CVEN5313

Environmental Fluid Mechanics

Section Topic: Open-Channel Flow

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Fall 2010

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Contents

References

- *Jain, S.C. (2001) "Open-Channel Flow," John Wiley & Sons.
- Chanson, H. (1999) "The Hydraulics of Open Channel Flow," John Wiley & Sons.
- Chaudhry, M.H. (1993) "Open-Channel Flow," Prentice Hall.
- Sturm, T.W. (2001) "Open-Channel Hydraulics," McGraw Hill.

*Principal reference for this section of notes.

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1 Introduction

Examples
Rivers and streams
Canals
Pipes, Conduits, and Culverts (when not flowing full)

Flow Types
uniform vs. *nonuniform* (spatial variation)
steady vs. *unsteady* (temporal variation)

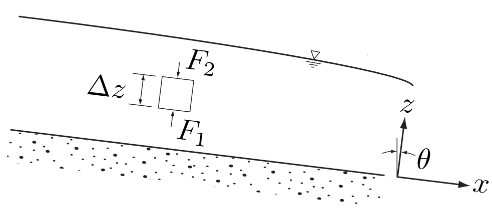
Solutions to open channel flow problems are complicated by the fact that the cross-sectional flow area is not known *a priori* (that is, the location of the free surface is a variable). 3

1.1 Assumptions

Incompressible Flow
$\rho = \text{constant} \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} = 0 \quad , \quad \nabla \rho = [0,0,0]$

Longitudinal Flow
$\vec{u} = [u(x,z,t),0,0]$

Hydrostatic Pressure Distribution
$p = \rho g(z_s - z)\cos\theta \qquad \frac{\partial p}{\partial z} = -\rho g \cos\theta \qquad$ (Typically $\cos\theta \approx 1$)



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1.2 Definitions (for reference)

Average Velocity
$$V \equiv \frac{1}{A}\int_A u \,\mathrm{d}A$$
where A is the cross-sectional area of the channel flow, in the y-z plane.

Bed Slope
$$S_0 \equiv -\frac{dz_0}{dx} = \sin\theta \approx \theta$$

Friction Slope
$$S_f \equiv \frac{\tau_0}{\gamma R_h} \qquad \text{where} \qquad R_h \equiv A/P_w$$

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Momentum Coefficient: allows momentum flux to be expressed in terms of V:
$$\int_A \rho u^2 \, dA = \beta \rho Q V \qquad \rightarrow \qquad \beta \equiv \frac{\int_A u^2 \,\mathrm{d}A}{V^2 A}$$

Energy Coefficient: allows K.E. flux to be expressed in terms of V:

$$\int_A \frac{\rho u^3}{2}\,\mathrm{d}A = \alpha \rho Q \frac{V^2}{2} \qquad \rightarrow \qquad \alpha \equiv \frac{\int_A u^3\,\mathrm{d}A}{V^3 A}$$

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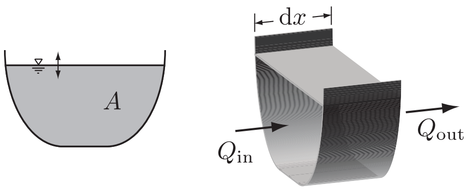
2 Conservation Laws: The Saint-Venant Equations

→ See Appendix for a formal derivation of these equations.

2.1 Conservation of Mass

Consider a fixed volume spanning the cross-section of an open channel:



The change in fluid mass in the volume in time $\mathrm{d}t$ can be expressed as

$$\Delta m = \rho(Q_{\mathrm{in}} - Q_{\mathrm{out}})\,\mathrm{d}t$$

and also as

$$\Delta m = \rho \frac{\partial A}{\partial t}\mathrm{d}t\ \mathrm{d}x$$

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Equating the two expressions for Δm gives:

$$\rho(Q_{\mathrm{in}} - Q_{\mathrm{out}})\,\mathrm{d}t = \rho \frac{\partial A}{\partial t}\mathrm{d}t\ \mathrm{d}x$$

or

$$\frac{Q_{\mathrm{out}} - Q_{\mathrm{in}}}{\mathrm{d}x} + \frac{\partial A}{\partial t} = 0$$

which, in the limit as $\mathrm{d}x \rightarrow 0$ becomes the Continuity Equation:

$$\boxed{\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0} \tag{1}$$

→ Equation 1 is the mass conservation statement ("Continuity") in open-channel flow, subject to the assumptions in Section 1.1.

Note that for steady flows, $\partial A/\partial t = 0$, and

$$\frac{\partial Q}{\partial x} = 0 \qquad \rightarrow \qquad Q_{\mathrm{in}} = Q_{\mathrm{out}}$$

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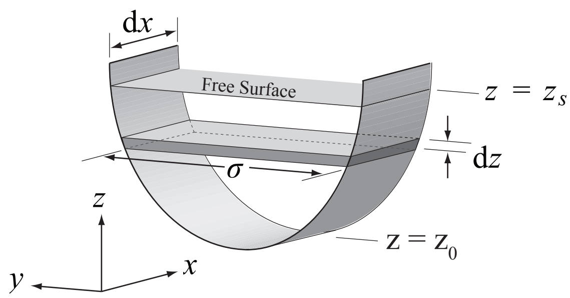
2.2 Conservation of Momentum

We begin with $F = ma$, considering only forces and accelerations in the x-direction (y and z accelerations are assumed negligible):

$$\Sigma F_x = f_p + f_g + f_s = m a_x$$

where f_p, f_g, and f_s are the x-direction pressure, gravitational, and shear forces acting on the channel section. On the differential volume shown below, we have

$$\mathrm{d}f_p + \mathrm{d}f_g + \mathrm{d}f_s = \mathrm{d}(m a_x)$$



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→ Pressure Forces f_p:

$$\mathrm{d}f_p = -\frac{\partial p}{\partial x}\,\mathrm{d}x\,\sigma\,dz$$

Recalling the hydrostatic pressure expression $p = \rho g (z_s - z)\cos\theta$, we have

$$\frac{\partial p}{\partial x} = \rho g \cos\theta \frac{\partial z_s}{\partial x} \quad \text{and thus} \quad df_p = -\rho g \cos\theta \frac{\partial z_s}{\partial x}\sigma\,dx\,dz$$

Now integrate over the depth to get the pressure forces on the entire channel section:

$$f_p = \int_A \mathrm{d}f_p\ \mathrm{d}A = -\rho g \cos\theta \frac{\partial z_s}{\partial x}\,\mathrm{d}x \underbrace{\int_0^{z_s} \sigma\,\mathrm{d}z}_{A}$$

$$f_p = -\rho g \cos\theta \frac{\partial z_s}{\partial x} A\,\mathrm{d}x$$

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→ Gravity Forces f_g:

$$\mathrm{d}f_g = \rho g \sin\theta\ \underbrace{\sigma\,\mathrm{d}x\,\mathrm{d}z}_{\mathrm{d}V}$$

Now integrate over the depth to get the force on the entire section:

$$f_g = \int_A \mathrm{d}f_g\ \mathrm{d}A = \rho g \sin\theta\,\mathrm{d}x \underbrace{\int_0^{z_s} \sigma\,\mathrm{d}z}_{A} = \rho g \sin\theta\,A\,\mathrm{d}x$$

If we define S_0 is the Bed Slope,

$$S_0 \equiv -\frac{dz_0}{dx} = \sin\theta \approx \theta$$

then we have

$$f_g = \rho g \, S_0 \, A \, \mathrm{d}x$$

→ Shear Forces f_s: 11

We define

$$P_w \equiv \text{ wetted perimeter}$$

and

$\tau_0 \equiv$ boundary shear stress $\qquad \tau_0 < 0$ on bottom faces of fluid elements

If assume that τ_0 is constant along P_w, then

$$f_s = -\tau_0 P_w \, dx$$

→ Acceleration ma_x: 12

The acceleration experienced by the differential mass $\rho \, dV$ is

$$\mathrm{d}\,(ma_x) = \rho \underbrace{\sigma \, dx \, dz}_{dV} \frac{\mathrm{d}u}{\mathrm{d}t} \qquad \text{where} \qquad \frac{du}{dt} \equiv \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \cancel{v \frac{\partial u}{\partial y}} + \cancel{w \frac{\partial u}{\partial z}}$$

The total acceleration, integrated over the depth is (see appendix):

$$ma_x = \left[\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left(\beta V^2 A \right) \right] \rho \, dx \tag{2}$$

where β is the momentum coefficient

$$\beta \equiv \frac{\int_A u^2 \, dA}{V^2 A}$$

We now can substitute expressions for f_p, f_g, f_s, and ma_x into the conservation of momentum relation 13

$$ma_x = f_p + f_g + f_s$$

The result is (after cancelling dx terms)

$$\rho \left[\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left(\beta V^2 A \right) \right] = -\rho g \cos\theta \frac{\partial z_s}{\partial x} A + \rho g \, S_0 A - \tau_0 P_w \tag{3}$$

The reader is left to show that this can be expressed equivalently as:

$$\boxed{\frac{1}{g} \frac{\partial V}{\partial t} + \frac{\partial H_\beta}{\partial x} = -\frac{\tau_0 P_w}{\rho g A} + (\beta - 1) \frac{V}{gA} \frac{\partial A}{\partial t} - \frac{V^2}{2g} \frac{\partial \beta}{\partial x}} \tag{4}$$

where

$$\boxed{H_\beta \equiv \beta \frac{V^2}{2g} + z_s \cos\theta + z_0 = \text{ total head}}$$

→ Equations 3 and 4 are two versions of the momentum conservation statement in open-channel flow, subject to the assumptions in Section 1.1. 14

2.3 Conservation of Energy

Multiplying the momentum equation by u before integrating over the depth leads to a (dependent) work-energy equation (see appendix for details).

$$\boxed{\frac{\beta}{g}\frac{\partial V}{\partial t} + \frac{\partial H_\alpha}{\partial x} = -\frac{\overline{\epsilon}}{\rho g Q} + (\alpha-\beta)\frac{V}{2gA}\frac{\partial A}{\partial t} - \frac{V}{2g}\frac{\partial \beta}{\partial t}} \tag{5}$$

where

$$\boxed{H_\alpha \equiv \alpha\frac{V^2}{2g} + z_s\cos\theta + z_0 = \text{total head}} \tag{6}$$

α is the energy coefficient

$$\alpha \equiv \frac{\int_A u^3\, dA}{V^3 A}$$

$\overline{\epsilon}$ is the viscous dissipation

$$\overline{\epsilon} \equiv -\int_0^{z_s} \sigma\tau\frac{\partial u}{\partial z}\,\mathrm{d}z$$

$\rightarrow$ ϵ is the rate of work done by internal shear forces. This rate of work cannot be converted back to mechanical energy, and is *dissipated* to heat. 15

3 Specific Energy

3.1 Basic Equation and Diagram

Definition

Specific Energy, E, is simply the total head *relative to the channel bottom*. It would be proper to call this quantity specific head instead of specific energy, the latter is used in common practice.

