rtron Add tic/toc to A* test **rtron** Add mtic/mtoc functions based on https://www.mathworks.com/ma how-can-i-profile-the-memory-usage-in-matlab-when-i-use-linux-and-mac-os and `memory` for Windows **rtron** Add optional question implementing path post-processing like θ^* **rtron** Add autograder checks with empty arrays **rtron** In solutions, remove `hw1_` prefix

Homework 4 (Python version)

ME570 - Prof. Tron

2021-11-16

In this homework, you will implement the `A*` graph search algorithms, and apply it to a discretization of the sphere world and the two-link manipulator from previous assignments.

General instructions

Programming For your convenience, together with this document, you will find a `zip` archive containing Python files with stubs for each of the questions in this assignment; each stub contains an automatically generated description and header of the function or class. You will have to complete these files with the requested code. The goal of this files is to save you a little bit of time, and to avoid misspellings in the function or argument names. The files for the parts marked as **provided** (see also the *Grading* paragraph below) contain already the body of the function.

Homework help For best coding practices, please refer to the guidelines on Blackboard under Class content/Programming Tips & Tricks/Python. For questions specific to the content of the homework, please post on the Blackboard discussion board.

Homework report Along the programming of the requested functions, prepare a PDF report containing one or two sentences of comments for each question marked as **report**, and including: embedded figures and outputs that are representative of those generated by your code. Include comments on the questions marked as **code** only to explain any difficulty you might have encountered.

A small amount of *beauty points* are dedicated to reward reports that present their content in a professional way (see the *Grading criteria* section in the syllabus).

Analytical derivations To include the analytical derivations in your report you can type them in LaTeX(preferred method), any equation editor or clearly write them on paper and use a scanner (least preferred method).

Submission

The submission will be on Gradescope through four separate assignments: two for the questions marked as **code**, and one for those marked as **report**, and one for providing feedback. Further details are explained below. You can submit as many times as you would like, up to the assignment deadline. Each question is worth 1 point unless otherwise noted. Please refer to the Syllabus on Blackboard for late homework policies.

Report Upload the PDF of you report, and then indicate, for each question marked as **report**, on which page it is answered (just follow the Gradescope interface). Note that some of the questions marked as **report** might include a coding component, which however will be evaluated from the output figures you include in the report. In general, these questions are intended as checkpoints for you to visually check the results of your functions.

Code questions Upload all the necessary `.py` files, both those written by you, and those provided with the assignment. You will see *two* assignments on Gradescope. In the one marked *Python files*, you will submit the `.py` files directly; Gradescope will run one test checking for completeness, and one test checking the style (using Pylint); these automated tests will not check the correctness your answers. In the other assignment marked *Python PDF listing*, please submit a PDF containing a print-out of the contents of the `.py` files (see the footnote[1] for how to generate such file). I will use this to manually grade the code. However, you can use the output of the test functions to judge if your code gives correct results or not.

Optional and provided questions. Questions marked as **optional** are provided just to further your understanding of the subject, and not for credit (if submitted, I will provide comments but it will not count toward your grade).

Hints

Some hints are available for some questions, and can be found at the end of the assignment (you are encouraged to try to solve the questions without looking at the hints first). If you use these hints, please state so in your report (your grading will not change based on this information, but it is a useful feedback for me).

Use of external libraries and toolboxes You are **not allowed** to use functions or scripts from external libraries or toolboxes (e.g., mapping toolbox), unless specifically instructed to do so (e.g., CVX).

Graph data structure and utilities

Both problems in this homework represent the configuration space as a graph. In practical terms, the graph will be represented by a list of dictionaries `graph_vector`, where each element of the array is a dictionary with fields:

- `neighbors` (dim. [`nb_neighbors` $\times$ 1]): an array containing the indexes (in `graph_vector`) of the vertices that are adjacent to the current one.
- `neighbors_cost` (dim. [`nb_neighbors` $\times$ 1]): an array, with the same dimension as the field `neighbors`, containing the cost to move to each neighbor.
- `g` (`float`): scalar variable to store the cost from the starting location along the path through the backpointer.

