ME 570: Robot Motion Planning

Homework 2 Report

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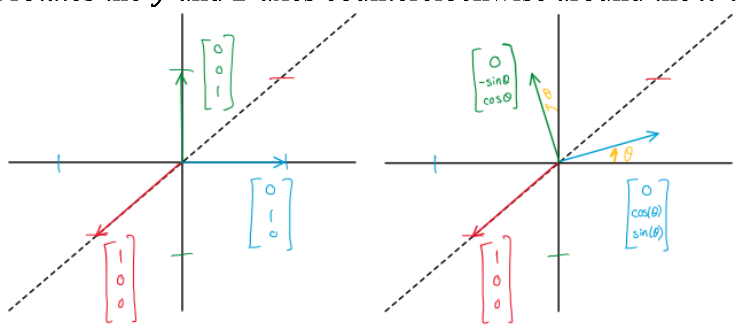
Problem 1: Rotations

Question 1.1 report – Geometric Reasoning of the Rotation Matrices

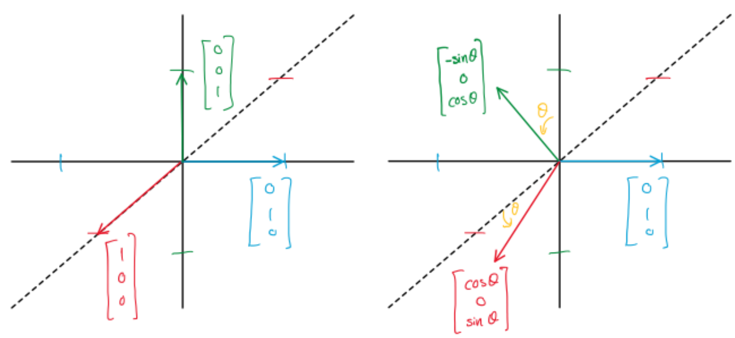
$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

This rotation matrix rotates the $\boldsymbol{y}$ and $\boldsymbol{z}$-axes counterclockwise around the $\boldsymbol{x}$-axis



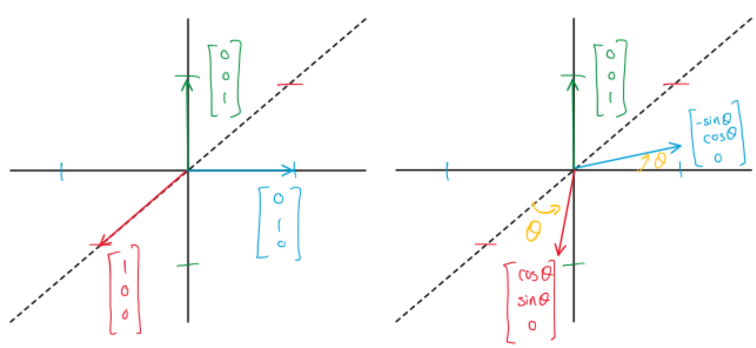
$$R_2(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

This rotation matrix rotates the $\boldsymbol{x}$ and $\boldsymbol{z}$-axes counterclockwise around the $\boldsymbol{y}$-axis



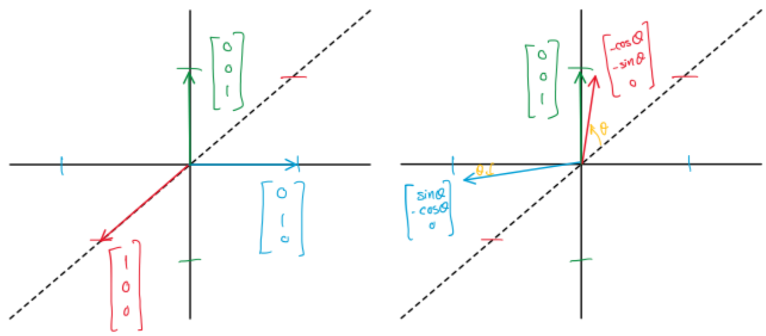
$$R_3(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This rotation matrix rotates the $\boldsymbol{x}$ and $\boldsymbol{y}$-axes counterclockwise around the $\boldsymbol{z}$-axis



$$R_4(\theta) = \begin{bmatrix} -\cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & -\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This rotation matrix reflects the $\boldsymbol{x}$ and $\boldsymbol{y}$-axes across the $\boldsymbol{z}$-axis and then rotates them counterclockwise.



