Unstructured adaptive mesh generation and sparse matrix storage applied to Stokes flow around cylinders

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E521
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Project Goals

1. Solve the 2D Steady Stokes equations via finite elements
2. Investigate the placement of cylindrical obstructions in the flow field
3. Adaptively generate the finite element mesh
4. Utilize sparse matrix storage

Stokes equations - Simplification

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And we arrive at the **Stokes equations**.
And for the continuity equation, assume incompressible,

$$\nabla \cdot \mathbf{V} = 0 \tag{2}$$

We have enough information to describe any **very** viscous and/or slow moving fluid ($RE \ll 1$,creeping flow).

Formal Problem Statement

Scalar Equations:

$$-\nu\frac{\partial^2 u}{\partial x^2} - \nu\frac{\partial^2 u}{\partial y^2} + \frac{\partial p}{\partial x} = 0$$

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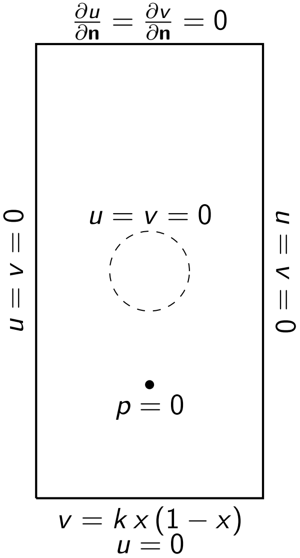
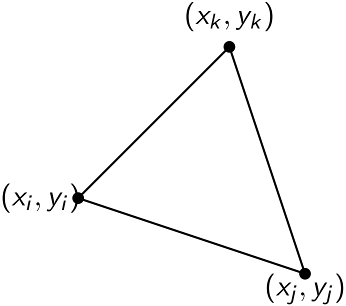
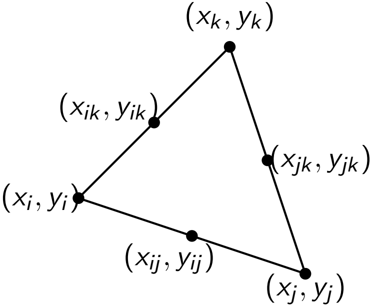


Basis Functions



Linear Pressure Element.



Quadratic Velocity Element.

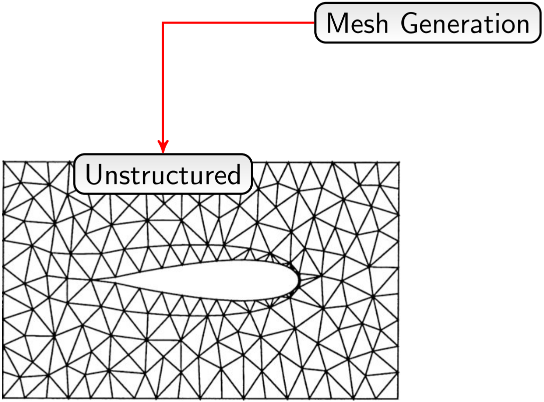
How, you ask, do we generate a mesh which accommodates holes and adaptively refines itself?

Constrained Adaptive Mesh Generation

Mesh Generation

[Basic Structured Grid Generation, Farrashkhalvat and Miles 2003]

Constrained Adaptive Mesh Generation



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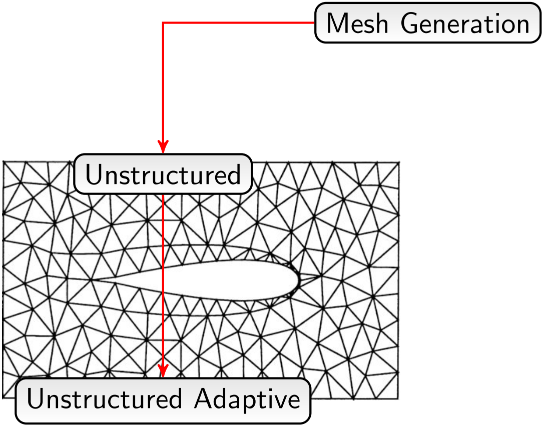