[1] The provided `.zip` file includes a `pygments_homework4.py` file that you can use to generate a HTML file with all the necessary pretty-printed listings via the command `python pygments_homework4.py`. The script requires the `pygments` Python package. You can then generate a PDF file using a browser and its *"Print to pdf"* functionality.

- `backpointer` (`float`): index of the previous vertex in the current path from the starting location.
- `x` (dim. [2 × 1]): the physical (x, y) coordinates of the vertex.

Note that, in the above, the dimension `nb_neighbors` is in general different for each element in `graph_vector`. The graph is defined by the fields `x`, `neighbors`, `neighbors_cost`; the fields `g` and `backpointer` will be added and used by the graph search algorithm, which will modify these fields while leaving the others constant.

To help you with the homework, the assignment includes a number of utilities.

> Class name: `Graph`
> File name: `me570_graph.py`
> Description: A class collecting a `graph_vector` data structure and all the functions that operate on a graph.
>
> > Method name: `__init__`
> > Description: Stores the arguments as internal attributes.
> > Input arguments
> >
> > - `graph_vector` (dim. [nb_nodes × 1], type `dictionary list`): A list of dictionaries as described above.

Since most of the functions that you will implement are methods of the class `Graph`, when the assignment refers to `graph_vector`, this corresponds to the internal attribute in the same class.

Question provided 0.1. The first utility is a function to plot the graph.

> Class name: `Graph`
> File name: `me570_graph.py`
>
> > Method name: `plot`
> > Description: The function plots the contents of the graph described by the `graph_vector` structure, alongside other related, optional data.
> > Optional arguments
> >
> > - `flag_edges` (default: `True`): Show the edges.
> > - `flag_labels` (default: `False`): Show the labels of the vertices.
> > - `flag_edge_weights` (default: `False`): Show weights for each vertex.
> > - `flag_backpointers` (default: `False`): Show arrows for the backpointer.
> > - `flag_backpointers_cost` (default: `False`): Show the value of `'g'` for the corresponding backpointer.
> > - `node_lists` (default: `None`): A list or list of lists, of nodes to be marked (e.g., start or goal nodes, a list of closed nodes). Different lists are shown with different markers (in the order `'d'`, `'o'`, `'s'`, `'*'`, `'h'`, `'^'`, `'8'`).

Question **provided** **0.2.** The function `graph_load_test_data`(`_`) described below allows you to load already-made graphs, stored in the variables `graph_vector` and `graph_vector_medium`. Additionally, you can access `graph_vector_solved`, and `graph_vector_medium`, which are the same as `graph_vector`, and `graph_vector_medium`, but with the fields `g` and `backpointer` populated.

> File name: `me570_graph.py`
> Function name: `graph_load_test_data`
> Description: Loads data from the file `graph_test_data.pkl`.
> Input arguments
> - `variable_name` (type `string`): name of the variable to load. Available names include: `closedMedium`, `graphVector`, `graphVectorMedium`, `graphVectorMedium_solved`, `graphVector_solved`.
>
> Returns arguments
> - `graph_vector` (dim. [`nb_nodes` $\times$ 1], type `dictionary list`): A list of dictionaries as described above, **or a list**.

You can visualize the data from `graph_load_test_data`(`_`) with the call `Graph(graph_load_test_data(name)).plot()`, where `name` is one of the available graphs.

Question **provided** **0.3.** The last provided utility allows you to find the nodes in the graph that are closest to a given point.

> Class name: `Graph`
> File name: `me570_graph.py`
>
> > Method name: `nearest_neighbors`
> > Description: Returns the k nearest neighbors in the graph for a given point.
> > Input arguments
> > - `x_query` (dim. [2 $\times$ 1]): coordinates of the point of which we need to find the nearest neighbors.
> > - `k_nearest` (dim. [1 $\times$ 1]): number of nearest neighbors to find.
> >
> > Output arguments
> > - `idx_neighbors` (dim. [`nb_neighbors` $\times$ 1]): indeces in `graph_vector` of the neighbors of `x`. Generally, `nb_neighbors=k`, except when `graph_vector` contains less than k vertices, in which case all vertices are returned.

In this homework, you will mainly use this function with $k = 1$ to find vertices that approximate start and goal locations.

Question **provided** **0.4.** This function should takes as input a discretized world and outputs the corresponding `graph_vector` structure.