Problem 2: Free Configuration Space for a Two-Link Manipulator

Question 2.1 report – Coordinate Transformations

1. The coordinates of the point given in β_1, denoted as ${}^{\beta_1}p$:

   $${}^{\mathcal{W}}p = {}^{\mathcal{W}}R_{\beta_1}\,{}^{\beta_1}p + {}^{\mathcal{W}}T_{\beta_1} \quad \text{where } {}^{\mathcal{W}}T_{\beta_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } {}^{\mathcal{W}}R_{\beta_1} \text{ is the simple rotation matrix}$$

   $${}^{\mathcal{W}}R_{\beta_1} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$$

   $$\begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} {}^{\beta_1}p_x \\ {}^{\beta_1}p_y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} {}^{\beta_1}p_x \cos(\theta_1) - {}^{\beta_1}p_y \sin(\theta_1) \\ {}^{\beta_1}p_x \sin(\theta_1) + {}^{\beta_1}p_y \cos(\theta_1) \end{bmatrix}$$

2. The coordinates of the point given in β_2 denoted as ${}^{\beta_2}p$:

   ${}^{\mathcal{W}}p = {}^{\mathcal{W}}R_{\beta_2}\,{}^{\beta_2}p + {}^{\mathcal{W}}T_{\beta_2}$: To compute this, we need to perform $\beta_2 \rightarrow \beta_1 \rightarrow \mathcal{W}$

   ${}^{\beta_1}p = {}^{\beta_1}R_{\beta_2}\,{}^{\beta_2}p + {}^{\beta_1}T_{\beta_2}$: Plugging this into the above, converting form $\beta_1 \rightarrow \mathcal{W}$

   $${}^{\mathcal{W}}p = {}^{\mathcal{W}}R_{\beta_1}\left({}^{\beta_1}R_{\beta_2}\,{}^{\beta_2}p + {}^{\beta_1}T_{\beta_2}\right) + {}^{\mathcal{W}}T_{\beta_1}$$

   $${}^{\mathcal{W}}p = \left({}^{\mathcal{W}}R_{\beta_1}\,{}^{\beta_1}R_{\beta_2}\right){}^{\beta_2}p + \left({}^{\mathcal{W}}R_{\beta_1}\,{}^{\beta_1}T_{\beta_2} + {}^{\mathcal{W}}T_{\beta_1}\right)$$

   where ${}^{\mathcal{W}}R_{\beta_2} = {}^{\mathcal{W}}R_{\beta_1}\,{}^{\beta_1}R_{\beta_2}$ **and** ${}^{\mathcal{W}}T_{\beta_2} = {}^{\mathcal{W}}R_{\beta_1}\,{}^{\beta_1}T_{\beta_2} + {}^{\mathcal{W}}T_{\beta_1}$

   Also important is ${}^{\beta_1}R_{\beta_2}$ is, again, the simple rotation matrix:

$$^{\beta_1}R_{\beta_2} = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix}$$

We now have:

$$^{w}p = \left(\begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix}\right) {}^{\beta_2}p + \left(\begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

$$^{w}p = \left(\begin{bmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & -\cos(\theta_1)\sin(\theta_2) - \sin(\theta_1)\cos(\theta_2) \\ \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) & -\sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2) \end{bmatrix}\right) \begin{bmatrix} {}^{\beta_2}p_x \\ {}^{\beta_2}p_y \end{bmatrix} + \left(\begin{bmatrix} 5\cos(\theta_1) \\ 5\sin(\theta_1) \end{bmatrix}\right)$$

$$^{w}p = \begin{bmatrix} {}^{\beta_2}p_x(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) - {}^{\beta_2}p_y(\cos(\theta_1)\sin(\theta_2) - \sin(\theta_1)\cos(\theta_2)) + 5\cos(\theta_1) \\ {}^{\beta_2}p_x(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)) - {}^{\beta_2}p_y(\sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2)) + 5\sin(\theta_1) \end{bmatrix}$$

Question 2.1 code – TwoLink.kinematic_map()