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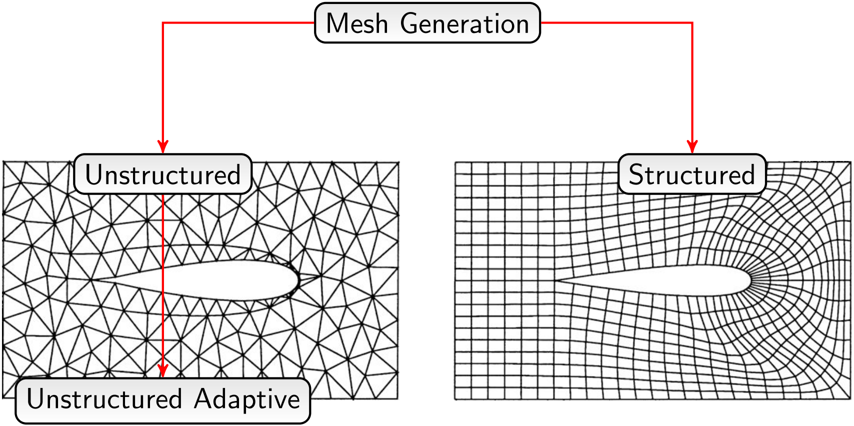

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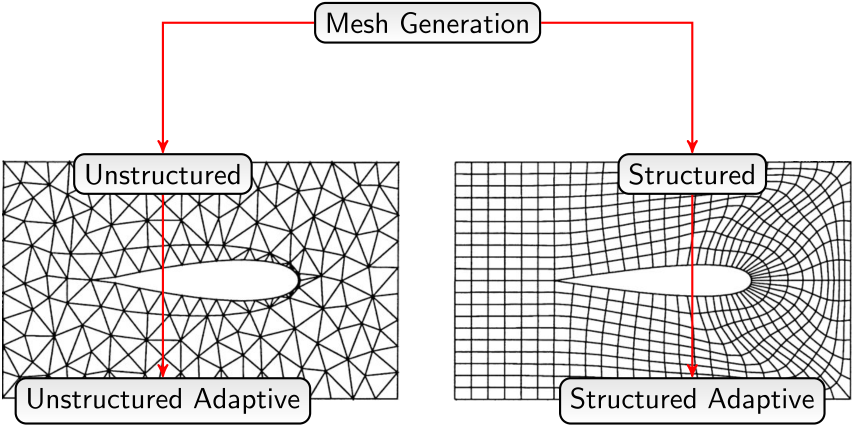
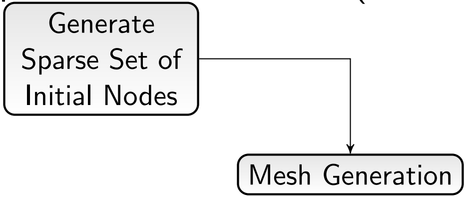


[Basic Structured Grid Generation, Farrashkhalvat and Miles 2003]

Adaptive mesh framework (a.k.a. Error controlled FE)

Generate
Sparse Set of
Initial Nodes

Adaptive mesh framework (a.k.a. Error controlled FE)



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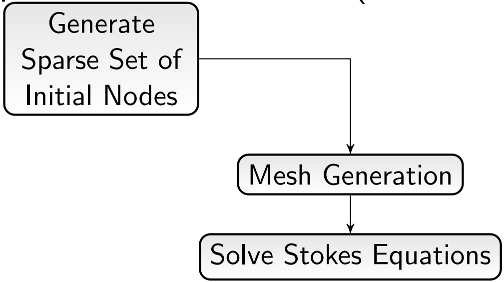

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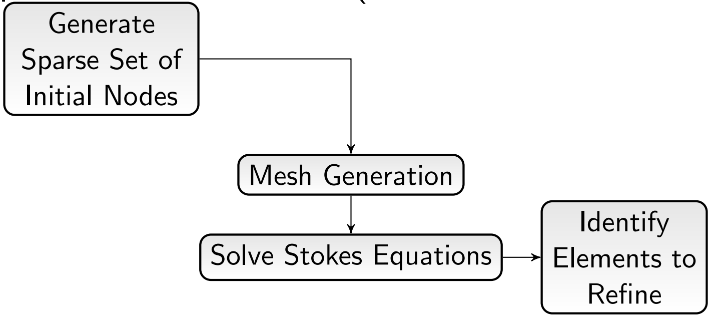

