Vibrations of a Membrane: A Derivation and Numerical Solution

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1 Introduction and Objective

The objective of this paper is to investigate the behavior of a vibrating membrane. In this report a derivation of the partial differential equation describing the vibration of a thin membrane is given. The derivation includes a dampening term and a source term in rectangular and polar coordinates. The equation was solved numerically using finite differences.

2 Methodology

A derivation of an equation describing the vibration of a thin membrane is carried out based on fundamental principals. All spacial and temporal variables are replaced by dimensionless versions ($x' = x/L$, $y' = y/H$ where H and L are the lengths of the sides of the membrane). The displacement of the membrane is $u(x, y, t)$.

2.1 Assumptions

1. The effect of gravity on the membrane is negligible.
2. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are small slopes (Haberman 2004).
3. Displacement is only in the vertical direction (Kaplan 1981).
4. The membrane is thin enough to neglect its volume and only consider its area.
5. The Magnitude of the tension (T_0) is constant.
6. ρ_0 (mass/area) is uniform throughout the membrane.

2.2 Derivation

To begin the derivation carry out a force balance on a free body of a differentially small area of the membrane by using Newton's second law (Figure 1).

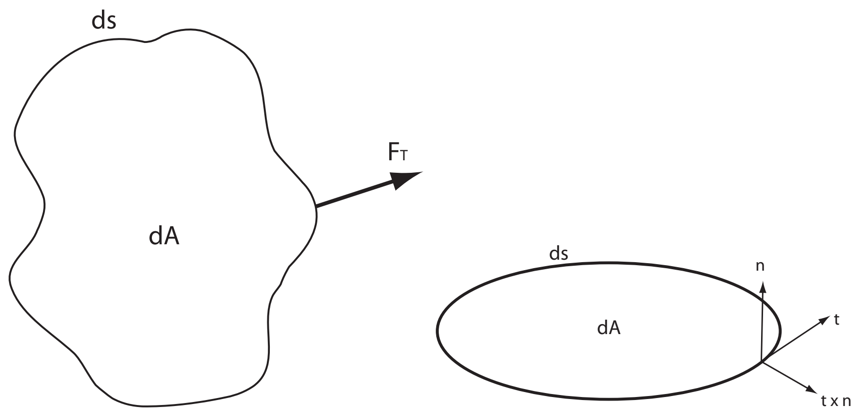
$$\sum \mathbf{F} = m\mathbf{a}$$

Where $\mathbf{F}$ is a force vector acting on the free body, m is mass and $\mathbf{a}$ is the acceleration vector.

The force balance in the vertical direction is

$$\mathbf{F}_T ds + \mathbf{Q} dA - \mathbf{V}\beta dA = m\mathbf{a}$$

where $\mathbf{F}_T$ is the tension force per unit length, $\mathbf{Q}$ is the vertical component of an arbitrary body force per unit area, $\mathbf{V}\beta$ is a dampening force which is proportional to the velocity $\mathbf{V}$ and acts in the opposite direction. The weight force is ignored by the assumption. The tension force acts perpendicular to the boundary in the direction $\hat{\mathbf{t}} \times \hat{\mathbf{n}}$ (Kaplan 1981) (Figure 1). $\hat{\mathbf{t}}$ is the unit tangent vector and $\hat{\mathbf{n}}$ is the unit



(a) Arbitrary area of membrane. (b) Tangent and normal vectors.

Figure 1: Tension force on membrane.

normal vector (Marsden and Tromba 2003). The magnitude of the tension force is constant by the assumption. The tension force is shown in figure 1. The vertical component of the tension force is given by

$$\mathbf{F}_T = T_0(\hat{\mathbf{t}} \times \hat{\mathbf{n}}) \cdot \hat{\mathbf{k}}$$

