Lecture 7, September 14, 2010 (Key Points)

This and next several lectures are taken from a research monograph, "Introduction to multi-scale hydrology" that I am writing

POWER LAWS AND THE MATHEMATICAL FOUNDATIONS OF SELF-SIMILARITY SPANNING MULTIPLE SCALES

Different empirical relationships for floods described in the previous lectures showed a power law form. Do these empirical relationships have a special theoretical significance? Can they be understood within a basic analytical and modeling framework in terms of physical processes that generate runoff from precipitation, and govern the transport of water and sediments in landscapes? These are examples of some of the key questions that have instigated the development of new geomorphic and hydrologic theories and models in the last two decades.

In simple terms, modeling refers to developing and/or solving an equation, or a set of equations, analytically or numerically. Examples include algebraic equations, difference or differential equations, statistical regression equations and so on. However, in contemporary science and engineering, modeling has acquired a broader context because of complexity of problems being posed, and because of concomitant advances in computing technology. *Models can be either foundational or utilitarian.* For example, conservation equations governing mass, momentum, energy, and entropy represent examples of *foundational models*. These equations typically require specification of 'constitutive flux relationships', as well as input-output relationships in the form of boundary conditions, and they can provide a general framework for binding together physical descriptions at different scales.

In the hydrology literature, the constitutive relationships have developed mostly at the laboratory scale because of ease of experimentation, and because each process could be studied in isolation under controlled conditions. Well-known examples include the Darcy equation for water flow in saturated porous media, the Buckingham-Darcy equation governing moisture flux in unsaturated soils, and momentum equations for open channel flows, e.g., Chezy and Manning equations (Brutsaert, 2005). At larger space-time scales than the laboratory, most hydrologic models have developed in response to the need to solve practical engineering and applied problems. They generally lack adequate observational and theoretical foundations, and this deficiency is masked through model calibration or parameters. These models are mostly *utilitarian* in nature.

For example, we have seen that empirical power-law relations manifest at much larger spatial and temporal scales than the laboratory scale. But a foundational understanding of these relations has lagged behind applications. Our goal is to develop a foundational understanding of empirical power laws at multiple scales involving appropriate conservation laws, and related flux equations, as well as input-output relationships in the form of boundary conditions. Such an understanding will enable us to benefit from the theoretical and practical strengths of power laws, including similitude of phenomena across multiple scales in space or time.

7.1 Power Laws: Geometric, Statistical and Dynamic Self-Similarity

Similitude, or similarity, is a very familiar concept to engineers and scientists. It has served as a basis for design since a long time. The classical notion of similitude is typically based on the concepts of geometric and dynamic similarity between a model and a prototype. In recent times, the concept of geometric similitude or similarity has been greatly extended to define notion of *geometric self-similarity* across many scales. The generalization is based on a functional equation that is introduced in Section 7.2. A power-law solution of this functional equation is covered in Section 7.3.

Dynamic similitude has found a broad range of applications through the well-known Buckingham-Pi Theorem. Vaschy and Riabouchinsky formulated and proved it separately, and Buckingham popularized it. They laid the foundations for defining the important concept of dimensionless numbers in fluid mechanics, such as, Reynolds, Froude, and Mach numbers, which are scale independent quantities. Barenblatt (1996) gives a derivation of the widely used result that the dimension function of a mechanical variable involving length (L), mass (M) and time (T) is a power law. *The derivation is based on the use of a functional equation that is also required in the context of geometric self-similarity*.

Dimensional analysis and dynamical similarity have been of an unquestionable relevance and importance in engineering design and for understanding and predicting many fluid dynamical phenomena. In modern times, this concept has been generalized to the notion of *dynamic self-similarity* (Barenblatt, 1996). We will not have the occasion to explore this important concept in this course, because the application of dynamic self-similarity to solving hydrologic problems is in a state of infancy. For example, in an on-going research effort, we are exploring ideas of dynamic self-similarity to solve a long-standing problem of hydraulic-geometry (HG) in channel networks. HG was first introduced more than half a century ago. I will cover HG in connection with water transport in river networks. I believe that applications of dynamic self-similarity to hydrology in due course would lead to major advances. It provides a promising theme for your MS or PhD research topics.

In hydrologic and many other natural systems, in addition to dynamic and geometric similitude, one needs to consider statistical similarity due to space-time fluctuations that manifest in such processes that cannot be attributed to measurement noise. However, similarity or similitude involving only two scales is not sufficient, because hydrologic processes evolve over multiple space and time scales. For example, natural river networks as branched hierarchical structures contain many spatial scales. The presence of multiple scales has instigated a need to generalize the notion of statistical similarity to that of *statistical self-similarity*. It involves a power-law solution of a functional equation that also arises in geometric and dynamic self-similarity. In the last twenty years, a substantial literature has evolved on applications of statistical self-similarity to hydrology, Geosciences and Biological sciences.

In summary, this brief introduction outlines the foundational role of power law as a solution of a functional equation in defining geometric, dynamic, and statistical self-similarity. Research within the past 15-20 years has shown that the fundamental concepts of geometric and statistical self-similarity provide a common unifying umbrella for developing new physical theories of different hydrologic phenomena spanning multiple space and time scales. We will introduce and explain statistical self-similarity with hydrologic applications to floods involving precipitation, runoff generation and transport. Self-similarity introduced here is called *simple scaling*. It is the simplest example of the general concept of self-similarity or scale invariance for multi-scale phenomena. More general mathematical notions of scale invariance than simple scaling are beyond the scope of this course.

