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# II. BUCKINGHAM PI THEOREM

The Buckingham Pi Theorem reduces a number of dimensional variables into a smaller number of dimensionless groups. The resulting dimensionless groups are <u>independent</u> but <u>not unique</u>. Additional dimensionless groups may be formed by taking products of the existing dimensionless groups.

The basic steps involved are listed below:

① Conjecture a physical relationship of the form

$$f(v_1, v_2, v_3, \ldots, v_m) = 0$$

where $v_i$ $1 \le i \le m$ are all of the dimensional parameters that have a bearing on the problem at hand.

② List the dimensions of each variable

$r$ = number of primary dimensions in the list

③ Form the matrix system

$$\begin{array}{c} \\ d_1 \\ d_2 \\ \vdots \\ d_r \end{array} \begin{array}{c} v_1 \; v_2 \; v_3 \; \cdots \; v_m \\ \left[ \begin{array}{c} \\ \\ \\ \\ \end{array} \right] \end{array} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{m-1} \\ a_m \end{bmatrix} = 0$$

$i^{th}$ dimension in the problem (e.g. $M, L, \theta, T$)

Note: You must order the columns such that columns $v_1$ through $v_r$ contain at least one instance of each of the $r$ dimensions.

④ Perform Gaussian elimination. If a zero row appears, $r \leftarrow r-1$

⑤ Write expressions $a_1$ , $a_r$ in terms of

, $a_{r+2}$, $a_{r+3}$, , $a_m$.

(6.) Write the equations from step (5) as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} \\ b^{I} \\ \\ \end{bmatrix} a_{r+1} + \begin{bmatrix} \\ b^{II} \\ \\ \end{bmatrix} a_{r+2} + \cdots + \begin{bmatrix} \\ b^{m-r} \\ \\ \end{bmatrix} a_m$$

(7.) There are m-r dimensionless groups defined by

$$V_1^{b_1} \, V_2^{b_2} \, V_3^{b_3} \cdots V_{m-1}^{b_{m-1}} \, V_m^{b_m}$$

Each $\vec{b}$ defines one dimensionless group.

# BUCKINGHAM PI EXAMPLE

Suppose we wish to investigate flow patterns in a lake, and we are interested in determining the nondimensional parameters associated with the problem.

Based on experience, we decide that the following dimensional variables are relevant:

| | |
|---|---|
| $D_1$ | length of lake ($x_1$-direction) |
| $D_2$ | width of lake ($x_2$-direction) |
| $D_3$ | depth of lake ($x_3$-direction) |
| $U_1$ | flow velocity in $x_1$ direction |
| $U_2$ | " " " $x_2$ " |
| $U_3$ | " " " $x_3$ " |
| $\rho$ | density of lake water |
| $f$ | coriolis parameter |
| $\nu$ | lake water viscosity |
| $g$ | gravitational constant |
| $t$ | time |
| $\Delta p$ | pressure |

Following the steps outlined before, we have

① $$f\left(D_1, D_2, D_3, U_1, U_2, U_3, \rho, f, \nu, g, t, \Delta p\right) = 0 \qquad \boxed{m = 12}$$

② $$[D_1] = [D_2] = [D_3] = L$$

$$[U_1] = [U_2] = [U_3] = LT^{-1}$$

$$[\rho] = ML^{-3}$$

$$[f] = T^{-1}$$

$$[\nu] = L^2T^{-1}$$

$$[g] = LT^{-2}$$

$$[t] = T$$

$$[\Delta p] = ML^{-1}T^{-2}$$

3 dimensions in the list $(M, L, T)$, so $\boxed{r = 3}$

③

$$\begin{array}{c} \\ L \\ T \\ M \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} D_1 & U_1 & \rho & f & \nu & g & t & \Delta p & D_2 & D_3 & U_2 & U_3 \end{array} \\
\left[\begin{array}{ccc:ccccccccc}
1 & 1 & -3 & 0 & 2 & 1 & 0 & -1 & 1 & 1 & 1 & 1 \\
0 & -1 & 0 & -1 & -1 & -2 & 1 & -2 & 0 & 0 & -1 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \end{bmatrix} = 0$$

④ Perform Gaussian elimination. Note that I have chosen $v_1$, $v_2$, and $v_3$ $(r=3)$ so that this step is trivial:

$$\begin{array}{c|cccccccccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} \\
L & 1 & 1 & -3 & 0 & 2 & 1 & 0 & -1 & 1 & 1 & 1 & 1 \\
T & 0 & 1 & 0 & 1 & 1 & 2 & -1 & 2 & 0 & 0 & 1 & 1 \\
M & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}$$

there are no zero rows, so $r$ is indeed 3.

⑤ from row 3 we have: $a_3 + a_8 = 0 \Rightarrow \boxed{a_3 = -a_8}$

from row 2 we have:

$$a_2 + a_4 + a_5 + 2a_6 - a_7 + 2a_8 + a_{11} + a_{12} = 0$$

$$\Rightarrow \boxed{a_2 = -a_4 - a_5 - 2a_6 + a_7 - 2a_8 - a_{11} - a_{12}}$$

from row 1 we have:

$$a_1 + a_2 - 3a_3 + 2a_5 + a_6 - a_8 + a_9 + a_{10} + a_{11} + a_{12} = 0$$

so $$a_1 = -\left(-a_4 - a_5 - 2a_6 + a_7 - 2a_8 - a_{11} - a_{12}\right) + 3(-a_8) - 2a_5 - a_6 + a_8 - a_9 - a_{10} - a_{11} - a_{12}$$

$$\Rightarrow \boxed{a_1 = a_4 - a_5 + a_6 - a_7 - a_9 - a_{10}}$$

since $r = 3$, we have written $a_1$, $a_2$, and $a_3$ in terms of $a_4, a_5, a_6, \ldots, a_{10}, a_{11}, a_{12}$.

(6.)

$$
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \end{bmatrix}
=
\overset{b^{I}}{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} a_4 +
\overset{b^{II}}{\begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} a_5 +
\overset{b^{III}}{\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} a_6 +
\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} a_7 +
\begin{bmatrix} 0 \\ -2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} a_8 +
\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} a_9 +
\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} a_{10} +
\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} a_{11} +
\overset{b^{IX}}{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} a_{12}
$$

(7.) There are $m - r = 9$ nondimensional groups

For example, the first one is $V_1^{1} V_2^{-1} V_4^{1} = \dfrac{V_1 V_4}{V_2} = \dfrac{D_1 f}{U_1}$

The complete set of dimensionless parameters is given below:
(Note that I have inverted some of them)

$\dfrac{U_1}{f D_1}$ "Rossby Number" ratio of inertial forces to Coriolis "forces"

$\dfrac{U_1 D_1}{\nu}$ "Reynolds Number" ratio of inertial forces to viscous diffusion forces

$\dfrac{U_1^2}{g D_1}$ "Froude Number" ratio of inertial forces to gravity forces

$\dfrac{D_1}{U_1 t}$ "Strouhal Number" ratio of inertial forces to forces from local accelerations (unsteadiness)

$\dfrac{\Delta p}{\rho U_1^2}$ "Euler Number" ratio of inertial forces to pressure forces.

$\left.\begin{matrix} \dfrac{D_2}{D_1} \\ \\ \dfrac{D_3}{D_1} \end{matrix}\right\}$ Aspect Ratios

$\left.\begin{matrix} \dfrac{U_2}{U_1} \\ \\ \dfrac{U_3}{U_1} \end{matrix}\right\}$ Velocity Ratios