CVEN5313: Environmental Fluid Mechanics

Navier-Stokes: Integral Theorems Due: 10/27/10

1. Let $\mathcal{R}(t)$ be a material volume bounded by the surface $\mathcal{S}(t)$. The volume of fluid within $\mathcal{R}(t)$ is $V(t)$.

    (a) Show from geometry that the volume $V(t)$ evolves as

    $$\frac{dV(t)}{dt} = \iint_{\mathcal{S}(t)} \vec{u} \cdot \hat{n} \; dS$$

    where $\vec{u}$ is the local flow velocity, dS is an area element on $S(t)$, and $\hat{n}$ is the outward-pointing unit normal on dS.

    (b) Use the divergence theorem to rewrite your answer from part (a) as

    $$\frac{dV(t)}{dt} = \iiint_{\mathcal{R}(t)} \nabla \cdot \vec{u} \; dV$$

    (c) Use your result from part (b) to intuit the physical interpretation of $\nabla \cdot \vec{u}$.

2. Derive the one-dimensional Leibnitz formula using the three-dimensional formulation given in lecture. Start by considering a function of only one dimension $f(x,t)$ that exists in a three-dimensional space. You can use an integration volume with a convenient shape (i.e. not a generalized potato).

3. Consider an incompressible fluid ($\nabla \cdot \vec{u} = 0$).

    (a) Use Gauss' theorem to show that the integral of the velocity flux over any closed surface is zero.

    (b) Use pictures and words to explain why the sum of the velocity fluxes over any closed surface must be zero.

4. Consider a vector field $\vec{u}(\vec{x},t)$ that describes the spatial and temporal variation of a fluid flow. We will see that an important kinematic (i.e. relating to motion) property of the flow is the vorticity field $\vec{\omega}(\vec{x},t)$. The vorticity is defined as the curl of the velocity, namely $\vec{\omega} = \nabla \times \vec{u}$. It can be shown that the integral of the vorticity flux vanishes over any closed surface. Note that an example of an arbitrary closed surface would be the skin of a potato.

(a) Draw a picture of an arbitrary three-dimensional volume (a potato). Indicate (schematically) a vorticity vector on the surface, the unit normal vector, and the vorticity flux through an elemental area dS.

(b) Use Stokes' theorem to show that the integral of the vorticity flux over the surface is zero. Hint: You might want to take a knife to your 'potato'.

(c) Repeat the exercise using Gauss'theorem. Hint: You will make use of things you learned in the first problem set.

5. Using the definition for vorticity, Stokes' theorem can be written as

$$\int \vec{\omega} \cdot d\vec{A} = \int \vec{u} \cdot d\vec{l}$$

(a) Using small element of fluid in cartesian coordinates in the x_1-x_2 plane, show that Stokes' theorem leads to the following expression for vorticity in the x_1-x_2 plane:

$$\omega_3 = \left[\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right]$$

(b) Using a small element of fluid in polar coordinates in the r-θ plane, show that Stokes' theorem leads to the following expression for vorticity in the r-θ plane:

$$\omega_z = \frac{1}{r} \left[\frac{\partial}{\partial r} (r u_\theta) - \frac{\partial u_r}{\partial \theta} \right]$$