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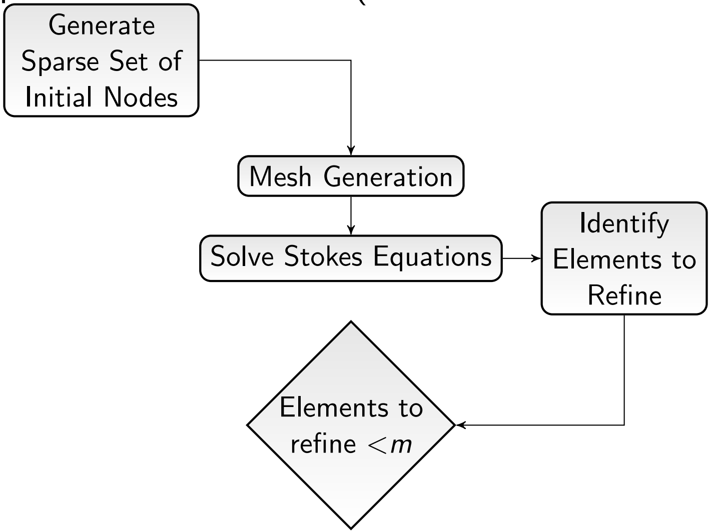

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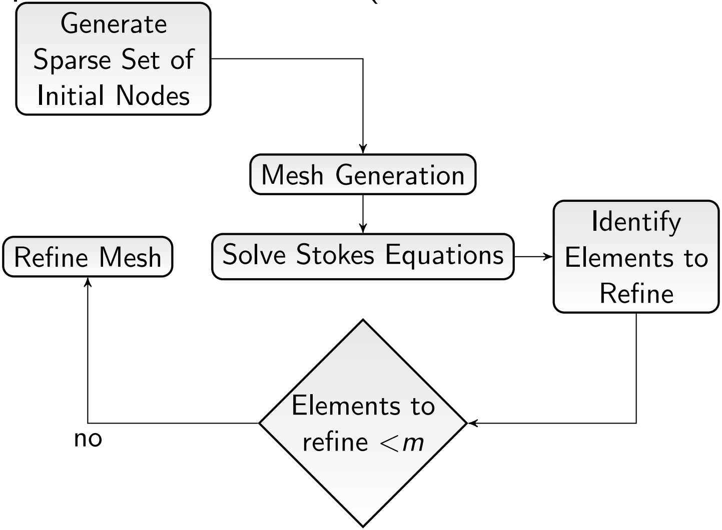

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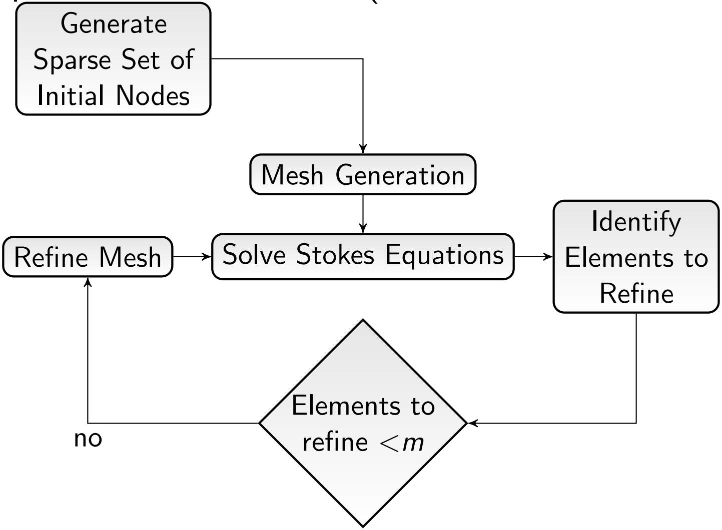