To expand the force balance to the entire membrane sum the body forces over the membrane and the tension force over the boundary. Making the substitutions $m = \rho_0 dA$, $\mathbf{a} = \frac{\partial^2 u}{\partial t^2}$, $\mathbf{V} = \frac{\partial u}{\partial t}$ and integrating gives

$$\oint T_0(\hat{\mathbf{t}} \times \hat{\mathbf{n}}) \cdot \hat{\mathbf{k}} \; ds + \iint \mathbf{Q} \; dA - \iint \frac{\partial u}{\partial t} \beta \, dA = \iint \rho_0 \frac{\partial^2 u}{\partial t^2} \; dA$$

where s is the length of the boundary and A is the area. Examining the tension force term we apply the vector triple product identity $((\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b})$ and rewrite the term as

$$\oint T_0(\hat{\mathbf{n}} \times \hat{\mathbf{k}}) \cdot \hat{\mathbf{t}} \; ds$$

Now apply stokes theorem which states that the intergral over the boundary of a region is equal to the surface integral of the curl of the vector field ($\iint \nabla \times \mathbf{d} \cdot \hat{\mathbf{n}} \; dA = \oint \mathbf{d} \cdot \hat{\mathbf{t}} \; ds$). The force term can now be written as

$$\iint [\nabla \times (\hat{\mathbf{n}} \times \hat{\mathbf{k}})] \cdot \hat{\mathbf{n}} \; dA$$

The force balance is now

$$\iint T_0[\nabla \times (\hat{\mathbf{n}} \times \hat{\mathbf{k}})] \cdot \hat{\mathbf{n}} \; dA + \iint \mathbf{Q} \; dA - \iint \frac{\partial u}{\partial t} \beta \, dA = \iint \rho_0 \frac{\partial^2 u}{\partial t^2} \; dA$$

Which can be simplified since all the terms are summed over the area of the membrane.

$$T_0[\nabla \times (\hat{\mathbf{n}} \times \hat{\mathbf{k}})] \cdot \hat{\mathbf{n}} + \mathbf{Q} - \frac{\partial u}{\partial t} \beta = \rho_0 \frac{\partial^2 u}{\partial t^2} \tag{1}$$

Let $\hat{\mathbf{t}}_x$ and $\hat{\mathbf{t}}_y$ be tangent unit vectors to the surface parameterized by (Kaplan 1981)

$$x = x \quad y = y \quad z = u$$

Then

$$\hat{\mathbf{t}}_x = \hat{\mathbf{i}} + \frac{\partial u}{\partial x}\hat{\mathbf{k}} \qquad \hat{\mathbf{t}}_y = \hat{\mathbf{j}} + \frac{\partial u}{\partial y}\hat{\mathbf{k}}$$

And

$$\mathbf{n} = \mathbf{t}_x \times \mathbf{t}_y = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & \frac{\partial u}{\partial x} \\ 0 & 1 & \frac{\partial u}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\hat{\mathbf{i}} - \frac{\partial u}{\partial y}\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

So the unit normal vector

$$\hat{\mathbf{n}} = \frac{\frac{\partial u}{\partial x}\hat{\mathbf{i}} - \frac{\partial u}{\partial y}\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + 1}}$$

here an approxamation is made because of the assumtion that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are small slopes.

$$\hat{\mathbf{n}} \approx \frac{\partial u}{\partial x}\hat{\mathbf{i}} - \frac{\partial u}{\partial y}\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

So

$$\hat{\mathbf{n}} \times \hat{\mathbf{k}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial u}{\partial x} & -\frac{\partial u}{\partial y} & 1 \\ 0 & 0 & 1 \end{vmatrix} = -\frac{\partial u}{\partial x}\hat{\mathbf{i}} - \frac{\partial u}{\partial x}\hat{\mathbf{j}}$$

And

$$\nabla \times (\hat{\mathbf{n}} \times \hat{\mathbf{k}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial u}{\partial x} & -\frac{\partial u}{\partial y} & 0 \end{vmatrix} = \frac{\partial^2 u}{\partial z \partial x}\hat{\mathbf{i}} + \frac{\partial^2 u}{\partial z \partial y}\hat{\mathbf{j}} + \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\hat{\mathbf{k}}$$

The first two components of the resultant vector are ≈ 0 because of the assumption that displacement is only in the vertical direction, so

$$\nabla \times (\hat{\mathbf{n}} \times \hat{\mathbf{k}}) = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\hat{\mathbf{k}} \tag{2}$$

Substituting equation 2 into equation 1 and noticing that $\hat{\mathbf{k}} \cdot \hat{\mathbf{n}} = 1$ gives

$$T_0\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \mathbf{Q} - \frac{\partial u}{\partial t}\beta = \rho_0 \frac{\partial^2 u}{\partial t^2} \tag{3}$$

Dividing by ρ_0 and making the substitution $c^2 = T_0/\rho_0$ gives

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\mathbf{Q}}{\rho_0} - \frac{\partial u}{\partial t}\frac{\beta}{\rho_0} \tag{4}$$

Equation 4 describes the damped vibrations of a membrane of any shape.

