# Permutation Groups and Transformation Semigroups
# Lecture 5: Idempotent generation

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In the final lecture, I will consider when the semigroup generated by a permutation group and one additional map of given rank $k$ is idempotent-generated. I will concentrate on the case $k = 2$, which gives rise to a challenging problem, the *Road closure conjecture*.

I will also say a little about some related problems.

## 1 Orbital graphs

Other lecturers have talked about orbital graphs; I will summarise here enough for my needs. I will consider only undirected graphs, although a richer theory (that of *coherent configurations* is obtained if directed graphs are used.

Let $G$ be a permutation group on $\Omega$. For convenience, I assume that $G$ is transitive. As we have seen, $G$ has an induced action on the set of 2-element subsets of $\Omega$. This set falls into a number of orbits for this action, say $O_1, O_2, \ldots, O_s$.

Now for each subset $I$ of $\{1, 2, \ldots, s\}$, there is a graph with vertex set $\Omega$ and edge set $\bigcup_{i \in I} O_i$. Since the orbits are preserved by $G$, this graph is invariant under $G$, and the action of $G$ on this graph is vertex-transitive. Moreover, every $G$-invariant graph arises in this way. In the case where $|I| = 1$, say $I = \{i\}$, we obtain a graph $\Gamma_i$ on which $G$ acts, transitively on the vertices and on the edges. The graphs $\Gamma_i$ are the *orbital graphs* for $G$.

Now Donald Higman showed:

**Theorem 1.1** *$G$ is primitive if and only if all the orbital graphs are connected.*

For, if $G$ is imprimitive, suppose that two vertices $x, y$ are related by a non-trivial $G$-invariant equivalence relation. If $O_i$ is the orbit containing $\{x, y\}$, then the orbital graph $\Gamma_i$ has the property that any edge is contained in an equivalence class of the relation, and so this graph is disconnected. Conversely, if there is a disconnected orbital graph, then its connected components are the classes of a non-trivial $G$-invariant equivalence relation.

From this, our earlier result that the 2-ut property is equivalent to primitivity follows. For 2-ut asserts that, for any 2-set $\{x,y\}$ and any 2-partition, there is an image of $\{x,y\}$ which is a transversal to the partition; so the corresponding orbital graph has an edge transversal to any 2-partition, and must be connected.

# 2 Idempotents and idempotent generation

**Theorem 2.1** *For a permutation group $G$ on $\Omega$, and positive integer $k$ with $2 \leq k \leq n/2$, the following are equivalent:*

*(a) for any map $f$ of rank $k$, $\langle G, f\rangle$ contains an idempotent of rank $k$;*

*(b) $G$ has the $k$-ut property.*

**Proof** Suppose that $G$ has the $k$-ut property, and that $f$ has rank $k$, image $A$, and kernel $P$. Then, there exists $g$ such that the image of $fg$ is a transversal for its kernel $P$. As we remarked in the second lecture, this guarantees that some power of $fg$ is an idempotent.

Conversely, let $A$ be a $k$-set and $P$ a $k$-partition. Choose a map $f$ with kernel $P$ and image $A$. By assumption, $\langle G, f\rangle$ contains an idempotent $e$ of rank $k$; without loss, $e = fg_1fg_2\cdots fg_r$. (If the expression for $e$ begins with an element of $g$, conjugate by this element to move it to the end.) Now the rank of $fg_1$ is equal to $k$, and so $Ag_1$ is a transversal to $P$, as required.

However, for $\langle G, f\rangle \setminus G$ to be idempotent-generated for all rank $k$ maps $f$ is a stronger condition. First note that the condition is empty for $k = 1$, since every rank 1 map is an idempotent; so $k = 2$ is the first non-trivial case.

In general, a combinatorial equivalent to idempotent generation is not known. (There is a condition, the *strong $k$-ut property*, which implies idempotent-generation, but is not equivalent to it.) There is such a condition in the case $k = 2$, leading to an interesting open problem in permutation groups. Recall first that the 2-ut property is equivalent to primitivity, so whatever our condition is, it must be stronger than primitivity.

Let $G$ be a primitive permutation group on $\Omega$. As we saw earlier, any orbital graph for $G$ is connected. Our stronger condition is as follows.

We say that $G$ has the *road closure property* if the following holds: for any orbit $O$ of $G$ on 2-sets, and any proper block of imprimitivity (smaller than $O$) for the action of $G$ on $O$, the graph with vertex set $\Omega$ and edge set $O \setminus B$ (obtained by deleting the edges in $B$ from the orbital graph) is connected.

Imagine that the graph represents a connected road network; we ask that, if workmen come along and dig up a proper block of imprimitivity for $G$, the graph remains connected.

