The Gunfight at the OK Coral: A Summary and Simulation Verification

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Introduction

The scene is the epic three way standoff between the Good, Bad and the Ugly in the classic spaghetti western. The paper *The Gunfight at the OK Coral* by James T. Sandefur (1989) proposes a stochastic model for this epic standoff in which one round consists of each shooter firing once. The results of the model give the probability that one shooter will win as well as the probability that none will be left standing and the expected number of rounds.

The proposed model uses an absorbing Markov chain and is solved using difference equations. In addition to a summary and verification (via computer simulation) of the results derived by Sandefur, this paper will carry out a sensitivity analysis.

Model & Analysis

In the model proposed by Sandefur, the Good is designated as G, the Bad is designated a B and the Ugly is designated a U. As previously mentioned, the fight consists of rounds in which all living shooters fire once. The shooters always fire at the better of their two opponents with a fixed probability of hitting. G hits with a 60% probability, U hits with a 50% probability and B hits with a 30% probability. The Markov chain is set up so that the states consist of which shooters are alive. The transition probabilities between states are composed of the chances of the various living shooters hitting or missing each other in one round (Appendix A).

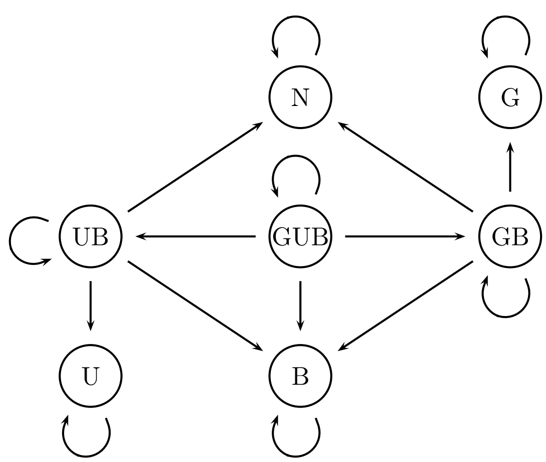

Figure 1: Gunfight Markov chain expressed as a digraph

A digraph representing the Markov chain is given in Figure 1. State N is the state that occurs when no one is left alive. GUB is the starting state where all three shooters are alive. As indicated by the lack of a probability of leaving, states G, U, B and N are absorbing states while GUB, UB and GB are transient states because there is a nonzero probability of never returning to them.

Additionally, Sandefur defines $p_1(n)$, $p_2(n)$, $p_3(n)$ and $p_4(n)$ as the probabilities of reaching absorbing state N, G, U, B respectively after n rounds of shooting. Also $q_1(n)$, $q_2(n)$ and $q_3(n)$ are defined as the probabilities of reaching nonabsorbing state GB, UB and GUB respectively after n rounds of shooting.

Based on the formulas given in Appendix A the transition probabilities for the Markov chain can be obtained. Using the transition probabilities, difference equations can be written for each $p_i(n+1)$ and $q_j(n+1)$ for $i = \{1,2,3,4\}$ and $j = \{1,2,3\}$. The difference equations are

$$p_1(n+1) = p_1(n) + .18q_1(n) + .15q_2(n),$$

$$p_2(n+1) = p_2(n) + .42q_1(n),$$

$$p_3(n+1) = p_3(n) + .35q_2(n),$$

$$p_4(n+1) = p_4(n) + .12q_1(n) + .15q_2(n) + .39q_3(n),$$

$$q_1(n+1) = .28q_1(n) + .21q_3(n),$$

$$q_2(n+1) = .35q_2(n) + .26q_3(n),$$

and

$$q_3(n+1) = .14q_1(n).$$

The last three difference equation can be written as a matrix equation

$$Q(n+1) = RQ(n) \tag{1}$$

where

$$Q = \begin{pmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{pmatrix}, \text{ and } R = \begin{pmatrix} .28 & 0 & .21 \\ 0 & .35 & .26 \\ 0 & 0 & .14 \end{pmatrix}.$$

Since the fight always starts with every shooter alive the Markov chain always starts in state GUB. Therefore the initial condition is $Q(0) = (0\ \ 0\ \ 1)^T$.