2.3 Dimensional Homogeneity

A unit analysis of equation 3 in the M, L, T (mass, length, time) unit system reveals that equation 3 is dimensionally homogeneous. The units of β are M/L^2T, the units of ρ_0 are M/L^2 and the units of Q are M/LT^2 (force per unit area).

$$\frac{M}{T^2}\left(\frac{L}{L^2} + \frac{L}{L^2}\right) + \frac{ML}{L^2T^2} - \left(\frac{L}{T}\right)\left(\frac{M}{L^2T}\right) = \left(\frac{M}{L^2}\right)\left(\frac{l}{T^2}\right)$$

$$\frac{M}{LT^2} = \frac{M}{LT^2} + \frac{M}{LT^2} + \frac{M}{LT^2}$$

2.4 Polar Coordinate Transformation

Solving the membrane equation on a domain other than a square one, would be cumbersome with cartesian coordinates. A circular domain is applicable in many situations including the vibrations of a drum head. One natural way to go about this is to transform equation 4 into polar coordinates.

The 2D Laplacian, $\nabla^2 u$, in polar coordinates is given by (Twizell 1984)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{5}$$

Substituting Equation 5 into equation 4 gives

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) + \frac{\mathbf{Q}}{\rho_0} - \frac{\partial u}{\partial t}\frac{\beta}{\rho_0} \tag{6}$$

Equation 6 is much easer to solve on a circular domain.

2.5 Finite Difference Approxamations

To obtain a numerical solution to equation 3 finite differences can be used. The general central difference approximations are are (Kaplan 1981)

$$\frac{\partial f}{\partial x} \sim \frac{f(x+\Delta x, y, t) - f(x-\Delta x, y, t)}{2\Delta x}$$
$$\frac{\partial^2 f}{\partial x^2} \sim \frac{f(x+\Delta x, y, t) - 2f(x, y, t) + f(x-\Delta x, y, t)}{\Delta x^2}$$

Analogous equations exist for derivatives with respect to y and t. To make these equations useful the domain of the membrane is discretized where any node in the mesh is given by (i, j, n) where $i, j, k \in \{1, 2, ...\}$. The point in space corresponding to the node (i, j, n) is $(i\Delta x, j\Delta y, n\Delta t)$ where the distance between points is $\Delta x, \Delta y, \Delta t$ respectively. If the function to be approximated is $u(x, y, t)$ then the displacement at any grid point (node) is given by $u(i, j, n)$ or $u_{i,j,n}$. The difference approximations now become finite difference approximations

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1,j,n} - u_{i-1,j,n}}{2\Delta x}$$
$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j,n} - 2u_{i,j,n} + u_{i-1,j,n}}{\Delta x^2}$$

As before analogous equations exist for derivatives with respect to y and t. Replacing the derivatives in equation 3 with their respective central difference approximations gives

$$\begin{aligned}\frac{u_{i,j,n+1} - 2u_{i,j,n} + u_{i,j,n-1}}{\Delta t^2} =& c^2\left(\frac{u_{i+1,j,n} - 2u_{i,j,n} + u_{i-1,j,n}}{\Delta x^2} + \frac{u_{i,j+1,n} - 2u_{i,j,n} + u_{i,j-1,n}}{\Delta y^2}\right) \\ &+ \frac{Q_i}{\rho_0} - \frac{u_{i,j,n+1} - u_{i,j,n-1}}{2\Delta t}\beta\end{aligned} \tag{7}$$

The only unknown in equation 7 is $u_{i,j,n+1}$. Substituting $p = c\Delta t/\Delta x$, $s = c\Delta t/\Delta y$ and $\alpha = 2 - 2p^2 - 2s^2 - \frac{\Delta t \beta}{2}$ gives

$$u_{i,j,n+1} = \frac{2}{(2+\Delta t\beta)}\left[\alpha u_{i,j,n} + \left(\frac{\beta}{2}-1\right)u_{i,j,n-1} + p^2(u_{i+1,j,n}+u_{i-1,j,n}) + s^2(u_{i,j+1,n}+u_{i,j-1,n}) + \frac{Q_{i,j,n}\Delta t^2}{\rho_0}\right] \tag{8}$$

Equation 5 is an explicit finite difference scheme and is only stable if $p, s \leq \frac{1}{\sqrt{2}}$ (Mitchell and Driffiths 1980).