From the Conservation of Energy Equation (Eq. 6), the total head is

$$H_\alpha = z_0 + z_s\cos\theta + \alpha\frac{V^2}{2g} = \text{total head}$$

and so the total head relative to the bed $z = z_0$ is simply

$$\boxed{E \equiv d\cos\theta + \alpha\frac{V^2}{2g}} \tag{7}$$

where we shall now refer to the flow depth as

$$d \equiv z_s$$

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Prototype Flow for Examples

- Rectangular cross section: $A = Bd$
- Horizontal bed: $\theta = 0$
- Uniform velocity: $\alpha = 1$

Simplified Specific Energy Equation
For prototype flow, we have

$$E = d + \frac{V^2}{2g}$$

or, defining specific discharge as $q \equiv Q/B$

$$E = d + \frac{q^2}{2gd^2}$$

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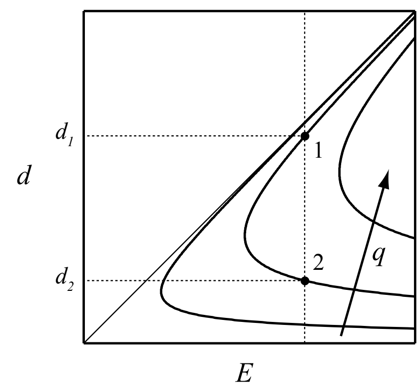

- Two possible flow depths d for a given value of E (but 3 roots to Eqn.)
- Curve shifts up and to the right as q increases

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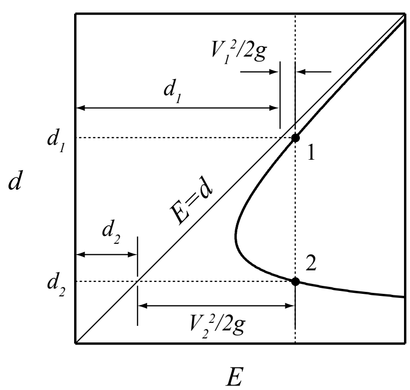
$$E = d + \frac{V^2}{2g}$$



- Upper branch (pt. 1) corresponds to deeper, slower flows
- Upper branch (pt. 2) corresponds to shallower, faster flows

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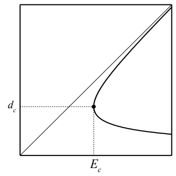
For any given specific discharge q, there is a minimum specific energy



Setting $\dfrac{\partial E}{\partial d} = 0$ leads to:

$$d_c = \left(\frac{q^2}{g}\right)^{1/3} = \frac{2}{3}E_c$$

$$E_c = \frac{3}{2}\left(\frac{q^2}{g}\right)^{1/3}$$

$$\frac{V_c^2}{2g} = \frac{d_c}{2}$$

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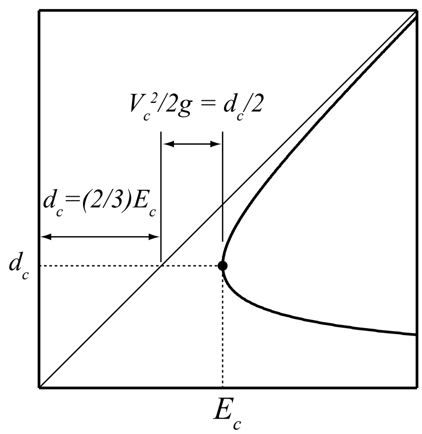

- Critical depth d_c corresponds to minimum specific Energy E_c
- For a given q, no possible flows corresponding to $E < E_c$

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The Froude Number

Since

$$\frac{V_c^2}{2g} = \frac{d_c}{2}$$

we have

$$V_c = \sqrt{g d_c}$$

We now define the Froude number Fr

$$\text{Fr} \equiv \frac{V}{\sqrt{gd}}$$

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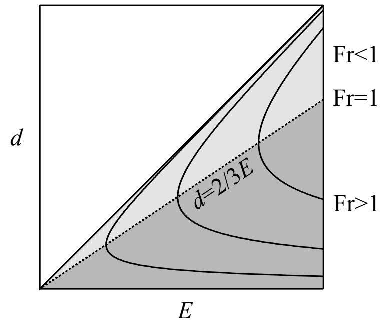

- Fr < 1 Flow is subcritical, $d > d_c$, $V < V_c$
- Fr $= 1$ Flow is critical, $d = d_c$, $V = V_c$
- Fr > 1 Flow is supercritical, $d > d_c$, $V > V_c$

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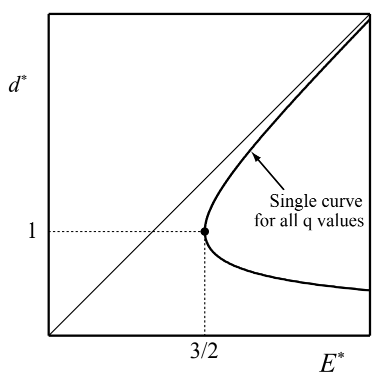
Non-Dimensional Specific Energy Equation

Combining

$$E = d + \frac{q^2}{2gd^2} \quad \text{with} \quad d_c = \left(\frac{q^2}{g}\right)^{1/3}$$

we can write:

$$E^* = d^* + \frac{1}{2d^{*2}} \quad \text{where} \quad E^* \equiv \frac{E}{d_c} \quad , \quad d^* \equiv \frac{d}{d_c}$$



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We can rewrite the non-dimensional specific energy equation

$$E^* = d^* + \frac{1}{2d^{*2}}$$

as

$$d^{*3} - e^* d^{*2} + \frac{1}{2} = 0$$

which is cubic in d^*, with two positive roots:

$$\begin{aligned} d^*_{\text{subcritical}} &= \frac{E^*}{3}\left[1 + 2\cos\left(\frac{\Gamma}{3}\right)\right] \\ d^*_{\text{supercritical}} &= \frac{E^*}{3}\left[1 + 2\cos\left(\frac{\Gamma}{3} + \frac{4\pi}{3}\right)\right] \end{aligned}$$

where

$$\Gamma \equiv \text{acos}\left[1 - \frac{27}{4}E^{*-3}\right] \quad \text{and} \quad E^* \geq \frac{3}{2}$$

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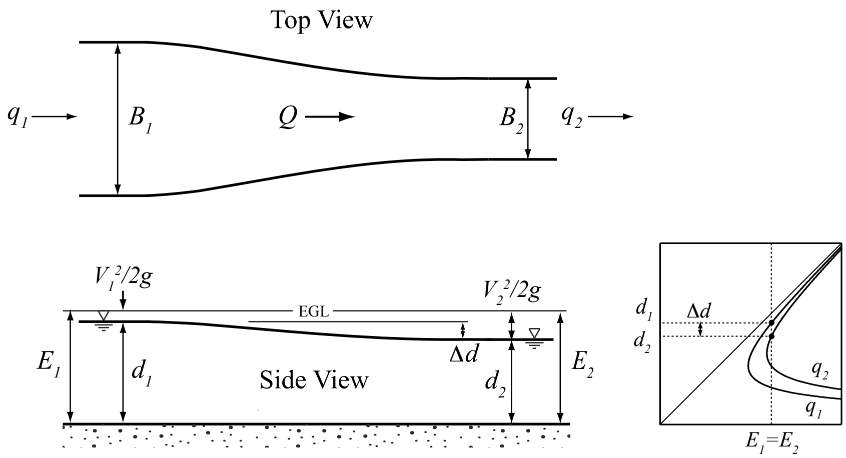
Example 1: Change in channel width $(B_2 < B_1 \;\rightarrow\; q_2 > q_1)$



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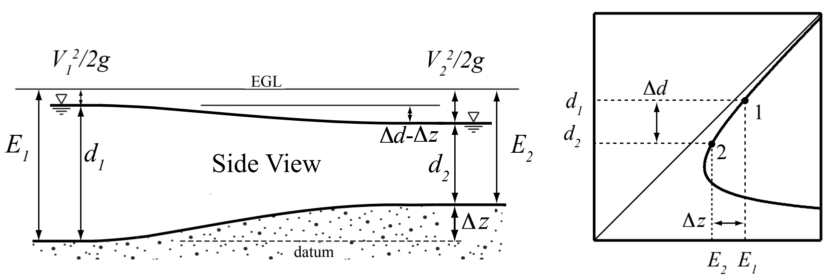
Example 2: Change in bed height $(E_1 = E_2 + \Delta z)$



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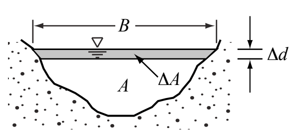
Non-rectangular channel sections

We now consider the more general case of a non-rectangular channel, with non-zero bed slope ($\theta \neq 0$) and non-uniform velocity ($\alpha \neq 1$).