Also the last four difference equations can be written in the matrix form

$$P(n+1) = P(n) + SQ(n) \tag{2}$$

where

$$P = \begin{pmatrix} p_1(n) \\ p_2(n) \\ p_3(n) \\ p_4(n) \end{pmatrix}, \text{ and } S = \begin{pmatrix} .18 & .15 & .0 \\ .42 & 0 & 0 \\ 0 & .35 & 0 \\ .12 & .15 & .39 \end{pmatrix}.$$

The fight will never start in one of the absorbing states so $P(0) = (0\ \ 0\ \ 0\ \ 0)^T$.

As a side note, from the transition probabilities (t_{ji}), the transition probability matrix (T) can be constructed of four submatricies where

$$T = \begin{pmatrix} R & \mathbf{0} \\ S & I \end{pmatrix}.$$

T can be partitioned in this way because the Markov chain is reducible.

Sandefur shows by induction that the solution to (1) is

$$Q(n) = R^n Q(0).$$

Substituting this result into (2) gives

$$P(n+1) = P(n) + SR^n Q(0) \tag{3}$$

which is a nonhomogeneous system of difference equations.

Sandefur uses the method of undetermined coefficient to find a general solution to (3). By this method we look for a nonhomogeneous solution of the form

$$P(n) = SR^n C$$

where C is the undetermined coefficient. Since GUB, UB and GB are transient states all the elements of $R^n \to 0$ as $n \to \infty$. Since $R^n \to 0$, $(I-R)$ is invertible so with some manipulation and substitution Sandefur shows

$$C = (R-I)^{-1} Q(0)$$

The general solution of (3) will be the linear combination of the homogeneous solution, $P(n) = A$, and the nonhomogeneous solution using the result from above. The general solution is

$$P(n) = A + SR^n (R-I)^{-1} Q(0).$$

Now use the initial condition to find

$$A = S(I-R)^{-1} Q(0)$$

And since $R^n \to 0$ as $n \to \infty$. The solution is

$$P = S(I-R)^{-1} Q(0) \tag{4}$$

Additionally, if $D(n)$ is the expected number of times we are in each absorbing state after n rounds, the expected number of rounds can calculated by summing the elements in D where

$$D = (I-R)^{-1} Q(0).$$

Summary of Results

The information we want about the probability of each shooter winning or of no one winning is given by the p_i's in (4). To verify the analytical results this information was also calculated using a computer simulation averaged over 1000 and 100000 fights. The simulation results given in table 1 compare favorably with the analytical results derived by Sandefur.

let g, u and b be probability of Good, Ugly and Bad hitting his target respectively. Note that $0 < b < u < g < 1$. This is necessary so that the shooters will still fire at the same opponents and the original form of the model still holds. To test the sensitivity of the system, each parameter, g, u or b, was varied within its limits while all other parameters remained fixed. The results of the sensitivity are summarized in figure 2, 3 and 4.

Table 1: Summary of calculated results

End State	Analytical	Sim. 1000 fights	Sim. 100,000 fights
None	0.1308	0.1305	0.1308
Good	0.1424	0.1417	0.1424
Ugly	0.1628	0.1632	0.1628
Bad	0.5640	0.5646	0.5640
Expected Rounds	1.96	1.99	1.97