Analogously, a polar coordinate scheme was derived:

$$u_{i,j,n+1} = \frac{1}{(1+\Delta t\beta)}\left[\gamma u_{i,j,n} + \left(\frac{\beta\Delta t}{2}-1\right)u_{i,j,n-1} + \left(p^2 - \frac{cp\Delta t}{2i}\right)u_{i-1,j,n} + \left(p^2 + \frac{cp\Delta t}{2i}\right)u_{i+1,j,n} + \frac{s^2}{i^2}(u_{i,j+1,n}+u_{i,j-1,n}) + \frac{Q_{i,j,n}\Delta t^2}{\rho_0}\right] \tag{9}$$

Where $\gamma = 2 - 2p^2 - 2s^2 - \frac{\Delta t\beta}{2}$. Equation 9 has not been analyzed to determine stability conditions.

3 Application

The vibrating membrane in question is fixed along all boundaries so the problem can be summarized

$$\ddot{u} = c^2\nabla^2 u + Q(x,y,t)/\rho_0 - \dot{u}\beta$$

$$\begin{aligned} \text{BC:}\quad & u(0,y,t)=0 \quad u(L,y,t)=0 \\ & u(x,0,t)=0 \quad u(x,H,t)=0 \\ \text{IC:}\quad & u(x,y,t)=0 \end{aligned}$$

If not for the forcing term Q, the membrane would never move because of the initial conditions.

In polar coordinates the problem formally is

$$\ddot{u} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) + Q(r,\theta,t)/\rho_0 - \dot{u}\beta$$

$$\begin{aligned} \text{BC:}\quad & u(R,\theta,t)=0 \\ \text{IC:}\quad & u(r,\theta,t)=0 \end{aligned}$$

The programming of the problem in cartesian coordinates is fairly strait forward because the computer logic and design is inherently rectangular. The problem in polar coordinates presents a more challenging task. The challenge lies in the transformation of a rectangular (r,θ) grid to a circular grid in cartesian coordinates.

The calculation for the polar coordinate problem can be summarized at each timestep by an $n_r \times n_\theta$ matrix. The matrix is partitioned into zones where the type of calculation in each zone is the same.

$$\begin{pmatrix} b & | & & \mathbf{a} & \\ -- & - & -- & -- & -- \\ & | & \mathbf{c} & | & \\ \mathbf{f_{left}} & | & -- & | & \mathbf{f_{right}} \\ & | & N & | & \\ & | & -- & -- & -- \\ & | & & \mathbf{f_{bc}} & = \mathbf{0} \end{pmatrix}$$

Where **a** is a row vector with $n_\theta - 1$ entries who's entries are used to calculate the center grid point (they are all the same). The value in **a** is the average of the calculated value for the center point at every angle $i\Delta\theta$. Entry b must reference the value in **a** and the center point from the previous timestep. Vector **c** is a row vector with $n_\theta - 1$ entries. It's values only reference the corresponding entries in **a**. The vector $\mathbf{f_{left}}$ is the same as $\mathbf{f_{right}}$ with $n_r - 1$ entries. Both vectors reference each other because that is where the grid wraps around and overlaps. N is a $n_r - 1 \times n_\theta - 1$ matrix where normal calculations take place. $\mathbf{f_{bc}}$ is the constant zero boundary condition (all entries=0).

4 Results

One sample result is given.

