CVEN5313

Environmental Fluid Mechanics

Section Topic: Navier-Stokes Equations

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Contents

References

- Panton, R.L. (2005) "Incompressible Flow," John Wiley & Sons.
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1 Introduction

The Navier-Stokes Equations describe conservation of

- mass (1 eqn, called the *continuity* equation)
- momentum (3 eqns, one each for each coordinate direction)

The mass conservation equation is based on

Continuity

$$\frac{d}{dt}(m) = 0$$

The three momentum equations are a form of Newton's second law

Newton II

$$\vec{F} = m\vec{a}$$

or

$$\vec{F} = \frac{d}{dt}(m\vec{u})$$

However, we must extend these concepts to account for: 3

- Continuum mechanics (i.e. mass is distributed, not discrete)
- Viscous diffusion of momentum ($\delta \sim \sqrt{\nu t}$)

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Navier Stokes Equations in Cartesian Coords

$$\begin{aligned}
\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} &= 0 \\
\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} + u_3\frac{\partial u_1}{\partial x_3} &= -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu\left[\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right] \\
\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} + u_3\frac{\partial u_2}{\partial x_3} &= -\frac{1}{\rho}\frac{\partial p}{\partial x_2} + \nu\left[\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2}\right] \\
\frac{\partial u_3}{\partial t} + u_1\frac{\partial u_3}{\partial x_1} + u_2\frac{\partial u_3}{\partial x_2} + u_3\frac{\partial u_3}{\partial x_3} &= -\frac{1}{\rho}\frac{\partial p}{\partial x_3} + \nu\left[\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2}\right] + g
\end{aligned}$$

To extend simple ideas like mass conservation and Newton II to the Navier Stokes Equations, it will be useful to first

- introduce some new notation (called index notation)
- review some calculus theorems (e.g., Gauss, Stokes, Leibnitz)

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2 Index Notation

2.1 Definitions

Scalar

- a simple magnitude
- independent of coordinate system
- examples: pressure $p(\vec{x}, t)$ and density $\rho(\vec{x}, t)$

Vector

- a magnitude with an associated direction
- dependent on coordinate system
- examples: velocity $\vec{u}(\vec{x}, t)$ and vorticity $\vec{\omega}(\vec{x}, t)$

Tensor

- No simple definition analogous to those for a scalar or vector...

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Let's go back to the vector...

A vector "associates a magnitude with a direction".

Simple Example
Given a two-dimensional vector

$$\vec{u} = [3 \quad 4]$$

The vector $\vec{u}$ "associates" the magnitude $\sqrt{3^2 + 4^2} = 5$ with a direction 60° above the horizontal.

General Example
Let $\hat{n}$ be a unit vector in some chosen direction.

A vector $\vec{u}$ "associates" a magnitude b to the direction $\hat{n}$ through the dot product operation given by:

$$\hat{n} \cdot \vec{u} = b$$

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Now we can extend this line of thinking as a way to define a tensor...

A tensor "associates a vector with a direction".

Example for a rank-2 tensor $\underline{\underline{A}}$ (more on rank later):
Let $\hat{n}$ be a unit vector in some chosen direction.

A rank-2 tensor $\underline{\underline{A}}$ "associates" a vector $\vec{b}$ to the direction $\hat{n}$ through the dot product operation given by:

$$\hat{n} \cdot \underline{\underline{A}} = \vec{b}$$

We will eventually define a rank-2 tensor $\underline{\underline{T}}(\vec{x}, t)$ called the *stress tensor*. If $\hat{n}$ is the unit normal to some fluid element face, the stress vector $\vec{r}$ on that face is

$$\hat{n} \cdot \underline{\underline{T}} = \vec{r}$$

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Index versus Vector Notation

Index notation (a.k.a. Einstein notation) is a powerful tool for manipulating multidimensional equations. However, there are times when the more conventional vector notation is more useful. It is therefore important to be able to easily convert back and forth between the two. These notes will use both index and vector formulations, and will adhere to the notation conventions summarized below:

	Vector Notation	Index Notation
scalar (rank=0)	a	a
vector (rank=1)	$\vec{a}$	a_i
tensor (rank=2)	$\underline{\underline{A}}$	A_{ij}
tensor (rank=3)	$\underline{\underline{\underline{A}}}$	A_{ijk}

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Free Indices

- A free index appears once and only once within each additive term in an expression. In the equation below, i is a free index:

$$a_i = \epsilon_{ijk} b_j c_k + D_{ij} e_j$$

- A free index implies three distinct equations. That is, the free index sequentially assumes the values 1, 2, and 3. Thus,

$$a_j = b_j + c_j \quad \text{implies} \quad \begin{cases} a_1 = b_1 + c_1 \\ a_2 = b_2 + c_2 \\ a_3 = b_3 + c_3 \end{cases}$$

- The same letter must be used for the free index in every additive term. The free index may be renamed if and only if it is renamed in every term.
- Terms in an expression may have more than one free index so long as the indices are distinct. For example the vector-notation expression $\underline{\underline{A}} = \underline{\underline{B}}^T$ is written $A_{ij} = (B_{ij})^T = B_{ji}$ in index notation. This expression implies nine distinct equations, since i and j are both free indices. 10

- The number of free indices in a term equals the *rank* of the term:

	Notation	Rank
scalar	a	0
vector	a_i	1
tensor	A_{ij}	2
tensor	A_{ijk}	3

Technically, a scalar is a tensor with rank 0, and a vector is a tensor of rank 1. Tensors may assume a rank of any integer greater than or equal to zero. You may only sum together terms with equal rank.

- The first free index in a term corresponds to the row, and the second corresponds to the column. A vector is written as

$$\vec{a} = a_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

and a rank-2 tensor is written as

$$\underline{\underline{A}} = A_{ij} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

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Dummy Indices

- A dummy index appears twice within an additive term of an expression. In the equation below, j and k are both dummy indices:

$$a_i = \epsilon_{ijk} b_j c_k + D_{ij} e_j$$

- A dummy index implies a summation over the range of the index:

$$a_{ii} \equiv a_{11} + a_{22} + a_{33}$$

- A dummy index may be renamed to any letter not currently being used as a free index (or already in use as another dummy index pair in that term). The dummy index is "local" to an individual additive term. It may be renamed in one term (so long as the renaming doesn't conflict with other indices), and it does not need to be renamed in other terms (and, in fact, may not necessarily even be present in other terms).

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The Kronecker Delta

The Kronecker delta is a rank-2 symmetric tensor defined as follows:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

or,

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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The Alternating Unit Tensor

- The alternating unit tensor is a rank-3 antisymmetric tensor defined as follows:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123,\ 231, \text{ or } 312 \\ 0 & \text{if any two indices are the same} \\ -1 & \text{if } ijk = 132,\ 213, \text{ or } 321 \end{cases}$$

The alternating unit tensor is positive when the indices assume any clockwise cyclical progression, as shown in the figure:
- We will use the following identity frequently:

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

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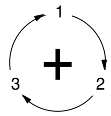

2.2 Commutation and Association in Vector and Index Notation

- In general, operations in vector notation do *not* have commutative or associative properties. For example,

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$$

- All of the terms in index notation are scalars (although the term may represent multiple scalars in multiple equations), and only multiplication/division and addition/subtraction operations are defined. Therefore, commutative and associative properties hold. Thus,

$$\epsilon_{ijk} a_j b_k = \epsilon_{ijk} b_k a_j$$

  and,

$$(a_i b_j) c_k = a_i (b_j c_k)$$

A caveat to the commutative property is that calculus operators *(discussed later) are not, in general, commutative.* 15

2.3 Vector Operations using Index Notation

- Multiplication of a vector by a scalar:

Vector Notation	Index Notation
$a\vec{b} = \vec{c}$	$ab_i = c_i$

  The index i is a free index in this case.
- Scalar product of two vectors (a.k.a. dot or inner product):

Vector Notation	Index Notation
$\vec{a} \cdot \vec{b} = c$	$a_i b_i = c$

  The index i is a dummy index in this case. The term "scalar product" refers to the fact that the result is a scalar.
- Scalar product of two tensors (a.k.a. inner or dot product):

Vector Notation	Index Notation
$\underline{\underline{A}} : \underline{\underline{B}} = c$	$A_{ij} B_{ji} = c$

  The two dots in the vector notation indicate that both indices are to be summed. Again, the result is a scalar. 16

- Tensor product of two vectors (a.k.a. dyadic product):

Vector Notation	Index Notation
$\vec{a}\vec{b} = \underline{\underline{C}}$	$a_i b_j = C_{ij}$

  The term "tensor product" refers to the fact that the result is a tensor.

- Tensor product of two tensors:

Vector Notation	Index Notation
$\underline{\underline{A}} \cdot \underline{\underline{B}} = \underline{\underline{C}}$	$A_{ij}B_{jk} = C_{ik}$

The single dot refers to the fact that only the inner index is to be summed. Note that this is *not* an inner product.

- Vector product of a tensor and a vector:

Vector Notation	Index Notation
$\vec{a} \cdot \underline{\underline{B}} = \vec{c}$	$a_i B_{ij} = c_j$

Given a unit vector $\hat{n}$, we can form the vector product $\hat{n} \cdot \underline{\underline{B}} = \vec{c}$. We say here that then tensor $\underline{\underline{B}}$ associates the vector $\vec{c}$ with the direction given by the vector $\hat{n}$. Also, note that $\vec{a} \cdot \underline{\underline{B}} \neq \underline{\underline{B}} \cdot \vec{a}$. 17

- Cross product of two vectors:

Vector Notation	Index Notation
$\vec{a} \times \vec{b} = \vec{c}$	$\epsilon_{ijk} a_j b_k = c_i$

Recall that

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$$

Now, note that the notation $\epsilon_{ijk} a_j b_k$ represents three terms, the first of which is

$$\begin{aligned}
\epsilon_{1jk} a_j b_k &= \epsilon_{11k} a_1 b_k + \epsilon_{12k} a_2 b_k + \epsilon_{13k} a_3 b_k \\
\\
&= \epsilon_{111} a_1 b_1 + \epsilon_{112} a_1 b_2 + \epsilon_{113} a_1 b_3 + \\
&\quad\ \epsilon_{121} a_2 b_1 + \epsilon_{122} a_2 b_2 + \epsilon_{123} a_2 b_3 + \\
&\quad\ \epsilon_{131} a_3 b_1 + \epsilon_{132} a_3 b_2 + \epsilon_{133} a_3 b_3 \\
\\
&= \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 \\
&= a_2 b_3 - a_3 b_2
\end{aligned}$$

- Contraction or Trace of a tensor (sum of diagonal terms):

Vector Notation	Index Notation
$\mathrm{tr}\left(\underline{\underline{A}}\right) = b$	$A_{ii} = b$

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2.4 Calculus Operations using Index Notation

- Time derivative of a scalar field $\phi(x_1, x_2, x_3, t)$:

$$\frac{\partial \phi}{\partial t} \equiv \partial_0 \phi \quad \text{or} \quad \frac{\partial \phi}{\partial t} \equiv \partial_t \phi$$

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- Gradient (spatial derivatives) of a scalar field $\phi(x_1, x_2, x_3, t)$:

$$\mathrm{grad}\,\phi \equiv \nabla \phi \equiv \begin{bmatrix} \dfrac{\partial \phi}{\partial x_1} \\ \dfrac{\partial \phi}{\partial x_2} \\ \dfrac{\partial \phi}{\partial x_3} \end{bmatrix}$$

In index notation, $\nabla\phi$ is written

$$\nabla\phi \equiv \frac{\partial\phi}{\partial x_i}$$

or, more compactly

$$\nabla\phi \equiv \partial_i\phi$$

Note that the gradient increases the rank of the expression on which it operates by one. Thus, the gradient of a scalar ϕ (rank=0) is the vector $\partial_i\phi$ (rank=1).

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- Gradient (spatial derivatives) of a vector field $\vec{a}(x_1, x_2, x_3, t)$:

$$\mathrm{grad}\,\vec{a} \equiv \nabla\vec{a} \equiv \frac{\partial\vec{a}}{\partial\vec{x}} \equiv \begin{bmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_2}{\partial x_1} & \frac{\partial a_3}{\partial x_1} \\ \frac{\partial a_1}{\partial x_2} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_3}{\partial x_2} \\ \frac{\partial a_1}{\partial x_3} & \frac{\partial a_2}{\partial x_3} & \frac{\partial a_3}{\partial x_3} \end{bmatrix}$$

In index notation, $\nabla\vec{a}$ is written

$$\nabla\vec{a} \equiv \frac{\partial a_i}{\partial x_j}$$

or, more compactly

$$\nabla\vec{a} \equiv \partial_j a_i$$

The compact form emphasizes the row and column structure of the tensor:

$$\partial_j a_i \equiv \begin{bmatrix} \partial_1 a_1 & \partial_1 a_2 & \partial_1 a_3 \\ \partial_2 a_1 & \partial_2 a_2 & \partial_2 a_3 \\ \partial_3 a_1 & \partial_3 a_2 & \partial_3 a_3 \end{bmatrix}$$

Again, the gradient increases the rank of the expression on which it operates by one: the gradient of a vector $\vec{a} = a_i$ (rank=1) is the tensor $\nabla\vec{a} = \partial_j a_i$ (rank=2).

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- Divergence of a vector field $\vec{a}(x_1, x_2, x_3, t)$:

$$\mathrm{div}\,\vec{a} \equiv \nabla\cdot\vec{a} \equiv \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}$$

In index notation, $\nabla\cdot\vec{a}$ is written

$$\nabla\cdot\vec{a} \equiv \frac{\partial a_i}{\partial x_i}$$

or, more compactly

$$\nabla\cdot\vec{a} \equiv \partial_i a_i$$

The divergence decreases by one the rank of the expression on which it operates by one. The divergence of a vector $\vec{a} = a_i$ (rank=1) is the scalar $\nabla\cdot\vec{a} = \partial_i a_i$ (rank=0). *It is not possible to take the divergence of a scalar.*

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- Curl of a vector field $\vec{a}(x_1, x_2, x_3, t)$:

$$\operatorname{curl}\vec{a} \equiv \nabla \times \vec{a} \equiv \begin{bmatrix} \dfrac{\partial a_3}{\partial x_2} - \dfrac{\partial a_2}{\partial x_3} \\ \dfrac{\partial a_1}{\partial x_3} - \dfrac{\partial a_3}{\partial x_1} \\ \dfrac{\partial a_2}{\partial x_1} - \dfrac{\partial a_1}{\partial x_2} \end{bmatrix} = \vec{b}$$

  In index notation, $\nabla \times \vec{a}$ is written

$$\nabla \times \vec{a} \equiv \epsilon_{ijk} \frac{\partial a_k}{\partial x_j} = b_i$$

  or, more compactly

$$\nabla \times \vec{a} \equiv \epsilon_{ijk} \partial_j a_k = b_i$$

  The curl does not change the rank of the expression on which it operates. *It is not possible to take the curl of a scalar.*

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- Laplacian of a vector field $\vec{a}(x_1, x_2, x_3, t)$:

$$\nabla^2 \vec{a} \equiv \nabla \cdot (\nabla \vec{a}) \equiv \begin{bmatrix} \dfrac{\partial^2 a_1}{\partial x_1^2} + \dfrac{\partial^2 a_1}{\partial x_2^2} + \dfrac{\partial^2 a_1}{\partial x_3^2} \\ \dfrac{\partial^2 a_2}{\partial x_1^2} + \dfrac{\partial^2 a_2}{\partial x_2^2} + \dfrac{\partial^2 a_2}{\partial x_3^2} \\ \dfrac{\partial^2 a_3}{\partial x_1^2} + \dfrac{\partial^2 a_3}{\partial x_2^2} + \dfrac{\partial^2 a_3}{\partial x_3^2} \end{bmatrix} = \vec{b}$$

  In index notation, $\nabla^2 \vec{a}$ is written

$$\nabla^2 \vec{a} \equiv \nabla \cdot (\nabla \vec{a}) \equiv \frac{\partial}{\partial x_i} \left(\frac{\partial a_j}{\partial x_i} \right) = b_j$$

  or, more compactly

$$\nabla^2 \vec{a} \equiv \nabla \cdot (\nabla \vec{a}) = \partial_i \partial_i a_j = b_j$$

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- The ordering of terms in expression involving calculus operators:

  Index notation is used to represent vector (and tensor) quantities in terms of their constitutive scalar components. For example, a_i is the i^{th} component of the vector $\vec{a}$. Thus, a_i is actually a collection of three scalar quantities that collectively represent a vector.

  Since index notation represents quantities of all ranks in terms of their scalar components, the order in which these terms are written within an expression is usually unimportant. This differs from vector notation, where the order of terms in an expression is often very important. An extremely important caveat to the above discussion on independence of order is to pay special attention to operators (e.g. div, grad, curl).