Can't use specific discharge q, since the channel width varies with depth. Instead, we write the Specific Energy equation in terms of Q and A:

$$E \;=\; d\cos\theta \;+\; \alpha\frac{V^2}{2g} \;=\; d\cos\theta \;+\; \alpha\frac{Q^2}{2gA^2}$$

Critical flow occurs when $\partial E/\partial d = 0$:

$$\frac{\partial E}{\partial d} = \cos\theta - \alpha\frac{Q^2}{gA^3}\frac{dA}{dd} = 0$$

Since $\Delta A = B\Delta d$, $dA/dd = B$, and

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$$\frac{\partial E}{\partial d} = \cos\theta - \alpha\frac{BQ^2}{gA^3} = 0 \tag{8}$$

which can be re-written

$$\cos\theta - \alpha\frac{1}{g}\frac{B}{A}\left(\frac{Q}{A}\right)^2 = \cos\theta - \alpha\frac{1}{g}\frac{B}{A}{V_c}^2 = 0$$

This leads to the following more general relationships for V_c and Fr:

$$V_c = \sqrt{g\cos\theta D/\alpha}$$

$$\mathrm{Fr} \equiv \frac{V}{V_c} = \frac{V}{\sqrt{g\cos\theta D/\alpha}} \tag{9}$$

where D is the "hydraulic depth", defined as $D \equiv \frac{A}{B}$ 29

The critical depth d_c associated with V_c can be determined from Eq. 8:

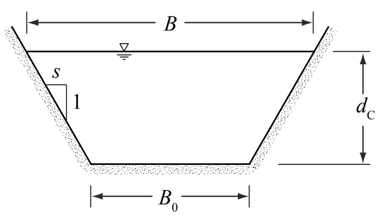
$$\frac{\partial E}{\partial d} = \cos\theta - \alpha\frac{BQ^2}{gA^3} = 0$$

or

$$\alpha\frac{BQ^2}{g\cos\theta A^3} = 1 \tag{10}$$

By rewriting A and B in terms of d_c, Eq. 10 can be solved numerically or graphically for d_c. 30

Critical Depth Example: Trapezoidal Channel



$$\begin{aligned} A &= B_0 d_C + s d_C^2 \\ B &= B_0 + 2 s d_C \end{aligned}$$

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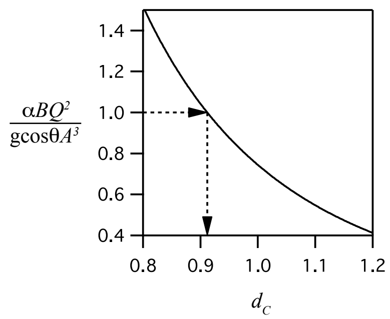
Channel and Flow Parameters:
$\alpha = 1$
$Q = 30\ \mathrm{m^3/s}$
$S_0 = 0.001 \quad \Rightarrow \quad \cos\theta \approx \cos(S_0) \approx 1$
$B_0 = 10$ m
$s = 2$

Solve $\alpha \dfrac{BQ^2}{g \cos\theta A^3} = 1$



solve graphically $\rightarrow d_C = 0.912$ m 32

3.2 Control Sections and Chokes

Note: We go back to simple rectangular channels for the examples in this section.

Control Section Definition
A Control Section is defined as a location in a channel flow where there is a unique relationship between discharge and depth. The flow at the control is critical (Fr = 1). These locations serve as boundary conditions for numerical simulations of open-channel flow.

Choke Definition
A Choke is a Control Section that influences the upstream flow conditions.

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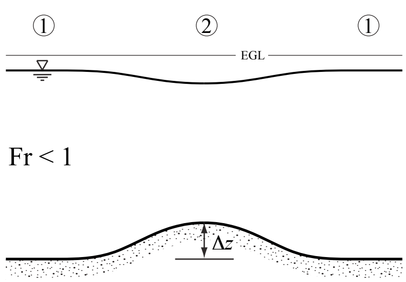
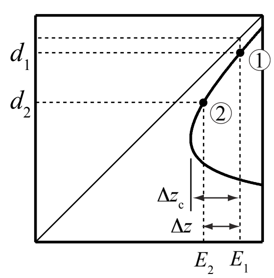
Flow over a bump with height Δz.





$\rightarrow$ Note that while q is constant, E changes due to the change in bed height.

How high must the bump be for the flow to become critical at section 2?

$$\Delta z = E_1 - E_2 \quad \rightarrow \quad \Delta z_c = E_1 - E_c$$

where, from before, we have

$$E_c = \frac{3}{2}\left(\frac{q^2}{g}\right)^{1/3}$$

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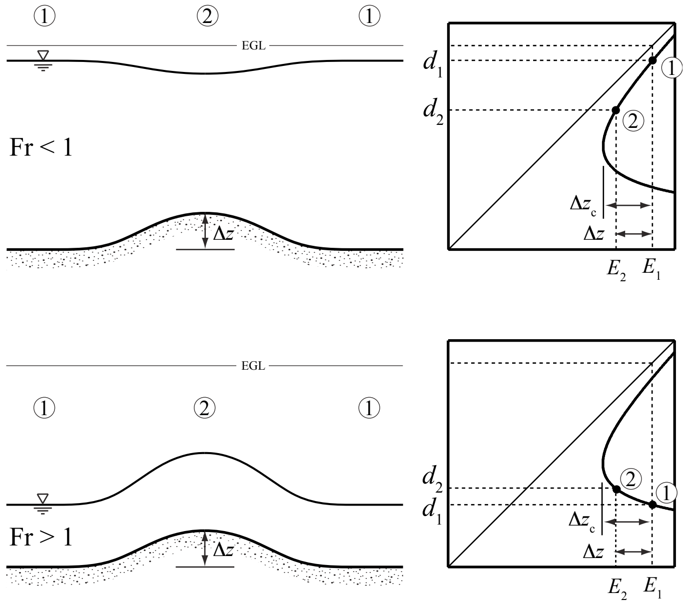
Flow over a bump, $\Delta z < \Delta z_c$



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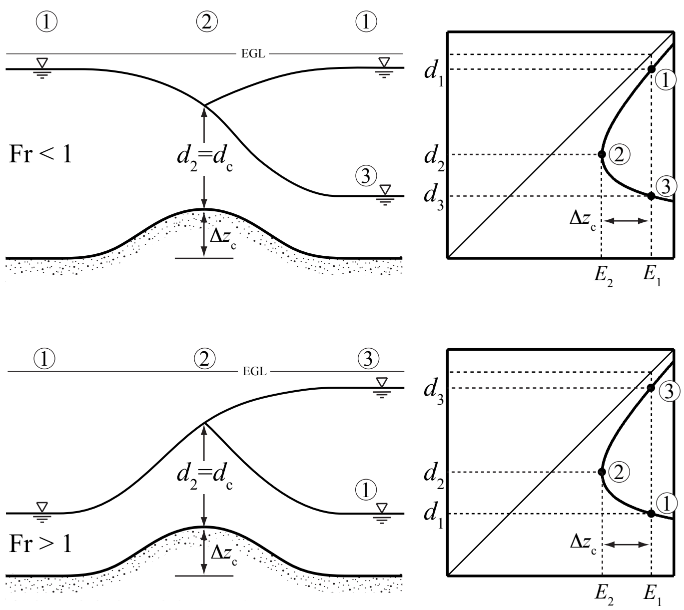
Flow over a bump, $\Delta z = \Delta z_c$:   A "Control Section"



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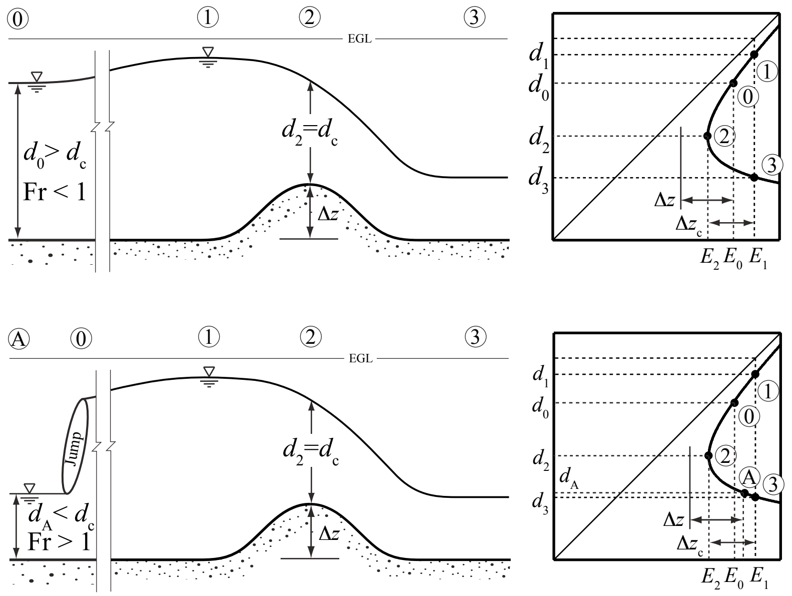
Flow over a bump, $\Delta z > \Delta z_c$: **A "Choke"**



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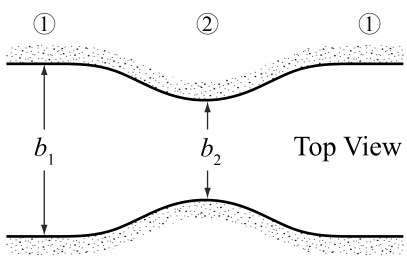
Flow through a width constriction where $b_2 < b_1$

Example:



Note that while E is constant, q changes due to the change in channel width.

