Lecture 12, September 30, 2010 (Key Points)

12.1 Tokunaga Model, Horton Laws, Fractal Dimension and Hack's Law

We derived a basic recursion equation for the Tokunaga networks in Lecture 11, which is reproduced below,

$$N_k^{\Omega} = 2N_{k+1}^{\Omega} + \sum_{j=1}^{\Omega-k} T_j N_{k+j}^{\Omega}, \ \ k = 1,2,...,\Omega - \omega \quad (12.1)$$

It can be solved either numerically or analytically if the tree generators are known. A key problem is to show using Eq. (12.1) that the Horton law of stream numbers is obeyed, and the second is to compute R_B.

Assume that the tree generators are given by Tokunaga mean self-similar assumption displayed in Eq. (11.3) with *a*=1, and *c*=2. It gives, $T_1 = 1, T_2 = 2,...,T_{n+1} = 2^n$. Take, Ω=10, and ω=9, so *k*=1,2,…,9. We will start the calculations with the largest value, *k*=9 and recursively calculate, $N_9^{10}, N_8^{10}, N_7^{10}$,.....etc. It follows form Eq. (12.1) that for $k = 9$,

$$N_9^{10} = 2N_{10}^{10} + 1.N_{10}^{10} \ \ \text{, or } \frac{N_9^{10}}{N_{10}^{10}} = 3 \quad (12.2)$$

For, *k*=8, in view of Eq. (12.2), it follows that,

$$\frac{N_8^{10}}{N_9^{10}} = 2 + 1 + 2\frac{N_{10}^{10}}{N_9^{10}} = 3 + (2/3) = 3.667 \quad (12.3)$$

Similarly, in view of Eqs. (12.2) and (12.3), it follows that,

$$\frac{N_7^{10}}{N_8^{10}} = 2 + 1 + 2\frac{N_9^{10}}{N_8^{10}} + 4\frac{N_{10}^{10}}{N_8^{10}} = 3 + (2/3.667) + (4/3.667X3) = 3.909 \quad (12.4)$$

Continuing in this manner a few more times, we get, $\frac{N_6^{10}}{N_7^{10}} = 3.97, \frac{N_5^{10}}{N_6^{10}} = 3.998$. The reader can check by calculating these ratios that the following pattern of convergence emerges for this example,

$$\frac{N_{\omega}^{\Omega}}{N_{\omega+1}^{\Omega}} = 2 + 1 + (1/2) + (1/2)^2 + (1/2)^3 + ... = 2 + \frac{1}{1-(1/2)} = 4, \ \ \omega = 1,2,...,\Omega \to \infty \quad (12.5)$$

This is same as the Horton law of stream numbers given in Eq. (9.1) (Lecture 9) that Horton introduced empirically nearly 60 years ago.

McConnell and Gupta (2008) generalized the above result to the entire set of parameters *a* and *c*. The derivation is a bit technical, and is beyond the scope of this course. But based on their result we assume that the Horton law of stream numbers holds for all of the Tokunaga parameters. Dividing both sides of Eq. (12.1) by N_{k+1}^{Ω}, substituting the expression for Tokunaga generators, $T_k = ac^{k-1}$, $k = 1,2,....$, into Eq. (12.1), and taking the limits as $\Omega \to \infty, \Omega - k \to \infty$

$$R_B = 2 + a\sum_{j=1}^{\infty}\left(\frac{c}{R_B}\right)^{j-1} \qquad (12.6)$$

The geometric series converges for, $R_B > c$. Summing the geometric series gives a quadratic equation in R_B,

$$R_B{}^2 - (2 + a + c)R_B + 2c = 0 \qquad (12.7)$$

Ignoring the root less than 2, because $R_B \geq 2$, the solution is,

$$R_B = \frac{(2 + a + c) + \sqrt{(2 + a + c)^2 - 8c}}{2} \qquad (12.8)$$

It is simple to check from Eq. (12.8) that $R_B = 4$ for $a = 1, c = 2$, which is the same as obtained in Eq. (12.5). These tree generators define the 'average Shreve tree', because the random topology model discussed earlier predicts, $R_B = 4$.

For the Powder River basin in Wyoming, Peckham (1995) gives the estimated values of the Tokunaga parameters as, $a = 1.2, c = 2.7$. Substituting these values in Eq. (12.8) gives, $R_B = 4.8$, which compares well with the empirical value of 4.73. Analysis of a dozen large river basins in Peckham (1995) showed that the bifurcation ratios are between 4.1 and 4.8, all of which are significantly larger than 4. Therefore, the random topology model does not describe the topology of these basins. This result provided a very important context to understand the role of the Horton law of stream numbers in discriminating among different network models. For example, It suggested that the random model is not appropriate for describing rive network topology. These results were instrumental in developing the RSN model and *replacing the assumption of 'topologic randomness' with that of ' random self-similarity',* which could accommodate the values of R_B observed in river networks (Veitzer and Gupta 2000).

