Permutation groups and transformation semigroups: 3. Regularity and idempotent generation

Peter J. Cameron, University of St Andrews

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You say I am repeating
Something I have said before. I shall say it again.
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T. S. Eliot

Where are we going?

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A semigroup S is regular if every $a \in S$ has a quasi-inverse $b \in S$ such that $aba = a$ (and we may assume, without loss, that also $bab = b$). An idempotent is an element e satisfying $e^2 = e$; we say S is idempotent-generated if it is generated by its idempotents.

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Note that if $aba = a$ then ab and ba are idempotents; so regularity implies the existence of idempotents, and idempotent-generation is stronger.

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- *$\langle G, t\rangle$ is regular for any non-permutation t if and only if either G is $\mathrm{Sym}(\Omega)$ or $\mathrm{Alt}(\Omega)$, or G is one of nine specific permutation groups with $n = 5, 6, 7, 8$ or 9.*

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In the second statement, we remove elements of G since non-trivial permutations cannot be products of idempotents.

Set-transitive permutation groups

This result reminded me of a theorem with an interesting history. A permutation group G on Ω is said to be set-transitive if, for any two subsets of Ω of the same size, there is an element of G carrying the first to the second. In his book on permutation groups, Wielandt attributes the classification of these groups to Bercov. But also von Neumann and Morgenstern, in their pioneering book on game theory, stated the classification problem, and in the second edition asserted that it had been solved by Chevalley.

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Theorem

Let G be a set-transitive permutation group on Ω, with $|\Omega| = n$. Then either G is $\mathrm{Sym}(\Omega)$ or $\mathrm{Alt}(\Omega)$, or G is one of four specific groups with $n = 5$, 6 or 9.

We note that this theorem long predates CFSG.

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In particular, the only k-transitive groups with $6 \leq k \leq |\Omega|/2$ are $\mathrm{Sym}(\Omega)$ and $\mathrm{Alt}(\Omega)$; for $k = 5$ there are additionally two Mathieu groups of degrees $n = 12$ and 24, and for $k = 4$ two further Mathieu groups with $n = 11$ and 23. For $k = 2$ or 3 there are infinitely many groups; but all are known.

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The basic strategy is to reformulate the hypotheses in a way which somehow resembles k-homogeneity.

The k-universal transversal property

A permutation group G on Ω is said to have the k-universal transversal property, or k-ut for short (where $1 \leq k \leq n-1$) if, given any k-subset A and k-partition P of Ω, there is an element $g \in G$ mapping A to a transversal for P.

Proposition

The permutation group G has the property that $\langle G, t \rangle$ is regular for all maps t of rank k if and only if G has the k-universal transversal property.

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But we have to take quite a journey to get there.

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Take $g \in G$ mapping $\mathrm{Im}(t)$ to a transversal for $\mathrm{Ker}(t)$. Then gt induces a permutation of $\mathrm{Im}(t)$, and so some power of it (say $(gt)^m$) is the identity on $\mathrm{Im}(t)$; then $t(gt)^m = t$. (Note in passing that $(gt)^m$ is an idempotent.)

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So if we knew that every element of rank at most k in $\langle G, t \rangle$ had a quasi-inverse, we would be done. In other words, we have to prove that k-ut implies $(k-1)$-ut (for $k \leq n/2$), and use induction.

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For let Γ be a G-invariant graph, and A an edge. If G has 2-ut, then for any 2-part partition, there is an image of A under G having one point in each part. Hence Γ is connected, and the result follows from Higman's theorem.

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Now we classify $(k-1, k)$-homogeneous groups; use combinatorics (including Ramsey's theorem) to show that such a group is 2-homogeneous (at least for $k \geq 3$) with a few known exceptions, and then appeal to the classification of such groups. Once they are all known, it is only necessary to check that they all have $(k-1)$-ut, and we are done.

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However, if we regard that answer as satisfactory, we could ask, what next? We can refine the question further by asking about groups G for which $\langle G, t \rangle$ is regular for just some of the maps of rank k. One such weakening is the k-et property, to which I turn next.

k-existential transversal property

A permutation group G has the k-existential transversal property, or k-et for short, if there is a k-subset A of the domain (called a witness) such that, for any map t with $\mathrm{Im}(t) = A$, the semigroup $\langle G, t \rangle$ is regular.