How narrow must the constriction be for critical flow at section 2?

$$E_1 = E_c = \frac{3}{2} d_c \qquad \text{where, from before,} \qquad d_c = \left(\frac{q^2}{g}\right)^{1/3} \quad , \quad q \equiv Q/b_c$$

Combining gives

$$b_c = \left(\frac{3}{2}\right)^{3/2} \frac{Q}{\sqrt{g E_1{}^3}}$$

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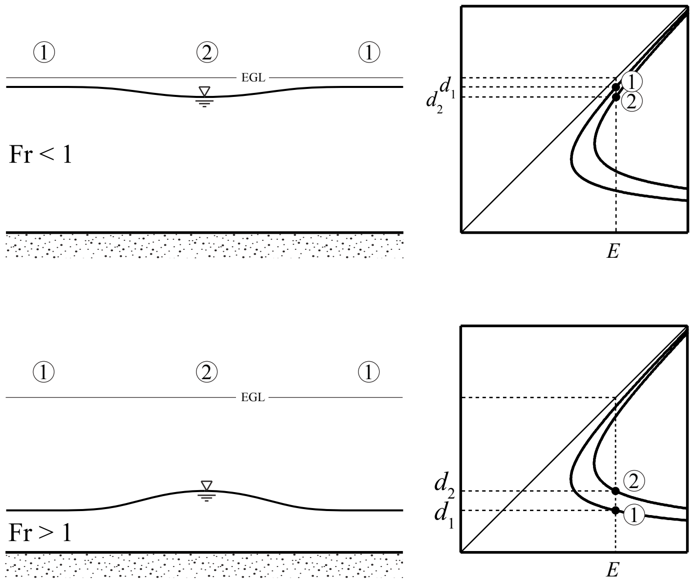
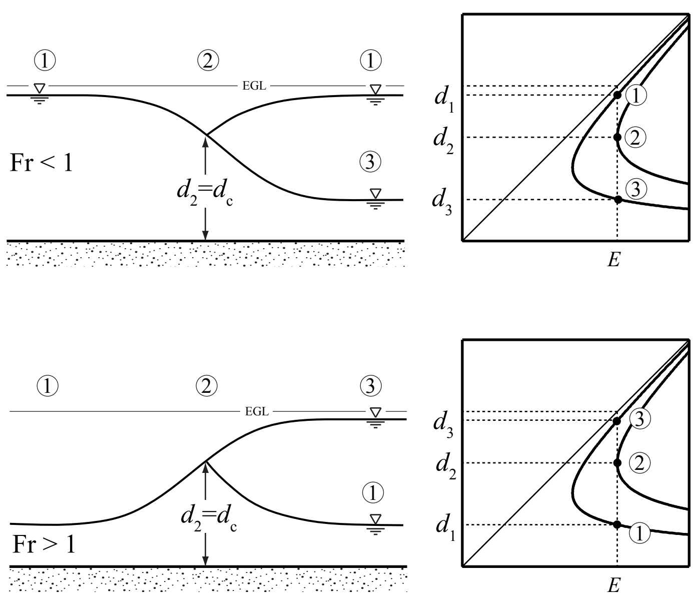
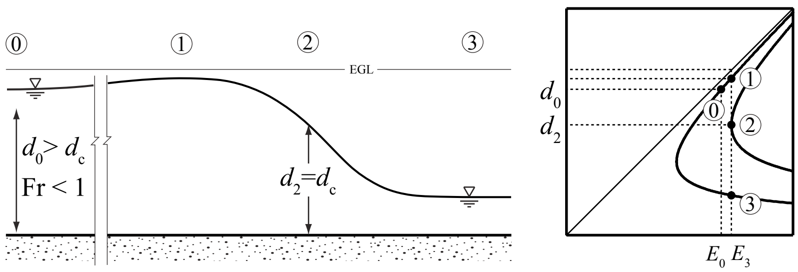
Flow through a constriction, $b_2 > b_c$

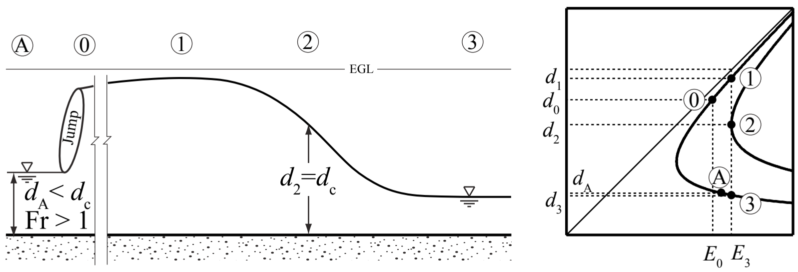


Flow through a constriction, $b_2 = b_c$ **:** **A "Control Section"**



Flow through a constriction, $b_2 < b_c$ **:** **A "Choke"**





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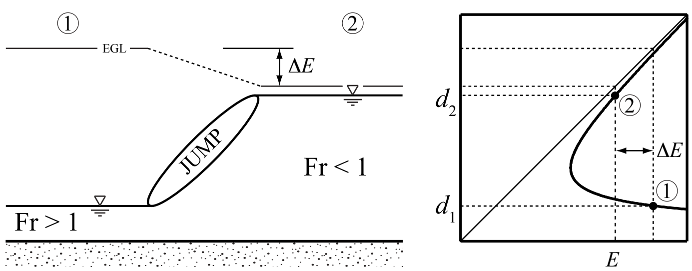
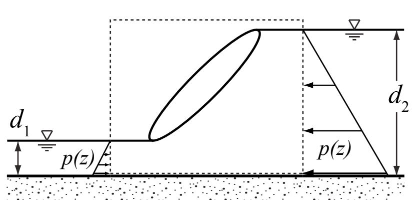
3.3 Hydraulic Jump

Definition
A supercritical flow can change abruptly to a subcritical flow through a feature known as a hydraulic jump. The jump itself is turbulent, resulting in significant energy loss ΔE across the jump.



The hydraulic 42
jump can be analyzed using a control volume and a simple algebraic form of the momentum equation (you used this equation in your undergraduate fluids course):

$$\Sigma F_{\text{ext}} = \rho Q \left[V_2 - V_1 \right]$$



Assuming a hydrostatic pressure distribution up and downstream of the jump, we have:

$$\Sigma F_{\text{ext}} = \frac{1}{2}\rho g d_1^2 B - \frac{1}{2}\rho g d_2^2 B = \rho Q \left[V_2 - V_1 \right]$$

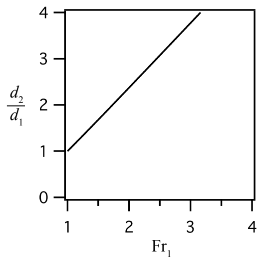
We can combine this with the continuity equation

$$V_1 d_1 B = V_2 d_2 B$$

and rearrange to get an expression for the ratio of depths across the jump: 43

Ratio of depths across a jump

$$\frac{d_2}{d_1} = \frac{1}{2}\left[\sqrt{1+8\mathrm{Fr}_1^2} - 1\right] \qquad \mathrm{Fr} \geq 1$$



The depth ratio across the jump is essentially linear with Fr. 44

The specific energy equation across the jump is

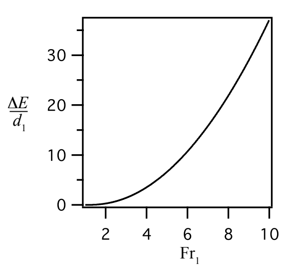
$$E_1 = E_2 + \Delta E \quad \text{where} \quad E_1 = d_1 + \frac{q^2}{2g d_1^2} \quad , \quad E_2 = d_2 + \frac{q^2}{2g d_2^2}$$

Solving for ΔE gives

$$\Delta E = \frac{(d_2 - d_1)^3}{4 d_1 d_2}$$

This can be rearranged to get a nondimensional expression for the energy loss across a jump:

$$\frac{\Delta E}{d_1} = \frac{1}{16}\frac{\left[\sqrt{1+8\mathrm{Fr}_1^2} - 3\right]^3}{\left[\sqrt{1+8\mathrm{Fr}_1^2} - 1\right]}$$



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Types of Hydraulic Jumps

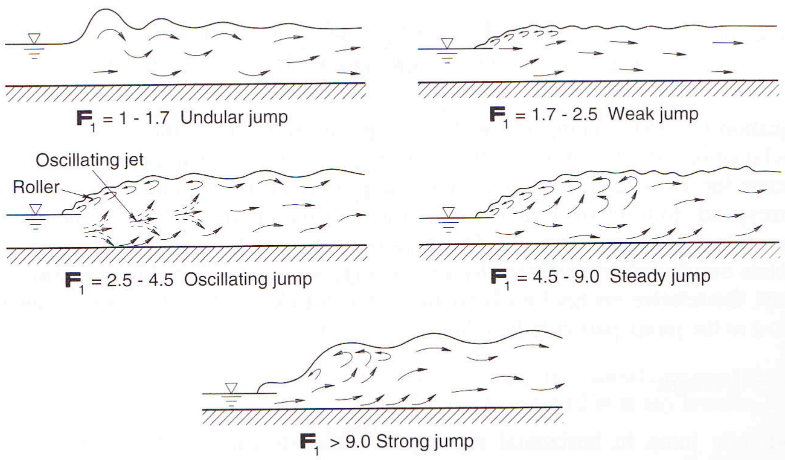

Image from Jain (2001), p. 304. 46

4 Steady Uniform Flow

4.1 Basic Equations

Steady Flow
Open-channel flow is steady when the two unknown flow quantities V and d do not vary with time t.

Uniform Flow
Open-channel flow is uniform when V and d do not vary spatially with x.

For steady, uniform flow, the continuity equation (Eq. 1)

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

reduces to simply:

$$\boxed{Q = VA = \text{constant}}$$

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For steady, uniform flow, the momentum equation (Eqs. 4 and 14)

$$\frac{1}{g}\frac{\partial V}{\partial t} + \frac{\partial H_\beta}{\partial x} = -\frac{\tau_0 P_w}{\rho g A} + (\beta - 1)\frac{V}{gA}\frac{\partial A}{\partial t} - \frac{V^2}{2g}\frac{\partial \beta}{\partial x}$$

$$H_\beta = \beta\frac{V^2}{2g} + z_s \cos\theta + z_0 = \text{total head}$$

reduces to simply:

$$\frac{dz_0}{dx} = -\frac{\tau_0}{\gamma R_h}$$

where $\gamma \equiv \rho g$, and $R_h \equiv A/P_w$ is the Hydraulic Radius of the channel section.

