Graphs on groups, rings, and maybe YBE solutions

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Groups, Rings and YBE
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I will start with what I know …

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The prototype is the commuting graph of a finite group G, where the vertex set is G (or possibly some subset), and g and h are joined by an edge if they commute.

This was used by Brauer and Fowler in 1955 to show that there are only finitely many finite simple groups with a given involution centraliser, one of the basic results in the Classification of Finite Simple Groups (leading to a large amount of work characterising particular simple groups by their involution centralisers, and yielding several new sporadic simple groups along the way.

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In fact, the word "graph" does not occur in the paper; but Brauer and Fowler carefully define the graph metric and use this instead.

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Much of the literature on these graphs consists of calculating various graph-theoretic parameters of these graphs. I will not cover most of this.

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2. Can we recognise old and new classes of groups by means of graphs?
3. Can we construct beautiful graphs in this way (possibly after some post-processing)?

I will give examples of all three.

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Recently, Parthajit Bhowal, Rajat Kanti Nath, Benjamin Sambale and I showed:

Theorem

There is a function f such that a finite group whose scc-graph has clique number k has order at most $f(k)$.

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Find such bounds!

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2. Choose two types of graph on groups, say t_1 and t_2, so that $t_1(G)$ is an induced subgraph of $t_2(G)$, and ask: *For which groups G is $t_1(G) = t_2(G)$?*

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There are several examples of each in the literature. I will concentrate on the second.

Two examples

We have seen the commuting graph ($g \sim h$ if $gh = hg$) and the power graph ($g \sim h$ if one of g and h is a power of the other). Between them is the enhanced power graph, with $g \sim h$ if there exists k such that g and h are powers of k.

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Let G be a finite group.

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2. *The enhanced power graph of G is equal to the commuting graph if and only if G contains no two commuting subgroups of the same prime order.*

I will briefly discuss the two classes.

Two classes of groups

The first class consists of EPPO groups, those in which every element has prime power order. (In other terminology these are groups whose Gruenberg–Kegel graph is null.) After pioneering work by Higman on solvable groups in the 1950s and Suzuki on simple groups in the 1960s, they were all determined by Brandl in a somewhat obscure paper in 1981.

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The second class consists of groups containing no subgroup $C_p \times C_p$ for p prime; in other words, all Sylow subgroups are cyclic or (if $p = 2$) generalized quaternon. Those with all Sylow subgroups cyclic are metacyclic of known structure; the others are determined by theorems of Glauberman and Gorenstein–Walter.

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All these results are without using CFSG.

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Bojan Kuzma and I investigated this graph, and proved (among other things)

Theorem

Let G be a finite group. Then the deep commuting graph is equal to the commuting graph if and only if the Bogomolov and Schur multipliers of G coincide.

Other classes

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In many other cases, work is in progress. For example, the power graph of any finite group is perfect (that is, every induced subgraph has clique number equal to chromatic number): this condition is equivalent to forbidding odd cycles (or length greater than 3) and their complements as induced subgraphs, according to the Strong Perfect Graph Theorem.

More on perfect graphs

There is no analogue for the enhanced power graph or commuting graph: these are universal (every finite graph occurs as an induced subgraph). We do not know which groups have one or other of these graphs perfect (this has been studied for the commuting graph by Britnell and Gill, who found all *perfect* groups for which this graph is perfect).

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3. Finding beautiful graphs

If you choose your favourite group and ask the computer to construct one of these graphs and tell you how many automorphisms it has, you are in for a shock. For example, the commuting group of the alternating group A_5 (a group of order 60) has 4770901323934635707596800000 automorphisms. In fact, most of this is rubbish; in the case of A_5 it is all rubbish. But sometimes there is a jewel buried in the heart of the lotus flower.

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Our graphs on groups tend to have many pairs of twins. If x and y generate the same cyclic subgroup of G, then they are twins in all the graphs mentioned so far, and essentially all others as well.

Twin reduction

Twin reduction is the process of choosing a pair of twins and identifying them, repeating the process until no twins remain. The resulting graph is (up to isomorphism) independent of the way the reduction is carried out. I will call it the cokernel of the original graph (no connection with homological algebra implied).

