Latin cubes

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What is a Latin square?

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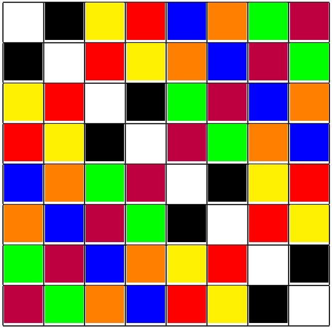
A Latin square of order 8



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Example
If Ω is the set of cells in a Latin square, then there are five natural uniform partitions of Ω:

R	each part is a row;
C	each part is a column;
L	each part consists of the those cells with a given letter;
U	the universal partition, with a single part;
E	the equality partition, whose parts are singletons.

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Draw a graph by putting an edge between two points if they are in the same part of P or the same part of Q. Then the parts of $P \vee Q$ are the connected components of the graph.

Hasse diagrams

Given a collection $\mathcal{P}$ of partitions of a set Ω,
we can show them on a Hasse diagram.

- Draw a dot for each partition in $\mathcal{P}$.
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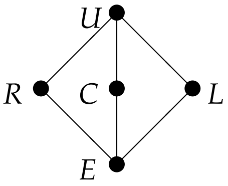
Here is the Hasse diagram for a Latin square.



An alternative definition of Latin square

Definition

Let P and Q be uniform partitions of a set Ω. Then P and Q are compatible if

- whenever ω_1 and ω_2 are points in the same part of $P \vee Q$, there are points α and β such that
  - ω_1 and α are in the same part of P,
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Comment

These definitions can be applied to finite or infinite sets.

Another nice family of partitions

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Suppose that P_1, P_2 and P_3 are partitions of a set Ω, none of which is U. Then $\{P_1, P_2, P_3\}$ is a Cartesian decomposition of Ω of dimension 3 if $|\Gamma_1 \cap \Gamma_2 \cap \Gamma_3| = 1$ whenever Γ_i is a part of P_i for $i = 1, 2, 3$.

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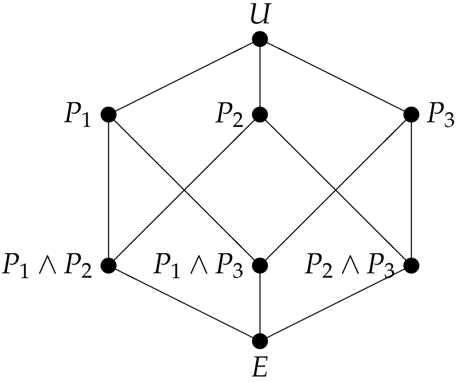
Taking infima gives a Cartesian lattice.



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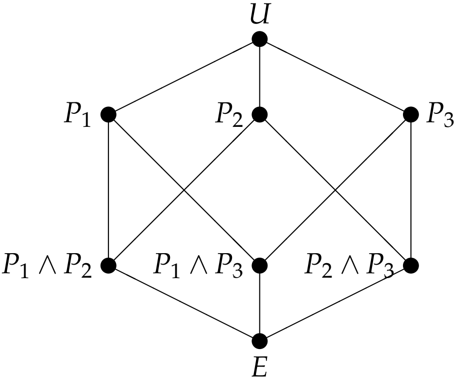

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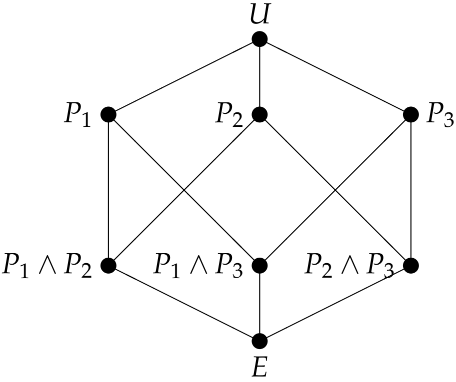

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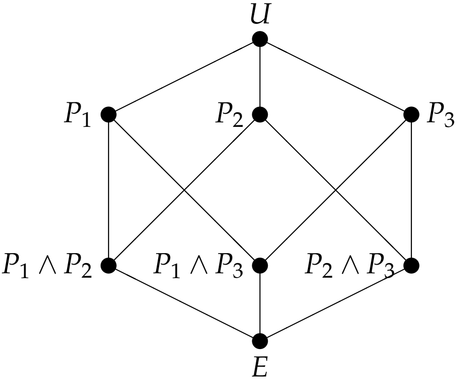

- Each partition is uniform.
- Each pair are compatible.
- Statisticians call this a completely crossed orthogonal block structure.

