Measuring triangle-free graphs

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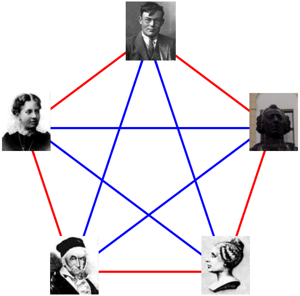

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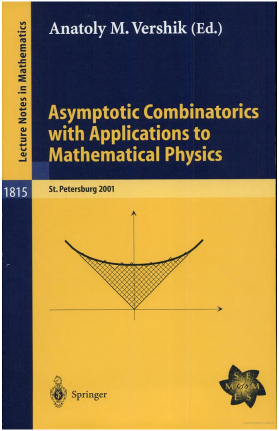

Happy birthday, Anatoly!

This talk was inspired by a conversation I had with Anatoly in Penderel's Oak, a pub in Holborn, London, about three years ago. The two of us had come to the problem of measuring triangle-free graphs from different directions: his solution (with Fedor Petrov) led to some very nice connections. But it doesn't completely answer my questions, so there is still more to be done!

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If we want to get a triangle-free graph by such a construction, it is necessary that we join blocks only if they are disjoint. The Higman–Sims graph is remarkable in that the converse holds.

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- it is highly symmetric: indeed, it is homogeneous (this means that any isomorphism between finite subgraphs extends to an automorphism);
- it is universal: it contains every finite or countable triangle-free graph as an induced subgraph.

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I became interested in the automorphism group of R.

Recognising homogeneous universal structures

Fraïssé gave a test for the existence of a homogeneous relational structure M which is universal for a given class $\mathcal{C}$ of finite structures. Briefly: $\mathcal{C}$ should be the class of finite structures embeddable in M; and if $A, B \in \mathcal{C}$ with $|B| = |A| + 1$, then any embedding of A into M can be lifted to an embedding of B into M.

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This is sometimes called the Alice's Restaurant property, since

> *You can get anything you want*
> *At Alice's Restaurant,*

according to Arlo Guthrie: you can "order" a new point with any consistent relationships with the finitely many points you have already.

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Indeed, if we replace "all distances rational" with "all distances 1 or 2", we obtain precisely the random graph!

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I have no idea what to do with these observations …

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Conversely we can extract the set S from the graph with its cyclic automorphism: number the vertices with integers so that the automorphism is the shift, and then let S be the set of positive neighbours of 0.
Now it is easy to show that two graphs with cyclic automorphisms give rise to the same set S if and only if

- the graphs are isomorphic;
- the cyclic automorphisms are conjugate.

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The proofs give more. For example, if R is a Cayley graph for G, then a random Cayley graph for G is isomorphic to R with probability 1.

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Questions remain, for example: which abelian groups can act in this way?

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- Add edges one at a time randomly, but only add an edge if it doesn't contain a triangle. The result depends on the order in which we consider pairs of vertices.

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These ideas can be applied to automorphisms. All the previous results about automorphisms of R can be proved using Baire category instead of measure.

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Moreover, the graph is isomorphic to Henson's graph if and only if S is sf-universal: this means that a given finite binary word w occurs in the characteristic function of S if (and only if) w does not contain 1s in positions whose distance belongs to S.
Now sf-universal sets are residual in the collection of sum-free sets. So Henson's graph has $2^{\aleph_0}$ non-conjugate cyclic automorphisms!

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It is not surprising to learn that the probability that S consists entirely of even numbers is zero. However, there is a surprise in store!

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This was the first theorem I proved with the help of a computer (a Sinclair Spectrum with 48 kilobytes of RAM and 3.5Khz clock speed).

More fragments of the space

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An example of the last case is the set of sum-free sets in which 2 is the only even number, which has probability somewhere round 10^{-6}.

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If $T \pmod{n}$ is complete sum-free, then elements of T occur with probability close to $1/2$, so the density is almost surely $|T|/2n$. This gives $1/4$ for sets of odd numbers, $1/5$ for sets contained in $\{1, 4\} \pmod{5}$ or $\{2, 3\} \pmod{5}$, etc.

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Plotting the density of large finite sum-free sets is like using a spectroscope: the longer you wait, the more accurate the plot should be. We expect a spectral line at $1/4$ with intensity $0.218\ldots$, and weaker lines at $1/5$, $3/16$, and so on.

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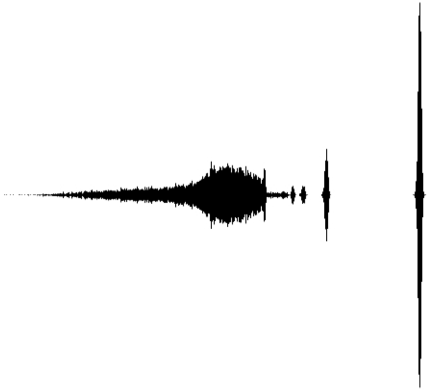

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One thing we do know. Thomasz Schoen proved:

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A sf-universal set has density 0.

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- Is the density positive almost surely?
- Is the spectrum discrete above $1/6$?
- What happens below $1/6$? Is there a continuous part to the spectrum, or is it many discrete parts smeared together?

One thing we do know. Thomasz Schoen proved:

Theorem

A sf-universal set has density 0.

So probably this model does not give information about Henson's graph.

Further development

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The method has been extended to a very wide range of homogeneous relational structures by Nate Ackerman, Cameron Freer and Rihanna Patel.

Cayley graphs for other groups

As mentioned earlier, there is a near-characterisation of countable groups which admit the randm graph as a Cayley graph; there are necessary and sufficient conditions are bit complicated to state, but all countable abelian groups of infinite exponent satisfy them. Moreover, if some Cayley graph is isomorphic to R, then almost all are.

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Recently Greg Cherlin showed that these graphs are Cayley graphs for some groups including non-abelian free groups.

What I would like

As hinted above, I want a measure on the set of pairs consisting of a triangle-free graph on $\mathbb{N}$ and a group acting on $\mathbb{N}$ by graph automorphisms, which is concentrated on the isomorphism class of Henson's graph. (I talked only about the infinite cyclic group above, but that was only the first step.)

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Cherlin's results, using bare-handed constructions, will probably be harder than the results for the triangle-free case.

The end

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