⇒ Simple force balance between gravity and viscous shear. 48

Recalling that the bed slope S_0 is defined as

$$S_0 \equiv -dz_0/dx$$

and if we define the friction slope S_f as

$$S_f \equiv \frac{\tau_0}{\gamma R_h}$$

then the momentum equation can be expressed

$$\boxed{S_0 = S_f}$$

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For steady, uniform flow, the energy equation (Eqs. 5 and 6)

$$\frac{\beta}{g}\frac{\partial V}{\partial t} + \frac{\partial H_\alpha}{\partial x} = -\frac{\bar{\epsilon}}{\rho g Q} + (\alpha - \beta)\frac{V}{2gA}\frac{\partial A}{\partial t} - \frac{V}{2g}\frac{\partial \beta}{\partial t}$$

$$H_\alpha = \alpha\frac{V^2}{2g} + z_s \cos\theta + z_0 = \text{total head}$$

reduces to simply:

$$S_0 = S_e$$

where

$$S_e \equiv \frac{\bar{\epsilon}}{\gamma Q}$$

⇒ Rate of work done by gravity equals the rate of viscous dissipation of energy So, for steady, uniform flow, the momentum and energy equations 50
imply

$$\boxed{S_0 = S_f = S_e}$$

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4.2 Flow Resistance

We expect the wall shear stress τ_0 to have a functional dependence:

$$\tau_0 = \phi(V, \rho, g, \nu, R_h, k, \xi)$$

where ϕ is some nondimensional function of :

- V is the flow velocity $[L\,T^{-1}]$
- ρ is the fluid density $[M\,L^{-3}]$
- g is the gravitational constant $[L\,T^{-2}]$
- ν is the fluid density $[L^2\,T^{-1}]$
- R_h is the channel hydraulic radius $[L]$
- k is the channel roughness scale $[L]$
- ξ is a nondimensional channel shape factor $[-]$

Dimensional analysis produces 4 Π groups, letting us write:

$$\frac{V}{\sqrt{\tau_0/\rho}} = \phi\left(\frac{VR_h}{\nu}, \frac{V}{\sqrt{R_h g}}, \frac{k}{R_h}\right)$$

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Material	k (mm)
Cement	0.30-1.2
Concrete	0.50-3.0
Gravel	5
Boulders	500

Using the momentum equation $S_0 = -\tau_0/\gamma R_h$, we can rewrite the left-hand-side as

$$\frac{V}{\sqrt{\tau_0/\rho}} = \frac{1}{\sqrt{g}} \frac{V}{\sqrt{R_h S_0}}$$

giving

$$\frac{V}{\sqrt{R_h S_0}} = \sqrt{g}\, \phi \left(\frac{V R_h}{\nu}, \frac{V}{\sqrt{R_h g}}, \frac{k}{R_h} \right)$$

If we now replace $\sqrt{g}\,\phi$ by a single variable C, we arrive at the Chézy formula

$$\boxed{V = C\sqrt{R_h S_0}}$$

where C is a dimensional coefficient $[L^{1/2}T^{-1}]$ that depends functionally on

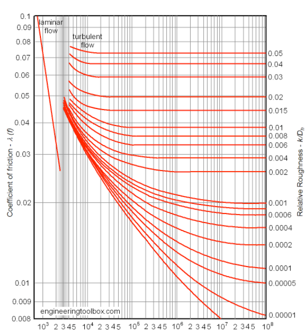
$$\frac{V R_h}{\nu}, \frac{V}{\sqrt{R_h g}}, \text{ and } \frac{k}{R_h}$$

⇒ So how do you get a value for C? 53

Method 1: Darcy-Weisbach

$$C = \sqrt{\frac{8g}{f}}$$

where f is the Darcy-Weisbach friction factor. For circular sections, f can be obtained from the moody diagram:



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Method 2: ASCE (1963)

$$C = 4\sqrt{2g}\log_{10}\left(\frac{12R_h}{k}\right)$$

where k values can be obtained from tables, eg:

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Material	n
Cement	0.011
Concrete	0.015
Gravel	0.022

Method 3: Manning's n

$$
\begin{aligned}
C &= \frac{R_h^{1/6}}{n} \quad \text{(SI units)} \\
C &= 1.49\frac{R_h^{1/6}}{n} \quad \text{(English units)}
\end{aligned}
$$

where n values can be obtained from tables, eg:

⇒ Method choice depends on your employer, previous studies, etc.
⇒ We will use Manning's n (and SI units) in the following examples. 56

4.3 Normal Depth

Definition
For uniform flow, the flow depth and velocity are called the normal depth, d_n and normal velocity V_n, respectively.

If discharge, Q, bed slope, S_0, and Manning's n are known, the normal depth d_n can be determined as follows:

The Chézy equation written in terms of Manning's n is

$$V = \underbrace{\frac{R_h^{1/6}}{n}}_{C} \sqrt{R_h S_0} = \frac{1}{n} R_h^{2/3} S_0^{1/2}$$

Combining this with continuity $Q = VA$ and rearranging gives

$$AR_h^{2/3} = \frac{nQ}{S_0^{1/2}} \tag{11}$$

The left hand side of this equation (the "Section Factor") can be written in terms of d_n (depending on channel geometry), and the equation solved for d_n.

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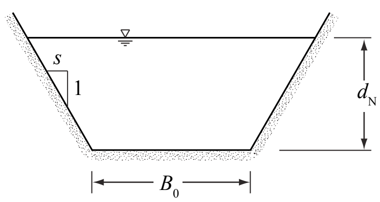
Normal Depth Example: Trapezoidal Channel



$$\begin{aligned} A &= B_0 d_N + s d_N^2 \\ P &= B_0 + 2\sqrt{s^2 d_N^2 + d_N^2} \end{aligned}$$

So the section factor is

$$AR_h^{2/3} = A\left(\frac{A}{P}\right)^{2/3} = \frac{A^{5/3}}{P^{2/3}} = \frac{\left[d_N\left(B_0 + s d_N\right)\right]^{5/3}}{\left[B_0 + 2 d_N \sqrt{s^2+1}\right]^{2/3}}$$

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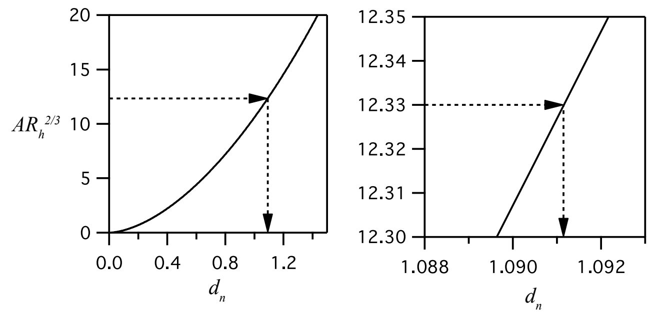
Channel and Flow Parameters:
$n = 0.013$
$Q = 30$ m^3/s
$S_0 = 0.001$
$B_0 = 10$ m
$s = 2$

$$\Rightarrow \frac{nQ}{S_0^{1/2}} = 12.33 \text{ m}^3/\text{s}$$

Solve graphically for d_N: Plot section factor $AR_h^{2/3}$ vs. d_N



$\Rightarrow$ $d_N = 1.09$ m $\qquad$ $\Rightarrow$ $V_N = Q/A = 2.26$ m/s 59

5 Gradually Varied Flow

We now consider steady flow with gradual streamwise variations in depth. Streamline curvature is small, so pressure distributions remain hydrostatic.

5.1 Basic Equations

For steady, gradually varied flow, the continuity equation (Eq. 1)

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

again reduces to simply:

$$\boxed{Q = VA = \text{constant}}$$

For steady, gradually varied flow, the momentum equation (Eqs. 4 and 60
14)

$$\frac{1}{g}\frac{\partial V}{\partial t} + \frac{\partial H_\beta}{\partial x} = -\underbrace{\frac{\tau_0 P_w}{\rho g A}}_{S_f} + (\beta - 1)\frac{V}{gA}\frac{\partial A}{\partial t} - \frac{V^2}{2g}\frac{\partial \beta}{\partial x}$$

$$H_\beta = \beta\frac{V^2}{2g} + z_s\cos\theta + z_0 = \text{total head}$$

reduces to:

$$\frac{\partial H_\beta}{\partial x} = -S_f - \frac{V^2}{2g}\frac{\partial \beta}{\partial x}$$

For steady, gradually varied flow, the energy equation (Eqs. 5 and 6) 61

$$\frac{\beta}{g}\frac{\partial V}{\partial t} + \frac{\partial H_\alpha}{\partial x} = -\underbrace{\frac{\bar{\epsilon}}{\rho g Q}}_{S_e} + (\alpha - \beta)\frac{V}{2gA}\frac{\partial A}{\partial t} - \frac{V}{2g}\frac{\partial \beta}{\partial t}$$

$$H_\alpha = \alpha\frac{V^2}{2g} + z_s\cos\theta + z_0 = \text{total head}$$

reduces to:

$$\frac{\partial H_\alpha}{\partial x} = -S_e$$

The momentum and energy equations 62

$$\frac{\partial H_\beta}{\partial x} = -S_f - \frac{V^2}{2g}\frac{\partial \beta}{\partial x}$$
$$\frac{\partial H_\alpha}{\partial x} = -S_e$$

are equivalent in the case where $\alpha = \beta =$ constant and $S_f = S_e$.

→ The governing equation for GVF commonly used in engineering practice is a hybrid of these two equations:

$$\boxed{\frac{\partial H_\alpha}{\partial x} = -S_f} \tag{12}$$

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In order to express S_F in terms of flow variables, it is common practice to approximate S_F by rewriting the Manning Eqn.

$$V = \frac{1}{n} R_h^{2/3} S_0^{1/2}$$

as

$$V = \frac{1}{n} R_h^{2/3} S_F^{1/2}$$

or

$$\boxed{S_F = \frac{n^2 V^2}{R_h^{4/3}}} \tag{13}$$

Equations 12 and 13 are the basis for computation of GVF free-surface profiles.