An example of a primitive group which fails to have the road closure property is the automorphism group of the square grid graph (the line graph of $K_{m,m}$: this is primitive for $m > 3$ (but of course not basic, since the grid is a Cartesian structure). See Figure 1.

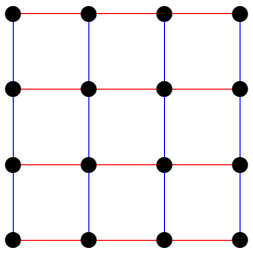

Figure 1: A grid

The automorphism group is transitive on the edges of this graph, and has two blocks of imprimitivity, the horizontal and vertical edges (coloured red and blue in the figure). If it is a road network, and if all the blue edges are closed, the network is disconnected: it is no longer possible to travel between different horizontal layers.

Using similar arguments it is possible to show that a primitive group which has the road closure property must be basic.

Here is an example of a basic primitive group which does have the road closure property. The group is $G = S_5$ acting on 2-sets; one orbital graph for it is the *Petersen graph* (Figure 2). Now the group $G$ acts transitively on the 15 edges, which fall into five groups of three mutually parallel or perpendicular edges in the standard drawing of the graph, as shown in the figure; these triples are the maximal blocks of imprimitivity. It is clear that, when the three edges shown in red are removed, the graph remains connected.

And here is a basic group which fails the road closure property. We take $G$ to be the group of automorphisms and dualities (maps which interchange points and lines but preserve incidence) of the *Fano plane* (Figure 3). $G$ acts on the *flags* of the Fano plane; the action is primitive. We consider the orbital graph in which two flags are joined if they share a point or a line. The edges fall into two types which are blocks of imprimitivity, depending on whether the two flags share a point or

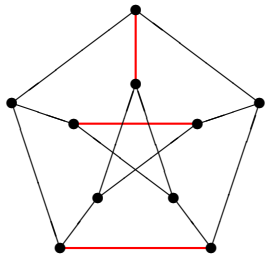

Figure 2: The Petersen graph

a line. If we remove the edges joining flags sharing a point, then from a given flag we can only move to the other two flags using the same line; so the graph is disconnected.

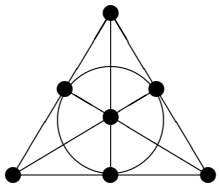

Figure 3: The Fano plane

The connection with our problem is:

**Theorem 2.2** *Let $G$ be a primitive permutation group on $\Omega$. Then the following are equivalent:*

*(a) $G$ has the road closure property.*

*(b) For any rank 2 map $f$, $\langle G, f\rangle \setminus G$ is idempotent-generated.*

The basic primitive groups which are known to fail the road closure property are rather few, and fall into two classes:

- groups which have an imprimitive normal subgroup of index 2 (the group associated with the Fano plane above is an example);

- a class of groups associated with the triality automorphism of the eight-dimensional orthogonal groups.

It is conjectured that this list is complete. This is the *Road closure conjecture*.

# 3 Partition transitivity and homogeneity

To conclude, here, more briefly, are a couple of related topics which are close to the ones already treated.

Let $f$ be a map of rank $k$ on $\Omega$, where $|\Omega| = n$, and $G$ a permutation group on $\Omega$, and consider the semigroup $S = \langle G, f\rangle \setminus G$. An element of $S$ of the form

$$s = fg_1fg_2f\cdots fg_rf$$

has the property that $\mathrm{Ker}(s)$ is a coarsening of $\mathrm{Ker}(f)$ (that is, any part of the latter is contained in a part of the former), while $\mathrm{Im}(s)$ is a subset of $\mathrm{Im}(f)$. Pre- and post-multiplying by elements of $G$, we see that the kernel of any element of $S$ is a $G$-image of a coarsening of $\mathrm{Ker}(f)$, while the image of any element of $S$ is a $G$-image of a subset of $\mathrm{Im}(f)$.

If $G = S_n$, then clearly the elements of maxumum rank $k$ in $S$ are all those whose kernels have the same shape as $\mathrm{Ker}(f)$, and whose images have the same cardinality as $\mathrm{Im}(f)$.

Consider the question: Which permutation groups $G$ have the property that

$$\langle G, f\rangle \setminus G = \langle S_n, f\rangle \setminus S_n.$$

Can we classify these groups? We see that this is equivalent to determining groups which are $k$-homogeneous and $\lambda$-homogeneous, where $\lambda$ is a partition of $n$ with $k$ parts: here, we say that a permutation group $G$ is *$\lambda$-homogeneous* if it acts transitively on partitions of $\Omega$ of shape $\lambda$.

