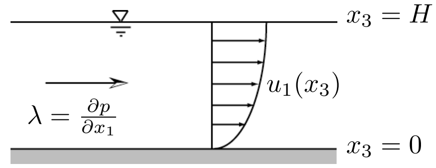
CVEN5313: Environmental Fluid Mechanics

Navier-Stokes 2   Due 5pm 12/10/09 (Friday)

Note: A general reminder that "sketch" means an informal, hand-drawn picture, and "plot" means a nice computer-generated graph.

1. Consider a river flow driven by a pressure gradient $\lambda = \partial p/\partial x_1$. The bed is at $x_3 = 0$ and the free surface is at $x_3 = H$. The resulting velocity distribution is $u_1(x_3)$. Note that when $\lambda < 0$, flow is to the right.

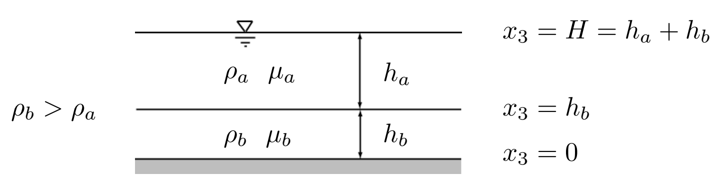


   (a) Starting with the x_1-direction momentum equation, derive the form of the velocity distribution $u_1(x_3)$. Clearly state all assumptions and list all boundary conditions. Hint: What must the shear be at $x_3 = H$?

   (b) Starting with the x_3-direction momentum equation, derive the form of the pressure distribution $p(x_3)$.

   (c) Carefully sketch profiles of velocity, pressure, and shear over the depth of the flow. Label all maxima and minima with their analytical values (in terms of the problem variables).

2. Now consider a two-layer flow such as you might find in an estuary. Again, the flow is driven by a pressure gradient $\lambda = \partial p/\partial x_1$. The upper layer has density ρ_a and viscosity μ_a, and the lower layer has density ρ_b and viscosity μ_b.



   (a) Starting with the x_1-direction momentum equation, derive the form of the velocity distributions $u_1^a(x_3)$ and $u_1^b(x_3)$. Clearly state all assumptions and list all boundary conditions. Hint: What must be true about velocities and shear stresses at the interface between the two fluids?

   (b) Starting with the x_3-direction momentum equation, derive the form of the pressure distribution $p(x_3)$ in each of the two layers.

   (c) Plot profiles of velocity, pressure, and shear over the entire depth of the flow. That is, plot profiles for both layers on a single plot that extends continuously from $x_3 = 0$ to $x_3 = H$, with x_3 on the vertical axis. As before, label all maxima and minima in terms of their analytical values. You will need to choose values for the densities and viscosities. For your plots, assume that $\rho_b = 2\rho_a$ and $\mu_b = 2\mu_a$.

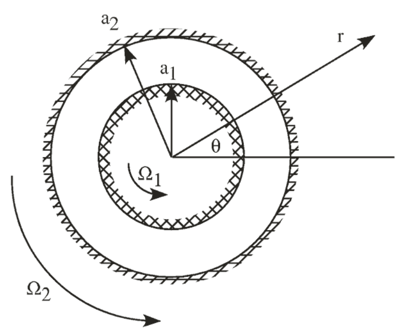
3. A winter wind blows over a lake 200km long and 3m deep, such that the free-surface velocity is 0.5 m/s. The lake contains water at 0 degrees C. Assume that the free-surface tilt is linear.

   (a) Draw a careful sketch of the problem, showing the assumed velocity profile, the static free surface location, and the tilted free surface.

   (b) Within the context of our P-C flow derivation, why is it important to assume that the free surface tilt is linear?

   (c) Calculate the steady-state water-level rise at the far end of the lake. Express your answer in mm.

4. Two coaxial cylinders with radii a_1 and a_2 rotate with angular rates Ω_1 and Ω_2, as shown in the top view below. The annular region between the cylinders is filled with a viscous fluid.



   (a) Write down the appropriate governing equations and boundary conditions for this flow.

   (b) Use the continuity equation to show that $u_r = 0$.

   (c) Solve the θ-direction momentum equation to determine the velocity field $u_\theta(r)$ between the cylinders. Check that your solution satisfies your boundary conditions!

   (d) Calculate the vorticity. Under what conditions is the flow irrotational? What previously-studied flow does this condition correspond to?

   (e) Use the r-direction momentum equation to calculate the pressure distribution between the cylinders. For convenience, let $p(r = a_1) = 0$.

   (f) Plot the radial distributions of velocity and pressure for the cases given below (overlay the two curves on the same plot, with velocity and pressure on the left and right-hand axes, respectively.). For each case, explain the behavior of the $p(r)$ distribution in relation to $u_\theta(r)$.

      i. $a_1 = 1$, $a_2 = 4$, $\Omega_1 = 1$, $\Omega_2 = 1$
      ii. $a_1 = 1$, $a_2 = 4$, $\Omega_1 = -1$, $\Omega_2 = -1$
      iii. $a_1 = 1$, $a_2 = 4$, $\Omega_1 = -6$, $\Omega_2 = 2$

5. The unsteady Taylor vortex is described by the angular circumferential distribution

$$v_\theta = \frac{Hr}{8\pi\nu t^2} \exp\left[\frac{-r^2}{4\nu t}\right]$$

where H is a constant and $v_r = 0$.

(a) Plot the velocity distribution for several values of time.

(b) Calculate the pressure distribution $p(r)$, assuming that $p(\infty) = 0$. Plot the pressure distribution for several values of time.

(c) Show that the pressure distribution in the vertical direction is just hydrostatic. Why do you suppose that this is true, even though the fluid is moving? Determine the shape of a radial profile of the free surface elevation, and plot.

(d) Calculate the vorticity. Plot the vorticity distribution for several values of time (on the same plot). Explain what is happening with respect to vorticity.

(e) Calculate the circulation in the flow around a circular contour with radius R. How much circulation is there in an infinitely large circle? What does this mean physically?

(f) What does an observer at a fixed location see (in terms of velocity and vorticity) as a function of time? Make a plot to explain your answer.