12. 2 Tokunaga Model and the Horton Law of Link Numbers

Let us introduce another very important topologic quantity called the link number, C_{ω}^{Ω} in a stream of order ω within a network of order Ω. It is intimately tied to the length of a stream of order ω. The number of links in a stream of order ω is one greater than the number of side tributaries, $T_{\omega,\omega-k} = T_k$, $k = 1,2,\ldots,\omega - 1$. Therefore,

$$C_{\omega}^{\Omega} = 1 + \sum_{k=1}^{\omega-1} T_k = 1 + \sum_{k=1}^{\omega-1} ac^{k-1} = \frac{a(c^{\omega} - 1)}{c - 1} \qquad (12.9)$$

Therefore, it follows from Eq. (12.9) that the link numbers satisfy a Horton law given by,

$$\lim_{\omega\to\infty}\frac{C_{\omega+1}}{C_\omega}=\frac{c^{\omega+1}-1}{c^{\omega}-1}=c=R_c \qquad (12.10)$$

Here R_C denotes the *Horton link-number ratio*. For the average Shreve tree, *c*=2, and R_C=2.

12.3 Topologic Fractal Dimension of Mean Self-Similar Networks

La Barbera and Rosso (1989) introduced the notion of the topologic fractal dimension of a network in terms of Horton numbers in the hydrology literature. Many others have investigated this concept quite extensively since that time. Typically, the empirical values of topologic fractal dimension for many river networks have been observed to lie between 1.7 and 1.8. Its physical significance is explained at the end of this section.

Dimension is a 'spatial concept', but the topologic description of a network does not need a spatial context. The 'topologic dimension', D_T for a network can be formally defined in a similar manner as the box counting dimension of a fractal object (Lecture 8),

$$D_T=\lim_{k\to\infty}\frac{\log n_k}{-\log\delta_k}\quad , \qquad (12.11)$$

where n_k is the total number of links (segments), and δ_k is the link length at the k^th^ stage of construction of a network.

Let Ω denote the Strahler order of a network. A mean self-similar network can be constructed in the same manner as any fractal geometrical object. Let $k=1,2,3,...$ denote various stages of construction. Let $N_j^{(k)}, j=1,2,...,k$ denote Strahler streams of different orders j at the $k-th$ stage of construction. Note that at each stage of construction the network order, $\Omega=k$, as we always begin the construction with Ω=$k=1$. As the construction proceeds to higher stages, the network order increases. Let $N_1^{(k)}$, denote the total number of source streams, or magnitude, at the $k-th$ stage of construction of a network. All binary branching trees have a topologic property that the total number of links in a network, n_k is related to the magnitude as (Shreve, 1967),

$$n_k=2N_1^{(k)}-1\approx 2R_B^{(k-1)} \qquad (12.12)$$

The Horton law of stream numbers can be used to compute $N_1^{(k)}$ for large k for a mean self-similar tree, but we will consider the example of Tokunaga trees shortly. First, let us consider δ_k, the divider length scale. At each stage of construction of a mean self-similar network, number of links in each the trunk stream of given Straher order, Ω=*k*, and all lower order streams, increases and the length scale decreases. Physically this decrease in the length scale is similar to an increase in the map resolution at which the network is being extracted. We assume that each of the side stream equally divides the trunk stream of unit length. Moreover, in assigning length to smaller order streams, we assume that each link throughout the network has the same length as the link length in the trunk stream. The natural length scale at the k-th stage of construction is determined by the total number of links in the trunk stream. In view of Eqs. (7.5.15) and (7.5.16), it is given by,

$$\lim_{k \to \infty} \delta_k = \frac{1}{1 + T_1 + T_2 + \ldots T_{k-1}} = \frac{(c-1)}{ac^k} \approx \frac{1}{ac^{(k-1)}} = \frac{1}{aR_C{}^{(k-1)}} \tag{12.13}$$

The length scale decreases as k increases, which is a property of all fractals.

As an example, let us specialize the computation of D_T to Tokunaga networks, because we have already obtained expressions for $N_j{}^{(k)}$ and δ_k, in terms of Horton laws of stream numbers and link numbers in section 7.5. These expressions are given by eq. (12.12) and eq. (12.13) respectively. Substituting these expressions into (12.11) gives the topological fractal dimension of self-similar Tokunaga networks as,

$$D_T = \lim_{k \to \infty} \frac{\log n_k}{-\log \delta_k} = \frac{\log R_B}{\log R_C} \tag{12.14}$$

or,

$$R_B = (R_C)^{D_T} \tag{12.15}$$

This is an important result because it shows that the two Horton ratios for Tokunaga networks are mutually related through the topologic fractal dimension.