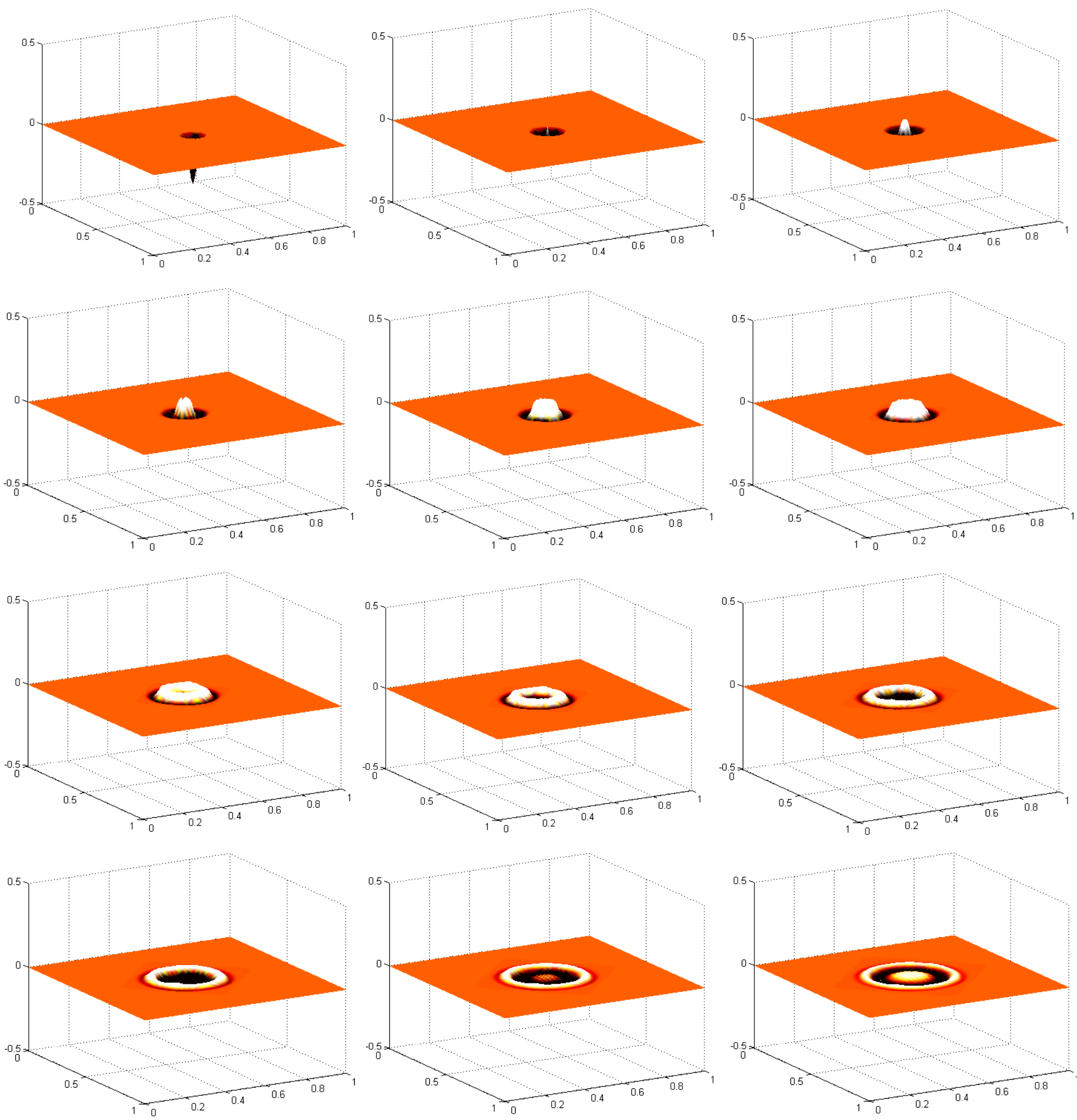


Figure 2: Sample vibration

The images given are for the rectangular domain. Considerable difficulty was encountered when