File name: `me570_graph.py`
Method name: `grid2graph`
Description: The function returns a `Graph` object described by the inputs. See Figure 1 for an example of the expected inputs and outputs.
Input arguments

- `grid` (type `dictionary`): An object of class `Grid`. The attribute `fun_evalued` should contain a logical array such that `fun_evalued[i,j]` is `true` if there is a cell (i.e., no collision) at the (`xx_grid[i]`, `yy_grid[j]`) location, and `false` otherwise.

Requirements: Note that the fields `xx` and `yy` in `grid` are to be intended as *generalized coordinate* pairs, and their interpretation could be different than x and y coordinates of points in $\mathbb{R}^2$. For instance, in Problem 3 below, which involves the two-link manipulator, they correspond to angles.

Question provided 0.5. This homework includes an updated version of the class `Grid` from Homework 2. The class now includes an internal attribute that contains the last value returned by the method `eval`( ). In the following questions, you will use this internal attribute to obtain data that was previously generated.

File name: `me570_geometry.py`
Class name: `Grid`
Description: A class to store the coordinates of points on a 2-D grid and evaluate arbitrary functions on those points.

Attribute name: `fun_evalued`
Description: Stores the value that was returned by the last call to the method `eval`( ). It is initially set to `None`.

Problem 1: Graph search

In this problem you will implement a graph search algorithm, and apply it to a graph obtained from a grid discretization of a free configuration space. In particular, you will apply this to the two-link manipulator from Homework 3.

The graph search function you will develop will be generic, because it can work on a `graph_vector` data structure in a way that is somewhat abstract from the actual problem. For instance, the function manipulates nodes in terms of their indexes in the data structure, instead of, say, using their coordinates. In this way, the same function can be applied to different problems (an occupancy graph in this problem and a roadmap in the next).

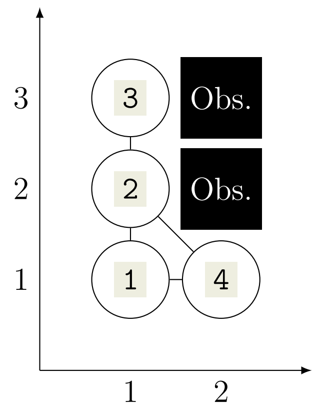
You will be required to implement the `A`* algorithm, for which the reference pseudo-code for the algorithm can be found on page 531 of the book, and is reproduced in Algorithm 1 with some additional minor clarifications.

Data structures The algorithm uses a priority queue O, and a list of closed edges C. For the priority queue O, you are expected to use the corresponding set of functions from Homework 1. For the list C, you should use a simple array. See Question code 1.5 for further details.

```
grid.xx=[1 2]
grid.yy=[1 2 3]
grid.F =
```

$$\begin{bmatrix} \texttt{true} & \texttt{true} & \texttt{true} \\ \texttt{true} & \texttt{false} & \texttt{false} \end{bmatrix}$$

(a) The input arguments.



(b) Graphical representation of the graph with 8-neighbors connectivity. The numbers represent the indexes in `graph_vector`. The name `flag_free_space_map` has been abbreviated to `flag` for convenience.

```
neighbors=[4;2]
neighborsCost=[1;1]
x=[1;1]
```

```
neighbors=[1;4;3]
neighborsCost=[1;1.414;1]
x=[1;2]
```

```
neighbors=[2]
neighborsCost=[1]
x=[1;3]
```

```
neighbors=[1;2]
neighborsCost=[1;1.414]
x=[2;1]
```

(c) The output `Graph.graph_vector` (the fields `g` and `backpointer` are not visualized). This particular `graph_vector` can be found in the provided data.

Figure 1: Example of the input and output arguments for `grid2graph`( ).

Debugging tips Since `A`* is a somewhat complex algorithm to implement, you should use the provided function `Graph.plot`( ) and the provided data `graph_testData.mat` to test the individual functions and check that the outputs are consistent with what you would expect. In particular, embedding `Graph.plot`( ) together with the `pause` command in the loop of `Graph.search`( ) during debugging is instructive (but remember to remove it in the final version or, even better, use an optional argument to enable it only when needed).

Question code 1.1.