For the average Shreve tree, $R_B = 4, R_C = 2$, and D_T=2. This result is intriguing, because it says a Shreve network embedded in the 2-dimensional plane would essentially "fill" it. It is referred at as the *space-filling* property of a network. This result is physically quite insightful. Recall that a very important function of river networks is to drain water, sediments, nutrients, and bio-geochemical variables such as carbon and nitrogen, from small to large basins from terrains into the oceans. A network with dimension D_T=1 would do this job rather poorly, because a geometric object with a line-like structure would either leave out a large region of two dimensional space, or the water would have to take a circuitous route to the outlet rather than taking a direct path. Stevens (1972) discusses this physical issue in a rather insightful but qualitative manner. He compares three patterns, spiral, radial, and branching. A spiral has the shortest total length but the largest mean length to the outlet. A radial pattern by contrast has the shortest mean length but the largest total length. A branching pattern combines the best of the two worlds, as it has the shortest total length but its mean length is not much greater than that of a radial pattern. One can try to make this illustration more quantitative by relating the total length and the mean length of Stevens to the Horton branching numbers. But we would not stop to explore this issue any further, because how the self-similar networks fill space remains an important unsolved problem. It is called the problem of 'spatial embedding' (Jarvis and Woldenberg, Part-III, 1984).

12.4 Hack's law and Tail Probability Exponent of Basin Areas for Tokunaga Networks

Two important empirical results for river networks have been obtained for the Tokunaga trees. First is the Hack's law discovered by Hack (1957). It says that the longest channel length from the basin outlet, L is related to the upstream drainage area A as,

$$L = cA^{\beta} \tag{12.16}$$

For river networks, the Hack exponent $\beta \approx 0.56$ rather than 0.5 as one would expect for regular geometrical objects (Mueller, 1973). A large literature has grown around the attempts to exhibit the Hack's law within the context of different models of river networks. For example, it has been shown analytically that the random model predicts $\beta = 0.5$, which does not agree with data. A derivation of this result is a bit technical, and is outside the scope of this course.

The second result concerns the probability distribution of basin areas upstream of any location on a network,

$$P(A > a) \sim a^{-\alpha} \tag{12.17}$$

where the 'tail probability exponent' $\alpha \approx 0.44$. Rodriguez-Iturbe et al. (1992) first published this result. Several theoretical models of channel networks have tried to obtain this result from their respective constructs. For example, the random model predicts that $\alpha = 0.5$.

Our main objective in this section is to point out that the above two empirical results are mutually related, and it has been have shown for the class of Tokunaga networks. In particular, the probability distribution of link magnitude M obeys,

$$P(M \geq m) \sim m^{-\alpha} \tag{12.18}$$

where,

$$\alpha = 1 - \frac{\log R_C}{\log R_B} = 1 - \frac{1}{D_T} \tag{12.19}$$

where, D_T is the topologic dimension. Similarly, Tokunaga networks obey the Hack's law, where the Hack exponent is given by,

$$\beta = 1 / D_T \tag{12.20}$$

Eqs. (12.19) and (12.20) show the important result that for Tokunaga networks,

$$\alpha + \beta = 1 \tag{12.21}$$

We will not stop to derive these two results. But the derivations are algebraic in nature and are not difficult to understand (Peckham and Gupta 1999).

References

Hack, J. T. Studies of longitudinal stream profiles in Virginia and Maryland, *US Geological Survey Prof. Paper 294-B*, 505 B, 1957.

Jarvis, R. S., and M. J. Woldenberg, *River networks*, Benchmark Papers in Geology, vol. 80, R. S. Jarvis and M. J. Woldenberg (Ed.), 386 pp., Van Nostrand, 1984.

La Barbera, P. and R. Rosso (1989): On the fractal dimension of stream networks, *Water Resour. Res.*, 25(4), 735-741.

Mueller, J. P. Re-evaluation of the relationship of master streams and drainage basins, *Geol. Soc. Am. Bulletin*, 84: 3127-3130, 1973.

Peckham, S. and Gupta V. K., A reformulation of Horton laws for large river networks in terms of statistical self-similarity, *Water Resour. Res*., 35(9), 2763-2777, 1999.

Peckham, S. New results for self-similar trees with applications to river networks. *Water Resour. Res*., 31(4): 1023-29, 1995.

Rodriguez-Iturbe, I, EJ Ijjász-Vásquez, RL Bras, DG Tarboton. Power law distributions of discharge mass and energy in river basins. *Water Resour. Res*., 28(4): 1089-1093, 1992.

Stevens, P. S. *Patterns in Nature*. Little, Brown and Co., 1972.

Veitzer, S. and V. K. Gupta, 2000: Random self-similar river networks and derivations of generalized Horton laws in terms of statistical simple scaling, *Water Resour. Res.,* 36(4): 1033-1048.