Using Ordinary Differential Equations to Model Fluid Drag on a Sphere

Cameron Bracken
Georgia Haas
Eric Schori

August 24, 2008

Abstract

Fluid drag is a complex physical phenomena that is not completely understood, but provided that certain assumptions are valid it is possible to use the methods of ordinary differential equations to create a mathematical model that is reasonably accurate. The ordinary differential equations are not difficult to set up. The dimensional analysis, and the integrals that result from the differential equation can both be rather difficult to solve.

Contents

List of Figures

1 Problem Description and Assumptions

1.1 Problem Description

In this paper we show how the behavior of a spherical object that is released from rest in a stream of moving liquid can be modeled using the methods of ordinary differential equations. This model will take into account the effects of fluid drag on a falling object, and so provide a better approximation than those models commonly seen in general calculus courses. The introductory part of this paper deals with how results from observations of experiments mathematically describe the factors that influence the motion of the sphere. The main body of the paper deals with the ordinary differential equations that model the behavior of the sphere, and how those equations are solved. The final section of the paper deals with the applications of the model that has been developed and what we learn from solving the problem.

1.2 Assumptions

The mathematical model that is described in this paper describes a specific type of physical situation and relies on certain assumptions being true in order to remain valid. The study of the movement of objects through liquid or gaseous fluids has been primarily through empirical observation, to which researchers have been able to fit mathematical models that approximate the observed behavior with reasonable accuracy(Donley 1991). We consider the case of a sphere falling through a body of moving water, and require the following conditions to be true in order for the math model to hold.

- The shape of the object must be that of a sphere
- The density of the fluid is constant
- The relative velocity of the sphere and the fluid is significantly less than the speed of sound in the fluid, in this way avoiding the complications of sonic shock waves (Donley 1991)

- The sphere is macroscopic, so is sufficiently large that Van der Waals and other molecular and atomic scale forces do not significantly interfere with the results predicted by Newtonian physics

2 Objectives of the Project

The objective of this paper is to develop a math model that will describe the velocity and position of a falling sphere in a stream of moving fluid such that the model can take account of the effects of the drag force of the fluid. In developing the model we will apply techniques of ordinary differential equations to develop the equations that are used in the model.

3 Preliminary Mathematics

The math model that is developed in this paper is different than those typically seen in general calculus courses in that it takes into account the force of drag. This means that we need a mathematical way to describe the drag force. Physicists have determined that for the situation we are investigating the drag force is determined by the following parameters (Donley 1991):

- The relative velocity v of the fluid and the sphere.
- The diameter D of the sphere.
- The density ρ of the fluid.
- The viscosity μ of the fluid.

Using the methods of dimensional analysis, which we will not discuss in depth here, it is possible to reduce the number of independent variables to two dimensionless variables; the coefficient of drag and the Reynold's number(Donley 1991).

The result of dimensional analysis taking the initial variables v, D, ρ and μ produces

Coefficient of Drag C_D

$$C_D = \frac{F_D}{.5\,\rho v^2 A}$$

Where F_d is the drag force and A is the cross sectional area of the sphere.

Reynolds Number R

$$R = \frac{\rho v D}{\mu}$$

And the important result stems from the relationship between these two variables that allows us to describe the drag force as

$$F_D = \frac{1}{2}\rho v^2 A f\left(\frac{\rho v D}{\mu}\right)$$

Where f is a function that is experimentally determined (Donley 1991).

There is a great deal physics involved in determining the function f that is not directly involved in developing the math model so long as we have the results from empirical studies. We need some way to represent the drag force so that we can incorporate it into the differential equations derived from Newton's Second Law (describing position and velocity of a falling object). When empirical results are evaluated one finds that there are equations that are produce from applying the observed f to our equation for the drag force. There are distinct equations for low Reynolds numbers $R < 0.5$ and for high Reynolds numbers $10^3 < R < 10^5$. The behavior of the drag force in the intermediate range and at very high Reynolds numbers is sufficiently complex to be beyond the scope of this paper.

For low Reynolds numbers the drag force is

$$F_D = 3\pi\mu D v = k_1 v$$

(Donley 1991).