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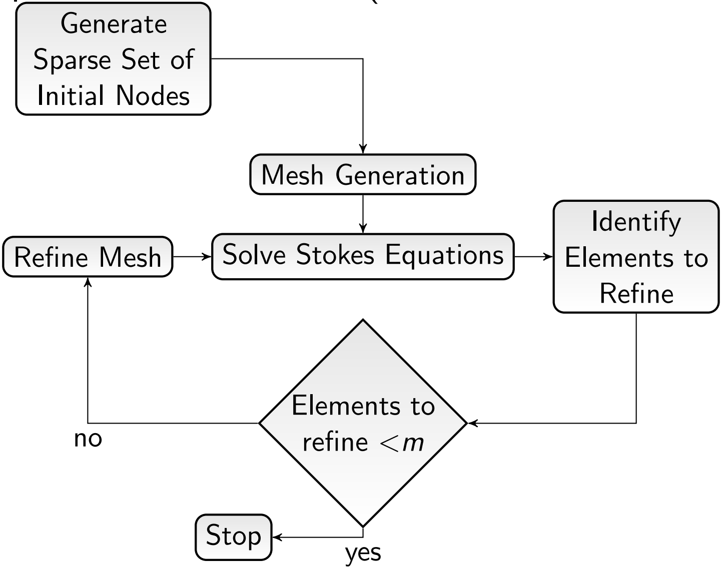

How is this efficient?

The Error Indicator and Relative Error Measurement

- Typically we take some measure of the solution gradient as an indicator of elemental error.

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Let θ^e be the indicator for an element e with vertex nodes i, j, k:

$$E_e = \frac{(\text{Maximum nodal value} - \text{Minimum Nodal Value})_e}{\text{Average difference in max and min nodal values over all elements}}$$

$$E_e = \frac{\max\left(\theta_i^e, \theta_j^e, \theta_k^e\right) - \min\left(\theta_i^e, \theta_j^e, \theta_k^e\right)}{\frac{1}{nele}\sum_{n=1}^{nele}\left[\max\left(\theta_i^n, \theta_j^n, \theta_k^n\right) - \min\left(\theta_i^n, \theta_j^n, \theta_k^n\right)\right]}$$

Simulation Model Results - Verification

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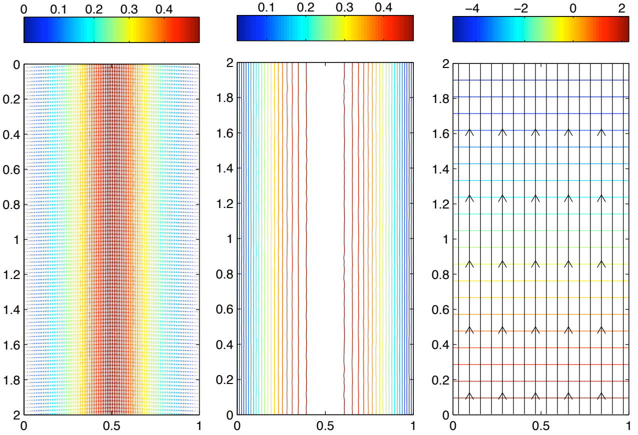
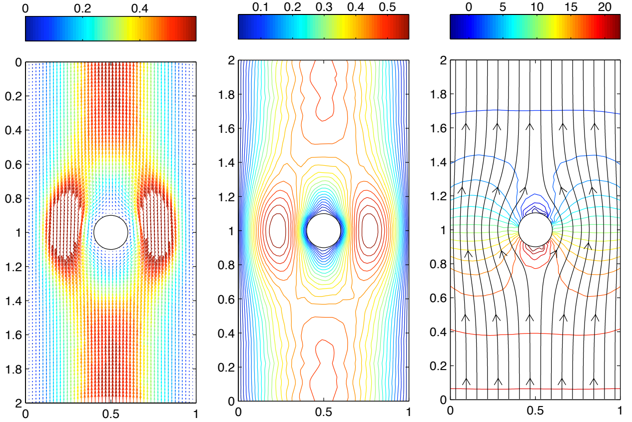