> Class name: `Graph`
> File name: `me570_graph.py`

Method name: `heuristic`
Description: Computes the heuristic `h` given by the Euclidean distance between the nodes with indexes `idx_x` and `idx_goal`.
Input arguments

- `idx_x` (type `int`), `idx_goal` (type `int`): indexes of the elements in `graph_vector` to use to compute the heuristic.

Output arguments

- `h_val` (type `float`): the heuristic (Euclidean distance) between the two elements.

Question code 1.2.

Class name: `Graph`
File name: `me570_graph.py`

Algorithm 1 The `A`* algorithm.

Add the starting node n_{start} to O, set $g(n_{start}) = 0$, and set the `backpointer` of x to be empty. ▷ Initialization
repeat
  Pick n_{best} from O such that $f(n_{best}) \leq f(n)$ for all $n \in O$.
  Remove n_{best} from O and add it to C.
  if $n_{best} = q_{goal}$ **then**
    Exit.
  end if
  for all $x \in \text{Star}(n_{best})$ that are not in C **do** ▷ Expand n_{best}
    if $x \notin O$ **then**
      Set the value of $g(x)$ to $g(n_{best}) + c(n_{best}, x)$.
      Set the `backpointer` of x to n_{best}.
      Add x to O with value $f(x)$.
    else if $g(n_{best}) + c(n_{best}, x) < g(x)$ **then**
      Update the value of $g(x)$ to $g(n_{best}) + c(n_{best}, x)$.
      Update the `backpointer` of x to n_{best}.
    end if
  end for
until O is empty

Method name: `get_expand_list`

Description: Finds the neighbors of element `idx_n_best` that are not in `idx_closed` (line 8 in Algorithm 1).

Input arguments

- `idx_n_best` (type `int`): the index of the element in `graph_vector` of which we want to find the neighbors.
- `idx_closed` (dim. [`nb_closed` × 1], type `list`): list of indexes containing the list of elements of `graph_vectors` that have been closed (already expanded) during the search.

Output arguments

- `idx_expand` (dim. [`nb_neighborsnotclosed` × 1], type `list`): array of indexes of the neighbors of element `idx_n_best` in `graph_vector`

Question code 1.3 (2 points). The function below uses the priority queue from Homework 1.

Class name: `Graph`

File name: `me570_graph.py`

Method name: `expand_element`

Description: This function expands the vertex with index `idx_x` (which is a neighbor of the one with index `idx_n_best`) and returns the updated versions of `graph_vector` and `pq_open`.

Input arguments

- `idx_n_best` (type `int`), `idx_x` (type `int`), `idx_goal` (type `int`): indexes in `graph_vector` of the vertex that has been popped from the queue, its neighbor under consideration, and the goal location.
- `pq_open` (type `Priority`): object of type `Priority` with the priority queue of the open nodes.

Output arguments

- `pq_open` (type `Priority`): Same as the homonymous input argument, but updated with the new nodes that have been opened.

Requirements: This function corresponds to lines 9–16 in Algorithm 1.

Question code 1.4. Implement a function that transforms the backpointers describing a path into the actual sequence of coordinates.

Class name: `Graph`

File name: `me570_graph.py`

Method name: `path`
Description: This function follows the backpointers from the node with index `idx_goal` in `graph_vector` to the one with index `idx_start` node, and returns the *coordinates* (not indexes) of the sequence of traversed elements.
Input arguments

- `idx_start` (type `int`), `idx_goal` (type `int`): indexes in `graph_vector` of the starting and end vertices.

Output arguments

- `x_path` (dim. [2 × `nb_path`]): array where each column contains the coordinates of the points obtained with the traversal of the backpointers (in reverse order). Note that, by definition, we should have `x_path[:,0]` equal to `graph_vector[idx_start]['x']` and `x_path[:,-1]` equal to `graph_vector[idx_goal]['x']`.

Question code 1.5. This question puts together the answers to Questions code 1.1–code 1.4.

Class name: `Graph`
File name: `me570_graph.py`

Method name: `search`
Description: Implements the `A`* algorithm, as described by the pseudo-code in Algorithm 1.
Input arguments

- `idx_start` (dim. [1 × 1]), `idx_goal` (dim. [1 × 1]): indexes in `graph_vector` of the starting and end vertices.