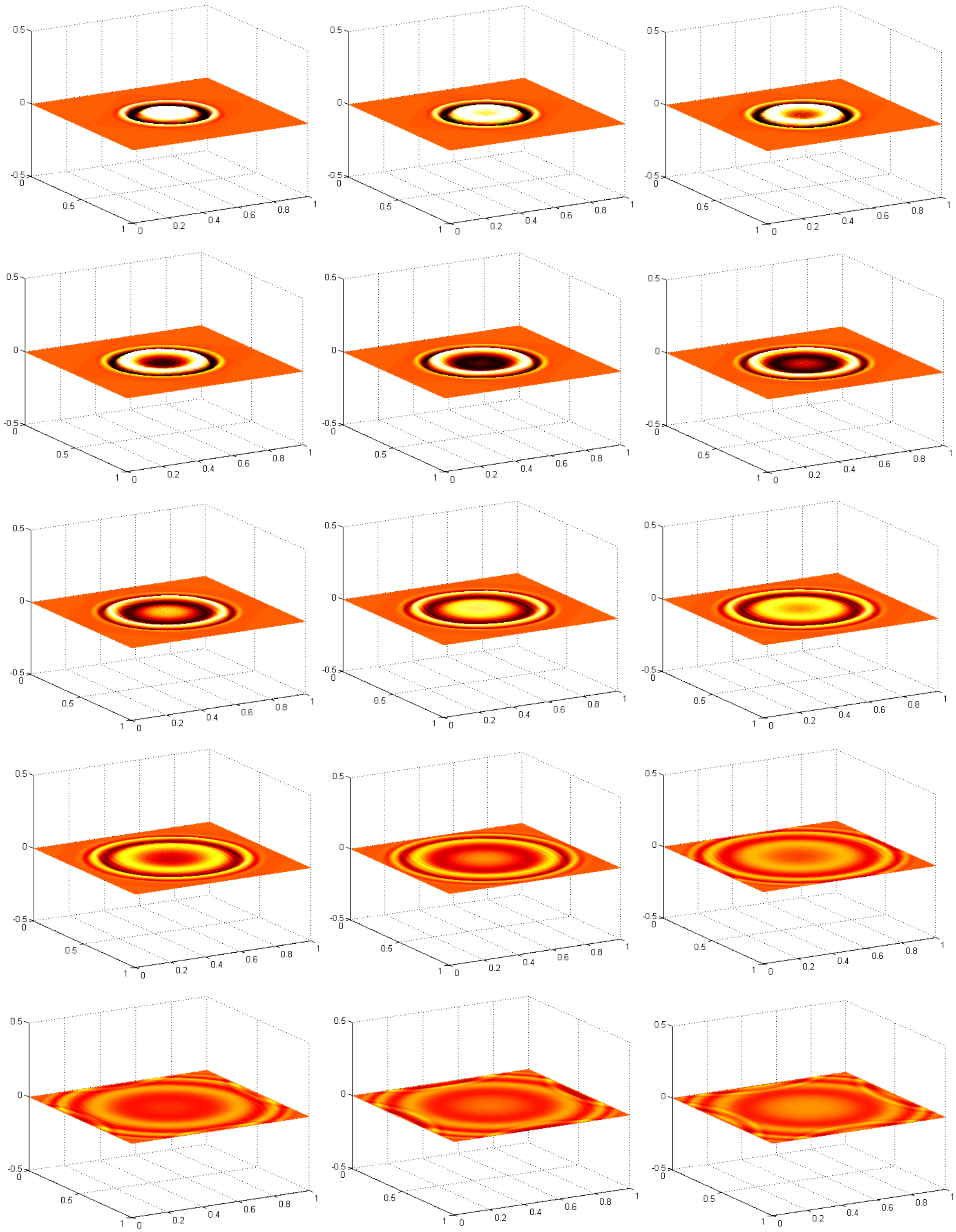


Figure 3: Sample vibration (cont.)

the polar grid was implemented. There is a chance that the finite difference scheme is unconditionally unstable.

5 Conclusion

In terms of further areas of research, the derivation of the equations used in this report is general enough to be applied to many physical situations. A modification of the boundaries could apply to a speaker head or a microphone. If the tension force on the membrane was the surface tension of water, then the equations could be used to animate a transient surface profile of water. There are vibrating membranes in the inner ear, on trampolines, and screen doors. The application are numerous. In terms of realness, well, not one scrap of real data was used in this report. The addition of empirical data would confirm the accuracy of the theoretical equations.

6 Further Research

To improve stability and gain the ability to use a larger time step, an implicit scheme could be derived. This would be especially helpful in the polar case where no numerical solution was able to be implemented.

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