(d) Velocity Field (e) Velocity Contours (f) Streamlines and Pressure Contours

1 Obstruction

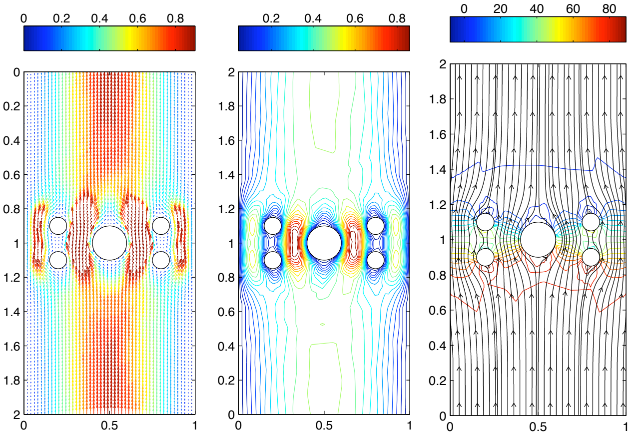


(a) Velocity Field

(b) Velocity Contours

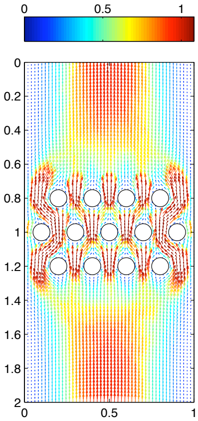
(c) Streamlines and Pressure Contours

5 Obstructions

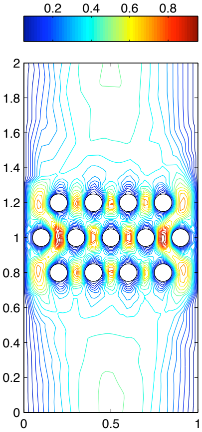


(a) Velocity Field (b) Velocity Contours (c) Streamlines and Pressure Contours

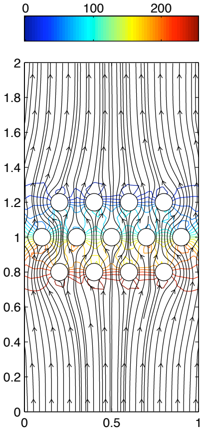
13 obstructions



(a) Velocity Field

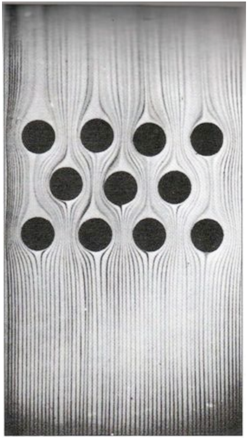


(b) Velocity Contours



(c) Streamlines and Pressure Contours

Additional Verification

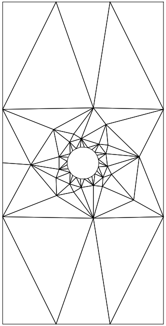


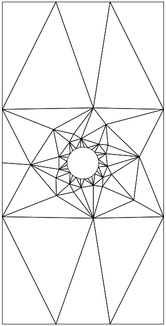
[TQ Education and Training Ltd.]