Output arguments

- `x_path` (dim. [2 × `nb_path`]): array where each column contains the coordinates of the points of the path found from `idx_start` to `idx_goal`.

Requirements: **optional** Set a maximum limit of iterations in the main `A`* loop on line 2 of Algorithm 1. This will prevent the algorithm from remaining stuck on malformed graphs (e.g., graphs containing a node as a neighbor of itself), or if you make some mistake during development.

For the purposes of this homework, you can assume that a path always exists (although this can be optionally relaxed in Question optional 1.1).

The function for the cost to use in the priority queue, denoted as $f(n)$ in the book and in Algorithm 1, is $f(n) = g(n) + h(n)$, where $g(n)$ is the cost of the path from node n to the start vertex (going through the backpointer), and $h(n)$ is the heuristic (the Euclidean distance between nodes, see below). The cost $c(n, x)$ between two vertices is the one stored in the `neighbors_cost` field.

Your implementation of `Graph.search( )` must contain the following elements:

- a priority queue `pq_open` of the *opened* vertices (the structure O in Algorithm 1); this structure must be an object instantiated from the class `Priority` from Homework 1; use the index of the vertex for the *key* and the function $f(n)$ described above for the `cost`.
- a list `idx_closed` (dim. [`nb_closed` × `1`]) containing the indexes of the closed vertices.

Question optional 1.1. Add conditions to return an empty `path` if `A*` cannot find a feasible path.

Question optional 1.2. Add an argument `method` containing a string that determines the behavior of the algorithm. The function $f(n)$ will then depend on the value of the argument `method`:

- $f(n) = g(n)$ if `method` is equal to `bfs`.
- $f(n) = h(n)$ if `method` is equal to `greedy`.
- $f(n) = g(n) + h(n)$ if `method` is equal to `astar`.

Question optional 1.3 (recommended). Make a function `graph_search_test`( ) that calls `Graph.search`( ) to find a path between arbitrary two nodes in the graphs provided by `graph_load_test_data`( ), and then plots the results. Visually inspect if the results make sense, try to also use your own graphs. This will give you the confidence that your `A*` algorithm works well. Since this routine is the backbone of the next questions, I strongly encourage you to make sure that it works properly before moving on.

Problem 2: Application of `A*` to the sphere world

In this problem you will apply the `A*` graph search function from Problem 1 to a discretized version of the sphere world used in Homework 3. The instructions below assume that you all the functions and data from Homework 3 (that you have developed or from the solution) are in the same directory.

> File name: `me570_graph.py`
> Class name: `SphereWorldGraph`
> Description: A discretized version of the `SphereWorld` from Homework 3 with the addition of a search function.

Question code 2.1. Create a function that initializes an object with a discretized version of the Sphere World environment.

> Class name: `SphereWorldGraph`
> File name: `me570_graph.py`

Method name: `__init__`
Description: The function performs the following steps:

1) Instantiate an object of the class `SphereWorld` from Homework 3 to load the contents of the file `sphereworld.mat`. Store the object as the internal attribute `sphereworld`.d
2) Initializes an object `grid` from the class `Grid` initialized with arrays `xx_grid` and `yy_grid`, each one containing `nb_cells` values linearly spaced values from `-10` to `10`.
3) Use the method `grid.eval`( ) to obtain a matrix in the format expected by `grid2graph`( ) in Question provided 0.4, i.e., with a `true` if the space is free, and a `false` if the space is occupied by a sphere at the corresponding coordinates. The quickest way to achieve this is to manipulate the output of `Total.eval`( ) (for checking collisions with the spheres) while using it in conjunction with `grid.eval`( ) (to evaluate the collisions along all the points on the grid); note that the choice of the attractive potential here does not matter.
4) Call `grid2graph`( ).
5) Store the resulting `graph` object as an internal attribute.

Input arguments

- `nb_cells` : Number of cells on one side of the grid used for the discretization.

optional It is suggested that you use `Graph.plot`( ) to check that the result is consistent with the map shown by `Sphereworld.plot`( ).

Question code 2.2. The function from this question is similar to (and is actually implemented using) the function `Graph.search`( ), except that the start and end locations are specified using actual coordinates instead of indeces to nodes in the graph.