⇒We will now develop two convenient forms of Eq 12 for computing free surface profiles in GVF in prismatic channel reaches ($\partial A/\partial x = 0$). 64

In Eq.7, we defined specific energy E as the head relative to the bed z_0:

$$E \equiv H_\alpha - z_0 \quad \Rightarrow \quad H_\alpha = E + z_0$$

So,

$$\frac{\partial H_\alpha}{\partial x} = \frac{\partial E}{\partial x} + \frac{\partial z_0}{\partial x} = \frac{\partial E}{\partial x} - S_0$$

And so we can re-write

$$\frac{\partial H_\alpha}{\partial x} = -S_f$$

as:

First Form of GVF Equation for Computations

$$\boxed{\frac{\partial E}{\partial x} = S_0 - S_f} \tag{14}$$

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Now use continuity $Q = VA$ to rewrite

$$H_\alpha = \alpha\frac{V^2}{2g} \;+\; z_s\cos\theta \;+\; z_0 \;=\; \alpha\frac{Q^2}{2gA^2} \;+\; z_s\cos\theta \;+\; z_0$$

then the GVF governing equation becomes

$$\frac{dH_\alpha}{dx} \;=\; \alpha\frac{Q^2}{2g}\underbrace{\frac{d}{dx}\left(\frac{1}{A^2}\right)}_{\Downarrow} \;+\; \cos\theta\frac{dz}{dx} + \underbrace{\frac{dz_0}{dx}}_{-S_0} \;=\; -S_f$$

$$\overbrace{\frac{d}{dx}\left(\frac{1}{A^2}\right)} = \frac{d}{dA}\left(\frac{1}{A^2}\right)\underbrace{\frac{dA}{dx}}_{\Downarrow} = -\frac{2}{A^3}\frac{dA}{dx}$$

$$\overbrace{\frac{dA}{dx}} = \cancel{\frac{\partial A}{\partial x}} + \underbrace{\frac{\partial A}{\partial z}}_{B}\frac{dz}{dx}$$

Combining gives:

$$\alpha\frac{Q^2}{2g}\left(-\frac{2B}{A^3}\frac{dz}{dx}\right) \;+\; \cos\theta\frac{dz}{dx} \;=\; S_0 - S_f$$

Rearranging gives: 66

$$\cos\theta\frac{dz}{dx}\left[1 - \frac{\alpha BQ^2}{gA^3\cos\theta}\right] \;=\; S_0 - S_f$$

or

$$\cos\theta\frac{dz}{dx}\left[1 - \underbrace{\frac{V^2}{g\cos\theta D/\alpha}}_{\text{Fr}^2\text{- See Eq. 9}}\right] \;=\; S_0 - S_f \qquad\qquad D \;\equiv\; \frac{A}{B}$$

So we can write

$$\frac{dz}{dx} \;=\; \frac{1}{\cos\theta}\frac{S_0 - S_f}{1 - \text{Fr}^2}$$

and since, for most practical problems, $\cos\theta \approx 1$, we have

Second Form of GVF Equation for Computations

$$\boxed{\frac{dz}{dx} \;=\; \frac{S_0 - S_f}{1 - \text{Fr}^2}} \qquad (15)$$

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5.2 Slope Classification

Steep Slope
$d_N < d_C$, supercritical uniform flow

Critical Slope
$d_N = d_C$, critical uniform flow

Mild Slope

$d_N > d_C$, subcritical uniform flow

Horizontal Slope
$\theta = 0 \quad (d_N = \infty)$

Adverse Slope
$dz_0/dx > 0$

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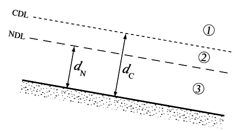
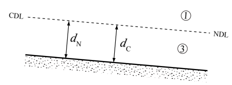
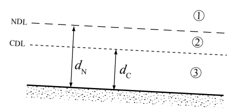
Steep



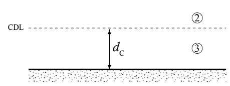
Critical



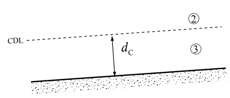
Mild



Horizontal



Adverse



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Flow Profile Summary

$$\frac{dz}{dx} = \frac{S_0 - S_f}{1 - \text{Fr}^2}$$

Profile	Depths	Slopes	Fr	dz/dx
S1	$d > d_C > d_N$	$S_f < S_0$	< 1	$+$
S2	$d_C > d > d_N$	$S_f < S_0$	> 1	$-$
S3	$d_C > d_N > d$	$S_f > S_0$	> 1	$+$
C1	$d > d_N = d_C$	$S_f < S_0$	< 1	$+$
C3	$d_N = d_C > d$	$S_f > S_0$	> 1	$+$
M1	$d > d_N > d_C$	$S_f < S_0$	< 1	$+$
M2	$d_N > d > d_C$	$S_f > S_0$	< 1	$-$
M3	$d_N > d_C > d$	$S_f > S_0$	> 1	$+$
H2	$d_N > d > d_C$	$S_f > S_0$	< 1	$-$
H3	$d_N > d_C > d$	$S_f > S_0$	> 1	$+$
A2	$d > d_C$	$S_f > S_0$	< 1	$-$
A3	$d_C > d$	$S_f > S_0$	> 1	$+$

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Asymptotic behavior of Flow Profiles

$$\frac{dz}{dx} = \frac{S_0 - S_f}{1 - \text{Fr}^2}$$

$$\text{as} \quad d \to d_N \quad S_f \to S_0 \quad \tfrac{dd}{dx} \to 0$$

$\Rightarrow$ When the free surface approaches the NDL, it does so asymptotically.

$$\text{as} \quad d \to d_C \quad \text{Fr} \to 1 \quad \tfrac{dd}{dx} \to \infty$$

$\Rightarrow$ When the free surface approaches the CDL, it does so steeply. (Slope never gets vertical - hydrostatic pressure assumption breaks down first).

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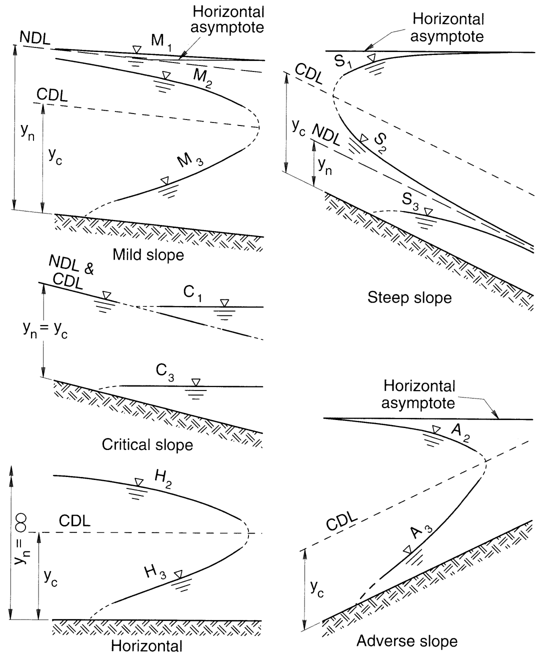
Surface Profiles in GVF



Source: Jain (2001)

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5.3 Sketching Profiles

To be done on the board in lecture... 73

5.4 Numerical Computation of Profiles

Direct Step Method
In this method, the depth is changed incrementally, and the resulting reach associated with that change in depth is calculated.

Starting with Eqn. 14

$$\frac{\partial E}{\partial x} = S_0 - S_f$$

and discretizing over a short channel reach Δx gives:

$$\Delta x = \frac{\Delta E}{S_0 - \overline{S_F}}$$

In finite-difference form, we have teh direct step implementation: 74

$$x_i = x_{i-1} + \frac{E_i - E_{i-1}}{S_0 - \overline{S_F}}$$

where

$$E_i = d_i + \alpha \frac{V_i^2}{2g}$$

$$\overline{S_F} \equiv \frac{1}{2}\left(S_{F[i]} + S_{F[i-1]}\right)$$

and S_F is calculated using Manning's n (Eq.13):

$$S_{F[i]} = \left(\frac{n^2V^2}{R_h^{4/3}}\right)_{[i]}$$

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Algorithm for Direct Step Method
$\Rightarrow$ Given Q $\Rightarrow$ Start at control location x_0 where depth d_0 is known $\Rightarrow$ Integrate upstream for subcritical flow and downstream for super-critical flow

- At step i and depth d_i
- Calculate A_i and R_{hi}
- Calculate $V_i = Q/A_i$
- Calculate $S_{F[i]} = (n_iV_i)^2/{R_h}_i^{4/3}$
- Calculate $E_i = d_i + \alpha\frac{V_i^2}{2g}$
- Using information from step $[i]$ and $[i-1]$, Compute

$$x_i = x_{i-1} + \frac{E_i - E_{i-1}}{S_0 - \frac{1}{2}\left(S_{F[i]} + S_{F[i-1]}\right)}$$

- Increment i and d_i
- Iterate until $d - d_N$ is sufficiently small

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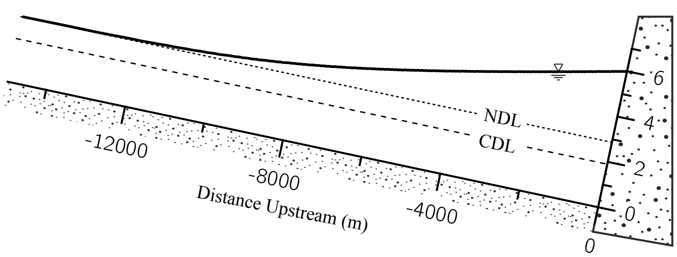
Example of Computed Surface Profile: Backwater Curve
Given: Trapezoidal Section $B_0 = 5$ m $s = 1$ $n = 0.013$ $Q = 50$ m^3/s $S_0 = 0.0004$ Computed Values:

$$\Rightarrow\ d_N = 2.87 \text{ m}$$

$$\Rightarrow\ d_C = 1.90 \text{ m}$$

If a dam at $x = 0$ backs up the water to a depth of 6 m, compute the upstream depth profile.