  Remember that the rules of calculus (e.g. product rule, chain rule) still apply.

  Example 1:

$$\frac{\partial}{\partial x_k}(a_i b_j) \equiv \partial_k (a_i b_j) = a_i \partial_k b_j + b_j \partial_k a_i \quad \text{(product rule)}$$

Example 2: Show that $\nabla \cdot \vec{a} \neq \vec{a} \cdot \nabla$

$$\nabla \cdot \vec{a} = \partial_i a_i = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} = \quad \text{a scalar}$$

whereas

$$\vec{a} \cdot \nabla = a_i \partial_i = a_1 \frac{\partial}{\partial x_1}() + a_2 \frac{\partial}{\partial x_2}() + a_3 \frac{\partial}{\partial x_3}() = \quad \text{an operator}$$

Thus,

$$\partial_i a_i \neq a_i \partial_i$$

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2.5 Tensor Decomposition into Symmetric and Antisymmetric Parts

- A tensor $\underline{\underline{Q}}$ *symmetric* if it is equal to its transpose:

  $$Q_{ij} = Q_{ji}$$

- A tensor $\underline{\underline{R}}$ *antisymmetric* if it is equal to the negative of its transpose:

  $$R_{ij} = -R_{ji}$$

  Note that the diagonal terms of R_{ij} must necessarily be zero. 26

- Any arbitrary tensor $\underline{\underline{T}}$ may be decomposed into the sum of a symmetric tensor (denoted $T_{(ij)}$) and an antisymmetric tensor (denoted $T_{[ij]}$).

  $$T_{ij} = T_{(ij)} + T_{[ij]}$$

  Proof. Start with $\underline{\underline{T}}$ and then add and subtract one-half of its transpose:

  $$T_{ij} = T_{ij} + \left(\frac{1}{2}T_{ji} - \frac{1}{2}T_{ji}\right)$$

  Now rearrange as:

  $$T_{ij} = \left(\frac{1}{2}T_{ij} + \frac{1}{2}T_{ij}\right) + \left(\frac{1}{2}T_{ji} - \frac{1}{2}T_{ji}\right)$$

  $$T_{ij} = \underbrace{\frac{1}{2}(T_{ij} + T_{ji})}_{symmetric} + \underbrace{\frac{1}{2}(T_{ij} - T_{ji})}_{antisymmetric}$$

  Thus,

  $$T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji}) \quad \text{and} \quad T_{[ij]} = \frac{1}{2}(T_{ij} - T_{ji})$$

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- It is also possible to show that the antisymmetric component of $\underline{\underline{T}}$ can be calculated as

$$T_{[ij]} = \frac{1}{2}\epsilon_{ijk}\epsilon_{klm}T_{lm}$$

- (PS#1) The scalar product of any symmetric and antisymmetric tensor is zero.

$$\underline{\underline{Q}} : \underline{\underline{R}} = \underline{\underline{R}} : \underline{\underline{Q}} = 0 \quad \text{if} \quad Q_{ij} = Q_{ji} \quad \text{and} \quad R_{ij} = -R_{ji}$$

- (PS#1) A more general form of the previous relationship can be stated as follows. The expression $A_{ijkl\ldots}B_{jklm\ldots} = 0$ is equal to zero if $\underline{\underline{\mathrm{A}}}$ and $\underline{\underline{\mathrm{B}}}$ are symmetric and antisymmetric (respectively) with respect to the same indices. For example,

$$A_{ijkl}B_{jklm} = 0$$

  if

$$A_{ijkl} = A_{ikjl} \quad \text{and} \quad B_{jklm} = -B_{kjlm}$$

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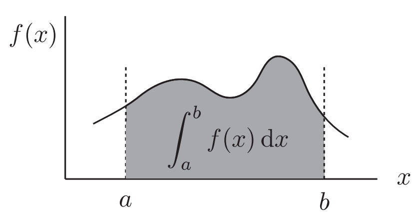
3 Integral Theorems

Recall the fundamental theorem of calculus:

$$\text{Given a function} \quad f = \frac{\mathrm{d}\phi}{\mathrm{d}x}$$

$$\int_a^b f(x)\,\mathrm{d}x \;=\; \int_a^b \frac{\mathrm{d}\phi}{\mathrm{d}x}\,\mathrm{d}x \;=\; \phi(b) \;-\; \phi(a)$$



- The integral of $f = \mathrm{d}\phi/\mathrm{d}x$ over the *continuous range* $a \le x \le b$ can be evaluated by evaluating $\phi = \int f\,\mathrm{d}x$ only ath the *endpoints* of the range.
- The theorems of Stokes and Gauss are loosely analogous to this idea, except that they apply in two and three-dimensional domains, respectively.

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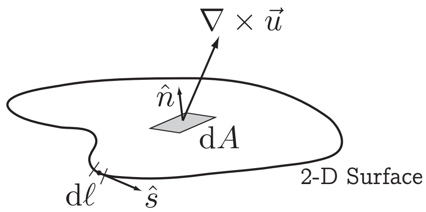
3.1 Stokes

Stokes' Theorem relates the *surface* integral of the curl of a vector $\nabla \times \vec{u}$ to the *line* integral of the vector $\vec{u}$ along the closed curve bounding the surface.

$$\iint_A \hat{n} \cdot (\nabla \times \vec{u})\;\mathrm{d}A \;=\; \oint_C \hat{s} \cdot \vec{u}\;\mathrm{d}\ell$$

or, equivalently

$$\iint_A (\nabla \times \vec{u}) \cdot \mathrm{d}\vec{A} = \oint_C \vec{u} \cdot \mathrm{d}\vec{\ell}$$



and, in index notation,

$$\iint_A n_i \epsilon_{ijk} \partial_j u_k \mathrm{d}S \;=\; \oint_C s_i u_i \mathrm{d}\ell$$

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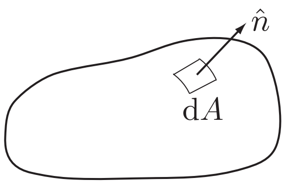
3.2 Gauss

Gauss' Theorem relates the *volume* integral of the gradient or divergence of a vector or a tensor to the *surface* integral of the vector or tensor on the surface bounding the surface.



3D Volume

gradient form: $$\iiint_R \partial_i \left(T_{jk\dots}\right) \mathrm{d}V \;=\; \iint_A n_i \left(T_{jk\dots}\right) \mathrm{d}A$$

divergence form: $$\iiint_R \partial_i \left(T_{ij\dots}\right) \mathrm{d}V \;=\; \iint_A n_i \left(T_{ij\dots}\right) \mathrm{d}A$$

The latter is the so-called *Divergence Theorem*:

$$\iiint_R \nabla \cdot \underline{\underline{T}}\, \mathrm{d}V \;=\; \iint_A \underline{\underline{T}} \cdot \mathrm{d}\vec{A}$$

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3.3 Leibnitz

Leibnitz's Theorem relates tells us how to handle dertivates of integrals when the limits of integration are changing.

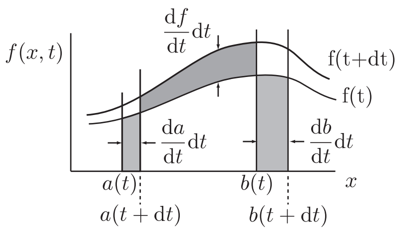
One-Dimensional version:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{a(t)}^{b(t)} f(x,t)\, \mathrm{d}x \;=\; \int_a^b \frac{\partial f}{\partial t} \mathrm{d}x \;+\; \frac{\mathrm{d}b}{\mathrm{d}t} f(b,t) \;-\; \frac{\mathrm{d}a}{\mathrm{d}t} f(a,t)$$

Three-Dimensional version:

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{R(t)} T_{ij\dots}(x_i,t)\, \mathrm{d}V \;=\; \iiint_V \frac{\partial T_{ij\dots}}{\partial t} \mathrm{d}V \;+\; \iint_A n_k u_k T_{ij\dots} \mathrm{d}A$$

where $\vec{u}$ is the local velocity of the surface of $R(t)$ 32



4 Kinematics

4.1 Definitions

Kinematics *n* [from the French *cinématique*, from the Greek *kinema* (motion)]

– a branch of dynamics describing motion (e.g. velocity, acceleration, strain), aside from considerations of what caused it (e.g., forces, mass).

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Uniformity and Steadiness

Uniform Flow

- $\frac{\partial \vec{u}}{\partial s} = 0$
- flow constant *along a streamline*

Steady Flow

- $\frac{\partial \vec{u}}{\partial t} = 0$
- flow constant *over time*

Non-uniform Flow

- $\frac{\partial \vec{u}}{\partial s} \neq 0$
- flow changing *along a streamline*

Unsteady Flow

- $\frac{\partial \vec{u}}{\partial t} \neq 0$
- flow changing *over time*

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Perspectives for describing motion

Eulerian

- description of the flow from an observer fixed in space.
- example: sitting on a river bank and describing $\vec{u}[\vec{x},t]$, $\vec{a}[\vec{x},t]$, etc.

Lagrangian

- description of the flow from an observer moving with a parcel of fluid.
- example: sitting in a raft and describing $\vec{u}[\vec{x}(t)]$, $\vec{a}[\vec{x}(t)]$, etc.

Why two perspectives?

- The Eulerian perspective is typically more intuitive and corresponds to the way we make (most) measurements in fluids.
- Accelerations due to *spatial* variation are not evident in the Eulerian perspective, so $\vec{F} = m\vec{a}$ cannot be directly applied.
- When deriving the N-S eqns, we will therefore formulate fluid accelerations in the Lagrangian perspective. 35

4.2 Fluid Acceleration

$$\vec{x} = [x_1 \ x_2 \ x_3]$$
$$\vec{u} = [u_1 \ u_2 \ u_3]$$

$$\vec{u} = \vec{u}(\vec{x}, t)$$

$$a_1 \equiv \frac{\mathrm{d}u_1}{\mathrm{d}t} \qquad a_2 \equiv \frac{\mathrm{d}u_2}{\mathrm{d}t} \qquad a_3 \equiv \frac{\mathrm{d}u_3}{\mathrm{d}t}$$

These are *not* partial derivatives!

Chain Rule for $f = f(x(t))$
if $f = f(x)$ and $x = x(t)$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t}$$

For example, 36

$$a_1 \equiv \frac{\mathrm{d}u_1}{\mathrm{d}t} = \frac{\partial u_1}{\partial t}\underbrace{\frac{\partial t}{\partial t}}_{=1} + \frac{\partial u_1}{\partial x_1}\underbrace{\frac{\partial x_1}{\partial t}}_{=u_1} + \frac{\partial u_1}{\partial x_2}\underbrace{\frac{\partial x_2}{\partial t}}_{=u_2} + + \frac{\partial u_1}{\partial x_3}\underbrace{\frac{\partial x_3}{\partial t}}_{=u_3}$$

So,

$$a_1 = \underbrace{\frac{\partial u_1}{\partial t}}_{\text{local accel}} + \underbrace{u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} + u_3\frac{\partial u_1}{\partial x_3}}_{\text{advective accel}}$$

- local acceleration is due to *unsteadiness* (an Eulerian metric)
- advective acceleration is due to *non-uniformity* (a Lagrangian metric)

Note that "advective" change and "convective" change are synonymous. 37

4.3 The Substantial Derivative

- generalizes the concepts of local and convective change

Consider some property F in a flow, such that

$$F = F(\vec{x}, t)$$

In a short time Δt, a fluid parcel moves distance $\Delta\vec{x}$. The value of F changes in response to both Δt and $\Delta\vec{x}$:

$$\Delta F = \frac{\partial F}{\partial t}\Delta t + \frac{\partial F}{\partial \vec{x}} \cdot \Delta\vec{x}$$

Dividing by Δt gives

$$\frac{\Delta F}{\Delta t} \equiv \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \vec{x}} \cdot \frac{\Delta \vec{x}}{\Delta t}$$

In the limit as $\Delta t \to 0$, $\Delta x \to 0$, and:

$$\frac{DF}{Dt} \equiv \frac{\partial F}{\partial t} + \vec{u} \cdot \frac{\partial F}{\partial \vec{x}}$$

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Substantial Derivative Notation

$$\frac{DF}{Dt} \equiv \frac{\partial F}{\partial t} + \vec{u} \cdot \frac{\partial F}{\partial \vec{x}}$$

$$\frac{DF}{Dt} \equiv \frac{\partial F}{\partial t} + \vec{u} \cdot \nabla F$$

$$\frac{DF}{Dt} \equiv \partial_0 F + u_i \partial_i F$$

In general:

$$\underbrace{\frac{D(\ )}{Dt}}_{\text{total change}} \equiv \underbrace{\frac{\partial}{\partial t}(\ )}_{\text{local change}} + \underbrace{\vec{u} \cdot \nabla(\ )}_{\text{advective change}}$$

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4.4 Pathlines, Streamlines, Streaklines

Pathline

- line connecting the position of an individual particle over time
- a Lagrangian perspective

Streamline

- line drawn tangent to instantaneous velocity vectors
- an Eulerian perspective

Streakline

- line traced by dye introduced continuously at a point

For steady flow, pathlines, streamlines, and streaklines are identical.

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Equations for calculating pathlines

Consider the motion of a fluid particle in time $\mathrm{d}t$:

$$\mathrm{d}x_i = u_i \mathrm{d}t$$

$$\mathrm{d}x_1 = u_1 \mathrm{d}t \qquad \mathrm{d}x_2 = u_2 \mathrm{d}t \qquad \mathrm{d}x_3 = u_3 \mathrm{d}t$$

Solving each equation for $\mathrm{d}t$ and equating gives:

$$\frac{\mathrm{d}x_1}{u_1} = \frac{\mathrm{d}x_2}{u_2} = \frac{\mathrm{d}x_3}{u_3}$$

or

$$\frac{\mathrm{d}x_i}{u_i} = \text{constant}$$

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4.5 Relative Motion

- It is possible to decompose any fluid motion into the superposition of **translation** and **relative motion**. In other words, relative motion is what remains in a coordinate system moving with the local flow.

- Relative motion can be divided into the superposition of **solid-body rotation** and **strain**.

- Strain can be categorized as either **linear strain**(also known as normal strain) or **angular strain** (also known as shear strain). However, this distinction turns out to be arbitrary, and depends only on the orientation of the chosen coordinate system.

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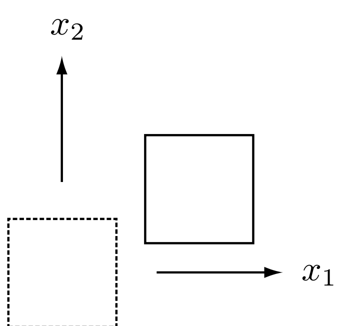
Translation



- no rotation
- no strain
- no volume change
- arbitrary - can be eliminated by reformulating the problem in terms of a coordinate system that moves with the flow.

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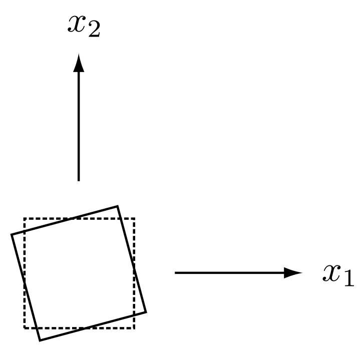
Solid-Body Rotation



- no strain
- no volume change

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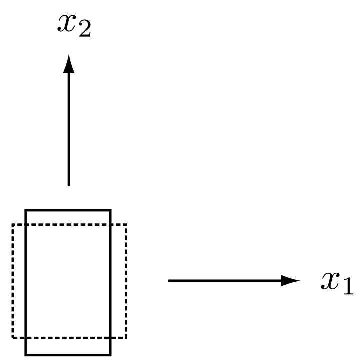
Strain (Linear or "Normal")



- corners remain at 90°
- no rotation
- volume change is possible

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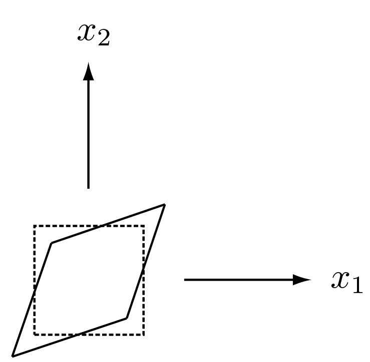
Strain (Angular or "Shear")



- corner angles change
- no net rotation
- no volume change

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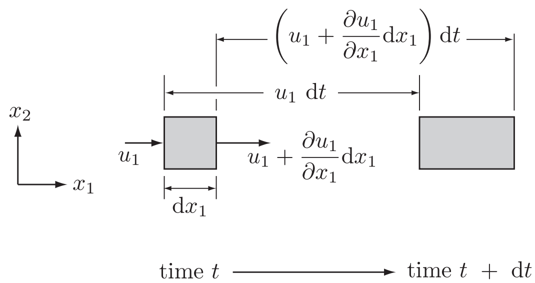
Linear (Normal) Strain Rates



Linear strain is the change in length *per unit length* (nondimensional).