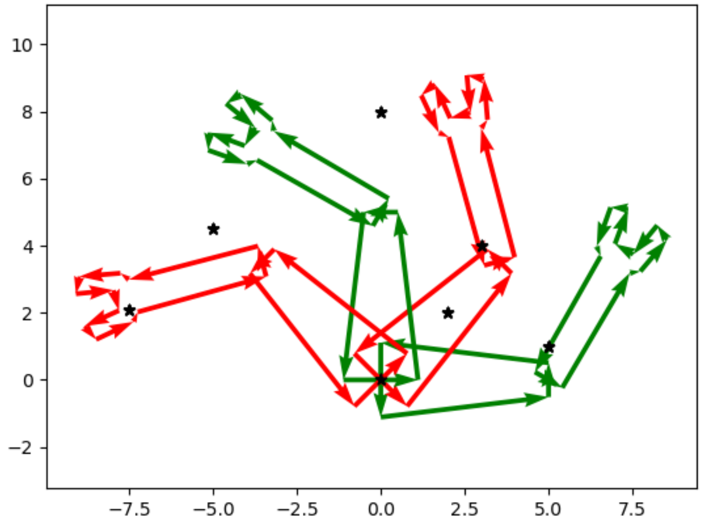
Code based on the previous report question worked very well

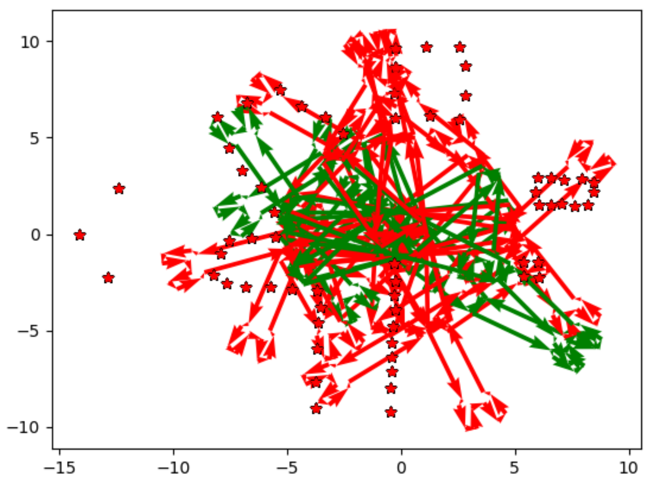
Question 2.1 optional – TwoLink.kinematic_map(i)

Question 2.2 code – TwoLink.is_collision()

I had a little bit of trouble with this method because the output was consistently incorrect. I decided to dig deeper into the python solution code that I am using, and I believe the Polygon.is_collision() method is returning the incorrect value. The method is using logical OR on flag_points based on if the point is_visible(). However, this logical OR is determining a flag_point to say there is a collision. The return of is_visible() is thus being used to indicate a collision instead of a non-collision, so I had to logically reverse the returned array to really determine if there was a collision.

Question 2.2 report – TwoLink.plot_collision()





Question 2.2 optional – TwoLink.free_space()

Problem 4: Charts for the Circle Using Rotations

Question 4.1 report – Show $R(\theta)$ is a rotation and that $\phi_{circle}(\theta) \in \mathbb{S}^1 \; \forall \theta \in \mathbb{R}$

To show that $R(\theta)$ is a rotation $\left(R(\theta) \in SO(2)\right)$, we have to show that $R^T R = I$ and $\det(R) = 1$

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad R^T(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R^T(\theta) R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta) & \cos^2(\theta) + \sin^2(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\det(R(\theta)) = \cos^2(\theta) + \sin^2(\theta) = 1$$

This demonstrates that $R(\theta)$ is a rotation

Given $R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$, having a map that is $R(\theta)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$, which by definition, is the coordinate space of the unit circle $\mathbb{S}^1$

Question 4.1 optional – Two Charts for a Circle

We need two charts for a circle because, by definition, a chart maps ***open regions*** in one space to ***open regions*** in another. Because it is an open region, there must be at least two regions such that their union covers the entire area.

Question 4.2 optional – Generate Vectors θ_1 and θ_2

Problem 5: Charts for the Torus Using Rotations

Question 5.1 code – Torus.phi()

Question 5.1 report – Create Atlas for Torus

The torus that we are interested in $\mathbb{T}^2$ is created via the product of 2 circles, $\mathbb{S}^1 \times \mathbb{S}^1$. Because each circle needs a minimum of 2 charts to cover, we thus determine that there needs to be a minimum of 4 charts to cover (using rectangular charts).