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Proposition

Let H and K be subgroups of a group G. The following hold.

1. P_H *is uniform.*
2. $P_H \wedge P_K = P_{H \cap K}$.
3. $P_H \vee P_K = P_{\langle H, K \rangle}$.
4. P_H *and* P_K *are compatible if and only if* $HK = KH$.

Latin squares and quasigroups

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Cayley table of cyclic group C_5

A	B	C	D	E
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Not a Cayley table of a group

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A Latin square satisfies the quadrangle criterion if, whenever there are 2×2 subsquares

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(Frolov was in the French army, and was unaware of the notion of "group".)

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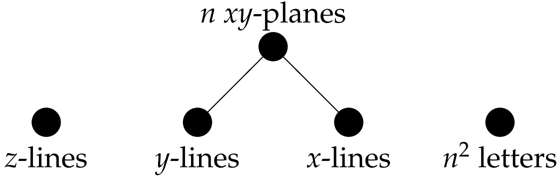

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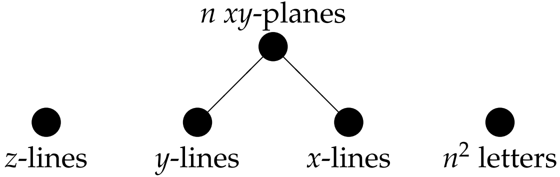

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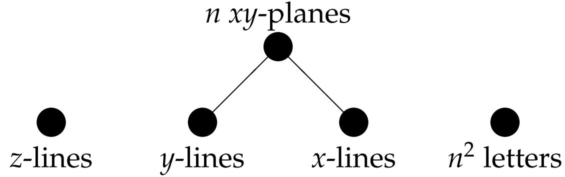

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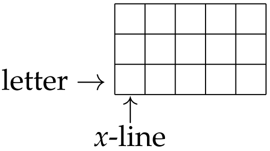

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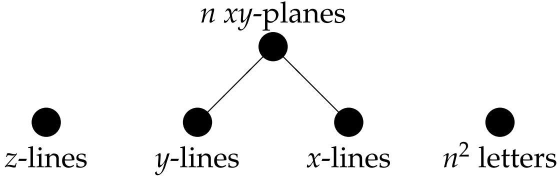

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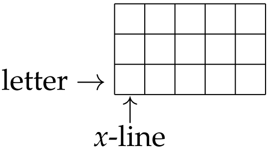

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Conditions (1) and (2) give one definition (among very many) of a Latin cube.

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Put $x = ab^{-1}$, $y = bc^{-1}$, $z = cd^{-1}$ and $t = xyz = ad^{-1}$.
Then $H = \langle x \rangle \times \langle y \rangle \times \langle z \rangle$ and the coset partitions of H defined by any 3 of $\langle x \rangle$, $\langle y \rangle$, $\langle z \rangle$ and $\langle t \rangle$ are the minimal non-trivial partitions in a Cartesian lattice of dimension 3.

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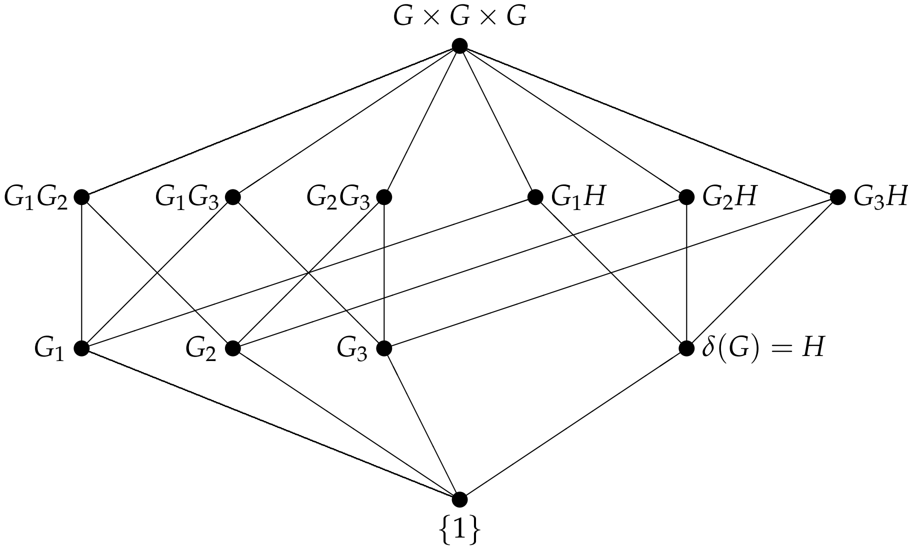
Each of these groups give a coset partition.