During the time $\mathrm{d}t$, the linear strain in the x_1-direction is:

$$\frac{\left(u_1 + \frac{\partial u_1}{\partial x_1} \mathrm{d}x_1\right) \mathrm{d}t \; - \; u_1 \mathrm{d}t}{\mathrm{d}x_1} = \frac{\partial u_1}{\partial x_1} \mathrm{d}t$$

And

$$\text{linear strain rate in } x_1 - \text{direction} = \frac{\partial u_1}{\partial x_1}$$

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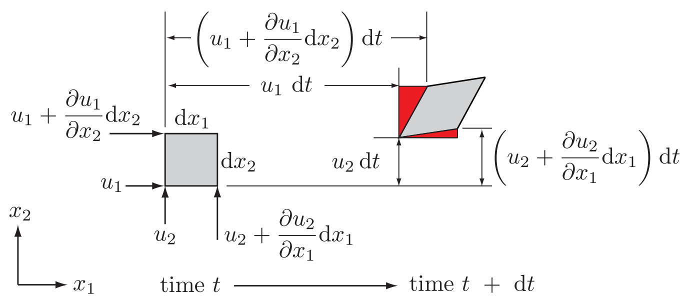
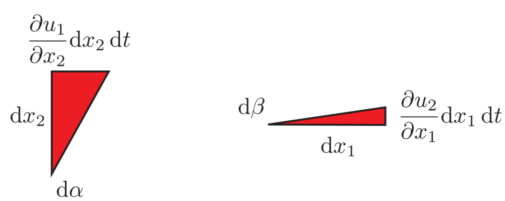
Angular (Shear) Strain Rates





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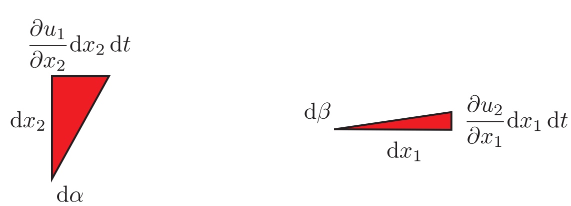
Angular (Shear) Strain Rates



Measuring angles as positive in the CCW direction, and noting that $\mathrm{d}\theta \approx \tan \mathrm{d}\theta$ for small $\mathrm{d}\theta$ (which is always true in the limit as $\mathrm{d}t \to 0$), we have:

$$\mathrm{d}\alpha = \tan\alpha = -\frac{\frac{\partial u_1}{\partial x_2} \cancel{\mathrm{d}x_2} \mathrm{d}t}{\cancel{\mathrm{d}x_2}} \qquad \mathrm{d}\beta = \tan\beta = \frac{\frac{\partial u_2}{\partial x_1} \cancel{\mathrm{d}x_1} \mathrm{d}t}{\cancel{\mathrm{d}x_1}}$$

and

$$\frac{\partial \alpha}{\partial t} = -\frac{\partial u_1}{\partial x_2} \qquad \frac{\partial \beta}{\partial t} = \frac{\partial u_2}{\partial x_1}$$

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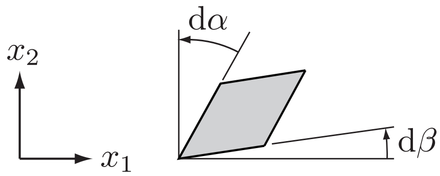
Angular (Shear) Strain Rates



The angular strain rate in the x_1-x_2 plane is the *difference* of the angular rates of change of $d\beta$ and $d\alpha$.

$$\text{angular strain rate in } x_1 - x_2 \text{ plane} = \frac{\partial \beta}{\partial t} - \frac{\partial \alpha}{\partial t}$$

and so

$$\text{angular strain rate in } x_1 - x_2 \text{ plane} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}$$

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Solid Body Rotation Rate

Following the analysis for angular strain rates, we found that the CCW rotation rate for horizontal lines was

$$\frac{\partial \beta}{\partial t} = \frac{\partial u_2}{\partial x_1}$$

and the CCW rotation rate for vertical lines was

$$\frac{\partial \alpha}{\partial t} = -\frac{\partial u_1}{\partial x_2}$$

The solid body rotation rate is simly the average of these two rates:

$$\text{SBR rate} = \frac{1}{2}\left(\frac{\partial \beta}{\partial t} + \frac{\partial \alpha}{\partial t}\right)$$

and so

$$\text{SBR rate} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

We will eventually define vorticity $\vec{\omega}$ as twice the SBR rate. 51

Summary of strain and SBR rates found so far

$$\text{linear strain rate in } x_1 - \text{direction} = \frac{\partial u_1}{\partial x_1}$$

$$\text{angular strain rate in } x_1 - x_2 \text{ plane} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}$$

$$\text{SBR rate in } x_1 - x_2 \text{ plane} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

Note that all types of *relative* motion (strain and SBR) depend only on velocity gradients. We will now proceed to relate relative motion to elements of the velocity gradient tensor $\nabla\vec{u}$:

$$\nabla\vec{u} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_3}{\partial x_1} \\ \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_3}{\partial x_2} \\ \dfrac{\partial u_1}{\partial x_3} & \dfrac{\partial u_2}{\partial x_3} & \dfrac{\partial u_3}{\partial x_3} \end{bmatrix} \qquad \text{or} \qquad \partial_j u_i \equiv \begin{bmatrix} \partial_1 u_1 & \partial_1 u_2 & \partial_1 u_3 \\ \partial_2 u_1 & \partial_2 u_2 & \partial_2 u_3 \\ \partial_3 u_1 & \partial_3 u_2 & \partial_3 u_3 \end{bmatrix}$$

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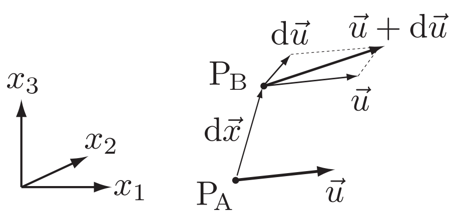
Consider the *relative* motion of two neighboring particles $\mathrm{P_A}$ and $\mathrm{P_B}$ separated a distance $\mathrm{d}\vec{x}$: Both particles move in the local (relative to $\mathrm{P_A}$)



velocity field $\vec{u}$. But particle $\mathrm{P_B}$, being a distance $\mathrm{d}\vec{x}$ away from $\mathrm{P_A}$, experiences an incremental velocity

$$\mathrm{d}\vec{u} = \frac{\partial\vec{u}}{\partial\vec{x}}\cdot\mathrm{d}\vec{x} = \nabla\vec{u}\cdot\mathrm{d}\vec{x}$$

The relative velocity increment $\mathrm{d}\vec{u}$ is responsible for relative motion:

- SBR
- Strain (linear and/or angular)

All information about $\mathrm{d}\vec{u}$ across distances $\mathrm{d}\vec{x}$ is carried in $\nabla\vec{u}$. 53

Decomposition of $\nabla\vec{u}$

Recall that any tensor can be decomposed into symmetric and antisymmetric components:

$$T_{ij} = T_{(ij)} + T_{[ij]}$$

$$T_{(ij)} = \frac{1}{2}\left(T_{ij} + T_{ji}\right) \quad \text{and} \quad T_{[ij]} = \frac{1}{2}\left(T_{ij} - T_{ji}\right)$$

In this case, our tensor is $\nabla\vec{u} = \partial_i u_j$, the decomposition becomes:

$$\begin{aligned} \partial_i u_j &= \partial_{(i}u_{j)} + \partial_{[i}u_{j]} \\ &= \frac{1}{2}\left[\partial_i u_j + \partial_j u_i\right] + \frac{1}{2}\left[\partial_i u_j - \partial_j u_i\right] \end{aligned}$$

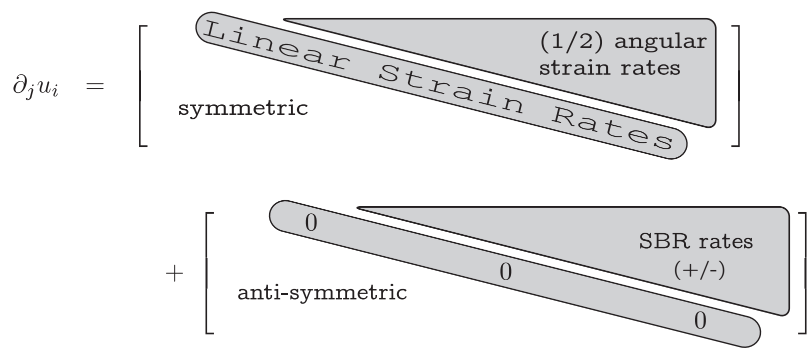
We will use the following notation for the decomposition:

$$\nabla\vec{u} = S_{ij} + r_{ij}$$

where S_{ij} is known as the **strain rate tensor**. 54

$$
\begin{aligned}
\partial_j u_i &= \begin{bmatrix} \partial_1 u_1 & \frac{1}{2}(\partial_1 u_2 + \partial_2 u_1) & \frac{1}{2}(\partial_1 u_3 + \partial_3 u_1) \\ \frac{1}{2}(\partial_2 u_1 + \partial_1 u_2) & \partial_2 u_2 & \frac{1}{2}(\partial_2 u_3 + \partial_3 u_2) \\ \frac{1}{2}(\partial_3 u_1 + \partial_1 u_3) & \frac{1}{2}(\partial_3 u_2 + \partial_2 u_3) & \partial_3 u_3 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & \frac{1}{2}(\partial_1 u_2 - \partial_2 u_1) & \frac{1}{2}(\partial_1 u_3 - \partial_3 u_1) \\ \frac{1}{2}(\partial_2 u_1 - \partial_1 u_2) & 0 & \frac{1}{2}(\partial_2 u_3 - \partial_3 u_2) \\ \frac{1}{2}(\partial_3 u_1 - \partial_1 u_3) & \frac{1}{2}(\partial_3 u_2 - \partial_2 u_3) & 0 \end{bmatrix}
\end{aligned}
$$

which, upon inspection, maps out informally as:



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Being a bit more formal, we can write:

$$
\nabla \vec{u} \quad = \quad S_{ij} \quad + \quad r_{ij}
$$

$$
\partial_j u_i = \begin{bmatrix} \partial_1 u_1 & S_{12} & S_{13} \\ & \partial_2 u_2 & S_{23} \\ & & \partial_3 u_3 \end{bmatrix} + \begin{bmatrix} 0 & r_{12} & r_{13} \\ & 0 & r_{23} \\ & & 0 \end{bmatrix}
$$

where the linear strain rates are

$$
S_{11} = \partial_1 u_1 \qquad S_{22} = \partial_2 u_2 \qquad S_{33} = \partial_3 u_3
$$

the half-angular strain rates are

$$
S_{12} = \frac{1}{2}(\partial_1 u_2 - \partial_2 u_1) \quad S_{13} = \frac{1}{2}(\partial_1 u_3 - \partial_3 u_1) \quad S_{23} = \frac{2}{3}(\partial_1 u_2 - \partial_3 u_2)
$$

and the SBR rates are

$$
\begin{aligned}
r_{12} &= \frac{1}{2}(\partial_1 u_2 - \partial_2 u_1) = \frac{1}{2}(\nabla \times \vec{u})_3 = \frac{1}{2}\omega_3 \\
r_{13} &= \frac{1}{2}(\partial_1 u_3 - \partial_3 u_1) = -\frac{1}{2}(\nabla \times \vec{u})_2 = -\frac{1}{2}\omega_2 \\
r_{23} &= \frac{1}{2}(\partial_2 u_3 - \partial_3 u_2) = \frac{1}{2}(\nabla \times \vec{u})_1 = \frac{1}{2}\omega_1
\end{aligned}
$$

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A note about linear and angular strain rates

- For any flow with non-zero strain rates, the appearance of these strain rates S_{ij} as linear strains (along the diagonal) or angular strains (off-diagonal) is simply a manifestation of the chosen coordinate system.
- Said another way, strain is strain. Strain appears linear, angular, or some combination of the two, depending on the observer's orientation (i.e. coordinate system).
- There is always a coordinate system in which all strains appaer linear (no angular strains), and S_{ij} becomes diagonalized. This coordinate system (found with an Eigenvalue analaysi) is called the *principal axes*, and the associated leinar strain rates are called the *principal strain rates.*
- When viewed from a coordinate system 45° (in either direction) from the principal axes, the flow will exhibit angular strains, with no linear strains.
- In between these two special coordinate systems (the general case), both linear and angular strains exist.

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4.6 Vorticity and Circulation

Vorticity

- a vector quantifying the rotation rate of fluid elements about their centers
- magnitude equal to twice the local SBR rate
- points in the direction of the spin axis (using the right-hand rule)
- a *local* quantity

$$\vec{\omega} = \nabla \times \vec{u} = \epsilon_{ijk}\partial_j u_k$$

$$\begin{aligned}
\omega_1 &= \partial_2 u_3 - \partial_3 u_2 \\
\omega_2 &= \partial_3 u_1 - \partial_1 u_3 \\
\omega_3 &= \partial_1 u_2 - \partial_2 u_1
\end{aligned}$$

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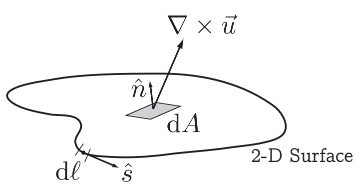
Circulation

- a scalar quantity equal to the total amount of vorticity in an area A
- more formally, equal to the flux of vorticity through a given area A

$$\Gamma = \iint_A \vec{\omega} \cdot \mathrm{d}\vec{A}$$



Stokes Theorem: $$\iint_A \hat{n} \cdot (\nabla \times \vec{u})\, \mathrm{d}A = \oint_C \hat{s} \cdot \vec{u}\, \mathrm{d}\ell$$

So,

$$\Gamma = \underbrace{\iint_A \vec{\omega} \cdot \mathrm{d}\vec{A}}_{\text{flux of } \vec{\omega} \text{ through } A} = \underbrace{\oint_C \vec{u} \cdot \mathrm{d}\vec{\ell}}_{\text{sum of velocities around } A}$$

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$$\Gamma = \iint_A \vec{\omega} \cdot \mathrm{d}\vec{A}$$

Differentiating both sides by $\mathrm{d}\vec{A}$ gives:

$$\frac{\partial \Gamma}{\partial \vec{A}} = \vec{\omega}$$

which gives an alternative definition for $\vec{\omega}$:

- Vorticity is the circulation per unit area (in the limit as the area $\to 0$)

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Example: Forced Vortex

- no strain
- pure SBR

Velocity field is given in polar coordinates by:

$$\begin{aligned} u_r &= 0 \\ u_\theta &= \Omega_0\, r \\ u_z &= 0 \end{aligned}$$

where Ω_0 is a rotation rate $[\mathrm{T}^{-1}]$

The vorticity is (see, for example, Panton, Appendix B):

$$\begin{aligned} \omega_r &= \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} = 0 \\ \omega_\theta &= \frac{\partial u_r}{\partial z} - \frac{\partial z}{\partial r} = 0 \\ \omega_z &= \frac{1}{r}\frac{\partial}{\partial r}(r\, u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} = 2\Omega_0 \end{aligned}$$

- Fluid elements rotate about their center at rate Ω_0
- Vorticity is *twice* the SBR rate
- Vorticity uniform across vortex

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Now, calculate the circulation for the forced vortex in an area A consisting of a circle with radius R, centered at $r = 0$:

$$\text{method 1:} \quad \Gamma \;=\; \iint_A \vec{\omega}\cdot \mathrm{d}\vec{A} \;=\; \vec{\omega} \iint_A \mathrm{d}\vec{A} \;=\; 2\Omega_0 \pi R^2$$

$$\text{method 2:} \quad \Gamma \;=\; \oint_C \vec{u}\cdot \mathrm{d}\vec{\ell} \;=\; \int_0^{2\pi} u_\theta \, R\mathrm{d}\theta \;=\; \int_0^{2\pi} \Omega_0 R^2 \mathrm{d}\theta \;=\; 2\Omega_0 \pi R^2$$

- circulation increases as R^2
- circulation linear with area A 62

Example: Ideal (or Free) Vortex

- no SBR
- pure strain

Velocity field is given in polar coordinates by:

$$\begin{aligned} u_r &= 0 \\ u_\theta &= C/r \\ u_z &= 0 \end{aligned}$$

The vorticity is:

$$\begin{aligned} \omega_r &= \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \;=\; 0 \\ \omega_\theta &= \frac{\partial u_r}{\partial z} - \frac{\partial z}{\partial r} \;=\; 0 \\ \omega_z &= \frac{1}{r}\frac{\partial}{\partial r}(r\, u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} \;=\; 0 \end{aligned}$$