The charts that I will use need to cover the entire space with a little overlap so that the open-set overlaps.

$$\begin{aligned} U_1 &= [0, 2\pi - 0.1] \times [0, 2\pi - 0.1] \\ U_2 &= [0.1, 2\pi] \times [-\pi, \pi - 0.1] \\ U_3 &= [-\pi, \pi - 0.1] \times [0, 2\pi - 0.1] \\ U_4 &= [-\pi + 0.1, \pi] \times [-\pi + 0.1, \pi] \end{aligned}$$

Question 5.2 report – Torus.plot_charts()

Question 5.2 optional – Plot All Charts Simultaneously

Question 5.3 report – Explain the Atlas for the Torus

Each chart should not overlap itself, and no part should be left uncovered due to the definition of a chart. Charts are homeomorphisms that map open regions in $\mathbb{R}^d$ to the topological space. Because these regions are open, they cannot overlap themselves

Question 5.4 report – Compute the Tangent $\dot{\theta}(t)$ of the curve $\theta(t)$

Tangent of this curve: $\frac{d}{dt}\left(\dot{\theta}(t)\right) = \frac{d}{dt}\left(a(1)t + b(1), a(2)t + b(2)\right) = \left(a(1), a(2)\right)$

Question 5.2 code – Torus.phi_push_curve()

Question 5.5 report – Torus.plot_charts_curves

Problem 6: Jacobians and End Effector Velocities

Question 6.1 report – Expression for $\frac{d}{dt}\left({}^{w}p_{eff}\right)$ as a function of $\dot{\theta} = \begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix}$

$$\frac{d({}^{w}p_{eff})}{dt} = \begin{bmatrix} 5[\cos(\theta_1(t))\cos(\theta_2(t)) - \sin(\theta_1(t))\sin(\theta_2(t))] + 5\cos(\theta_1(t)) \\ 5[\sin(\theta_1(t))\cos(\theta_2(t)) + \cos(\theta_1(t))\sin(\theta_2(t))] + 5\sin(\theta_1(t)) \end{bmatrix}$$

$$\frac{d({}^{w}p_{eff})}{dt} = 5\frac{d}{dt}\begin{bmatrix} \cos(\theta_1(t))\cos(\theta_2(t)) - \sin(\theta_1(t))\sin(\theta_2(t)) + \cos(\theta_1(t)) \\ \sin(\theta_1(t))\cos(\theta_2(t)) + \cos(\theta_1(t))\sin(\theta_2(t)) + \sin(\theta_1(t)) \end{bmatrix}$$

$$5\begin{bmatrix} \frac{\partial p_{eff_1}}{\partial\theta_1} & \frac{\partial p_{eff_1}}{\partial\theta_2} \\ \frac{\partial p_{eff_2}}{\partial\theta_1} & \frac{\partial p_{eff_2}}{\partial\theta_2} \end{bmatrix}\begin{bmatrix} \frac{d\theta_1}{dt} \\ \frac{d\theta_2}{dt} \end{bmatrix}$$

$$5\begin{bmatrix} -\sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2) - \sin(\theta_1) & -\cos(\theta_1)\sin(\theta_2) - \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1) \\ \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + \cos(\theta_1) & -\sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1) \end{bmatrix}\begin{bmatrix} \frac{d\theta_1}{dt} \\ \frac{d\theta_2}{dt} \end{bmatrix}$$

Question 6.1 optional – Find the Jacobian from Previous Question

$$5\begin{bmatrix} -\sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2) - \sin(\theta_1) & -\cos(\theta_1)\sin(\theta_2) - \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1) \\ \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + \cos(\theta_1) & -\sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1) \end{bmatrix}$$

Question 6.1 code – TwoLink.jacobian()

Question 6.2 report – Find $\frac{d}{dt}\left({}^{w}p_{eff}\right)$ for Set of θ Values

Question 6.3 report – me570_hw2.torus_twolink_plot_jacobian()

Question 6.4 report – Relationship between report 5.5 and report 6.3