Covers of sets of groups

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More about this later …

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We are about to submit a revised version to the arXiv.

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In particular, let us call an $\mathcal{F}$-cover G cover minimal if no proper subgroup of G is an $\mathcal{F}$-cover, and minimum if no group of smaller order is an $\mathcal{F}$-cover. We are particularly interested in minimal and minimum $\mathcal{F}$-covers.

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For any set $\mathcal{F}$, there is a minimal $\mathcal{F}$-cover: take the direct product of the groups in $\mathcal{F}$ (this is a cover), and take a subgroup minimal with respect to embedding all the groups in $\mathcal{F}$.

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In particular, if $n = p^m$ with p prime, then the Sylow p-subgroup of S_n is a p^m-cover, of order $p^{(p^m-1)/(p-1)}$.

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We know that the values for $p^m = 2^3, 2^4, p^3$ (p an odd prime) are respectively 2^5, 2^8, and p^6 respectively.

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For, by Dirichlet's Theorem, there are infinitely many primes p congruent to 1 (mod q) and to -1 (mod r). Then $G = \mathrm{PSL}(2, p)$ is a $\{C_q, C_r\}$-cover. If $r > 5$, then the only maximal subgroup of G containing C_r is D_{p+1}, which does not contain C_q. The argument is similar in the other case.

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Remaining pairs of primes have not yet been settled.

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- Suzuki groups can't occur since their orders are not divisible by 3. The others all involve $\mathrm{PSL}(2, p)$, where q is a power of p.
- $\mathrm{PSL}(2, 3) \cong A_4$, while for other p they have D_6 as a subgroup.

Minimal p^m-covers

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Here SD_{2^m} is the semi-dihedral group

$$\langle a, b : a^{2^{m-1}} = b^2 = 1, b^{-1}ab = a^{2^{m-2}-1} \rangle.$$

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Of course we cannot say that all minimum $\mathcal{F}$-covers are nilpotent. For example, if $\mathcal{F} = \{(C_2)^2, C_3, C_5\}$, then a minimum $\mathcal{F}$-cover has order 60, and any group of order 60 having these as its Sylow subgroups is an $\mathcal{F}$-cover, including the simple group A_5.

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Any such group has a canonical form

$$G = C_{p^{a_1}} \times \cdots \times C_{p^{a_r}}$$

where $a_1 \geq \cdots \geq a_r$. Now given a set of finite abelian p-groups, we can write them all in canonical form and assume that the value of r is the same for each (by adding trivial factors if necessary). Then the smallest abelian cover has canonical form whose ith factor is the largest group occurring as the ith factor of one of the groups in $\mathcal{F}$.

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This function was considered by Dirichlet, who showed that $f(n) = n(\log n + 2\gamma - 1) + O(\sqrt{n})$, where γ is the Euler–Mascheroni constant. It is given as sequence A006218 in the On-Line Encyclopedia of Integer Sequences, where a number of occurrences of it are noted. But the one given here seems to be new.

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This is proved by applying the preceding theorem, first computing that the largest size of the kth component in the canonical form of an abelian group of order p^n is $p^{\lfloor n/k \rfloor}$.

Questions and answers

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For soluble groups, take $\mathcal{F} = \{D_{10}, A_4\}$. The least common multiple of their orders is 60, and it is an easy exercise to show that the only group of order 60 containing both is A_5.

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Example Let $M = A_5$ and $N = \mathrm{PSL}(2, 8)$. The orders of these groups are 60 and 504. Their least common multiple is 2520 and their product is 30240. The only simple groups with order divisible by 2520 and not greater than 30240 are A_7, A_8 and $\mathrm{PSL}(3, 4)$; none of these embed $\mathrm{PSL}(2, 8)$. By the theorem, the unique minimum $\{M, N\}$-cover is $M \times N$.

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Example Let $M = A_6$ and $N = \mathrm{PSL}(2, 7)$. Their orders are 360 and 168, with least common multiple 2520. There is a unique simple group of order 2520, namely A_7, which embeds both M and N; so A_7 is the unique minimum $\{M, N\}$-cover.

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In the latter case, there are infinitely many triples of simple groups $\{L, M, N\}$ with $|L| = |M| \cdot |N|$. (The smallest is $\{A_6, \mathrm{PSL}(2,8), A_9\}$.) Can we determine all such triples? And can both L and $M \times N$ be minimum covers of the same pair of simple groups?

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Theorem

Suppose that $\mathcal{X}$ is a subgroup-closed class of finite groups. Let $\mathcal{F}$ be a finite set of finite groups, none of which has a non-trivial $\mathcal{X}$-group as a quotient, and let G be a minimal $\mathcal{F}$-cover. Then G has no non-trivial $\mathcal{X}$-group as a quotient.

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Proof.

Suppose that $G/N \in \mathcal{X}$. Then for any group $H \in \mathcal{F}$, $H/H \cap N \cong HN/N \leq G/N \in \mathcal{X}$. So $H/H \cap N \in \mathcal{X}$ which implies $H \subseteq N$. By minimality of G, we have $N = G$. $\square$

Applications

- Let $\mathcal{X}$ be the class of finite abelian groups. The condition that G has no non-trivial homomorphism to an $\mathcal{X}$-group means that G is perfect. So we deduce that, if every group in $\mathcal{F}$ is perfect, then any minimal $\mathcal{F}$-cover is perfect.

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- Let $\mathcal{X}$ be the class of finite solvable groups. The condition that G has no non-trivial homomorphism to an $\mathcal{X}$-group means that G is equal to its solvable residual. So, if every group in $\mathcal{F}$ is equal to its solvable residual, then the same is true of any minimal $\mathcal{F}$-cover.

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Suppose that $\mathcal{X}$ is a subgroup-closed class of finite groups. Let $\mathcal{F}$ be a finite set of finite groups, and suppose that no group in $\mathcal{F}$ has a non-trivial normal $\mathcal{X}$-subgroup. Let G be a co-minimal cover of $\mathcal{F}$. Then G has no non-trivial normal $\mathcal{X}$-subgroup.

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- Similarly with "soluble" replacing "abelian".

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We can dualise the entire set-up. Given a finite set $\mathcal{F}$ of finite groups, say that G is a dual cover for $\mathcal{F}$ if every group in $\mathcal{F}$ is a homomorphic image of G.

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But for non-abelian groups, essentially nothing is known.

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What happens for semigroups?

… for your attention.