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As before, calculate the circulation for the ideal vortex in an area A consisting of a circle with radius R, centered at $r = 0$:

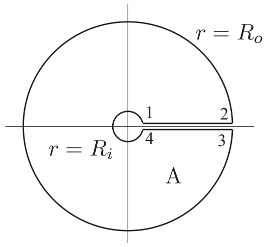
$$\Gamma \;=\; \oint_C \vec{u}\cdot \mathrm{d}\vec{\ell} \;=\; \int_0^{2\pi} u_\theta \, R\mathrm{d}\theta \;=\; \int_0^{2\pi} \frac{C}{R} R\mathrm{d}\theta \;=\; 2\pi C$$

→ What's going on? Why is $\Gamma \neq 0$ even though we found $\vec{\omega} = 0$? 64

→ Singularity at $r = 0$

Calculate the circulation for the area A below, which excludes the origin:

$$\begin{aligned} \Gamma \;=\; \oint_C \vec{u}\cdot \mathrm{d}\vec{\ell} \;&=\; \cancel{\int_{1\to 2} \vec{u}\cdot \mathrm{d}\vec{\ell}} \;+\; \int_{2\to 3} \vec{u}\cdot \mathrm{d}\vec{\ell} \;+\; \cancel{\int_{3\to 4} \vec{u}\cdot \mathrm{d}\vec{\ell}} \;+\; \int_{4\to 1} \vec{u}\cdot \mathrm{d}\vec{\ell} \\ &=\; \int_0^{2\pi} \frac{C}{R_o} R_o \mathrm{d}\theta \;-\; \int_0^{2\pi} \frac{C}{R_i} R_i \mathrm{d}\theta \end{aligned}$$



$$= \quad 2\pi C \quad - \quad 2\pi C$$
$$= \; 0$$

→ Spike of vorticity at the origin! 65

4.7 Streamfunctions and Streamlines (2-D)

Consider a 2-D, incompressible flow:

$$\text{two-dimensional} \quad \rightarrow \quad \vec{u} = [u_1(x_1, x_2), u_2(x_1, x_2)]$$
$$\text{incompressible} \quad \rightarrow \quad \nabla \cdot \vec{u} \; = \; \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \; = \; 0$$

Under these constraints, we can always define a streamfunction

$$\Psi = \Psi(x_1, x_2)$$

such that

$$u_1 \equiv \frac{\partial \Psi}{\partial x_2} \quad \text{and} \quad u_2 \equiv -\frac{\partial \Psi}{\partial x_1}$$

Note that under this definition, the flow *must* satisfy continuity:

$$\begin{aligned} \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \quad &= \quad \frac{\partial}{\partial x_1}\left(\frac{\partial \Psi}{\partial x_2}\right) \; + \; \frac{\partial}{\partial x_2}\left(-\frac{\partial \Psi}{\partial x_1}\right) \\ &= \quad \frac{\partial^2 \Psi}{\partial x_1 \partial x_2} \; - \; \frac{\partial^2 \Psi}{\partial x_1 \partial x_2} \; = \; 0 \end{aligned}$$

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The curve described by Ψ =constant is a streamline

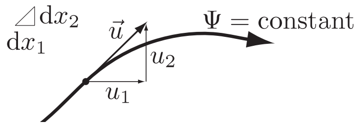
$$\mathrm{d}\Psi = \frac{\partial \Psi}{\partial x_1}\mathrm{d}x_1 + \frac{\partial \Psi}{\partial x_2}\mathrm{d}x_2$$

Noting that $\dfrac{\partial \Psi}{\partial x_1} = -u_2$ and $\dfrac{\partial \Psi}{\partial x_2} = u_1$ gives

$$\mathrm{d}\Psi = -u_2 \mathrm{d}x_1 + u_1 \mathrm{d}x_2$$

If Ψ =constant, $\mathrm{d}\Psi = 0$, which leads to $u_2 \mathrm{d}x_1 = u_1 \mathrm{d}x_2$, or

$$\underbrace{\frac{u_2}{u_1}}_{\text{slope of } \vec{u}} \; = \; \underbrace{\left.\frac{\mathrm{d}x_2}{\mathrm{d}x_1}\right|_{\Psi=\text{constant}}}_{\text{slope of } \Psi}$$



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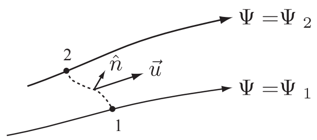
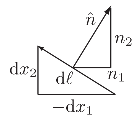
The volume flowrate between streamlines $\Psi = \Psi_1$ and $\Psi = \Psi_2$ is $Q = \Psi_2 - \Psi_1$





$$
\begin{aligned}
n_1 \mathrm{d}\ell &= \mathrm{d}x_2 \\
n_2 \mathrm{d}\ell &= -\mathrm{d}x_1
\end{aligned}
$$

$$Q_{12} = \int_{1\to 2} n_i u_i \,\mathrm{d}\ell = \int_{1\to 2} (n_1 u_1 + n_2 u_2)\mathrm{d}\ell$$

which, from the similar triangles above, can be written

$$Q_{12} = \int_{1\to 2} (u_1 \mathrm{d}x_2 - u_2 \mathrm{d}x_1)$$

From the previous slide, the integrand $u_1 \mathrm{d}x_2 - u_2 \mathrm{d}x_1 = \mathrm{d}\Psi$, so

$$Q_{12} = \int_{1\to 2} \mathrm{d}\Psi = \Psi_2 - \Psi_1$$

→ Facing in the direction of $\vec{u}$, Ψ increases to the left. 68

Streamline Example #1: Stagnation Flow

Given: $\Psi = x_1 x_2$

then

$$u_1 = \frac{\partial \Psi}{\partial x_2} = x_1 \qquad \text{and} \qquad u_2 = -\frac{\partial \Psi}{\partial x_1} = x_2$$

Check continuity:

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 1 - 1 = 0$$

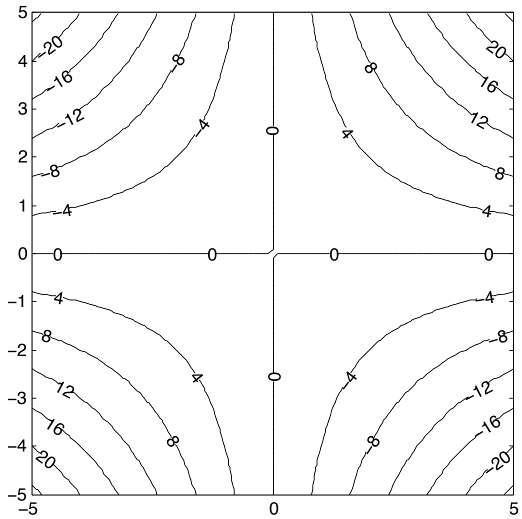
Plot streamlines:

Method 1: Solve for x_2 (easy in this case):

$$x_2 = \frac{\Psi}{x_1}$$

and plot hyperbolas for different values of Ψ. 69

Method 2: Use contouring software to plot contours of $\Psi(x_1, x_2)$:



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MATLAB Code

```
xsize=5;
xstep=.1;
ysize=5;
ystep=.1;
contour_range=24;
contour_step=4;

[X,Y] = meshgrid(-xsize:xstep:xsize,-ysize:ystep:ysize); % 2-D grid
Psi=Y.*X; % calculate Psi on grid

 v=(-contour_range:contour_step:contour_range); % contour levels
[C,h]=contour(X,Y,Psi,v,'-k'); % make the contour plot

clabel(C,h) % label the contours made in the previous step
axis equal % make the x and y axes have the same scale
axis([-xsize xsize -ysize ysize]) % sets the axis range
```

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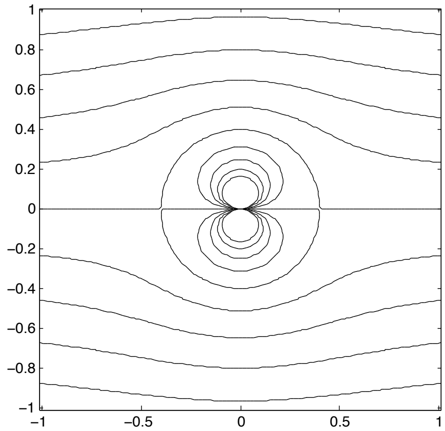
Streamline Example #2: Inviscid flow around a cylinder with radius R

$$\text{Given:}\quad \Psi = U_0\, r \sin\theta \left[1 - \frac{R^2}{r^2}\right]$$

$$u_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} \qquad \text{and} \qquad u_\theta = -\frac{1}{r}\frac{\partial \Psi}{\partial r}$$



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MATLAB Code

```
xsize=1.01; xstep=.02;
ysize=1.01; ystep=.02;
contour_range=8; contour_step=2;

U0=10; rSphere=0.4;

[X,Y] = meshgrid(-xsize:xstep:xsize,-ysize:ystep:ysize); %2-D grid
[THETA,RHO] = cart2pol(X,Y);

Psi=U0*(RHO.*sin(THETA)).*(1-((rSphere./RHO).^2)); % Psi on polar grid

v=(-contour_range:contour_step:contour_range); % contour levels
contour(X,Y,Psi,v,'-k');
axis equal
axis([-xsize xsize -ysize ysize])
```

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4.8 Potential Flows

Irrotational flows ($\nabla \times \vec{u} = 0$) are often termed Potential Flows because their velocity fields can always be expressed in terms of a scalar function ϕ called a velocity potential.

Consider a 2-D, irrotational flow $\vec{u} = (u_1, u_2)$. Since ($\nabla \times \vec{u} = 0$), we have

$$\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 0$$

There then exists a scalar function ϕ defined as $\vec{u} = \nabla \phi$ such that

$$u_1 \equiv \frac{\partial \phi}{\partial x_1} \quad \text{and} \quad u_2 \equiv \frac{\partial \phi}{\partial x_2}$$

Note that this will automatically satisfy ($\nabla \times \vec{u} = 0$).

Equipotential lines ($\phi = C$) are perpendicular to streamlines ($\Psi = C$):

$$\begin{aligned} \frac{\partial \phi}{\partial x_1} &= \frac{\partial \Psi}{\partial x_2} = u_1 \\ \frac{\partial \phi}{\partial x_2} &= -\frac{\partial \Psi}{\partial x_1} = u_2 \end{aligned}$$

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For flows that are both incompressible ($\nabla \cdot \vec{u} = 0$) and irrotational ($\nabla \times \vec{u}$), the velocity potential formulation leads to the Laplace equation:

$$\text{irrotational} \;\rightarrow\; \text{velocity potential:} \quad \vec{u} = \nabla\phi \quad \text{or} \quad u_i = \partial_i \phi$$

$$\text{incompressible}: \qquad \nabla \cdot \vec{u} = 0 \quad \text{or} \quad \partial_i u_i = 0$$

Combining gives:

$$\partial_i u_i \;=\; \partial_i(\partial_i \phi) \;=\; \partial_i \partial_i \phi \;=\; 0$$

In traditional vector notation, this is written shorthand as:

$$\nabla^2 \phi = 0 \qquad \text{Laplace's Equation}$$

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5 Conservation Laws

5.1 Conservation of Mass

Material Region
A Material Region (MR) is a type of control volume where the control surface surface (the bounding surface of the control volume) is prescribed to move with the local fluid velocity. Thus, no mass ever crosses the control surface. The material inside the volume is unique and constant.

We state the mass conservation principle as:

The amount of mass in a material region is constant.

Mathematically, this is

$$\frac{\mathrm{d}}{\mathrm{d}t} m_{\mathrm{MR}} \;=\; \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\mathrm{MR}} \rho \, \mathrm{d}V \;=\; 0$$

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We want to bring the time derivative inside the volume integral:

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\mathrm{MR}} \rho \, \mathrm{d}V \;=\; 0$$

but the integration region MR changes with time.

→ Use Leibnitz's theorem, remembering that the surface of the MR moves with the local fluid velocity $\vec{u}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathrm{MR}}\rho\,\mathrm{d}V \;=\; \iiint_V \frac{\partial\rho}{\partial t}\,\mathrm{d}V \;+\; \iint_A \rho\,\vec{u}\cdot\mathrm{d}\vec{A} \;=\; 0$$

A technical note: The integrations regions on the RHS no longer need be restricted to material regions. In principle, the region could be redefined at each instant in time (perhaps to remain fixed in space). The inclusion of the surface velocity $\vec{u}$ ensures that the book-keeping is done properly. 77

Now use Gauss' Theorem to rewrite the lone area integral as a volume integral:

$$\iint_A \rho\,\vec{u}\cdot\mathrm{d}\vec{A} \;=\; \iiint_V \nabla\cdot(\rho\vec{u})\;\mathrm{d}V$$

Substituting gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathrm{MR}}\rho\,\mathrm{d}V \;=\; \iiint_V \frac{\partial\rho}{\partial t}\,\mathrm{d}V \;+\; \iiint_V \nabla\cdot(\rho\vec{u})\;\mathrm{d}V \;=\; 0$$

And then combining the two integrals on the RHS into a single integral gives

$$\iiint_V \frac{\partial\rho}{\partial t} \;+\; \nabla\cdot(\rho\vec{u})\;\mathrm{d}V \;=\; 0$$

Since the integral must be zero for *any* integration region, the integrand itself must be zero:

$$\frac{\partial\rho}{\partial t} \;+\; \nabla\cdot(\rho\vec{u}) \;=\; 0$$

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Converting to index notation, this is

$$\frac{\partial\rho}{\partial t} \;+\; \nabla\cdot(\rho\vec{u}) \;=\; \partial_t\rho \;+\; \partial_i(\rho u_i) \;=\; 0$$

or

$$\partial_t\rho \;+\; u_i\,\partial_i\rho \;+\; \rho\,\partial_i u_i \;=\; 0$$

Now, if the flow is *incompressible*, then $\partial_t\rho = 0$ and $\partial_i\rho = 0$, leaving

$$\boxed{\partial_i u_i \;=\; 0}$$

or

$$\boxed{\nabla\cdot\vec{u} \;=\; 0}$$

→ No *net* change in fluid volume 79

5.2 Conservation of Momentum: The Cauchy Equation

We seek a continuum analogue to Newton's second law for a point mass:

$$\Sigma\vec{F} = \frac{d}{dt}(m\vec{u})$$

The analogue will take the form

$$\iiint_{\text{MR}} \text{Forces } \mathrm{d}V \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{MR}} \text{Momentum } \mathrm{d}V$$

where the LHS is the sum of external forces acting on a Material Region, and the RHS is the resulting rate of change of total momentum in the Material Region. 80

Forces

We consider two types of forces:

Body Forces

- body forces act on the bulk of the material in the MR
- gravity, electromagnetism

$$F_i \equiv \text{body force per unit mass of fluid}$$

Surface Forces

- surface forces act on the bounding surface of the MR
- pressure, viscous forces

$$R_i \equiv \text{surface force per unit area}$$

We can express the total force acting on the MR as

$$\iiint_{\text{MR}} \text{Forces } \mathrm{d}V \;=\; \iiint_{\text{MR}} F_i\,\rho\,\mathrm{d}V \;+\; \iint_{\text{MR}} R_i\,\mathrm{d}A$$ 81

Momentum Change

We now turn to thr RHS of the Newton II analogue:

$$\iiint_{\text{MR}} \text{Forces } \mathrm{d}V \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{MR}} \text{Momentum } \mathrm{d}V$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{MR}} \text{Momentum } \mathrm{d}V \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{MR}} u_i\,\rho\,\mathrm{d}V$$

Since the MR boundaries move with time, we use Leibnitz to bring the derivative inside the integral:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{MR}} \rho\,u_i\,\mathrm{d}V \;=\; \iiint_{\text{MR}} \frac{\partial}{\partial t}(\rho\,u_i)\,\mathrm{d}V \;+\; \iint_{\text{MR}} n_j u_j\,(\rho\,u_i)\,\mathrm{d}A$$

The surface integral can be converted to a volume integral using Gauss:

$$\iint_{\text{MR}} n_j\,(\rho\,u_j u_i)\,\mathrm{d}A \;=\; \iiint_{\text{MR}} \partial_j\,(\rho\,u_j u_i)\,\mathrm{d}V$$ 82

Putting the pieces together, the RHS of the Newton II analogue becomes:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathrm{MR}} \text{Momentum } \mathrm{d}V &= \iiint_{\mathrm{MR}} \frac{\partial}{\partial t}(\rho u_i)\ \mathrm{d}V + \iiint_{\mathrm{MR}} \partial_j(\rho u_j u_i)\ \mathrm{d}V \\ &= \iiint_{\mathrm{MR}} [\partial_t(\rho u_i) + \partial_j(\rho u_j u_i)]\ \mathrm{d}V\end{aligned}$$