$\Rightarrow$ M1 Curve



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6 Appendix: Derivation of the Saint-Venant Equations

6.1 Conservation of Mass

The integral form of the conservation of mass law for a fixed volume V is

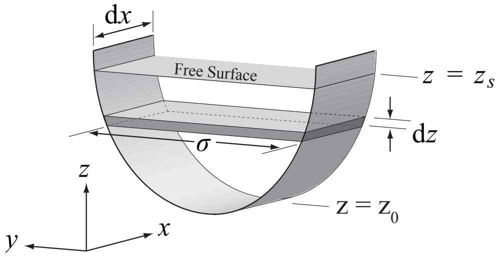
$$\int_V \frac{\partial\rho}{\partial t}dV = -\int_A \rho\vec{u}\cdot d\vec{A}$$

where $d\vec{A} = \hat{n}\,dA$. For incompressible flows (ρ = const) this reduces to simply

$$\int_A \rho\vec{u}\cdot d\vec{A} = 0$$

The total flow of mass across the entire volume boundary must sum to zero. 78

Consider a fixed volume spanning the cross-section of an open channel:



The conservation of mass equation can be written as the sum of mass fluxes across each of the six faces on the differential volume $\sigma \times dx \times dz$:

$$\int_{\text{front}} \rho\vec{u}\cdot d\vec{A} \;+\; \int_{\text{back}} \rho\vec{u}\cdot d\vec{A} \;+\; \int_{\text{top}} \rho\vec{u}\cdot d\vec{A} \;+\; \int_{\text{bottom}} \rho\vec{u}\cdot d\vec{A} \;+\; \int_{\text{sides}} \rho\vec{u}\cdot d\vec{A} = 0$$

where

$$\begin{aligned}
\int_{\text{front}} \rho\vec{u}\cdot d\vec{A} &= -\rho\, u\sigma\, dz \\
\int_{\text{back}} \rho\vec{u}\cdot d\vec{A} &= \rho\left[\left(u + \frac{\partial u}{\partial x}dx\right)\left(\sigma + \frac{\partial \sigma}{\partial x}dx\right)\right]dz = \rho\left[u\sigma + \frac{\partial}{\partial x}(u\sigma)dx\right]dz \\
\int_{\text{top}} \rho\vec{u}\cdot d\vec{A} &= \int_{\text{bottom}} \rho\vec{u}\cdot d\vec{A} \;=\; 0 \\
\int_{\text{sides}} \rho\vec{u}\cdot d\vec{A} &= 0
\end{aligned}$$

The mass balance on the boundary of the differential volume $\sigma \times dx \times dz$ becomes: 79

$$\int \rho\vec{u}\cdot d\vec{A} \;=\; -\rho\, u\sigma\, dz \;+\rho\left[u\sigma + \frac{\partial}{\partial x}(u\sigma)dx\right]dz \;=\; 0$$

Which reduces to

$$\frac{\partial}{\partial x}(u\sigma) \;=\; 0 \qquad (16)$$

and the *total* mass balance is found by integrating the previous expression over the depth from $z = 0$ to $z = z_s$

$$\int_0^{z_s} \frac{\partial}{\partial x}(u\sigma)\; dz \;=\; 0$$

Note that $z_s = z_s(x,t)$. The x-dependence means we cannot simply swap the order of integration and differentiation. However, we can take the derivative outside the integral by using a rearranged form of the 1-D Leibnitz rule

$$\int_{a(x,t)}^{b(x,t)} \frac{\partial f(x,t)}{\partial x}\, dz = \frac{\partial}{\partial x}\int_{a(x,t)}^{b(x,t)} f(x,t)\, dz - \frac{\partial b}{\partial x} f(b) + \frac{\partial a}{\partial x} f(a)$$

$$\int_0^{z_s} \frac{\partial}{\partial x}(u\sigma)\; dz \;=\; \frac{\partial}{\partial x}\int_0^{z_s} u\sigma\, dz \;-\; \frac{\partial z_s}{\partial x}\,[u\sigma]_{z=z_s} \;=\; 0$$

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$$\underbrace{\frac{\partial}{\partial x}\int_0^{z_s} u\,\overbrace{\sigma dz}^{dA}}_{=Q} - \frac{\partial z_s}{\partial x}\underset{\substack{\uparrow\\ u(z=z_s)}}{u_s}\overset{\substack{\sigma(z=z_s)\\ \downarrow}}{B} = 0$$

$$\frac{\partial Q}{\partial x} - u_s B\frac{\partial z_s}{\partial x} = 0 \tag{17}$$

We now have an expression that relates streamwise changes in flowrate (Q) and free-surface height (z_s).

To continue working with the right-hand side, we must consider the free-surface kinematic boundary condition. 81

Free-surface definition
The free surface is a material boundary for which a particle initially on the boundary will remain on the boundary

Mathematical description of free-surface
Let the location of the free surface $z = z_s$ be given by the function

$$F(x,y,z,t) = z - z_s = 0$$

Free-surface kinematic boundary condition
Moving with a particle on the free surface, we experience no change in the location of the free surface. Mathematically, this is

$$\left.\frac{DF}{Dt}\right|_{z=z_s} = 0$$

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z}$$

So, the kinematic boundary condition is: 82

$$\left.\frac{DF}{Dt}\right|_{z=z_s} = 0$$

where

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z}$$

and where

$$F(x,y,z,t) = z - z_s$$

This gives:

$$\frac{\partial(z-z_s)}{\partial t} + u\frac{\partial(z-z_s)}{\partial x} + v\frac{\partial(z-z_s)}{\partial y} + w\frac{\partial(z-z_s)}{\partial z} = 0$$

so:

$$-\frac{\partial z_s}{\partial t} - u\frac{\partial z_s}{\partial x} - v\frac{\partial z_s}{\partial y} + w = 0$$

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Because of our Longitudinal Flow assumption

$$\frac{\partial z_s}{\partial y} = 0 \qquad \text{and} \qquad w = 0$$

$$-\frac{\partial z_s}{\partial t} - u\frac{\partial z_s}{\partial x} - \cancel{v\frac{\partial z_s}{\partial y}} + \cancel{w} = 0$$

leaving

$$\frac{\partial z_s}{\partial x} = -\frac{1}{u}\frac{\partial z_s}{\partial t} \tag{18}$$

which we can insert into our prior mass-conservation statement (Eq. 17):

$$\frac{\partial Q}{\partial x} \; - \; u_s B\frac{\partial z_s}{\partial x} \; = \; 0$$

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This gives

$$\frac{\partial Q}{\partial x} \; + \; B\frac{\partial z_s}{\partial t} \; = \; 0 \tag{19}$$

where Q is the volumetric flowrate and B is the free-surface channel width.

Noting that $B\,dz_s = dA$, where A is the cross-sectional flow area, this can also be written

$$\frac{\partial Q}{\partial x} \; + \; \frac{\partial A}{\partial t} \; = \; 0 \tag{20}$$

Equations 19 and 20 are two versions of the mass conservation statement ("Continuity") in open-channel flow, subject to the assumptions in Section 1.1. 85

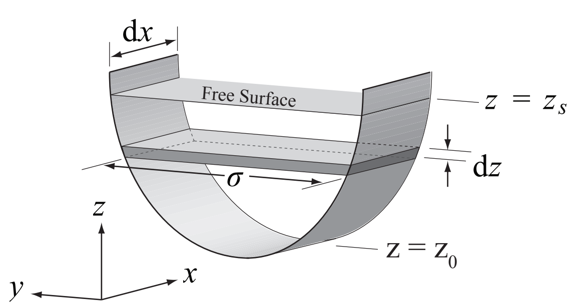
6.2 Conservation of Momentum

We begin with $F = ma$, considering only forces and accelerations in the x-direction (y and z accelerations are assumed negligible):

$$\Sigma F_x = f_p + f_g + f_s = ma_x$$

where f_p, f_g, and f_s are the x-direction pressure, gravitational, and shear forces acting on the channel section. On the differential volume, we have

$$df_p + df_g + df_s = d(ma_x)$$



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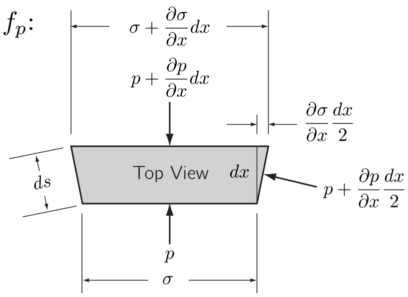
Pressure Forces f_p:



$$df_p = df_p^{\text{front}} + df_p^{\text{back}} + df_p^{\text{sides}}$$

$$df_p = p\,\sigma dz - \left(p + \frac{\partial p}{\partial x} dx\right)\left(\sigma + \frac{\partial \sigma}{\partial x} dx\right) dz + 2\,ds\left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)\underbrace{\frac{\frac{\partial \sigma}{\partial x}\frac{dx}{2}}{ds}}_{\cos\theta}\,dz$$

$$df_p = -\frac{\partial p}{\partial x}\sigma\,dx\,dz$$

Recalling the hydrostatic pressure expression $p = \rho g(z_s - z)\cos\theta$, we have: 87

$$df_p = -\frac{\partial p}{\partial x}\sigma\,dx\,dz = -\rho g\cos\theta\frac{\partial z_s}{\partial x}\sigma\,dx\,dz$$

Now integrate over the depth to get the pressure forces on the entire channel section:

$$f_p = \int_A df_p\,dA = -\rho g\cos\theta\frac{\partial z_s}{\partial x}\,dx\underbrace{\int_0^{z_s}\sigma\,dz}_{A}$$

$$f_p = -\rho g\cos\theta\frac{\partial z_s}{\partial x}A\,dx \tag{21}$$

Gravity Force f_g: 88

$$df_g = \rho g\sin\theta\,\underbrace{\sigma\,dx\,dz}_{dV}$$

Define the Bed Slope S_0

$$S_0 \equiv -\frac{dz_0}{dx} = \sin\theta \approx \theta$$

$$df_g = \rho g\,S_0\,\sigma\,dx\,dz$$

Now integrate over the depth to get the gravity force on the entire channel section:

$$f_g = \int_A df_g\,dA = \rho g\,S_0\,dx\underbrace{\int_0^{z_s}\sigma\,dz}_{A}$$

$$f_g = \rho g\,S_0 A\,dx \tag{22}$$

Shear Forces f_s: 89

$$df_s = df_s^{\text{bottom}} + df_s^{\text{top}} + df_s^{\text{sides}}$$

$$df_s = -\tau\sigma\, dx + \left(\tau + \frac{\partial \tau}{\partial z} dz\right)\left(\sigma + \frac{\partial \sigma}{\partial z} dz\right) dx - \tau_0 \underbrace{2\, dP_w\, dx}_{\text{wetted area}}$$

$$df_s = -\cancel{\tau\sigma\, dx} + \cancel{\tau\sigma\, dx} + \left(\tau \frac{\partial \sigma}{\partial z} + \sigma \frac{\partial \tau}{\partial z}\right) dx\, dz + \cancel{\frac{\partial \tau}{\partial z}\frac{\partial \sigma}{\partial z} dx\, dz^2} - 2\,\tau_0\, dP_w\, dx$$

leaving

$$df_s = \frac{\partial(\sigma\tau)}{\partial z}\, dx\, dz - 2\tau_0\, dP_w\, dx$$

