QUEEN MARY, UNIVERSITY OF LONDON

M. Sc. Examination Specimen

MTHM A30 Permutation groups

Duration: 3 hours

Date and time:

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

Calculators are NOT permitted in this examination.

Question 1 Let H be a subgroup of a group G. Explain how G acts as a permutation group on the *coset space* $H\backslash G$ (the set of all right cosets of H in G). Prove that

- the stabiliser of the coset Hg in this action is $g^{-1}Hg$;
- the action is faithful if and only if

$$\bigcap_{g\in G} g^{-1}Hg = 1;$$

- the action is primitive if and only if H is a maximal proper subgroup of G.

A *double coset* of H in G is a set of the form $HgH = \{h_1gh_2 : h_1, h_2 \in H\}$. Prove that a double coset is a union of right cosets, and that these right cosets form an orbit of H in its action on the coset space $G\backslash H$. Hence show that G is 2-transitive if and only if $G = H \cup HgH$ for $g \notin H$.

Question 2 Let G be a transitive permutation group on Ω, and let N be a regular normal subgroup of G.

Prove that there is a bijection ϕ between Ω and N which is an isomorphism of N-spaces, where N acts on itself by right multiplication.

Prove that, if $\alpha \in \Omega$ corresponds to the identity element of N under this bijection, then ϕ is also an isomorphism of G_α-spaces, where G_α acts on N by conjugation.

Hence show that, if Ω is finite, then

- if G is 2-transitive then N is an elementary abelian p-group for some prime p;
- if G is 3-transitive and $|\Omega| > 3$ then N is an elementary abelian 2-group;
- if G is 4-transitive then $|\Omega| = 4$.

Do these conclusions hold if Ω is infinite? Give brief reasons.

Show that a 4-transitive group other than S_4 cannot have a sharply 2-transitive normal subgroup.

Question 3 Let G be the subgroup of S_7 generated by the permutations $a = (1\,2\,3)(4\,7\,5)$ and $b = (1\,5\,7)(2\,6\,4)$. Let H be the stabiliser in G of the point 1.

(a) Is G transitive?

(b) Find a set of coset representatives for H in G, both as permutations and as words in a and b.

(c) Find generators for H.

(d) Find the order of G.

(e) Does the permutation $(1\,2\,3)(4\,5\,6)$ belong to G?

Question 4 (a) Define the *alternating group* A_n and sketch a proof that A_5 is simple.

(b) Prove that A_5 has a 2-transitive action of degree 6.

(c) Construct a primitive action of A_5 of degree 10. What is its rank?

(d) Construct a primitive permutation group isomorphic to $A_5 \times A_5$.

(e) Is there a primitive permutation group isomorphic to $A_5 \times A_5 \times A_5$? Give brief reasons.

Question 5 (a) What is an *oligomorphic* permutation group?

(b) Prove that a permutation group G on a set Ω is oligomorphic if and only if

- G has only finitely many orbits on Ω; and
- the stabiliser of any point of Ω is oligomorphic.

(c) Let the oligomorphic group G have f_n orbits on the set of n-element subsets of Ω, and F_n orbits on the set of n-tuples of distinct elements of Ω. Prove that

$$f_n \leq F_n \leq n!\, f_n,$$

and give examples to show that both bounds can be attained.

(d) Let Ω be the disjoint union of two infinite subsets Ω_1 and Ω_2, and let G be the direct product of symmetric groups on Ω_1 and Ω_2. Calculate the numbers f_n and F_n defined in the preceding part.

 END OF EXAMINATION