And the complete Newton II analogue becomes

$$\iiint_{\mathrm{MR}} [\partial_t(\rho u_i) + \partial_j(\rho u_j u_i)]\ \mathrm{d}V = \iiint_{\mathrm{MR}} \rho F_i\ \mathrm{d}V + \iint_{\mathrm{MR}} R_i\ \mathrm{d}A$$

→ We would like to convert $\iint_{\mathrm{MR}} R_i\ \mathrm{d}A$ to a volume integral so that we can combine all terms under a single integral

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We assume that there is a stress tensor $\underline{\underline{T}}$ that assigns values to the stress vector $\vec{R}$ on any stated surface whose normal direction is $\hat{n}$:

$$\vec{R} = \hat{n}\cdot\underline{\underline{T}}$$

or, in index notation,

$$R_i = n_j T_{ji}$$

Then, the surface integral term becomes

$$\iint_{\mathrm{MR}} R_i\ \mathrm{d}A = \iint_{\mathrm{MR}} n_j T_{ji}\ \mathrm{d}A$$

which can be written as a volume integral using Gauss:

$$\iint_{\mathrm{MR}} n_j T_{ji}\ \mathrm{d}A = \iiint_{\mathrm{MR}} \partial_j T_{ji}\ \mathrm{d}V$$

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Finally, the Newton II analogue becomes

$$\iiint_{\mathrm{MR}} [\partial_t(\rho u_i) + \partial_j(\rho u_j u_i)]\ \mathrm{d}V = \iiint_{\mathrm{MR}} F_i\ \rho\mathrm{d}V + \iiint_{\mathrm{MR}} \partial_j T_{ji}\ \mathrm{d}V$$

or

$$\iiint_{\mathrm{MR}} [\partial_t(\rho u_i) + \partial_j(\rho u_j u_i) - \rho F_i - \partial_j T_{ji}]\ \mathrm{d}V = 0$$

Since the integral must be equal to zero for *any* choice of MR, the ingrand itself must be zero:

$$\partial_t(\rho u_i) + \partial_j(\rho u_j u_i) - \rho F_i - \partial_j T_{ji} = 0$$

The result is known as the Cauchy Equation:

Cauchy Equation

$$\partial_t(\rho u_i) + \partial_j(\rho u_j u_i) = \partial_j T_{ji} + \rho F_i$$

or

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \nabla\cdot(\rho\vec{u}\vec{u}) = \nabla\cdot\underline{\underline{T}} + \rho\vec{F}$$

→ $\nabla\cdot\underline{\underline{T}}$ is known as the *stress divergence* 85

5.3 The Newtonian Constitutive Relation

In order to transform the Cauchy Equation into the Navier-Stokes momentum equation, we seek an expression for the stress tensor $\underline{\underline{T}}$ that appears in the stress divergence term $\nabla\cdot\underline{\underline{T}}$ in the Cauchy Equation. In particular, we seek to relate $\underline{\underline{T}}$ to the strain rate tensor $\underline{\underline{S}}$.

Here, we must take a short departure from first principles, because the nature of the relationship between the stress tensor $\underline{\underline{T}}$ and the strain rate tensor $\underline{\underline{S}}$ depends on the class of fluids under consideration.

In this course, we will examine the most common class of fluids, called Newtonian fluids.

Newtonian Fluids
A Newtonian Fluid has a linear relationship between stress and strain rate, with the constant of proportionality being the dynamic viscosity μ.

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First, consider a static fluid with no strains. In this case, there are only nomal stresses in the fluid due to the weight of the fluid. These static normal stresses are called pressures, acting inwards:

$$T_{ji} = -p\,\delta_{ji} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

If the fluid is in motion, additional stresses (both normal and shear) can develop due to relative motion:

$$T_{ji} = -p\,\delta_{ji} + \tau_{ji}$$

where τ_{ji} is the viscous stress tensor.

For incompressible, Newtonian Fluids,

$$\tau_{ji} = 2\mu S_{ji}$$

where S_{ji} is the symmetric portion of the velocity gradient tensor $\partial_j u_i$:

$$S_{ji} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

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This leaves us with the Newtonian Constitutive Relation:

$$T_{ji} = -p\,\delta_{ji} + 2\mu S_{ji}$$

We can now introduce this relation into the stress divergence term $\nabla \cdot \underline{\underline{T}}$ in the Cauchy equation. 88

5.4 Navier Stokes

We can now derive the Navier Stokes momentum equation from the Cauchy equation, given by:

$$\partial_t\,(\rho\,u_i) + \partial_j\,(\rho\,u_j u_i) = \partial_j T_{ji} + \rho\,F_i$$

or

$$\frac{\partial}{\partial t}\,(\rho\vec{u}) + \nabla\cdot(\rho\vec{u}\vec{u}) = \nabla\cdot\underline{\underline{T}} + \rho\vec{F}$$

We make the following assumptions:

- Incompressible, Newtonian fluid

$$\nabla\cdot\vec{u} = 0$$

$$T_{ji} = -p\,\delta_{ji} + 2\mu S_{ji}$$

- The only body force is gravity, in the $-x_3$ direction

$$\begin{aligned} \vec{F} &= \vec{g} \\ F_i &= -g\,\delta_{i3} \end{aligned}$$

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The stress divergence term $\nabla \cdot \underline{\underline{T}}$ in the Cauchy equation becomes

$$\begin{aligned} \partial_j T_{ji} &= \partial_j\left[-p\,\delta_{ji} + 2\mu S_{ji}\right] \\ &= \partial_j\left[-p\,\delta_{ji} + 2\mu\,\frac{1}{2}\,(\partial_i u_j + \partial_j u_i)\right] \end{aligned}$$

Distributing the derivative gives (noting that $\partial_j[-p\,\delta_{ji}] = -\partial_i p$)

$$\begin{aligned} \partial_j T_{ji} &= -\partial_i p + \mu\,(\partial_j\partial_i u_j + \partial_j\partial_j u_i) \\ &= -\partial_i p + \cancel{\mu\partial_i\partial_j u_j} + \mu\partial_j\partial_j u_i \end{aligned}$$

Leaving

$$\nabla\cdot\underline{\underline{T}} = -\nabla p + \mu\nabla\cdot\nabla\vec{u}$$

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The LHS of the Cauchy equation can be simplified as

$$
\begin{aligned}
\partial_t(\rho u_i) + \partial_j(\rho u_j u_i) &= \rho\partial_t u_i + \rho\partial_j(u_j u_i) \\
&= \rho\partial_t u_i + \cancel{\rho u_i \partial_j u_j} + \rho u_j \partial_j u_i \\
&= \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u}
\end{aligned}
$$

Substituting this relation and the stress divergence relation into the Cauchy equation gives (after dividing by ρ):

$$\partial_t u_i + u_j \partial_j u_i = -\frac{1}{\rho}\partial_i p + \nu \partial_j \partial_j u_i + g\delta_{i3}$$

where $\nu = \mu/\rho$ is the kinematic viscosity

This momentum equation, along with the continuity equation, are together known as the Navier-Stokes equations. 91

The Navier Stokes equations
continuity:

$$\boxed{\nabla \cdot \vec{u} = 0}$$

momentum:

$$\boxed{\underbrace{\frac{\partial \vec{u}}{\partial t}}_{\text{local accel}} + \underbrace{\vec{u} \cdot \nabla \vec{u}}_{\text{advective accel}} = \underbrace{-\frac{1}{\rho}\nabla p}_{\text{pressure}} + \underbrace{\nu \frac{\partial^2 \vec{u}}{\partial \vec{x}^2}}_{\text{viscous}} + \underbrace{\vec{g}}_{\text{gravity}}}$$

→ The LHS terms are accelerations.
→ The RHS terms are forces (per unit mass)

$$\vec{a} = \vec{F}/m$$

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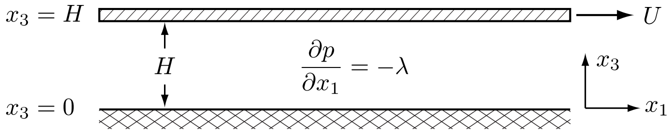
5.5 Example: Poiseuille-Couette Flow

We now use the Navier-Stokes equations to determine the steady state velocity profile of a flow driven both by apressure gradient $\frac{\partial p}{\partial x_1} = -\lambda$ and a prescribed velocity U at $x_3 = H$ (the latter condition amounts to the application of a shear stress).

Assumptions:

- Steady-state $\frac{\partial}{\partial t} = 0$
- One-dimensional flow: $u_2 = u_3 = 0$, $\frac{\partial u_1}{\partial x_1} = \frac{\partial u_1}{\partial x_2} = 0$



Boundary Conditions:
$u_1(x_3 = 0) = 0$
$u_1(x_3 = H) = U$

→ Solve for the steady-state velocity profile $u_1(x_3)$ 93

Solve the x_1-direction momentum equation:

$$\cancel{\frac{\partial u_1}{\partial t}} + \cancel{u_1 \frac{\partial u_1}{\partial x_1}} + \cancel{u_2 \frac{\partial u_1}{\partial x_2}} + \cancel{u_3 \frac{\partial u_1}{\partial x_3}} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu\left(\cancel{\frac{\partial^2 u_1}{\partial x_1^2}} + \cancel{\frac{\partial^2 u_1}{\partial x_2^2}} + \frac{\partial^2 u_1}{\partial x_3^2}\right) + \cancel{g_1}$$

which leaves

$$\boxed{\frac{\partial p}{\partial x_1} = \mu \frac{\partial^2 u_1}{\partial x_3^2}}$$

→ balance between pressure and viscous shear forces

Since $\frac{\partial p}{\partial x_1} = -\lambda$, we solve simply by integrating the equation

$$-\frac{\lambda}{\mu} = \frac{\partial^2 u_1}{\partial x_3^2}$$

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Integrate once:

$$-\int \frac{\lambda}{\mu}\,\mathrm{d}x_3 = \int \frac{\partial^2 u_1}{\partial x_3^2}\,\mathrm{d}x_3$$

to get

$$-\frac{\lambda}{\mu}\,x_3 + C_1 = \frac{\partial u_1}{\partial x_3}$$

Integrate a second time to get

$$u_1(x_3) = -\frac{\lambda}{2\mu}\,{x_3}^2 + C_1\,x_3 + C_2$$

→ Apply two boundary conditions to determine C_1 and C_2

$$u_1(x_3 = 0) = 0 \quad \Rightarrow \quad C_2 = 0$$

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$$u_1(x_3) \;=\; -\frac{\lambda}{2\mu}\,{x_3}^2 \;+\; C_1\,x_3 \;+\; \cancelto{=\,0}{C_2}$$

$$u_1(x_3 = H) = U \qquad \Rightarrow \qquad U \;=\; -\frac{\lambda}{2\mu}\,H^2 \;+\; C_1\,H \qquad \Rightarrow \qquad C_1 = \frac{U}{H} + \frac{\lambda H}{2\mu}$$

This leaves us with the dimensional solution to the P-C flow problem:

$$\boxed{u_1(x_3) \;=\; -\frac{\lambda}{2\mu}\,{x_3}^2 \;+\; \left(\frac{U}{H} + \frac{\lambda H}{2\mu}\right) x_3}$$

What does the solution look like?
Tricky to plot, since there are so many parameters.

→ Nondimensionalize!

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One way to nondimensionalize the solution is

$$\frac{u_1(x_3)}{U} \;=\; -\frac{\lambda H^2}{2\mu U}\left(\frac{x_3}{H}\right)^2 \;+\; \left(1 + \frac{\lambda H^2}{2\mu U}\right)\frac{x_3}{H}$$

or

$$\boxed{{u_1}^* \;=\; -4\,\mathbb{P}\,{x_3}^{*2} \;+\; (1 + 4\,\mathbb{P})\,{x_3}^*}$$

where

$$\mathbb{P} \equiv \frac{\lambda H^2}{8\mu U}, \qquad {u_1}^* \equiv \frac{u_1}{U} \quad \text{and} \quad {x_3}^* \equiv \frac{x_3}{H}$$

$\mathbb{P}$ is a nondimensional measure of the relative strength of the pressure gradient λ and the shear stress due to the upper plate moving at velocity U.

$$\begin{aligned} \mathbb{P} &= 0 & \Rightarrow \quad & \text{Pure shear-driven flow} \\ \mathbb{P} &= \pm\infty & \Rightarrow \quad & \text{Pure pressure-driven flow} \end{aligned}$$

→ We can now plot ${u_1}^*$ vs. ${x_3}^*$ with $\mathbb{P}$ as a single nondimensional parameter. 97

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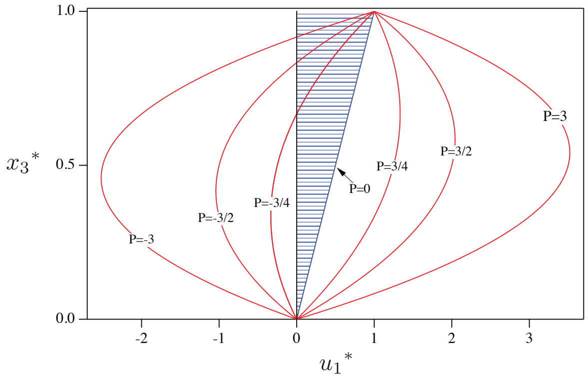
$$u_1{}^* \;=\; -4\,\mathbb{P}\,x_3{}^{*2} \;+\; (1+4\,\mathbb{P})\,x_3{}^*$$



→ What is the net horizontal flow as a function of $\mathbb{P}$?

Define q as the total horizontal fluid flux per unit width:

$$q = \int_0^H u_1(x_3)\,\mathrm{d}x_3$$

We can calculate a nondimensional horizontal flux as

$$q^* = \int_0^1 u_1{}^*(x_3{}^*)\,\mathrm{d}x_3{}^*$$

$$q^* \;=\; \int_0^1 -4\,\mathbb{P}\,x_3{}^{*2} + (1+4\,\mathbb{P})\,x_3{}^*\;\mathrm{d}x_3{}^* \;=\; \left[-\frac{4\,\mathbb{P}}{3}\,x_3{}^{*3} + \frac{1+4\,\mathbb{P}}{2}\,x_3{}^{*2}\right]_0^1$$

$$=\; -\frac{4\,\mathbb{P}}{3} + \frac{1+4\,\mathbb{P}}{2} \;=\; \frac{3+4\,\mathbb{P}}{6}$$

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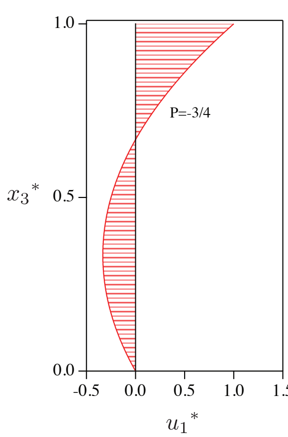
→ What value of $\mathbb{P}$ produces the case of zero net horizontal flow?

$$q^* \;=\; \frac{3+4\,\mathbb{P}}{6} \;=\; 0 \qquad \Rightarrow \qquad \mathbb{P} = -\frac{3}{4}$$



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Now solve the x_3-direction momentum equation:

$$\cancel{\frac{\partial u_3}{\partial t}} + u_1\cancel{\frac{\partial u_3}{\partial x_1}} + u_2\cancel{\frac{\partial u_3}{\partial x_2}} + u_3\cancel{\frac{\partial u_3}{\partial x_3}} = -\frac{1}{\rho}\frac{\partial p}{\partial x_3} + \nu\left(\cancel{\frac{\partial^2 u_3}{\partial x_1^2}} + \cancel{\frac{\partial^2 u_3}{\partial x_2^2}} + \cancel{\frac{\partial^2 u_3}{\partial x_3^2}}\right) + g_3$$

which leaves (after setting $g_3 = -g$ and letting $\gamma \equiv \rho g$)

$$\frac{\partial p}{\partial x_3} = -\gamma$$

Integrating gives

$$\int \mathrm{d}p = -\gamma \int \mathrm{d}x_3 \quad \Rightarrow \quad p = -\gamma x_3 + C$$

applying an (arbitrary) free-surface pressure condition $p(x_3 = H) = 0$ to solve for C leads to

$$\boxed{p = \gamma(H - x_3)}$$

$\rightarrow$ Hydrostatic pressure distribution! 101

5.6 Conservation of Vorticity

The NS momentum-conservation equation (written in terms of $\vec{u}$) can be transformed into vorticity-conservation equation (written in terms of $\vec{\omega} = \nabla \times \vec{u}$) by taking the curl of each term in the momentum equation.

momentum

$$\underbrace{\frac{\partial \vec{u}}{\partial t}}_{\text{I}} + \underbrace{\vec{u}\cdot\nabla\vec{u}}_{\text{II}} = \underbrace{-\frac{1}{\rho}\nabla p}_{\text{III}} + \underbrace{\nu\frac{\partial^2 \vec{u}}{\partial \vec{x}^2}}_{\text{IV}} + \underbrace{\vec{g}}_{\text{V}}$$

- Term I

$$\nabla \times \left(\frac{\partial \vec{u}}{\partial t}\right) = \epsilon_{ijk}\partial_j(\partial_t u_k) = \partial_t(\epsilon_{ijk}\partial_j)u_k = \frac{\partial \vec{\omega}}{\partial t}$$

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- Term II

$$\nabla \times (\vec{u}\cdot\nabla\vec{u}) = \vec{u}\cdot\nabla(\nabla\times\vec{u}) + (\nabla\cdot\vec{u})(\nabla\times\vec{u}) - (\nabla\times\vec{u})\cdot(\nabla\vec{u})$$

This is an identity that you proved on Problem Set 1.