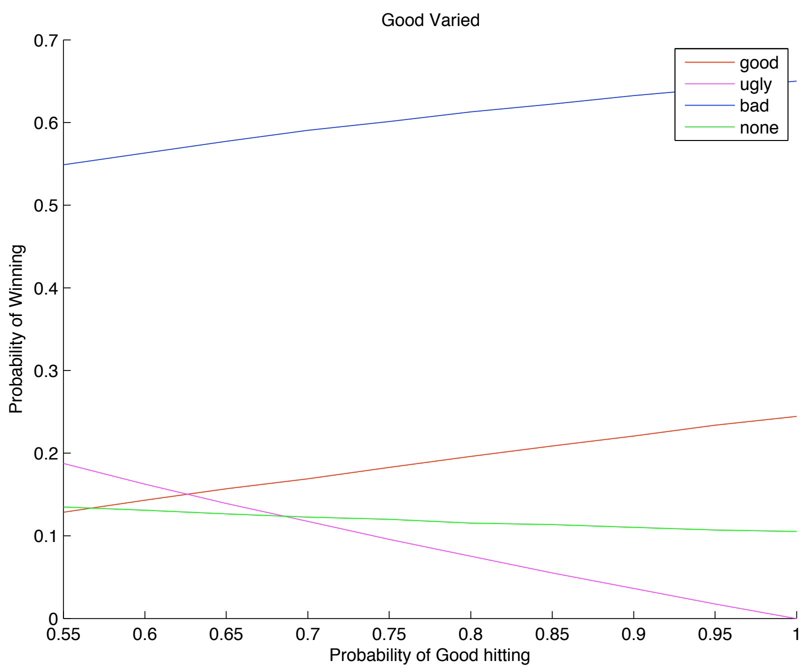

Figure 2: Sensitivity of g.

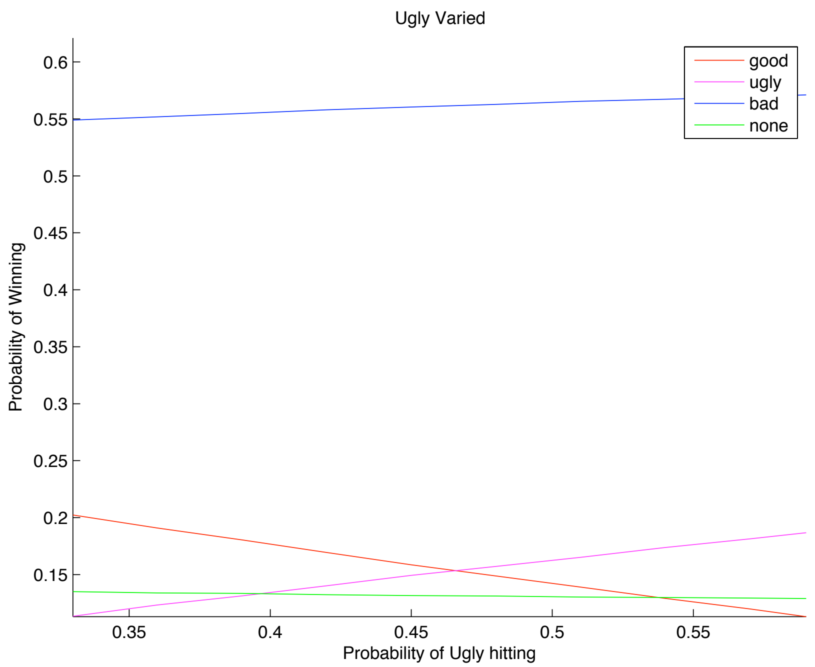

Figure 3: Sensitivity of u.