Note that a similar concept, $\lambda$-transitive, related to $\lambda$-homogeneous much as $k$-transitive is to $k$-homogeneous, was introduced by Martin and Sagan.

The $\lambda$-homogeneous permutation groups have been classified, and the problem posed above was solved. The $\lambda$-homogeneous groups were independently classified by Dobson and Malnič.

A related question concerns groups $G$ for which

$$\langle G, f\rangle \setminus G = \langle g^{-1}fg : g \in G\rangle.$$

Groups for which this property holds for all non-permutations $f$ are called *normalizing groups*. They have been determined: only the symmetric and alternating groups, the trivial group, and finitely many others have this property. The next question in this direction would be to determine the $k$-normalizing groups, those for which the above two semigroups are equal for all maps of rank $k$.

# 4 Automorphisms

Perhaps the single most surprising fact about finite groups is the following.

**Theorem 4.1** *The only symmetric group (finite or infinite) which admits an outer automorphism is $S_6$.*

An *outer automorphism* of a group is an automorphism not induced by conjugation by a group element. In the case of symmetric groups, the group elements are all the permutations, and so an outer automorphism is one which is not induced by a permutation.

The outer automorphism of $S_6$ was known in essence to Sylvester; it arguably lies at the root of constructions taking us to the Mathieu groups $M_{12}$ and $M_{24}$, the Conway group $Co_1$, the Fischer–Griess Monster, and the infinite-dimensional Monster Lie algebra. Here is a sketch of Sylvester's construction (in his own idiosyncratic terminology).

Begin with $A = \{1,\ldots,6\}$, so $|A| = 6$. A *duad* is a 2-element subset of $A$; so there are 15 duads. A *syntheme* is a set of three duads covering all the elements of $A$; there are also 15 synthemes. Finally, a *total* (or *synthematic total*) is a set of five synthemes covering all 15 duads. It can be shown that there are 6 totals. Let $B$ be the set of totals.

Then any permutation on $A$ induces permutations on the duads and on the synthemes, and hence on $B$; this gives a map from the symmetric group on $A$ to the symmetric group on $B$ which is an outer automorphism of $S_6$.

This outer automorphism has order 2 modulo inner automorphisms. For any syntheme lies in two totals, so we can identify synthemes with duads of totals; any duad lies in three synthemes covering all the totals, so we can identify duads with synthemes of totals; and finally, each element of $A$ lies in 5 duads whose corresponding synthemes of totals form a total of totals!

There are other examples of this phenomenon too. For example, in the second stage of the above process, the Mathieu group $M_{12}$ has an outer automorphism which is not induced by a permutation.

Does anything similar happen for transformation semigroups?

Sullivan proved the following theorem 40 years ago:

**Theorem 4.2** *A finite transformation semigroup $S$ containing all the rank 1 maps has the property that all its automorphisms are induced by permutations.*

We observe that the rank 1 maps are the minimal idempotents (and so are mapped among themselves by any automorphism), and are naturally in one-to-one correspondence with the points on which the semigroup acts. So what is required is just a proof that only the identity automorphism can fix all the rank 1 maps.

As a corollary, we see:

**Corollary 4.3** *Let $S$ be a transformation semigroup which is not a permutation group, whose group of units is a synchronizing permutation group. Then $\mathrm{Aut}(S)$ is contained in the symmetric group; that is, all automorphisms of $S$ are induced by conjugation in its normaliser in the symmetric group.*

For, since $S$ contains a synchronizing group $G$, it contains at least one element of rank 1; and since $G$ is transitive, it contains them all.

And there matters stayed for a long time! But it is tempting to wonder whether we can replace "synchronizing" by "primitive" here.

Recently a small step has been taken. Recall that, if $G$ is not synchronizing, then the smallest possible rank of an element in a non-synchronizing monoid with $G$ as its group of units is 3.

**Theorem 4.4** *Let $S$ be a transformation semigroup containing an element of rank at most 3, and whose group of units is a primitive permutation group. Then the above conclusion holds: all automorphisms of $S$ are induced by conjugation in its normaliser in the symmetric group.*

For this, it is necessary to reconstruct the points of $\Omega$ from the images and kernels of maps of rank 3 in a way which is invariant under automorphisms of $S$. This is achieved by counting endomorphisms with various properties. For example, consider the images, which are maximal cliques in a graph $\Gamma$ with $S \leq \mathrm{End}(\Gamma)$. It is not hard to show that no two such cliques can intersect in two points; we distinguish pairs of cliques intersecting in a point from disjoint pairs of cliques by properties of the idempotents. We refer to the paper for more details.