For an incompressible fluid, $\nabla\cdot\vec{u} = 0$, and we are left with

$$\nabla \times (\vec{u}\cdot\nabla\vec{u}) = \vec{u}\cdot\nabla\vec{\omega} - \vec{\omega}\cdot(\nabla\vec{u})$$

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- Term III

Note that we will allow for spatial variations in density ($\nabla\rho \neq 0$) even though the flow is incompressible ($\nabla \cdot \vec{u} = 0$)

$$
\begin{aligned}
\nabla \times \left(-\frac{1}{\rho}\nabla p\right) &= \epsilon_{ijk}\partial_j\left[-\rho^{-1}\partial_k p\right] \\
&= \epsilon_{ijk}\left[\rho^{-1}\partial_j\partial_k p \;+\; \rho^{-2}\partial_j(\rho)\partial_k(p)\right] \\
&= \frac{1}{\rho}\cancel{\epsilon_{ijk}\partial_j\partial_k p} \;+\; \frac{1}{\rho^2}\epsilon_{ijk}\partial_j(\rho)\partial_k(p) \\
&= \frac{\nabla\rho \times \nabla p}{\rho^2}
\end{aligned}
$$

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- Term IV

$$
\nabla \times \left(\nu\frac{\partial^2\vec{u}}{\partial\vec{x}^2}\right) = \epsilon_{ijk}\nu\partial_j\partial_l\partial_l u_k = \nu\partial_l\partial_l\epsilon_{ijk}\partial_j u_k = \nu\frac{\partial^2\vec{\omega}}{\partial\vec{x}^2}
$$

$$
\text{where} \quad \nu\frac{\partial^2\vec{\omega}}{\partial\vec{x}^2} \equiv \nu\nabla^2\vec{\omega}
$$

- Term V

$$
\nabla \times \vec{g} = \epsilon_{ijk}\cancel{\partial_j g_k} = 0
$$

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Putting it all together gives an the equation for Conservation of Vorticity

$$
\boxed{\underbrace{\frac{\partial\vec{\omega}}{\partial t}}_{\text{local change}} + \underbrace{\vec{u}\cdot\nabla\vec{\omega}}_{\text{advective change}} = \underbrace{\vec{\omega}\cdot\nabla\vec{u}}_{\text{tilting/stretching}} + \underbrace{\nu\frac{\partial^2\vec{\omega}}{\partial\vec{x}^2}}_{\text{viscous diffusion}} + \underbrace{\frac{1}{\rho^2}\nabla\rho\times\nabla p}_{\text{baroclinic}}}
$$

For the common case where $\nabla\rho = 0$, we have

$$
\underbrace{\frac{D\vec{\omega}}{Dt}}_{\text{total change}} = \underbrace{\vec{\omega}\cdot\nabla\vec{u}}_{\text{tilting/stretching}} + \underbrace{\nu\frac{\partial^2\vec{\omega}}{\partial\vec{x}^2}}_{\text{viscous diffusion}}
$$

- No production terms!
- $\vec{\omega}$ produced at solid boundaries (in the presence of ∇p or acceleration)
- $\vec{\omega}$ supplied to the conservation equation via boundary conditions

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Tilting and Stretching of Vortex Lines (3D Flows only)

$$\vec{\omega} \cdot \nabla \vec{u}$$

- Tilting *reorients* vortex lines
- Stretching *intensifies* vortex lines

$$\vec{\omega} \cdot \nabla \vec{u} = \omega_j \partial_j u_i$$

e.g., for $i = 1$, we have

$$(\vec{\omega} \cdot \nabla \vec{u})_1 = \underbrace{\omega_1 \partial_1 u_1}_{\text{stretching}} + \underbrace{\omega_2 \partial_2 u_1}_{\text{tilting}} + \underbrace{\omega_3 \partial_3 u_1}_{\text{tilting}}$$

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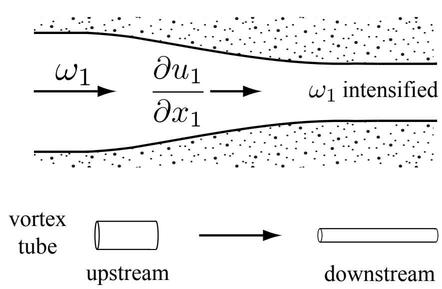
Vortex Stretching Example

Consider flow in the x_1 direction, with a superimposed swirl component ω_1. If a contraction imposes a streamwise acceleration $\partial u_1/\partial x_1$, the vorticity is stretched via $\omega_1 \partial_1 u_1$:



→ Consider flow going down a drain...

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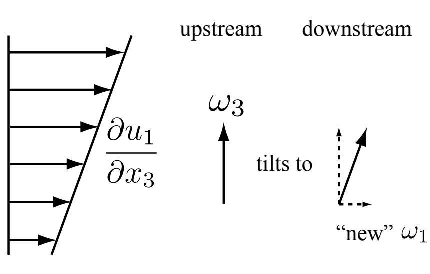
Vortex Tilting Example

Consider flow in the x_1 direction with shear $\partial u_1/\partial x_3$. If there is ω_3 vorticity in the flow, it is tilted via $\omega_3 \partial_3 u_1$:



This results in the addition of ω_1 in the x_1 vorticity equation, but it comes at the expense of the original ω_3 magnitude.

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Baroclinic Production of $\vec{\omega}$

$$\frac{1}{\rho^2}\nabla\rho\times\nabla p$$

Note that this is the only case where vorticity is produced *within* a flow.

Vorticity is produced when isobars (p contours) are not parallel with isopycnals (ρ contours).

The classic example is the lock-exchange problem. 110

5.7 Conservation of Energy

Recall that the momentum equation was derived from an equation relating the rate of change of momentum with applied body and surface forces. We go back to an intermediate step in that derivation:

$$\iiint_{\mathrm{MR}}\left[\partial_t\left(\rho u_i\right)+\ \partial_j\left(\rho u_j u_i\right)\right]\ \mathrm{d}V\ =\ \iiint_{\mathrm{MR}} F_i\ \rho\mathrm{d}V\ +\ \iint_{\mathrm{MR}} n_j T_{ji}\ \mathrm{d}A$$

To derive an energy equation, first consider what happens if we take the dot product of $\vec{u}$ with the integrand of the momentum term:

$$u_i\left[\partial_t\left(\rho u_i\right)+\ \partial_j\left(\rho u_j u_i\right)\right]\ =\ u_i\partial_t\left(\rho u_i\right)+\ u_i\partial_j\left(\rho u_j u_i\right)$$

The two terms on the RHS become, respectively:

$$\begin{aligned}
u_i\partial_t\left(\rho u_i\right) &= \frac{1}{2}u_i\partial_t\left(\rho u_i\right)+\frac{1}{2}u_i\partial_t\left(\rho u_i\right)\ =\ \partial_t\left(\frac{1}{2}\rho u_i u_i\right)\ =\ \frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right)\\
u_i\partial_j\left(\rho u_j u_i\right) &= \frac{1}{2}u_i\partial_j(\rho u_j u_i)+\frac{1}{2}u_i\partial_j(\rho u_j u_i)\ =\ \partial_j\left(\rho u_j\frac{1}{2}u_i u_i\right)\ =\ \nabla\cdot\left(\vec{u}\frac{1}{2}\rho u^2\right)
\end{aligned}$$

$\rightarrow$ these terms are the local and advective change in KE $\frac{1}{2}\rho u^2$ 111

Since energy can move back and forth between KE and thermal energy, we want a conservation equation in terms of

$$\text{energy per unit volume}\ =\ E\ =\ \rho\left(e\ +\ \frac{1}{2}u^2\right)$$

where e is thermal energy per unit mass.

So the LHS of our energy equation becomes (following the steps on the previous slide):

$$\iiint_{\mathrm{MR}}\partial_t\left[\rho\left(e+\frac{1}{2}\rho u_i u_i\right)\right]+\ \partial_j\left[\rho u_j\left(e+\frac{1}{2}u_i u_i\right)\right]\ \mathrm{d}V$$

Similarly, taking the dot product of $\vec{u}$ the RHS of the momentum equation gives

$$\iiint_{\mathrm{MR}} u_i F_i \ \rho \,\mathrm{d}V \ + \ \iint_{\mathrm{MR}} n_j (T_{ji} u_i) \ \mathrm{d}A \ - \ \iint_{\mathrm{MR}} n_i q_i \ \mathrm{d}A$$

where we have added a surface heat flux $\vec{q}$ for completeness. 112

Following the usual steps (converting to volume integrals, combining integrals to a single integral, setting the integrand equal to zero), we are left with

$$\partial_t(\rho E) + \ \partial_j \left(\rho u_j E\right) \ = \ \rho u_i F_i \ + \ \partial_j (T_{ji} u_i) \ - \ \partial_i q_i$$

or

$$\underbrace{\frac{\partial}{\partial t}(\rho E) \ + \ \nabla \cdot (\rho \vec{u} E)}_{\text{total rate of change of } E} \ = \ \nabla \cdot (\underline{\underline{T}} \cdot \vec{u}) \ + \ \rho \vec{u} \cdot \vec{g} \ - \ \nabla \cdot \vec{q}$$

The terms on the RHS are:

- $\nabla \cdot \vec{q}$ flux of heat across boundaries. $\rightarrow$ changes thermal energy
- $\rho \vec{u} \cdot \vec{g}$ work done by gravity. $\rightarrow$ changes kinetic energy
- $\nabla \cdot (\underline{\underline{T}} \cdot \vec{u})$ work done by surface forces. $\rightarrow$ changes both!

$\rightarrow$ We need to investigate $\nabla \cdot (\underline{\underline{T}} \cdot \vec{u})$ 113

$$\nabla \cdot (\underline{\underline{T}} \cdot \vec{u}) \ = \ \partial_j (T_{ji} u_i) \ = \ \underbrace{u_i \partial_j T_{ji}}_{I} \ + \ \underbrace{T_{ji} \partial_j u_i}_{II}$$

where

$$T_{ji} \ = \ -p\delta_{ji} + \tau_{ji} \ = \ -p\delta_{ji} + 2\mu S_{ji}$$

Term I is the product of the velocity and a force gradient (force imbalance). $\rightarrow$ describes KE changes due to pressure and viscous forces

Term II is the product of force and a fluid deformation. $\rightarrow$ describes thermal energy changes due to fluid deformation

$$\begin{aligned} T_{ji} \partial_j u_i \ &= \ -p\delta_{ji} \partial_j u_i \ + \ 2\mu S_{ji} \underbrace{(S_{ji} + r_{ji})}_{\partial_j u_i} \\ &= \ -p \partial_i u_i \ + \ 2\mu S_{ji} S_{ji} \end{aligned}$$

$\rightarrow$ $-p \nabla \cdot \vec{u}$ is *reversible* heating/cooling due to p-induced comp./exp.

$\rightarrow$ $2\mu S_{ji} S_{ji}$ is *irreversible* heating due to viscous straining $\rightarrow$ Dissipation per unit volume Φ_ν 114

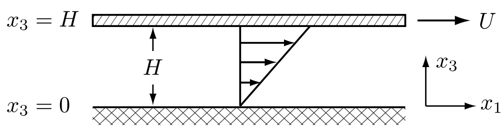
Noting that S_{ji} is symmetric, we can rewrite the viscous dissipation term as

$$\Phi_\nu = 2\mu S_{ji} S_{ji} = 2\mu S_{ji} S_{ij} = 2\mu \underline{\underline{S}} : \underline{\underline{S}} = 2\mu S^2$$

$$\rightarrow \text{ Always Positive!}$$

Viscous flows with fluid deformations dissipate energy irreversibly as heat. 115

Viscous Dissipation Example: Couette Flow



The steady-state Couette flow solution for the example shown in the figure is

$$u_1(x_3) = \frac{U}{H} x_3$$

$\rightarrow$ How much power is required to keep the top plate in motion?

The stress τ_{31} of the plate on the fluid $\tau_{31} = \mu \dfrac{\partial u_1}{\partial x_3} = \mu \dfrac{U}{H}$

The force per area A applied to the plate is $F = \tau_{31} A = \mu \dfrac{U}{H} A$

And the power (force $\cdot$ distance / time) applied per area A is

$$P = F \cdot U = \mu \frac{U^2}{H} A$$

$\rightarrow$ Where does this energy input go? 116

$\rightarrow$ Calculate dissipation of energy by viscous deformations:

$$\begin{aligned} \Phi_\nu &= 2\mu S_{ji} S_{ij} = 2\mu S^2 \\ &= 2\mu \left(S_{11}^2 + S_{12}^2 + S_{13}^2 + S_{21}^2 + S_{22}^2 + S_{23}^2 + S_{31}^2 + S_{32}^2 + S_{33}^2 \right) \end{aligned}$$

The only non-zero components of S_{ij} are

$$S_{13} = S_{31} = \frac{1}{2} \left(\cancelto{0}{\frac{\partial u_3}{\partial x_1}} + \frac{\partial u_1}{\partial x_3} \right) = \frac{U}{2H}$$

So

$$\Phi_\nu = 2\mu \left(S_{13}^2 + S_{31}^2 \right) = 2\mu \left(\frac{U^2}{4H^2} + \frac{U^2}{4H^2} \right) = \mu \frac{U^2}{H^2}$$

And the power dissipated in a column of water spanning the depth H with cross-sectional area A is

$$\iiint \Phi_\nu \, \mathrm{d}V \;=\; \Phi_\nu \int_0^H \mathrm{d}x_3 \iint_A \mathrm{d}A \;=\; \Phi_\nu H A \;=\; \mu \frac{U^2}{H} A$$

$\rightarrow$ All power input is dissipated out as heat 117

6 Scaling

To make inferences about the relative importance of various terms in a differential equation, it is necessary to *nondimensionalize* and *normalize* the equation.

- *nondimensionalization* simply removes the dimensions from the equation variables, using general scaling parameters to nondimensionalize the equation variables

- *normalization* uses scaling parameters specific to the problem at hand to ensure that the nondimensionalized equation variables are all order unity. This permits direct comparison of the leading nondimensional parameters that scale the variables. 118

6.1 Nondimensional Navier-Stokes

We begin by choosing general scales for the variables in the N-S equations:

Scaling Parameters

L characteristic length, units [L] U characteristic velocity, units [LT^{-1}] T characteristic timescale, units [T] ΔP characteristic pressure difference, units [ML^{-1}T^{-2}]

These scales are used to define nondimensional variables for the equations:

Nondimensional Variables

$$\begin{aligned}
\vec{x}^* &= \vec{x}/L \\
\vec{u}^* &= \vec{u}/U \\
t^* &= t/T \\
p^* &= p/\Delta p \\
\nabla^* &= L\nabla \\
\nabla^{*2} &= L^2\nabla^2
\end{aligned}$$

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Dimensional momentum equation:

$$\underbrace{\frac{\partial \vec{u}}{\partial t}}_{\text{local accel}} + \underbrace{\vec{u}\cdot\nabla\vec{u}}_{\text{advective accel}} = \underbrace{-\frac{1}{\rho}\nabla p}_{\text{pressure}} + \underbrace{\nu\nabla^2\vec{u}}_{\text{viscous}} + \underbrace{\vec{g}}_{\text{gravity}}$$

Substituting the nondimensional variables into the dimensional momentum equation gives:

$$\left(\frac{U}{T}\right)\frac{\partial \vec{u}^*}{\partial t^*} + \left(\frac{U^2}{L}\right)\vec{u}^*\cdot\nabla^*\vec{u}^* = -\left(\frac{\Delta p}{\rho\,L}\right)\nabla^* p^* + \left(\frac{\nu\,U}{L^2}\right)\nabla^{*2}\vec{u}^* + \vec{g}$$

Each additive term in the above equation has units LT^{-2}, with the units being carried by the parenthetical expressions.