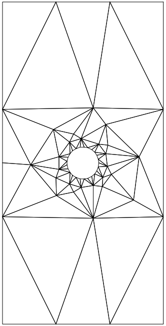
Meshing Results

Pressure Horizontal velocity Vertical velocity

Initial Mesh



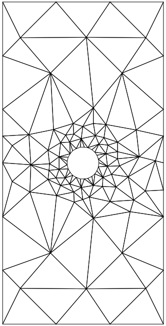


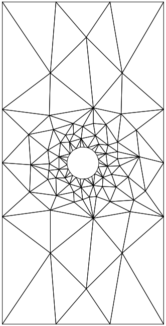


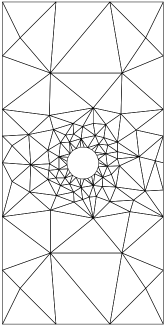
Meshing Results

Pressure Horizontal velocity Vertical velocity

Refinement 1



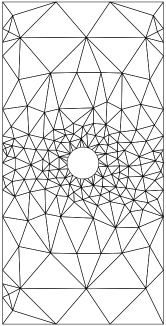


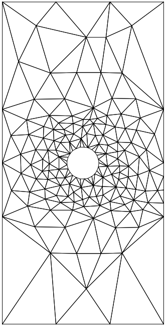


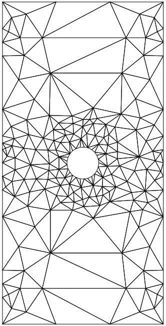
Meshing Results

Pressure Horizontal velocity Vertical velocity

Refinement 2



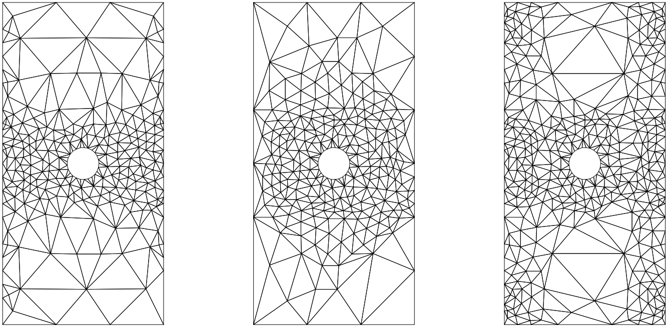




Meshing Results

Pressure Horizontal velocity Vertical velocity

Refinement 3

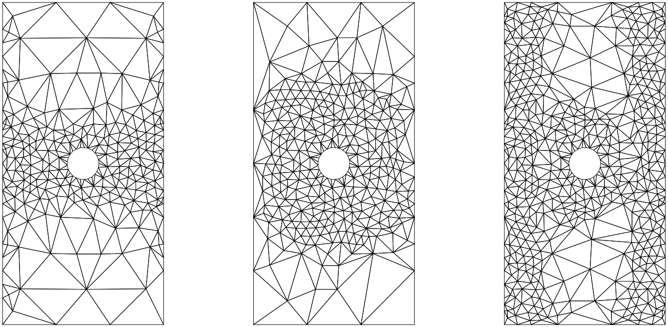


Done

Meshing Results

Pressure Horizontal velocity Vertical velocity

Refinement 4

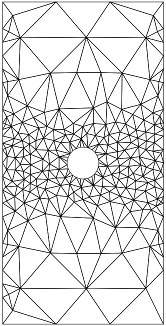


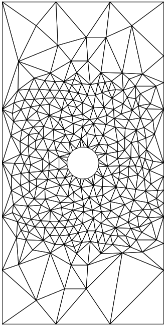
Done

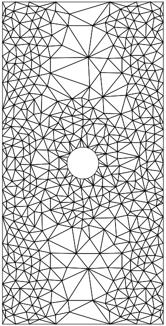
Meshing Results

Pressure Horizontal velocity Vertical velocity

Refinement 5





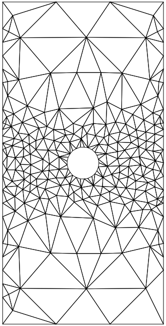


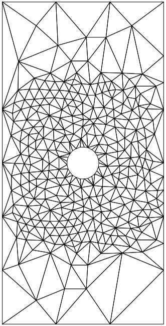
Done Done

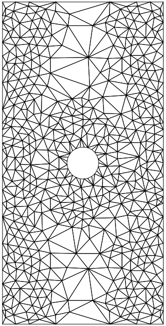
Meshing Results

Pressure Horizontal velocity Vertical velocity

Refinement 6







Done Done Done

Meshing Results 5 Hole

