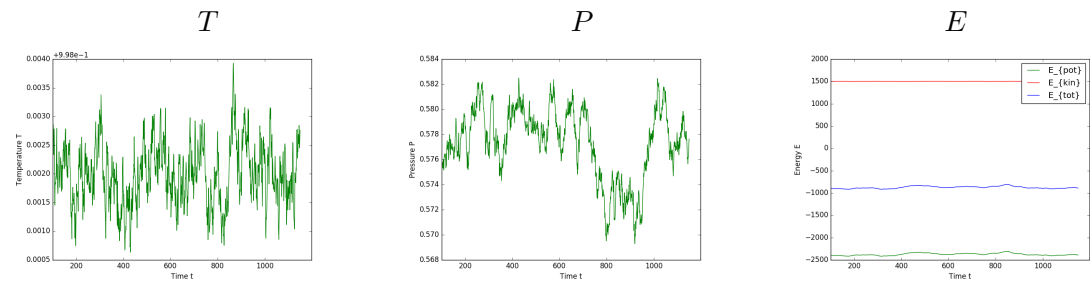


Fig. 3: Desired Temperature:  $T_0 = 1.0$ . Average values of temperature, pressure and energies from the left to the right.

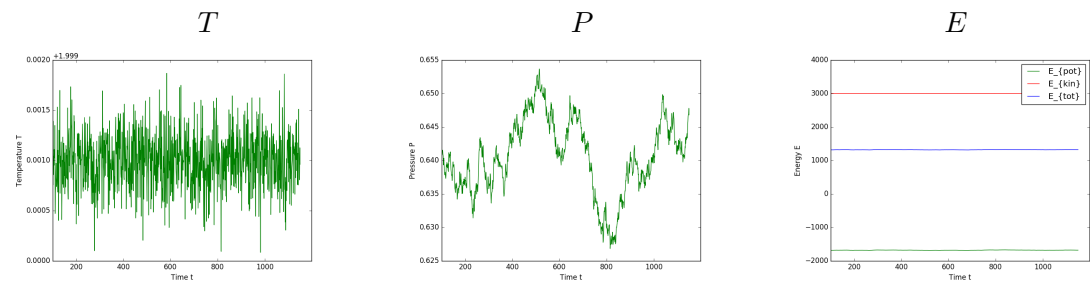


Fig. 4: Desired Temperature:  $T_0 = 2.0$ . Average values of temperature, pressure and energies from the left to the right.

The commands only differed in the thermostat temperature and the datafilename to be analyzed. Finally we plot the average RDF for the three different temperatures after equilibrium and compute the mean equilibrium values of the observables. The command to perform this analysis is:

```
python ljanalyze.py --datafile ../dat/ljsim_T0d3.F20d0.dat --teq 400
```

Here we have assumed that our system has equilibrated by timestep 400. We measured the following average values:

$T_0$	$E_{\text{pot}}$	$E_{\text{kin}}$	$E_{\text{tot}}$	$T$	$P$
0.3	-5754.9	450.0	-5304.9	0.300	0.281
1.0	-2367.3	1500.1	-867.3	1.000	0.577
2.0	-1679.0	3000.0	1321.0	2.000	0.642

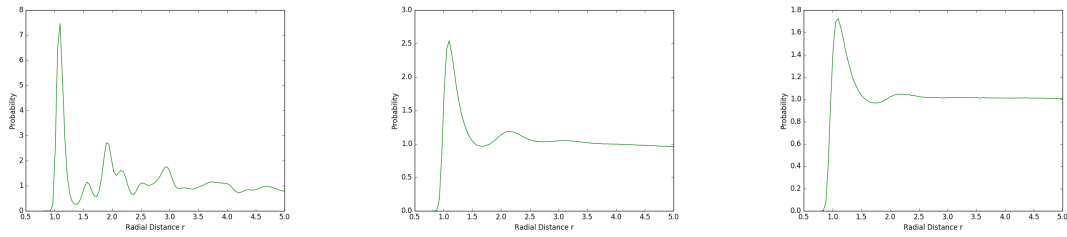


Fig. 5: Radial distribution function (RDF) in equilibrium for desired temperature of  $T_0 = \{0.3, 1.0, 2.0\}$  from the left to the right.