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Proposition

A graph is a cograph if and only if its cokernel is the 1*-vertex graph.*

The search

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Partial answers are known in some cases. In particular, Pallabi Manna, Ranjit Mehatari and I have determined the finite simple groups whose power graph is a cograph; Xuanlong Ma, Natalia Maslova and I have done the same for the commuting graph.

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Partial answers are known in some cases. In particular, Pallabi Manna, Ranjit Mehatari and I have determined the finite simple groups whose power graph is a cograph; Xuanlong Ma, Natalia Maslova and I have done the same for the commuting graph. The simplest results are for what I will call the difference graph, whose edges are those in the enhanced power graph but not in the power graph.

Some results

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- the difference graph is a cograph, so its cokernel has a single vertex;
- the cokernel of the difference graph has many very small connected components, all isomorphic;
- the cokernel is connected; its full automorphism group is the same as the automorphism group of the simple group with which we began; and the graph has nice properties (for example, large girth).

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For example, if G is the Matheu group M_{11}, then the cokernel of the difference graph is bipartite, with blocks of size 165 and 220; the valencies of vertices in the two blocks are 4 and 3 respectively; the graph is connected, with diameter and girth 10; and its automorphism group is M_{11}.

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More exploration remains to be done . . .

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To begin at the beginning: the set-theoretic Yang–Baxter equation is an equation for a function $r : X \times X \to X \times X$ satisfying

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23},$$

where this equation refers to maps on $X \times X \times X$, and r_{ij} replaces the pair (x,x_j) by the pair of coordinates of $r(x_i, x_j)$.

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- r is an involution (this implies that it is a bijection);
- r is non-degenerate (see next slide).

Actions

We can write $r(x, y)$ as $(f_x(y), g_y(x))$, where, for any $x, y \in X$, the functions f_x and g_y map X to X. We say that our solution is non-degenerate if these functions are bijections for all choices of x and y.

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Now we regard the permutations f_x and g_y as generators of a group $G(r)$ acting on X. Warning: It is customary to regard the f_x as acting on the left and the g_y on the right: as a mnemonic, $r(x, y)$ is often written as $({}^x y, x^y)$.

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The YBE implies that the gs can be written in terms of the fs, and *vice versa*; so the groups generated by the fs and by the gs are equal. This is the Yang–Baxter permutation group associated with the solution.

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The YBE implies that the gs can be written in terms of the fs, and *vice versa*; so the groups generated by the fs and by the gs are equal. This is the Yang–Baxter permutation group associated with the solution.

Note: we should certainly be open to relaxing the non-degeneracy condition and working with monoids rather than groups; but their theory is less developed.

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Problem

Does the Yang–Baxter permutation group permute the orbits among themselves? In other words, if $r(x, y) = (u, v)$ and $f_z(x) = u$ for some z, is it the case that $f_z(y) = v$?

First attempt

The function r is an involution on $X \times X$, and fixes all the points on the diagonal. Its remaining orbits have sizes 1 or 2, so they consist of edges or pairs of directed edges on X.
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Problem

Does the Yang–Baxter permutation group permute the orbits among themselves? In other words, if $r(x,y) = (u,v)$ *and* $f_z(x) = u$ *for some* z*, is it the case that* $f_z(y) = v$*?*

Variant: Since the trivial solution of YBE is the transposition $r(x,y) = (y,x)$, it would be sensible to change this to replace r by its composition with transposition. This should not change the problem too much.

Second attempt

Problem

Is there a set of ordered pairs of elements of X, naturally defined in terms of r, and invariant under the Yang–Baxter permutation group?

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If this is the case, then we have a graph to which we can hope to apply some of the ideas I have described earlier.

Third attempt

The function $x \mapsto f_x$ maps X into $\mathrm{Sym}(X)$, and its image is the set of generators for the Yang–Baxter group.

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So we are in the territory of Cayley graphs, and the structure on X is just what it gets as the generating set of a finite subgroup.
In particular this is not a group structure on X.
What can be achieved from this approach?

What to do?

Another question: Is it possible to use beautiful combinatorial objects (perhaps graphs on groups) to define interesting solutions of the YBE.

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… for your attention.