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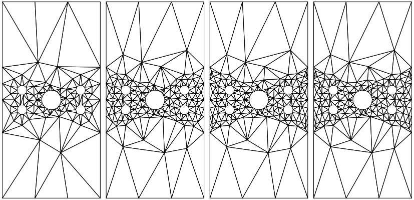
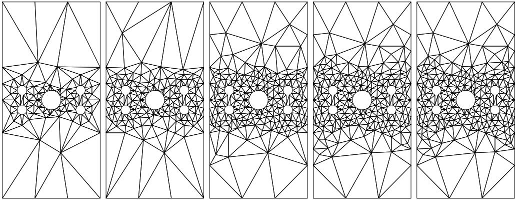
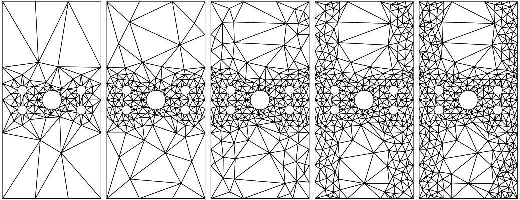
Pressure as error indicator



Horizontal velocity as error indicator

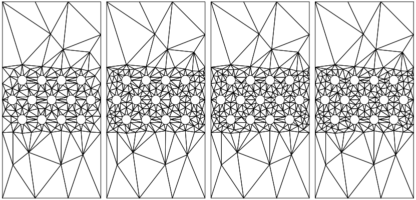


Vertical velocity as error indicator

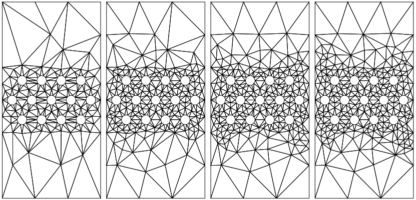


Meshing Results 13 Holes

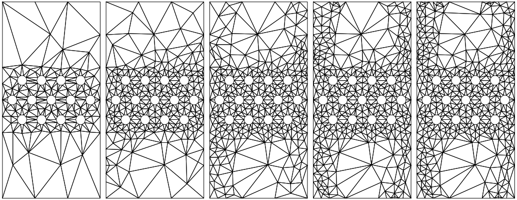
Pressure as error indicator

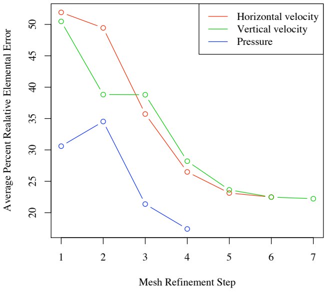


Horizontal velocity as error indicator

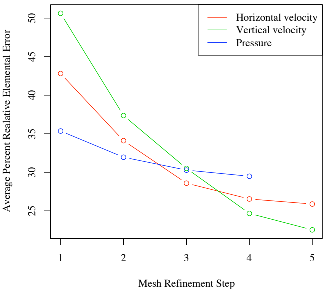


Vertical velocity as error indicator

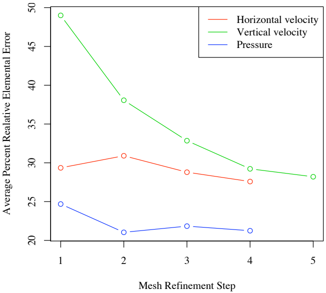




1 Obstruction



5 Obstructions

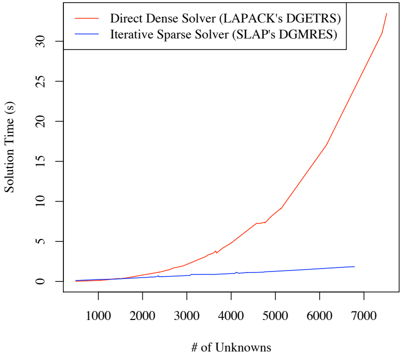


13 Obstructions

Error Results

Computation Time Results - Sparse vs. Dense Solvers

Computation Time Results - Sparse vs. Dense Solvers



Sparseness

Computation Time

Conclusions

- Reproduced well know flow patterns around cylinders at very low Reynolds numbers.
- Implemented unstructured adaptive meshing algorithm.
  - Pressure performed well as an error indicator
  - Different Error indicators produced very different meshes.
- Sparse matrix storage and equation solver drastically reduced solution time.