Dividing the equation by *any* of the parenthetical terms will result in a nondimensional equation. 120

For example, dividing by the parenthetical expression U^2/L in front of the advective acceleration term gives:

$$\left(\frac{L}{UT}\right)\frac{\partial \vec{u}^*}{\partial t^*} + (1)\,\vec{u}^*\cdot\nabla^*\vec{u}^* = -\left(\frac{\Delta p}{\rho\,U^2}\right)\nabla^* p^* + \left(\frac{\nu}{U L}\right)\nabla^{*2}\vec{u}^* + \left(\frac{\vec{g}L}{U^2}\right)$$

which can be written as

$$\mathrm{St}\,\frac{\partial \vec{u}^*}{\partial t^*} + (1)\,\vec{u}^*\cdot\nabla^*\vec{u}^* = -\mathrm{Eu}\,\nabla^* p^* + \mathrm{Re}^{-1}\,\nabla^{*2}\vec{u}^* + \mathrm{Fr}^{-2}$$

where

$$\begin{aligned} \mathrm{St} &= \text{Strouhal number} \\ \mathrm{Eu} &= \text{Euler number} \\ \mathrm{Re} &= \text{Reynolds number} \\ \mathrm{Fr} &= \text{Froude number} \end{aligned}$$

As an example, if $\mathrm{Re} \gg 1$, can we eliminate the the viscous term $\mathrm{Re}^{-1}\nabla^{*2}\vec{u}^*$ as small relative to the advective acceleration term $(1)\vec{u}^*\cdot\nabla^*\vec{u}^*$?

→ No. Not until we normalize the equations. 121

6.2 Normalized Navier-Stokes

Normalizing the Navier-Stokes equations is simply a more restrictive form of nondimensionalization using scaling parameters that are appropriate to the specific problem. The goal of normalization is to choose the scaling parameters such that the maximum magnitude of each of the nondimensional variables is order unity.

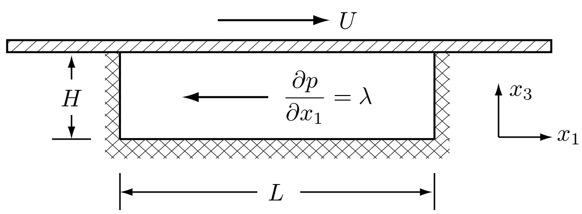
→ We will demonstrate the technique via an example problem. 122

Example Problem

Consider a modified version of the P-C flow, with the following changes:

- finite length L in the x_1 direction
- lid starts moving at $t = 0$, we are solving for the solution at time $t = T$

Note that, for simplicity, we only consider flow in the $x_1 - x_3$ plane.



Under what conditions can we get s simple solution for $u_1(x_3)$? 123

- Step I: Choose scales specific to this problem:

$$
\begin{aligned}
x_1 &= L\, {x_1}^* \\
x_3 &= H\, {x_3}^* \\
u_1 &= U\, {u_1}^* \\
u_3 &= W\, {u_3}^* \qquad (W \text{ unknown at this point}) \\
t &= T\, t^*
\end{aligned}
$$

Note that we don't need to scale p since $\partial p / \partial x_1 = \lambda$ is known.

- Step II: Relate velocity scales through continuity:

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3} = 0$$

$$\left(\frac{U}{L}\right) \frac{\partial {u_1}^*}{\partial {x_1}^*} + \left(\frac{W}{H}\right) \frac{\partial {u_3}^*}{\partial {x_3}^*} = 0$$

So

$$\frac{U}{L} \sim \frac{W}{H} \qquad \rightarrow \qquad W \sim \frac{H}{L} U$$

Now W is known in terms of the parameters for this problem. 124

- Step III: Nondimensionalize x_1-momentum equation using specific scales

$$\left(\frac{U}{T}\right) \frac{\partial {u_1}^*}{\partial t^*} + (U)\left(\frac{U}{L}\right) {u_1}^* \frac{\partial {u_1}^*}{\partial {x_1}^*} + \left(\frac{H}{L} U\right)\left(\frac{U}{H}\right) {u_3}^* \frac{\partial {u_1}^*}{\partial {x_3}^*} =$$

$$-\frac{\lambda}{\rho} + \nu \left[\left(\frac{U}{L^2}\right) \frac{\partial^2 {u_1}^*}{\partial {x_1^*}^2} + \left(\frac{U}{H^2}\right) \frac{\partial^2 {u_1}^*}{\partial {x_3^*}^2} \right]$$

which can be rewritten as

$$\left(\frac{U}{T}\right)\frac{\partial {u_1}^*}{\partial t^*} + \left(\frac{U^2}{L}\right)\left[{u_1}^*\frac{\partial {u_1}^*}{\partial {x_1}^*} + {u_3}^*\frac{\partial {u_1}^*}{\partial {x_3}^*}\right] =$$

$$-\frac{\lambda}{\rho} + \frac{\nu U}{H^2}\left[\left(\frac{H}{L}\right)^2\frac{\partial^2 {u_1}^*}{\partial {{x_1}^*}^2} + \frac{\partial^2 {u_1}^*}{\partial {{x_3}^*}^2}\right]$$

→ advective accelerations always share the same scaling group

→ viscous terms will only differ by aspect ratio(s) 125

- Step IV: Divide by a dominant scale

  Here, we choose the viscous scale $\nu U/H^2$:

$$\left(\frac{H^2}{\nu T}\right)\frac{\partial {u_1}^*}{\partial t^*} + \left(\frac{UL}{\nu}\right)\left(\frac{H}{L}\right)^2\left[{u_1}^*\frac{\partial {u_1}^*}{\partial {x_1}^*} + {u_3}^*\frac{\partial {u_1}^*}{\partial {x_3}^*}\right] =$$

$$-\frac{\lambda H^2}{\rho\nu U} + \left(\frac{H}{L}\right)^2\frac{\partial^2 {u_1}^*}{\partial {{x_1}^*}^2} + \frac{\partial^2 {u_1}^*}{\partial {{x_3}^*}^2}$$

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- Step V: Compare size of terms relative to unity (the dominant scale)

→ if $\left(\frac{H}{L}\right)^2 \ll 1$, then x_1 diffusion of x_1 momentum $\frac{\partial^2 {u_1}^*}{\partial {{x_1}^*}^2}$ is small

→ if $\left(\frac{UL}{\nu}\right)\left(\frac{H}{L}\right)^2 \ll 1$, then both advective accelerations are small
Note that this condition is $\left(\frac{UL}{\nu}\right) \ll \left(\frac{L}{H}\right)^2$

→ if $\frac{H^2}{\nu T} \ll 1$, then the unsteadiness term is small → steady-state
Note that this condition is $T \gg \frac{H^2}{\nu}$ 127

If all three of these condiitons are met, then the equation

reduces to

$$\frac{\partial^2 {u_1}^*}{\partial {{x_3}^*}^2} = 2\mathbb{P} \qquad \text{where } \mathbb{P} = \frac{\lambda H^2}{2\mu U}$$

- Step VI: Nondimensionalize boundary and initial conditions, as appropriate

$$\begin{aligned} {u_1}^*({x_3}^* = 0) &= 0 \\ {u_1}^*({x_3}^* = 1) &= 1 \end{aligned}$$

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- Step VII: Solve

Integrating twice gives:

$$u_1^*(x_3^*) \;=\; \mathbb{P}\, x_3^{*2} \;+\; C_1 x_3^* \;+\; C_2$$

Apply B.C.'s:

$$u_1^*(x_3^* = 0) = 0 \quad\rightarrow\quad C_2 = 0$$
$$u_1^*(x_3^* = 1) = 1 \quad\rightarrow\quad C_1 = 1 - \mathbb{P}$$

which leaves the nondimensional solution

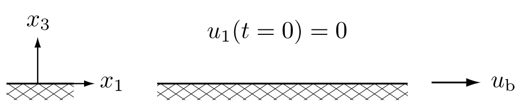
$$u_1^*(x_3^*) \;=\; P\, x_3^{*2} \;+\; (1-P)x_3^* \qquad \text{where}\;\; \mathbb{P} = \frac{\lambda H^2}{2\mu U}$$

or, dimensionally,

$$u_1(x_3) \;=\; U\left[\mathbb{P}\left(\frac{x_3}{H}\right)^2 \;+\; (1-\mathbb{P})\left(\frac{x_3}{H}\right)\right]$$ 129

7 Unsteady Solutions

7.1 Stokes I: Impulsively started boundary



$$u_\mathrm{b} \;=\; \begin{cases} 0 & t < 0 \\ U & t \geq 0 \end{cases}$$

→ Solve for $u_1(t)$ for $t \geq 0$, $x_3 \geq 0$.

Assumptions:

- One-dimensional flow: $u_2 = u_3 = 0$, $\dfrac{\partial u_1}{\partial x_1} = \dfrac{\partial u_1}{\partial x_2} = 0$

Boundary and Initial Conditions:

$u_1(x_3 = 0) = U$
$u_1(x_3 = \infty) = 0$
$u_1(t = 0) = 0$ 130

Solve the x_1-direction momentum equation:

$$\frac{\partial u_1}{\partial t} + \cancel{u_1\frac{\partial u_1}{\partial x_1}} + \cancel{u_2\frac{\partial u_1}{\partial x_2}} + \cancel{u_3\frac{\partial u_1}{\partial x_3}} \;=\; \cancel{-\frac{1}{\rho}\frac{\partial p}{\partial x_1}} + \nu\left(\cancel{\frac{\partial^2 u_1}{\partial x_1^2}} \;+\; \cancel{\frac{\partial^2 u_1}{\partial x_2^2}} \;+\; \frac{\partial^2 u_1}{\partial x_3^2}\right) + \cancel{g_1}$$

which leaves

$$\boxed{\underbrace{\frac{\partial u_1}{\partial t}}_{\text{unsteady horiz. accel.}} \;=\; \underbrace{\nu\frac{\partial^2 u_1}{\partial x_3^2}}_{\text{vertical diffusion of horiz. mom.}}}$$

→ balance between inertia and viscous shear forces 131

Nondimensionalize the problem

- Choose scales specific to this problem:

$$
\begin{aligned}
u_1 &= U\, {u_1}^* \\
x_3 &= H\, {x_3}^* \quad (H \text{ unknown at this point}) \\
t &= T\, t^* \quad (T \text{ unknown at this point})
\end{aligned}
$$

$$\left(\frac{U}{T}\right) \frac{\partial {u_1}^*}{\partial t^*} = \left(\frac{U}{H^2}\right) \nu \frac{\partial^2 {u_1}^*}{\partial {x_3^*}^2}$$

or

$$\frac{\partial {u_1}^*}{\partial t^*} = \left(\frac{\nu T}{H^2}\right) \frac{\partial^2 {u_1}^*}{\partial {x_3^*}^2}$$

If we choose $H = \sqrt{\nu T}$ then the PDE becomes parameter-free:

$$\boxed{\frac{\partial {u_1}^*}{\partial t^*} = \frac{\partial^2 {u_1}^*}{\partial {x_3^*}^2}}$$

→ Our choice of $H = \sqrt{\nu T}$ implies that the vertical scale of interest grows with time, specifically, as $t^{1/2}$. 132

So we are now trying to solve (for $t^* \geq 0$) the non-dimensional PDE

$$\frac{\partial {u_1}^*}{\partial t^*} = \frac{\partial^2 {u_1}^*}{\partial {x_3^*}^2}$$

subject to

$$
\begin{aligned}
{u_1}^*(0, t^*) &= 1 \\
{u_1}^*(\infty, t^*) &= 0 \\
{u_1}^*({x_3}^*, 0) &= 0
\end{aligned}
$$

Although the governing equation is a PDE, the $H = \sqrt{\nu T}$ scaling suggest that the solution does not depend on x_3 and t independently:

$$\frac{x_3}{\sqrt{t}} \sim \frac{H}{\sqrt{T}} = \sqrt{\nu} = \text{constant}$$

→ This motivates us to search for a solution in terms of a similarity variable η, where

$$\eta \equiv \frac{{x_3}^*}{\sqrt{t^*}}$$

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Similarity Solution

The goal of the similarity solution is to transform the PDE (and associated constraints) to an ODE as follows:

$${u_1}^* = {u_1}^*({x_3}^*, t^*) \qquad \Rightarrow \qquad {u_1}^* = {u_1}^*(\eta) \quad \text{where} \quad \eta \equiv \frac{{x_3}^*}{\sqrt{t^*}}$$

→ Use chain rule to transform PDE derivatives (in terms of ${x_3}^*, t^*$) to ODE derivatives (in terms of η):

$$\frac{\partial {u_1}^*}{\partial t^*} = \frac{\mathrm{d}{u_1}^*}{\mathrm{d}\eta}\frac{\partial \eta}{\partial t^*} = -\frac{1}{2}\,{x_3}^*\,{t^*}^{-3/2}\ \frac{\mathrm{d}{u_1}^*}{\mathrm{d}\eta}$$

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$$\begin{aligned}
\frac{\partial^2 {u_1}^*}{\partial {{x_3}^*}^2} &= \frac{\partial}{\partial {x_3}^*}\left(\frac{\partial {u_1}^*}{\partial {x_3}^*}\right) = \frac{\partial}{\partial {x_3}^*}\left(\frac{\mathrm{d}{u_1}^*}{\mathrm{d}\eta}\frac{\partial \eta}{\partial {x_3}^*}\right) \\
&= \frac{\mathrm{d}{u_1}^*}{\mathrm{d}\eta}\cancelto{0}{\frac{\partial}{\partial {x_3}^*}\left(\frac{\partial \eta}{\partial {x_3}^*}\right)} + \frac{\partial \eta}{\partial {x_3}^*}\frac{\partial}{\partial {x_3}^*}\left(\frac{\mathrm{d}{u_1}^*}{\mathrm{d}\eta}\right) \\
&= \frac{\partial \eta}{\partial {x_3}^*}\frac{\partial}{\partial \eta^*}\left(\frac{\mathrm{d}{u_1}^*}{\mathrm{d}\eta}\right)\frac{\partial \eta}{\partial {x_3}^*} \\
&= \left(\frac{\partial \eta}{\partial {x_3}^*}\right)^2\ \frac{\partial^2 {u_1}^*}{\partial \eta^2} \\
&= {t^*}^{-1}\ \frac{\partial^2 {u_1}^*}{\partial \eta^2}
\end{aligned}$$

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Substituting these two expressions back into the PDE

$$\frac{\partial {u_1}^*}{\partial t^*} = \frac{\partial^2 {u_1}^*}{\partial {{x_3}^*}^2}$$

gives

$$-\frac{1}{2}\,{x_3}^*\,{t^*}^{-3/2}\ \frac{\mathrm{d}{u_1}^*}{\mathrm{d}\eta} = t^{-1}\ \frac{\partial^2 {u_1}^*}{\partial \eta^2}$$

or

$$\frac{\partial^2 {u_1}^*}{\partial \eta^2} + \frac{1}{2}\underbrace{\frac{{x_3}^*}{\sqrt{t^*}}}_{\eta}\frac{\mathrm{d}{u_1}^*}{\mathrm{d}\eta} = 0$$

So we are left with an ODE in terms of η, which means the similarity solution is working!

$$\frac{\partial^2 {u_1}^*}{\partial \eta^2} + \frac{\eta}{2}\frac{\mathrm{d}{u_1}^*}{\mathrm{d}\eta} = 0$$

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If we let

$$f(\eta) \equiv \frac{\mathrm{d}{u_1}^*}{\mathrm{d}\eta}$$

then the second-order ODE can be transformed to a first-order ODE:

$$\frac{\partial^2 {u_1}^*}{\partial \eta^2} + \frac{\eta}{2}\frac{\mathrm{d}{u_1}^*}{\mathrm{d}\eta} = 0 \qquad \Rightarrow \qquad \frac{\mathrm{d}f}{\mathrm{d}\eta} + \frac{\eta}{2}f$$

Now separate variables,

$$\frac{\mathrm{d}f}{f} = -\frac{1}{2}\eta f$$

integrate,

$$\ln f = -\frac{1}{4}\eta^2 + C$$

and raise both sides to e to get

$$f = A \exp\left[-\frac{\eta^2}{4}\right]$$

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So we are left to solve:

$$\frac{\mathrm{d}{u_1}^*}{\mathrm{d}\eta} = A \exp\left[-\frac{\eta^2}{4}\right]$$

subject to the reduced set of constraints in terms of η:

$$\begin{aligned} {u_1}^*(\eta = \infty) &= 0 \\ {u_1}^*(\eta = 0) &= 1 \end{aligned}$$

Integrate from η to one of the boundary condition locations (∞):

$$\int_\eta^\infty \mathrm{d}\,{u_1}^* = A \int_\eta^\infty \exp\left[-\frac{\xi^2}{4}\right] \mathrm{d}\xi$$

$$\cancelto{0}{{u_1}^*(\eta = \infty)} - {u_1}^*(\eta) = A \int_\eta^\infty \exp\left[-\frac{\xi^2}{4}\right] \mathrm{d}\xi$$