7.2 Functional equation, Self-Similarity and Fractals

Consider a continuous curve whose straight-line distance between two furthest points is d. Let N_δ^d denote the number of segments of the broken line with side length δ which represents the curve. The quantity N_δ^d is dimensionless; therefore, standard dimensional analysis gives,

$$N_\delta^d = f(d/\delta) \tag{7.1}$$

f(.) denotes a function. Now consider approximating the curve with another broken line with lesser length segments, $\lambda < \delta$. Consider the portion of the second broken line contained between two neighboring points of the first broken line. We shall assume that the curve has the properties of *homogeneity* and *self-similarity*. *Homogeneity means that all portions of a basic curve between neighboring points of the first broken line generate equal number of segments of the second broken line. Self-similarity says that,* N_λ^δ*, the number of segments of the broken line with side length λ that are placed between two neighboring points of length* δ *depend only on the ratio* δ/λ*, and not on the individual values of λ and δ. This is the basic idea of geometric similitude that is expressed mathematically in Eq. (7.1).*

When the segment length is *d*, eq. (7.1) shows that $N_d^d = f(1) = 1$. On one hand, each segment of length *d* contains $f(d/\lambda)$ segments of length λ. On the other hand, each segment of length *d* contains $f(d/\delta)$ segments of length δ, and by homogeneity, each segment of length δ contains $f(\delta/\lambda)$ segments of length λ. Hence, the total number of segments of length λ within a segment of length *d* is given by, $f(d/\delta)f(\delta/\lambda)$. Equating these two expressions gives *a functional equation*,

$$f(d/\lambda) = f((d/\delta)(\delta/\lambda)) = f(d/\delta)f(\delta/\lambda), \tag{7.2}$$

It will be shown in Section 7.3 that a power law is a 'unique' solution of Eq. (7.2). Therefore,

$$f(x) = x^D \tag{7.3}$$

where, *D* is a constant. Substituting Eq. (7.3) into Eq. (7.1) gives the length of a broken curve of segments of length *s* as,

$$L_s = sN_s^d = s(d/s)^D = C(s)^{1-D} \tag{7.4}$$

where, $C = d^D$ is a constant. Applications of Eq. (7.4) to constructing geometrically self-similar objects, called *fractals* (Feder, 1988) are covered in Lecture 8.

The assumptions of homogeneity and self-similarity as defined above are two restrictive. While they apply to highly idealized geometrical objects, they cannot be expected to hold for physical objects, such as the coastline of a country. Barenblatt (1996, pp. 340-341) explains that much less restrictive assumptions requiring only *local homogeneity* and *local self-similarity* are enough for a curve to be a fractal. However, a discussion of this technical issue is beyond the scope of this course.

7.3 Power Law as Unique Solution of a Functional Equation

An equation governing a function, say $g(t)$, with respect to a change in the parameter t is called a functional equation. The first functional equation is,

$$g(t+s) = g(t)g(s), \quad t,s > 0 \tag{7.5}$$

It arises in a wide variety of applications (Parzen, 1967, p. 121). Its unique solution is given by,

$$g(t) = e^{-\nu t}, t > 0 \tag{7.6}$$

for some constant ν. A mathematically rigorous proof of Eq. (7.6) is long and technical, so we will not go discuss it here. The interested reader may refer to Parzen (1967, pp. 121-122).

Let us consider a second functional equation given by,

$$g(t+s) = g(t) + g(s), \ t,s > 0 \tag{7.7}$$

Most of you have seen this equation, because its unique solution, $g(t) = ct$, for some constant $c > 0$, is a linear function. The interested reader may consult Parzen (1960, p. 263) for a mathematically rigorous proof.

Our third functional equation is

$$g(ts) = g(t)g(s) \tag{7.8}$$

This equation has a unique power law solution. It arises naturally in a wide range of applications. We saw how it arises in empirical annual flood frequency analyses. We will see many more applications in later lectures. Power laws are intimately connected to the fundamental idea of *self-similarity* introduced in Section 7.2. They have gained wide applicability in hydrology and many other natural sciences and engineering.

Solution of Eq. (7.8) is based on the solution of Eq. (7.7). Define, $z(\log t) = \log g(t)$, and take logarithms of Eq. (7.8). Then

$$z[\log(t) + \log(s)] = z[\log(t)] + z[\log(s)] \tag{7.9}$$

It is same as the functional Eq. (7.7). Therefore, the solution is $z(\log t) = c \log t$ for some constant c. The unique solution of Eq. (7.8) follows, $\log g(t) = c \log t$, which is a *power law*,

$$g(t) = t^{c} \tag{7.10}$$

The fundamental parameter c is called a *scaling exponent*. It can take either positive or negative values, and integer or non-integer values. Eq. (7.10) represents a very important result, because it shows how power law is connected with a functional equation that arises for self-similar geometrical objects illustrated in Section 7.2.

For the sake of completeness, we mention a <u>fourth functional equation</u> given by,

$$g(ts) = g(t) + g(s) \tag{7.11}$$

whose solution is $g(t) = b \log t,\ t > 0$ for some constant b. These four functional equations are collectively known as the *Cauchy Equations*.

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