For high Reynolds numbers the drag force is

$$F_D = \frac{1}{2}\rho v^2 A C_D = k_2 v^2$$

(Donley 1991)

In the next section we will use Newton's second law $\sum F = \sum ma$ and rewrite accleration as

$$a = \frac{dv}{dt}$$

to develop our ordinary differential equations. They will be of the general form: mass times the derivative of velocity = mass time gravitational accleration plus drag force.

4 Ordinary Differential Equations

By analyzing a freebody of the sphere parametric ODE's can be developed that describe the acceleration of the sphere.

Low Reynolds number($Re < .5$)
for x

$$m\frac{dv_x}{dt} = k_1 v_x \tag{1}$$

and for y

$$m\frac{dv_y}{dt} = -mg + kv_y \tag{2}$$

High Reynolds Number ($10^3 < Re < 10^5$)
for x

$$m\frac{dv_x}{dt} = -kv_x^2 \tag{3}$$

and for y

$$m\frac{dv_y}{dt} = -mg + kv_y^2 \tag{4}$$

All of these equations are independent of the dependent variable t so they can be solved using separation of variables. The solutions to these equations are given in the solutions section.

5 Finding Solutions, and Applications of the Model

For a sphere moving under the influence of gravity the solutions to the differential equations can be shown to be (v_0,x_0 and y_0 represent the initial velocity, initial x position and y position, respectively):

For low Reynolds number($Re < .5$)
x-velocity

$$v_x = e^{\frac{-k}{m}(t+v_0)} \tag{5}$$

x-position

$$x = \frac{m}{k}\left(e^{\frac{k}{m}(t+v_0)}\right) + x_0 \tag{6}$$

y-velocity

$$v_y = \frac{1}{k_1}\left(e^{\frac{k_1}{m}(t+v_0)} - gm\right) \tag{7}$$

y position

$$y = \frac{m}{k_1^2}\left(e^{\frac{k_1}{m}(t-v_0)} - gk_1t\right) + y_0 \tag{8}$$

For high Reynolds ($10^3 < Re < 10^5$)
x velocity

$$v_x = \frac{m}{k_2(t+v_0)} \tag{9}$$

x position

$$x = \frac{k_2}{m}\ln(t+v_0) + x_0 \tag{10}$$

y velocity

$$v_x = \sqrt{\frac{mg}{k_2}}\tanh\left(-\sqrt{\frac{k_2g}{m}}t + \tan^{-1}\sqrt{\frac{k_2v_0}{mg}}\right) \tag{11}$$

y position

$$y = y_0 - \frac{m}{k_2}\ln\left(\sqrt{1-\frac{k_2}{mg}v_0^2}\cosh\left(-\sqrt{\frac{k_2g}{m}}t + \tanh^{-1}\sqrt{\frac{k_2}{mg}}v_0\right)\right) \tag{12}$$

To show how the sphere acts as it travels through the water stream we can plot parametric equations for the position of the particle as a function of time. The start of the graph represents the stream height and 0 position represents the stream floor. The graph uses an initial y position of 600 cm and no initial velocity. The sphere in question has a mass of 5 grams and a diameter of 1 cm (Figure 1,2).

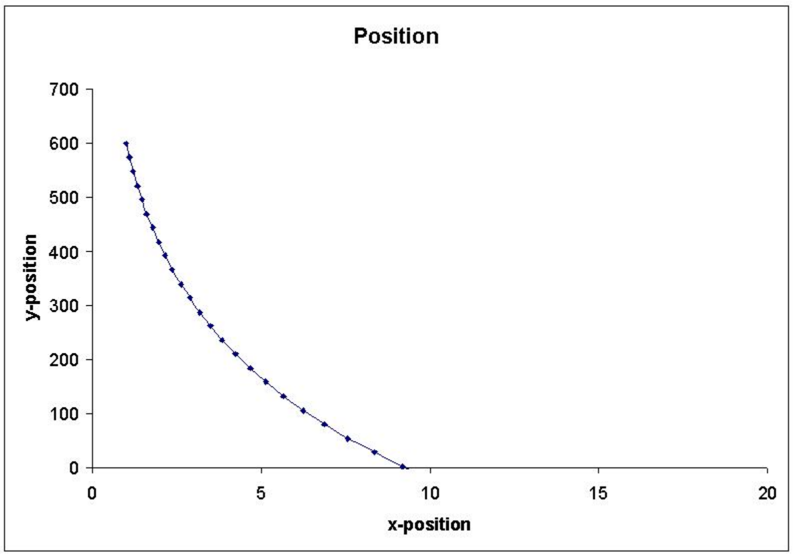

Figure 1: Path of sphere (Low Re)