These have the desired property that
any 3 of them are the minimal non-trivial partitions
in a Cartesian lattice of dimension 3.

Hasse diagram for subgroups involved



Hasse diagram for subgroups involved

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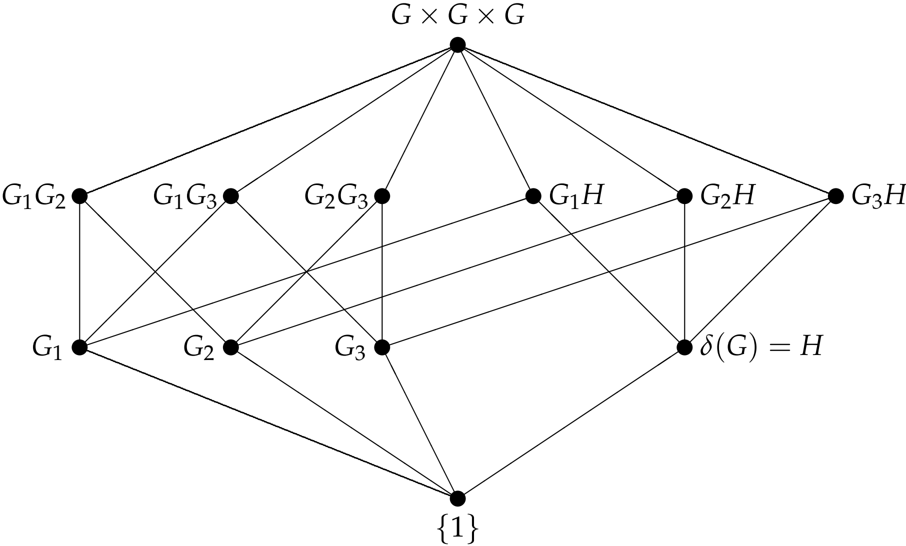

Hasse diagram for subgroups involved

Each partition is uniform.

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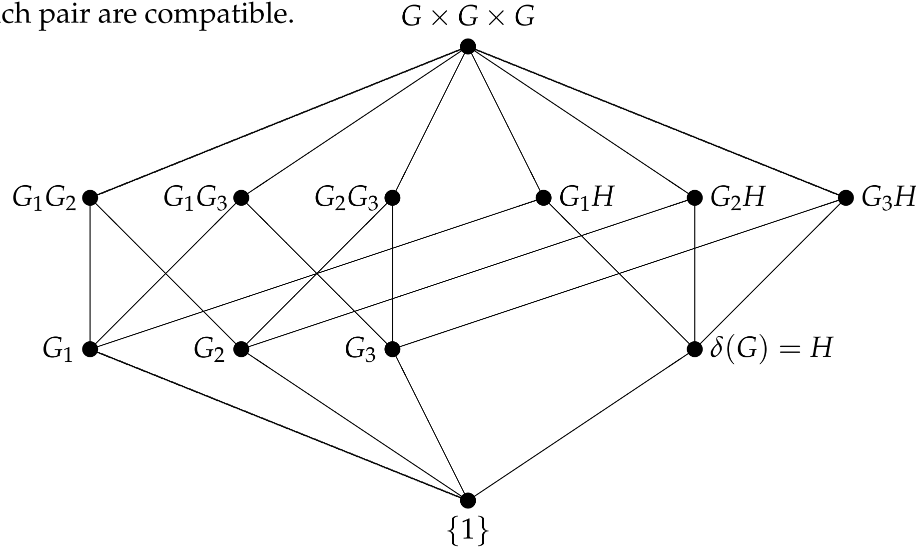