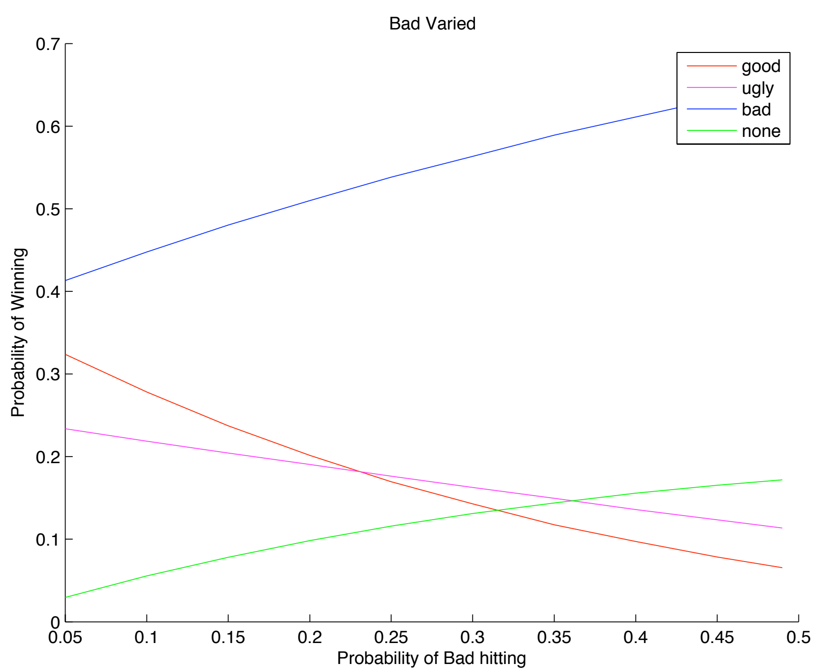

Figure 4: Sensitivity of b.

Discussion

In the current model setup, the B wins over half of the time. This is due to the rule of the shooters always firing at the better of their two opponents. In the beginning state GUB, no one fires at B and two men fire at G. This is why GU is not a possible state. So despite his having the highest hit rate, G still has the lowest probability of winning.

It is interesting to note that the same result can be arrived at by taking powers of the transition probability matrix T. The probabilities from submatrix R "leak out" to submatrix S because eventually the probability of transitioning to a transient state is zero. For large powers of T the probabilities approach the analytical result.

The sensitivity analysis provides some interesting results. Any increase in g, u or b causes B to win more often. This is because no one is firing at B to begin with so increasing his opponents chance of hitting effectively allows his opponents to kill each other more efficiently.

A possible way to improve the model would be to allow shooters to shoot at variable targets. That is, one shooter may fire at one opponent most of the time but occasionally would fire at a different opponent. Also, the shooters could be allowed to move physical positions after firing which would be controlled by a measurement of quickness.

References

Linda J. S. Allen. *An Introduction to Stochastic Processes*. Pearson Prentice Hall, 2003.

James T. Sandefur. Gunfight at the ok corral. *Mathematics Magazine*, 62:119–124, April 1989.

Appendix A Transition probabilities

A shorthand notation will be adopted here. Let $P(A \to B)$ be the probability of transitioning from state A to state B in a single round. Also let g, u and b be probability of Good, Ugly and Bad hitting his target in one round respectively. The shooters hit each other independently so from basic probability of independent and mutually exclusive events

$$P(GUB \to B) = gu(1-b) + gub + g(1-u)b$$

$$P(GUB \to GB) = g(1-u)(1-b)$$

$$P(GUB \to UB) = (1-g)ub + (1-g)u(1-b) + (1-g)(1-u)b$$

$$P(GUB \to GUB) = (1-g)(1-u)(1-b)$$

$$P(UB \to B) = (1-u)b$$

$$P(GUB \to U) = u(1-b)$$

$$P(UB \to UB) = (1-u)(1-b)$$

$$P(UB \to N) = ub$$

$$P(GB \to G) = g(1-b)$$

$$P(GB \to U) = (1-g)b$$

$$P(GB \to GB) = (1-g)(1-b)$$

$$P(GB \to N) = (1-g)(1-b)$$

Using $g = .6$, $u = .5$ and $b = .3$ gives all the probabilities found in S and R from Sandefur.