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Now apply second B.C. ${u_1}^*(\eta = 0) = 1$:

$$\underbrace{{u_1}^*(\eta = 0)}_{=1} = -A \int_0^\infty \exp\left[-\frac{\xi^2}{4}\right] \mathrm{d}\xi \qquad \Rightarrow \qquad A = -\frac{1}{\displaystyle\int_0^\infty \exp\left[-\frac{\xi^2}{4}\right] \mathrm{d}\xi} = -\frac{1}{\sqrt{\pi}}$$

so

$$\begin{aligned} {u_1}^*(\eta) &= \frac{1}{\sqrt{\pi}} \int_\eta^\infty \exp\left[-\frac{\xi^2}{4}\right] \mathrm{d}\xi \\ &= \frac{1}{\sqrt{\pi}} \left\{ \underbrace{\int_0^\infty \exp\left[-\frac{\xi^2}{4}\right] \mathrm{d}\xi}_{=\sqrt{\pi}} - \int_0^\eta \exp\left[-\frac{\xi^2}{4}\right] \mathrm{d}\xi \right\} \end{aligned}$$

leaving

$$ {u_1}^*(\eta) = 1 - \frac{1}{\sqrt{\pi}} \int_0^\eta \exp\left[-\frac{\xi^2}{4}\right] \mathrm{d}\xi$$

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Finally, use the variable substitution $\xi = 2\theta$, $\mathrm{d}\xi = 2\mathrm{d}\theta$ to get :

$$u_1{}^*(\eta) \;=\; 1 \;-\; \underbrace{\frac{2}{\sqrt{\pi}} \int_0^{\eta/2} \exp\left[-\theta^2\right] \mathrm{d}\theta}_{\equiv\,\mathrm{erf}(\eta/2)}$$

giving the final nondimensional solution (in terms of η):

$$\boxed{u_1{}^*(\eta) \;=\; 1 \;-\; \mathrm{erf}(\eta/2)}$$

or, dimensionally (in terms of x_3 and t):

$$\boxed{u_1(x_3, t) \;=\; U\left[1 \;-\; \mathrm{erf}\left(\frac{x_3}{\sqrt{4\nu t}}\right)\right]}$$

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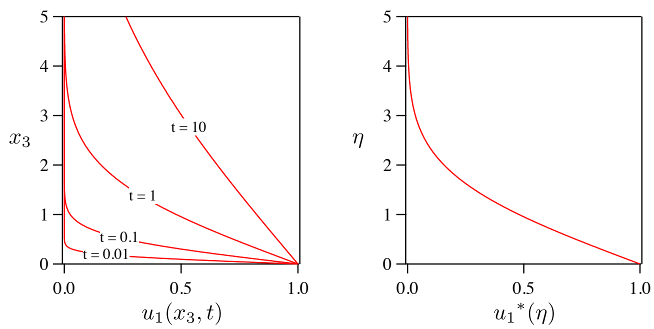
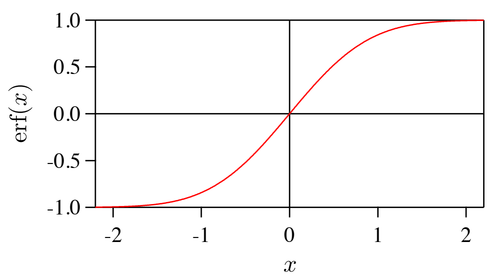
Solution to Stokes I



Note: The dimensional plot (left) is for $U = 1$ and $\nu = 1$. 141

The Error Function

$$\mathrm{erf}(x) \;\equiv\; \frac{2}{\sqrt{\pi}} \int_0^{x} \exp\left[-\xi^2\right] \mathrm{d}\xi$$



- $\mathrm{erf}(0) \;=\; 0$
- $\mathrm{erf}(x) \;=\; -\mathrm{erf}(-x)$
- $\mathrm{erf}(x) \approx 1$ when $x > 2$, $\qquad \mathrm{erf}(x) \approx -1$ when $x < -2$
- $\mathrm{erfc}(x) \;\equiv\; 1 - \mathrm{erf}(x)$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{erf}(x) \;=\; \dfrac{2}{\sqrt{\pi}} \exp\left[-x^2\right]$

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A vorticity-based interpretation to Stokes I

$$\vec{\omega} = \nabla \times \vec{u}$$

$$\omega_2 = \frac{\partial u_1}{\partial x_3} - \cancelto{0}{\frac{\partial u_3}{\partial x_1}} = -\frac{2}{\sqrt{\pi}}\frac{U}{\sqrt{4\nu t}}\exp\left[-\frac{{x_3}^2}{4\nu t}\right]$$

Note that

$$\lim_{t\to 0}\,\omega_2 = \begin{cases} 0 & x_3 > 0 \\ -\infty & x_3 = 0 \end{cases}$$

- An initial spike of vorticity forms at the boundary at $t = 0$,
- Vorticity subsequently diffuses into the flow.
- No vorticity formed or destroyed after $t = 0$.
- Circulation over the total depth (total vorticity) is constant.

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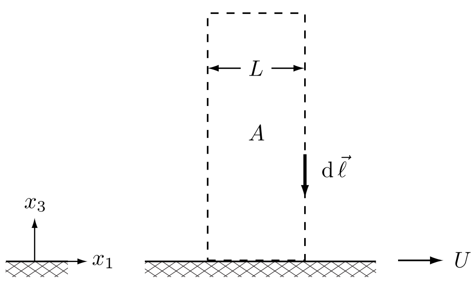

$$\Gamma = \iint_A \omega_2 \,\mathrm{d}\,A = \oint_C \vec{u}\cdot\mathrm{d}\vec{\ell} = -U\,L = \text{constant}$$

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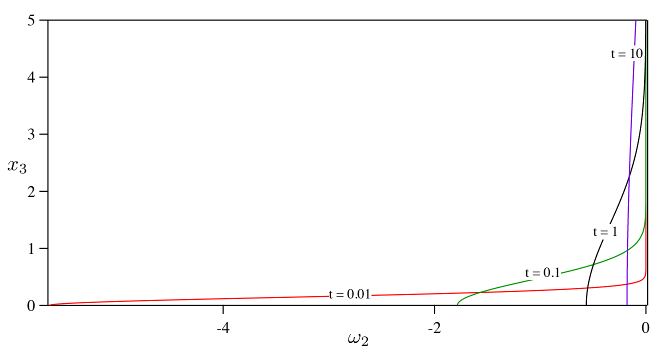

- Vorticity diffuses away from boundary
- Total vorticity conserved.

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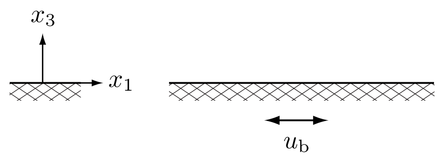

7.2 Stokes II: Oscillating Boundary

$$u_\mathrm{b} \;=\; U \sin \Omega t$$

→ Find quasi-steady state solution $u_1(t)$ for $x_3 \geq 0$.

Assumptions:

- One-dimensional flow: $u_2 = u_3 = 0$, $\dfrac{\partial u_1}{\partial x_1} = \dfrac{\partial u_1}{\partial x_2} = 0$

Boundary Conditions:
$u_1(x_3 = 0) = U \sin \Omega t$
$u_1(x_3 = \infty) = 0$ 146

Nondimensionalize the problem

The governing equation is the same as for Stokes I:

$$\boxed{\underbrace{\frac{\partial u_1}{\partial t}}_{\text{unsteady horiz. accel.}} \quad = \quad \underbrace{\nu \frac{\partial^2 u_1}{\partial x_3^2}}_{\text{vertical diffusion of horiz. mom.}}}$$

- Choose scales specific to this problem:

$$\begin{aligned} u_1 &= U\, {u_1}^* \\ t &= \frac{1}{\Omega}\, t^* \\ x_3 &= \left(\frac{\nu}{\Omega}\right)^{1/2} {x_3}^* \end{aligned}$$

$$(U\,\Omega)\,\frac{\partial {u_1}^*}{\partial t^*} \;=\; \left(\frac{U}{\nu/\Omega}\right) \nu \frac{\partial^2 {u_1}^*}{\partial {x_3^*}^2}$$

or

$$\frac{\partial {u_1}^*}{\partial t^*} \;=\; \frac{\partial^2 {u_1}^*}{\partial {x_3^*}^2} \qquad \rightarrow \text{parameter-free!}$$

B.C.'s $\quad {u_1}^*(0, t^*) = \sin t^* \qquad {u_1}^*(\infty, t^*) = 0$ 147

Solve using complex variables

Recall that $e^{it} \;=\; \cos t \;+\; i \sin t$

Assume an oscillating solution of the form

$$\boxed{{u_1}^* \;=\; \underbrace{f({x_3}^*)}_{\text{shape}}\, \underbrace{\exp(it^*)}_{\text{oscillation}}}$$

with the boundary conditions written as

$$\begin{aligned} {u_1}^*(0,t^*) &= \sin t* \;=\; \mathrm{Im}\left\{\exp(it^*)\right\} \\ {u_1}^*(\infty,t^*) &= 0 \end{aligned}$$

→ Since the imaginary part of the B.C. is driving the problem, we will take the imaginary part of the resulting solution as our asnwer. 148

Substituting the assumed form ${u_1}^* = f({x_3}^*)\exp(it^*)$ into the governing equation gives

$$\frac{\partial {u_1}^*}{\partial t^*} = \frac{\partial^2 {u_1}^*}{\partial {{x_3}^*}^2} \qquad \Rightarrow \qquad (f'' - if)\exp(it^*) = 0$$

This has a solution of the form

$$\boxed{f \;=\; A\exp\left(a{x_3}^*\right)}$$

where A and a may be complex.

If we choose

$$a \;=\; \pm\sqrt{i} \;=\; \pm\frac{1+i}{\sqrt{2}}$$

then $(f'' - if) = 0$, which guarantees that $(f'' - if)\exp(it^*) = 0$. 149

Thus we have a solution that looks like

$${u_1}^* = \mathrm{Im}\left\{A\exp\left[\pm\frac{1+i}{\sqrt{2}}{x_3}^*\right]\exp(it^*)\right\}$$

In order to satisfy the first B.C. ${u_1}^*(0,t^*) = \mathrm{Im}\left\{\exp(it^*)\right\}$, we have $A = 1$.

In order to satisfy the second B.C. ${u_1}^*(\infty,t^*) = 0$, choose the negative a.

This leaves the final solution

$${u_1}^* = \mathrm{Im}\left\{\exp\left[-\frac{1+i}{\sqrt{2}}{x_3}^*\right]\exp(it^*)\right\}$$

or

$$\boxed{{u_1}^* = \exp\left(-\frac{{x_3}^*}{\sqrt{2}}\right)\sin\left(t^* - \frac{{x_3}^*}{\sqrt{2}}\right)}$$

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- Motion is confined to the region near the boundary.
- For ${x_3}^* > 6.5$ the velocity $|{u_1}^*| < 0.01$

We can define the depth of the resulting boundary layer as:

$$\delta_{01}^* \;=\; 6.5 \qquad \text{or} \qquad \delta_{01} \;=\; 6.5\left(\frac{\nu}{\Omega}\right)^{1/2}$$

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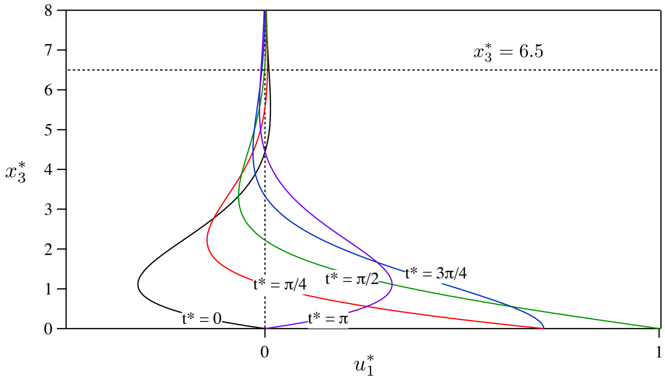


7.3 Stokes II: Tidal Flow

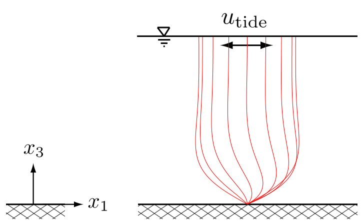
The Stokes II problem as solved was for

- a stationary freestream
- an oscillating boundary

Through a simple coordinate transformation, we can create a solution for

- an oscillating freestream
- a fixed boundary

The latter case cooresponds to a tidal flow in an estuary:



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A freestream tidal flow of

$$u_{\text{tide}} = U \sin t$$

corresponds to the Stokes II problem with a wall motion of $u_\text{b} = -U \sin t$:

$$u_1{}^* = -\exp\left(-\frac{x_3{}^*}{\sqrt{2}}\right) \sin\left(t^* - \frac{x_3{}^*}{\sqrt{2}}\right)$$

To this, we add a flow of $u_1 = U \sin t$ to accomplish the transformation:

$$\boxed{u_1{}^* = -\exp\left(-\frac{x_3{}^*}{\sqrt{2}}\right) \sin\left(t^* - \frac{x_3{}^*}{\sqrt{2}}\right) \;+\; \sin t^*}$$

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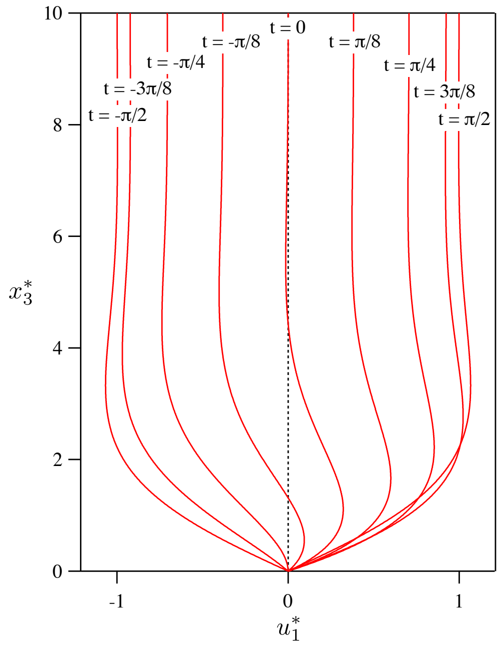
Velocity profiles for one-half of a tidal cycle:



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8 Appendix: Equations in Cartesian and Cylindrical Coordinates

8.1 Continuity (for constant ρ)

$$\nabla \cdot \vec{u} = 0$$

Cartesian (x_1, x_2, x_3):

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0$$

Cylindrical (u_r, u_θ, u_z):

$$\frac{1}{r}\frac{\partial}{\partial r}(r\, v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

8.2 Momentum (for constant ρ and μ)

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho}\nabla p + \nu \frac{\partial^2 \vec{u}}{\partial \vec{x}^2} + \vec{g}$$

Cartesian (x_1, x_2, x_3):

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} + u_3\frac{\partial u_1}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right) + g_1$$

$$\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} + u_3\frac{\partial u_2}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_2} + \nu\left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2}\right) + g_2$$

$$\frac{\partial u_3}{\partial t} + u_1\frac{\partial u_3}{\partial x_1} + u_2\frac{\partial u_3}{\partial x_2} + u_3\frac{\partial u_3}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_3} + \nu\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2}\right) + g_3$$

Cylindrical (u_r, u_θ, u_z):

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{{v_\theta}^2}{r} + v_z\frac{\partial v_r}{\partial z} = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\, v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + g_r$$

$$\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r\, v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\, v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + g_\theta$$

$$\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + g_z$$

8.3 Components of the Vorticity Vector

$$\vec{\omega} = \nabla \times \vec{u}$$

Cartesian (x_1, x_2, x_3):

$$\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}$$

$$\omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}$$

$$\omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

Cylindrical (u_r, u_θ, u_z):

$$\omega_r = \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}$$

$$\omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}$$

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r\, u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

8.4 Components of the Rate of Strain Tensor

$\underline{\underline{S}}$ is the symmetry portion of the tensor $\nabla \vec{u}$

Cartesian (x_1, x_2, x_3):

$$S_{11} = \frac{\partial u_1}{\partial x_1}$$

$$S_{22} = \frac{\partial u_2}{\partial x_2}$$

$$S_{33} = \frac{\partial u_3}{\partial x_3}$$

$$S_{12} = S_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)$$

$$S_{13} = S_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)$$

$$S_{23} = S_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right)$$

Cylindrical (u_r, u_θ, u_z):

$$S_{rr} = \frac{\partial u_r}{\partial r}$$

$$S_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{v_r}{r}$$

$$S_{zz} = \frac{\partial u_z}{\partial z}$$

$$S_{r\theta} = S_{\theta r} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$S_{rz} = S_{zr} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)$$

$$S_{\theta z} = S_{z\theta} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$