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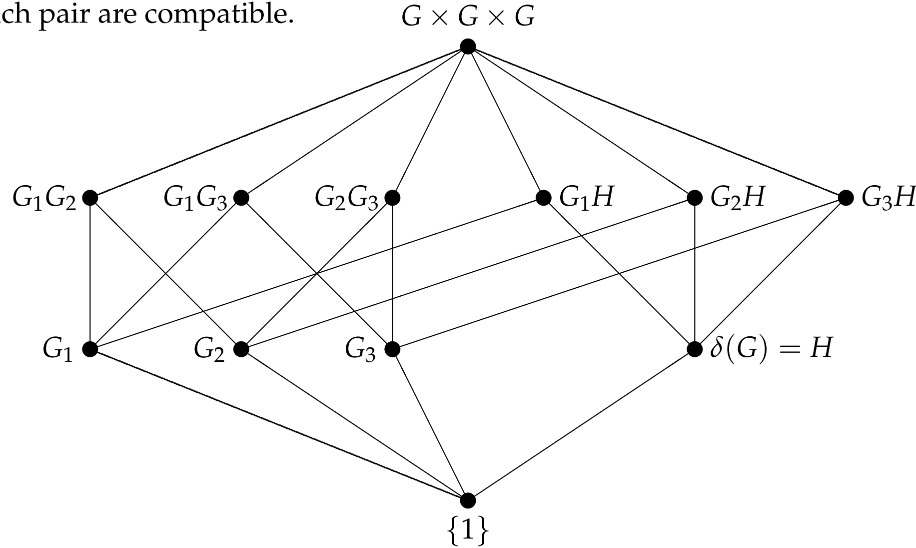


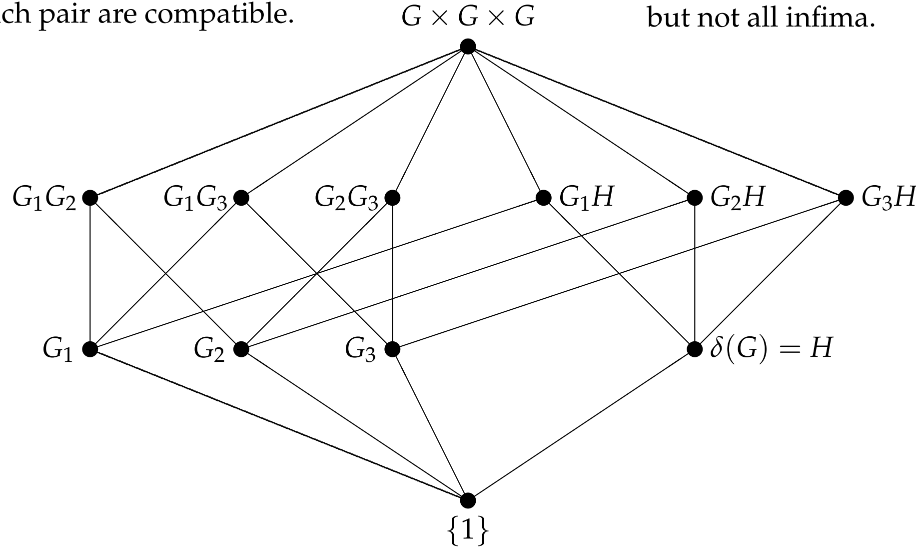
Hasse diagram for subgroups involved

Each partition is uniform.

Each pair are compatible.

All suprema are included,

but not all infima.



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2. In 1984, Danish statistician Tue Tjur pointed out that, for statistical purposes, closure under suprema is more important than closure under infima, and that such closure does not destroy compatibility.

Theorem about diagonal semilattices

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Let $\mathcal{Q}$ be a set of $m+1$ partitions of the same set Ω, where $m \geq 2$. Suppose that every subset of m of the partitions in $\mathcal{Q}$ form the minimal non-trivial partitions in a Cartesian lattice of dimension m.

(a) *If $m = 2$ then there is a Latin square on Ω, unique up to paratopism, such that $\mathcal{Q} = \{R, C, L\}$.*

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For $m > 2$, the combinatorial assumptions in the statement of the theorem force the existence of a group.

Final comments

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2. The rest of the proof followed by rather careful induction on the dimension.
3. Later, in joint work with Michael Kinyon, we extended these results to the multidimensional equivalent of sets of mutually orthogonal Latin squares.