Appendix B Simulation Source Code

```
program gunfight
  implicit none
  integer::nfights,round,i,j
  double precision::randnum,Gwin,Bwin,Uwin,Nwin,GUBtoGUB,GUBtoB,&
        GUBtoGB,GUBtoUB,UBtoU,UBtoB,UBtoN,UBtoUB,GBtoG,GBtoB,GBtoN,GBtoGB
  double precision, dimension(31)::g,u,b
  double precision,dimension(31,5)::win
  character(len=3)::survivors,change

    open(11,file="gunfight_input.in")
    open(12,file="gunfight.out")
    do i=1,31
      read(11,*)g(i),u(i),b(i)  %read in file containing prob of hit
    end do

    nfights=100000   %total number of fights to average

do j=1,31

    call random_seed  %initialize pseudo random number generator

    Bwin=0  %initialize win counts
    Gwin=0
    Uwin=0
    Nwin=0
    round=0
                 %transition probabilities
    GUBtoGUB=(1-g(j))*(1-u(j))*(1-b(j))
    GUBtoB=g(j)*u(j)*(1-b(j))+g(j)*u(j)*b(j)+g(j)*(1-u(j))*b(j)
    GUBtoGB=g(j)*(1-u(j))*(1-b(j))
    GUBtoUB=(1-g(j))*u(j)*b(j)+(1-g(j))*u(j)*(1-b(j))+(1-g(j))*(1-u(j))*b(j)
    UBtoU=u(j)*(1-b(j))
    UBtoB=(1-u(j))*b(j)
    UBtoN=u(j)*b(j)
    UBtoUB=(1-u(j))*(1-b(j))
    GBtoG=g(j)*(1-b(j))
    GBtoB=(1-g(j))*b(j)
    GBtoN=g(j)*b(j)
    GBtoGB=(1-g(j))*(1-b(j))

  do i=1,nfights
    survivors="GUB"
    do
      call random_number(randnum)
      if(survivors=="GUB")then
        if(randnum<=GUBtoGUB)then
          survivors="GUB"
        else if(randnum>GUBtoGUB.and.randnum<=(GUBtoGUB+GUBtoUB))then
          survivors="UB"
        else if(randnum>(GUBtoGUB+GUBtoUB).and.&
        randnum<=(GUBtoGUB+GUBtoUB+GUBtoGB))then
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```
          survivors="GB"
        else if(randnum>(1-GUBtoB))then
          survivors="B"
        end if
      else if(survivors=="UB")then
        if(randnum<=UBtoUB)then
          survivors="UB"
        else if(randnum>UBtoUB.and.randnum<=(UBtoUB+UBtoU))then
          survivors="U"
        else if(randnum>(UBtoUB+UBtoU).and.randnum<=(UBtoUB+UBtoU+UBtoB))then
          survivors="B"
        else if(randnum>=(1-UBtoN))then
          survivors="N"
        end if
      else if(survivors=="GB")then
        if(randnum<=GBtoGB)then
          survivors="GB"
        else if(randnum>GBtoGB.and.randnum<=(GBtoGB+GBtoG))then
           survivors="G"
        else if(randnum>(GBtoGB+GBtoG).and.randnum<=(GBtoGB+GBtoG+GBtoB))then
           survivors="B"
        else if(randnum>(1-GBtoN))then
           survivors="N"
        end if
      else if(survivors=="G")then   %good wins
        Gwin=Gwin+1
        exit
      else if(survivors=="B")then   %bad wins
        Bwin=Bwin+1
        exit
      else if(survivors=="U")then   %ugly wins
        Uwin=Uwin+1
        exit
      else if(survivors=="N")then   %none win
        Nwin=Nwin+1
        exit
      end if
      round=round+1
    end do
  end do

      win(j,1)=dble(Gwin)/dble(nfights)
      win(j,2)=dble(Uwin)/dble(nfights)
      win(j,3)=dble(Bwin)/dble(nfights)
      win(j,4)=dble(Nwin)/dble(nfights)
      win(j,5)=dble(round)/dble(nfights)

end do
    do i=1,31
      write(12,"(100f9.5)")g(i),u(i),b(i),(win(i,j),j=1,5)
    end do
    stop
end program
```