Class name: `Graph`
File name: `me570_graph.py`

Method name: `search_start_goal`

Description: This function performs the following operations:

1) Identifies the two indexes `idx_start`, `idx_goal` in `graph.graph_vector` that are closest to `x_start` and `x_goal` (using `Graph.nearestNeighbors`( ) twice, see Question provided 0.3).

2) Calls `Graph.search`( ) to find a feasible sequence of points `x_path` from `idx_start` to `idx_goal`.

3) Appends `x_start` and `x_goal`, respectively, to the beginning and the end of the array `x_path`.

Input arguments

- `x_start` (dim. [2 × 1]), `x_goal` (dim. [2 × 1]): vectors describing the initial and final points for the path search.

Output arguments

- `x_path` (dim. [2 × `nb_path`]): a sequence of pairs of points describing a feasible path. By definition `x_path[:,0]=x_start`, `x_path[:,-1]=x_goal`, and all the other columns are those returned by `Graph.search`( ).

Question report 2.1. Pick three values of `nb_cells` such that, after discretization:

1) Some or all of the obstacles fuse together (`nb_cells` is too low);

2) The topology of the Sphere World is well captured (`nb_cells` is "just right");

3) The graph is much finer than necessary (`nb_cells` is too high).

Include the three values in your report, together with a visualization of the corresponding graphs (using `Graph.plot`( )).

Question report 2.2. Create the following function:

Class name: `SphereWorldGraph`

File name: `me570_graph.py`

Method name: `run_plot`

Description:

1) Load the variables `x_start`, `x_goal` from the internal attribute `sphereworld`.

2) For each goal in `x_goal`:

(a) Run `search_start_goal`( ) from every starting location in `x_start` to that goal.

(b) Plot the world using `Sphereworld.plot`( ), together with the resulting trajectories.

Create three objects from `SphereWorldGraph` with the three values of `nb_cells` from the previous question, and call `run_plot`( ) for each one of them. In total, you should produce six different images (three choices for `nb_cell` times two goals). Include all the images in the report. Please make sure that images from different choices of `nb_cell` but the same goal appear together in the same page (to help comparisons).

Question **report** **2.3.** Comment on the behavior of the A* planner with respect to the choice of `nb_cell`.

Question **report** **2.4.** Comment on the behavior of the A* planner with respect to the potential planner from Homework 3.

Problem 3: Application of A* to the two-link manipulator

In this problem you will apply the graph search function you implemented in Problem 1 to the two-link manipulator from Homework 2. In this case, the coordinates in the field `'x'` of `graph_vector` will represent the pairs of angles (θ_1, θ_2) for the two links (as was specified in Homework 2).

The file `twolink_freeSpace_data.mat` contains a dictionary `grid` that describes the configurations of angles for the two-link manipulator that collide with the set of points in `twolink_testData.mat` (see Question provided 0.4 for the format used in `Grid`). This structure is essentially the result of an optional question from Homework 2 (please reread that question for details).

Question **provided** **3.1.** This method loads the data from the file `twolink_freeSpace_data.mat`

> File name: `me570_robot.py`
> Class name: `TwoLinkGraph`
> Description: A class for finding a path for the two-link manipulator among given obstacle points using a grid discretization and A*.

> File name: `me570_robot.py`
> Method name: `load_free_space_grid`
> Description: Loads the contents of the file `twolink_freeSpace_data.mat`
> Returns arguments
> - `grid` (type `Grid`): a grid where the attributes `xx_grid`, `yy_grid`, `fun_evalued` are set to the value loaded from `twolink_freeSpace_data.mat`.

Question **report** **3.1.** For this question, you need to implement the following functions:

> Class name: `TwoLinkGraph`
> File name: `me570_robot.py`

Method name: `load_free_space_graph`
Description: The function performs the following steps

1) Calls the method `load_free_space_grid`( ).
2) Calls `grid2graph`( ).
3) Stores the resulting `graph` object of class `Grid` as an internal attribute.

Class name: `TwoLinkGraph`
File name: `me570_robot.py`

Method name: `plot`
Description: Use the method `Graph.plot`( ) to visualize the contents of the attribute `graph`.