Now integrate over the depth to get the gravity force on the entire channel section: 90

$$f_s = \int_A df_s\, dA = dx \int_0^{z_s} \frac{\partial(\sigma\tau)}{\partial z}\, dz - \tau_0\, dx \underbrace{\int_0^{z_s} 2\, dP_w}_{P_w}$$

$$\text{since} \qquad \int_0^{z_s} \frac{\partial(\sigma\tau)}{\partial z}\, dz = (\sigma\tau)\Big|_0^{z_s} = 0 \qquad \text{we have}$$

$$f_s = -\tau_0 P_w\, dx \tag{23}$$

Acceleration ma_x: 91

Now that we have expressions for the forces f_p, f_g, and f_s, we turn to the right-hand side of the $F = ma$ equation:

$$\Sigma F_x = f_p + f_g + f_s = ma_x$$

The acceleration experienced by the differential mass $\rho\, dV$ is

$$d\,(ma_x) = \rho \underbrace{\sigma\, dx\, dz}_{dV} \frac{du}{dt} \qquad \text{where} \qquad \frac{du}{dt} \equiv \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}}$$

$$d\,(ma_x) = \rho\, dx \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) \sigma\, dz$$

Noting that $\sigma \neq \sigma(t)$, this can be written

$$d\,(ma_x) = \rho\, dx \frac{\partial(u\sigma)}{\partial t}\, dz \;+\; \rho\, dx\, u\frac{\partial u}{\partial x}\, \sigma\, dz$$

Now integrate over the depth, just as we did with the differential forces :

$$ma_x = \int_A d\,(ma_x)\; dA = \rho\, dx \int_0^{z_s} \frac{\partial(u\sigma)}{\partial t}\, dz \;+\; \rho\, dx \int_0^{z_s} \sigma\, u \frac{\partial u}{\partial x}\, dz$$

A quick aside: The second integrand can be rewritten as 92

$$\sigma\, u \frac{\partial u}{\partial x} = \frac{\partial\left(\sigma u^2\right)}{\partial x}$$

Proof. From the conservation of mass section, we showed (Eq. 16):

$$\frac{\partial(u\sigma)}{\partial x} = 0$$

So, we can add this term to the integrand without changing anything:

$$\begin{aligned}
\sigma\, u\frac{\partial u}{\partial x} &= \sigma u\frac{\partial u}{\partial x} + u\frac{\partial(\sigma u)}{\partial x} \\
&= \sigma u\frac{\partial u}{\partial x} + \sigma u\frac{\partial u}{\partial x} + u^2\frac{\partial\sigma}{\partial x} \\
&= 2\sigma u\frac{\partial u}{\partial x} + u^2\frac{\partial\sigma}{\partial x} \\
&= \sigma\frac{\partial u^2}{\partial x} + u^2\frac{\partial\sigma}{\partial x} \\
&= \frac{\partial\left(\sigma u^2\right)}{\partial x}
\end{aligned}$$

□

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$$ma_x = \rho\, dx \int_0^{z_s} \frac{\partial(u\sigma)}{\partial t}\, dz + \rho\, dx \int_0^{z_s} \frac{\partial\left(\sigma u^2\right)}{\partial x}\, dz$$

Use a simplified form of the 1-D Leibnitz rule to evaluate the integrals

$$\int_0^{b(x,t)} \frac{\partial f(x,t)}{\partial t}\, dz = \frac{\partial}{\partial t}\int_0^{b(x,t)} f(x,t)\, dz - \frac{\partial b}{\partial t} f(b)$$

First integral:

$$\begin{aligned}
\int_0^{z_s} \frac{\partial(u\sigma)}{\partial t}\, dz &= \frac{\partial}{\partial t}\int_0^{z_s} (u\sigma)\, dz - \frac{\partial z_s}{\partial t}\,(u\sigma)\big|_{z=z_s} \\
&= \frac{\partial Q}{\partial t} - \frac{\partial z_s}{\partial t}\, u_s B
\end{aligned}$$

Second integral (now with x derivatives instead of t): 94

$$\begin{aligned}
\int_0^{z_s} \frac{\partial\left(\sigma u^2\right)}{\partial x}\, dz &= \frac{\partial}{\partial x}\int_0^{z_s} \left(\sigma u^2\right) dz - \frac{\partial z_s}{\partial x}\left(\sigma u^2\right)\big|_{z=z_s} \\
&= \frac{\partial}{\partial x}\left(\beta V^2 A\right) - \frac{\partial z_s}{\partial x}\, {u_s}^2 B
\end{aligned}$$

where β is the momentum coefficient

$$\beta = \frac{\int_A u^2\, dA}{V^2 A}$$

as defined in Section 1.2. 95

Now, putting the two integral results back into our equation for ma_x, we have

$$ma_x = \left[\frac{\partial Q}{\partial t} - \frac{\partial z_s}{\partial t}\, u_s B + \frac{\partial}{\partial x}\left(\beta V^2 A\right) - \frac{\partial z_s}{\partial x}\, {u_s}^2 B\right]\rho\, dx$$

which can be rearranged as

$$ma_x = \left[\frac{\partial Q}{\partial t} \; + \; \frac{\partial}{\partial x}\left(\beta V^2 A\right) \; - \; u_s B \underbrace{\left(\frac{\partial z_s}{\partial t} \; + \; u_s \frac{\partial z_s}{\partial x} \right)}_{=0} \right] \rho \, dx$$

Noting that (from our free-surface kinematic B.C., Eq. 18)

$$\frac{\partial z_s}{\partial t} \; + \; u_s \frac{\partial z_s}{\partial x} \; = \; \frac{D(z_s)}{Dt} = 0$$

we are then left with

$$ma_x = \left[\frac{\partial Q}{\partial t} \; + \; \frac{\partial}{\partial x}\left(\beta V^2 A\right) \right] \rho \, dx \tag{24}$$

We now can substitute expressions for f_p (Eq. 21), f_g (Eq. 22), f_s 96
(Eq. 23), and ma_x (Eq. 24) into the conservation of momentum relation

$$ma_x = f_p + f_g + f_s$$

The result is (after cancelling dx terms)

$$\rho \left[\frac{\partial Q}{\partial t} \; + \; \frac{\partial}{\partial x}\left(\beta V^2 A\right) \right] \; = \; -\rho g \cos\theta \frac{\partial z_s}{\partial x} A \; + \; \rho g \, S_0 A \; - \; \tau_0 P_w \tag{25}$$

The reader is left to show that this can be expressed equivalently as:

$$\frac{1}{g}\frac{\partial V}{\partial t} \; + \; \frac{\partial H_\beta}{\partial x} \; = \; -\frac{\tau_0 P_w}{\rho g A} \; + \; (\beta - 1)\frac{V}{gA}\frac{\partial A}{\partial t} \; - \; \frac{V^2}{2g}\frac{\partial \beta}{\partial x} \tag{26}$$

where

$$H_\beta = \beta \frac{V^2}{2g} \; + \; z_s \cos\theta \; + \; z_0 \; = \; \text{total head} \tag{27}$$

Equations 25 and 26 are two versions of the momentum conservation
statement in open-channel flow, subject to the assumptions in Section 1.1. 97

6.3 Conservation of Energy

The momentum equation developed in the previous section can be used to derive a (dependent) work-energy equation.

Starting again with

$$df_p + df_g + df_s = d(ma_x)$$

but this time multiplying each term by u (and dividing by dx) before integrating over the depth as before, we get

$$\begin{aligned}
\frac{1}{dx}\int_0^{z_s} u \, df_p \quad &= \quad -\rho g \cos\theta \frac{\partial z_s}{\partial x}\int_0^{z_s} u\sigma \, dz \; = \; -\rho g \cos\theta \frac{\partial z_s}{\partial x} Q \\
\frac{1}{dx}\int_0^{z_s} u \, df_g \quad &= \quad \rho g S_0 \int_0^{z_s} u\sigma \, dz \; = \; \rho g S_0 \, Q
\end{aligned}$$

Skipping some math (see Jain [2001] for details), we also get

$$\frac{1}{dx}\int_0^{z_s} u\,df_s \;=\; -\int_0^{z_s} \sigma\tau\frac{\partial u}{\partial z}\,dz \;\equiv\; -\bar{\epsilon}$$

where ϵ is the rate of work done by internal shear forces. This rate of work cannot be converted back to mechanical energy, and is *dissipated* to heat. 98

Again skipping some math (see Jain [2001] for details), we get

$$\frac{1}{dx}\int_0^{z_s} u\,d(ma_x) \;=\; \frac{\rho}{2}\left[\frac{\partial}{\partial t}(\beta V^2 A) + \frac{\partial}{\partial x}(\alpha V^3 A)\right]$$

where α is the energy coefficient

$$\alpha = \frac{\int_A u^3\,dA}{V^3 A}$$

Combining, we get (after some manipulations)

$$\frac{\beta}{g}\frac{\partial V}{\partial t} \;+\; \frac{\partial H_\alpha}{\partial x} \;=\; -\frac{\bar{\epsilon}}{\rho g Q} \;+\; (\alpha-\beta)\frac{V}{2gA}\frac{\partial A}{\partial t} \;-\; \frac{V}{2g}\frac{\partial \beta}{\partial t} \tag{28}$$

where

$$H_\alpha = \alpha\frac{V^2}{2g} \;+\; z_s\cos\theta \;+\; z_0 \;=\; \text{total head} \tag{29}$$

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