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One curious feature is that the going-down result almost holds:

Theorem

Suppose that G is a permutation group of degree n, which has the k-et property for $2 \leq k \leq n/2$. Then either G has the $(k-1)$-et property, or G is one of two affine groups of degree 16, with $k = 6$.

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There are other strange beasts too: for example, the Higman–Sims group and its automorphism group (with $n = 100$) are the only groups with 4-et and $n \geq 8$ which are not 2-homogeneous.

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In particular, a regular permutation group G (that is, acting on itself by right multiplication) is fully imprimitive if and only if it is not cyclic.

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One could also play the game in other famous semigroups such as the Brauer monoid and the partition monoid. This has yet to be done.

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- *The (k,l)-ut property becomes weaker as l increases from k to n; so, if G has k-ut, then there is a threshold $t(G,k)$ such that G has (k,l)-ut if and only if $t(G,k) \leq l \leq n$.*

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The value of $T(G, k)$ is known for most groups G with k-ut.

Idempotent generation

As we saw, the k-ut property guarantees that $\langle G, t \rangle$ contains idempotents when $\mathrm{rank}(t) = k$. Asking for it to be generated by idempotents is a stronger condition on G which we call k-idempotent generated (for short, k-id).

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Since we have a reasonably good, if not complete, account of k-ut groups, we should be able to read off the k-id groups, yes? In fact the problem is that, apart from $k = 2$, we don't understand these groups. We have a condition called strong k-ut, and k-id is sandwiched between k-ut and strong k-ut, but we don't have a purely combinatorial or group-theoretic condition for k-id.

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The Road Closure Property is a strengthening of this. G has the Road Closure Property (RCP for short) if, whenever E is an orbit of G on 2-sets, and B a proper block of imprimitivity for the action of G on E, the graph with edge set $E \setminus B$ is connected.
In other words, if all the roads in a block are closed for roadworks, it is still possible to get around the network.

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In other words, if all the roads in a block are closed for roadworks, it is still possible to get around the network.

Theorem

The permutation group G has the 2-id property if and only if it has the Road Closure Property.

Examples

Clearly a group with the Road Closure Property is primitive. Moreover, it is basic. For a Hamming graph $H(m, q)$ on Ω gives it the structure of an m-dimensional cube (with q points on each edge); the m directions in the cube form a system of imprimitivity. If we delete all edges in one direction, the graph falls apart into m layers.

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Another family is exemplified by the group of collineations and dualities of a projective plane over a finite field, acting on the set of flags (incident point-line pairs) in the plane; edges consist of pairs of flags sharing a point or a line. There is a natural system of two blocks of edges; if we delete, say, all edges consisting of pairs of flags sharing a point, then we cannot move from flags through one point to flags through another.

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There are a few other examples too: Pablo Spiga found what is now known to be the smallest example with more than two blocks, the group $\mathrm{P\Gamma U}(3, 5)$ acting on 6000 points.

Comments

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Comments

We conjecture that the examples we have seen are typical: that is, a basic primitive group which fails the RCP has two or three blocks of imprimitivity on the edges of some orbital graph. However, we are still some way from a proof.

The RCP, like synchronization, is a strengthening of primitivity. It is tempting to think that there might be a connection between these two properties, but none is currently known or suspected.

References

- J. Araújo, J. P. Araújo, W. Bentz, P. J. Cameron and P. Spiga, A transversal property for permutation groups motivated by partial transformations, *J. Algebra* **573** (2021), 741–759.
- João Araújo, Wolfram Bentz and Peter J. Cameron, The existential transversal property: a generalization of homogeneity and its impact on semigroups, *Trans. Amer. Math. Soc.* **374** (2021), 1155–1195.
- João Araújo and Peter J. Cameron, Two generalizations of homogeneity in groups with applications to regular semigroups, *Trans. Amer. Math. Soc.* **368** (2016), 1159–1188.
- João Araújo and Peter J. Cameron, Primitive groups, road closures and idempotent generation, arXiv `1611.08233`.