Method name: `search_start_goal`
Description: Use the method `Graph.search`( ) to search a path in the graph stored in `graph`.
Input arguments

- `theta_start` (dim. [2×1]), `theta_goal` (dim. [2×1]): vectors describing the initial and final joint angles for the path search.

Output arguments

- `theta_path` (dim. [$2 \times$ `nb_path`]): a sequence of pairs of angles describing a feasible path.

Question optional 3.1. Modify the functions from the previous problems to work with the topology of the configuration space of the two-link manipulator by following the steps below:

1) Modify `grid2graph`( ) to allow an additional optional argument `mode='torus'`. If this argument is passed to `grid2graph`( ), in the final graph the vertices on the left edge become neighbors of those on the right edge, and the vertices on the bottom edge become neighbor of those on the top edge. With this option, we change the topology of the space from $\mathbb{R}^2$ to $\mathbb{S}^1 \times \mathbb{S}^1$, that is, from the plane to the torus.
2) Modify `Graph.heuristic`( ) to allow an additional optional argument `mode='torus'`. With this argument, the heuristic will use a mod-2π arithmetic to compute the distance between pairs of angles instead of the Euclidean distance (look at the function `Edge.angle`( ) from Homework 1 for inspiration). For instance, with this option the heuristic between the pairs of angles $\begin{bmatrix} 2\pi - 0.1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$ should be 0.2 instead of $2\pi - 0.4$.
3) Modify `Graph.search`( ) with an optional argument that enables the use of `graph_heuristic` with the `'torus'` option (you can either introduce an option `'torus'`, or allow passing the heuristic as a function).

4) Modify the method `TwoLinkGraph.search_start_goal()` that uses the modified `Graph.search()` from the previous point.

Question **report** **3.2.** Plot the points `obstacle_points` in `twolink_testData.mat` (see the code the was provided for Question provided 2.2 in Homework 2), call the function `TwoLinkGraph.search_start_goal()`, and then the method `TwoLink.plotAnimate()`, for the following start/goal configurations:

- *Easy*: `theta_start=`$\begin{bmatrix}0.76\\0.12\end{bmatrix}$, `theta_goal=`$\begin{bmatrix}0.76\\6.00\end{bmatrix}$.
- *Medium*: `theta_start=`$\begin{bmatrix}0.76\\0.12\end{bmatrix}$, `theta_goal=`$\begin{bmatrix}2.72\\5.45\end{bmatrix}$.
- **optional** *Hard*: `theta_start=`$\begin{bmatrix}3.30\\2.34\end{bmatrix}$, `theta_goal=`$\begin{bmatrix}5.49\\1.07\end{bmatrix}$. For this case, the planner will find a feasible path only if you implement and pass the `'torus'` option.

Note that all values for the angles are in radians. Every time the graph search finds a feasible path, you should see the manipulator move between the obstacle points, where each configuration that is plotted is not .

Question **report** **3.3 (2 points).** For the *Easy* case in the question above, comment on the *unwinding* phenomenon that appears if you do not use the `'torus'` option (that is, why the planner does not find the straightforward path that keeps the first link fixed). To obtain full marks, make sure to include the relation between your answer and the visualization of the configuration space from Homework 2. Include all the final figures in your report.

Question **report** **3.4.** Comment on how close the planner goes to the obstacles, and what you could do about it in a practical situation.

Question **optional** **3.2.** Notice that the majority of the time during planning is spent in checking collisions while generating the free space graph, but most of the graph is never actually explored during search. To significantly speed up the planner, you can use *lazy evaluation*. Lazy evaluation performs collision checking when looking for neighbors in the expansion of a node (line 8 in Algorithm 1), instead of performing it for all the nodes at the beginning. Make a method `TwoLinkGraph.search()` that is the same as `Graph.search()` but:

- The attribute `graph_vector` does not contain neighbor information (the fields `neighbors` and `neighbors_cost` are not used).
- The subfunction `getExpandList()` uses `TwoLink.is_collision()` to find the neighbors of the node being expanded.

Run the function `TwoLinkGraph.search()` on the problems above, and compare the computation times with the previous implementation.

Hint for question code 1.2: Since each element in `graph_vector` already contains a list of indexes of neighbors for each node, this function reduces to compute a set difference (see the `setdiff` Matlab function).