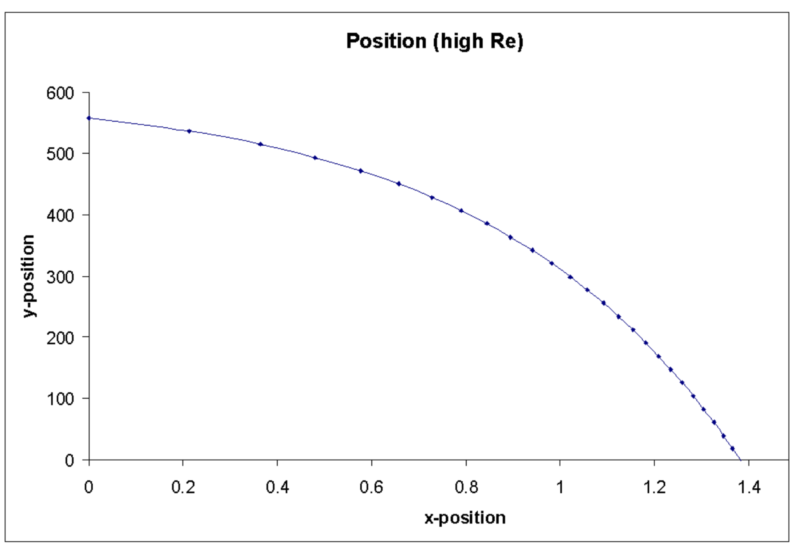

Figure 2: Path of sphere (High Re)

Because the mass is constant, Newtons second law tells us that the downward acceleration of the sphere will approach zero when the weight equals the drag force. By plotting the y velocity as a function of time it is clear that v_y is approaching a terminal velocity (Figure 3).

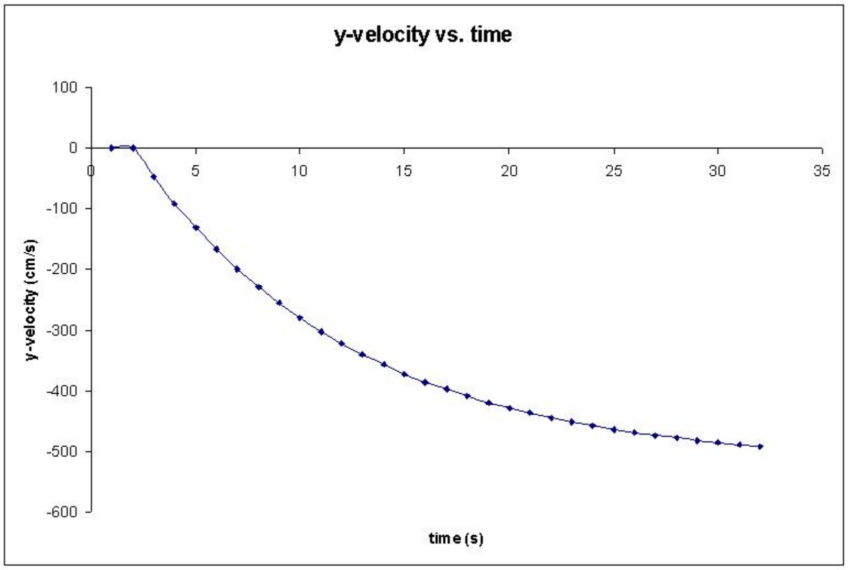

Figure 3: Position graph

6 Conclusion

The ODE technique of separation of variables involved in solving these equations was quite basic. In the effort to produce the equations necessary to model position and velocity we found that it is often not the ODE's which are hard, but the integration that the techniques require. We gained a large amount of knowledge about fluid mechanics of stream flow and fluid viscosity. This knowledge will be valuable in later classes.

7 References

H.E., Donley, The Drag Force On A Sphere, UMAP Unit 712 1991