Finding where you are: Automata, graph endomorphisms, and de Bruijn graphs

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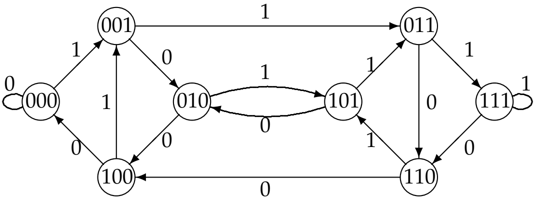

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I will give a brief introduction, and then talk about some work on each of these two problems. (Time does not permit a complete survey of either!)

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An automaton can be represented combinatorially by a directed graph (whose vertices are the states) with edges labelled by symbols of the alphabet, so that there is exactly one edge with each label *leaving* each vertex.

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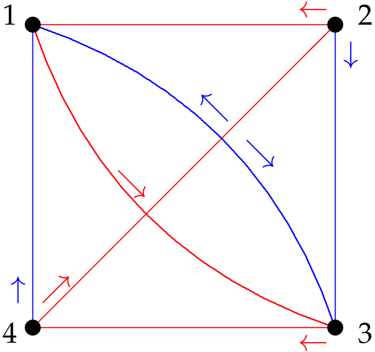

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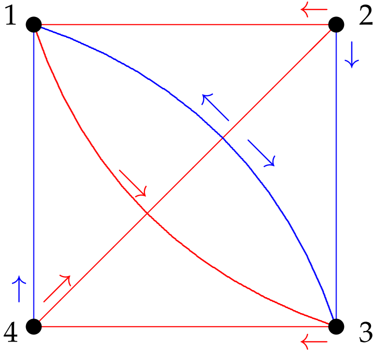

You can check that (Blue, Red, Blue) takes you to room 1, no matter where you start. So, if this were the map of a dungeon in which you were lost, you could use the map to find your way to the exit.

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The main open problem about synchronizing automata is the Černý conjecture, which states that, if an n-state automaton is synchronizing, then it has a reset word of length at most $(n-1)^2$. If true, this would be best possible. But I am not going to talk about this, although what I am going to say was motivated by the conjecture.

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Indeed, this gives us an cubic upper bound for the length of such a word: $n-1$ iterations of a process which takes $O(n^2)$ steps.
The bad news is that finding the shortest synchronizing word is NP-hard.

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Our next task is to describe the maximal non-synchronizing automata (or monoids).

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An endomorphism of Γ is a homomorphism from Γ to itself. The endomorphisms form a monoid $\mathrm{End}(\Gamma)$ (the endomorphism monoid of Γ).

Synchronization and graph endomorphisms

Theorem

A transformation monoid M on Ω is non-synchronizing if and only if there is a non-null graph Γ on the vertex set Ω such that $M \leq \mathrm{End}(\Gamma)$. We may assume that the clique number and chromatic number of Γ are equal.

One direction is trivial: if the clique number and chromatic number are equal, then Γ has non-trivial endomorphisms f, which are not synchronized by G.

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I won't prove the other direction; it is not difficult, but it is the foundation of our study of synchronizing automata, and is the basis of the most efficient test of synchronization.

Synchronizing groups

There are very few examples of synchronizing automata attaining the Černý bound. All of them have the property that one generator is a cyclic permutation, so the monoid contains a transitive permutation group as a subgroup. This motivates looking at monoids generated by a group and one further element.

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The hope is that we can use knowledge of permutation groups (acquired since the Classification of Finite Simple Groups) to understand synchronizing groups.

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The converse is false, but it is thought that the difference between primitivity and synchronization is not all that great.

A conjecture

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João Araújo conjectured that a primitive group synchronizes all non-uniform maps.
This conjecture has just been refuted.

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Our first example was found in the following way. Suppose we could find a graph with clique number and chromatic number 3, which has a primitive automorphism group, and has a homomorphism onto the butterfly:

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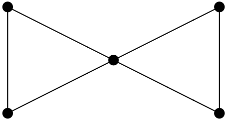

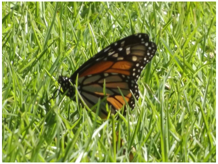

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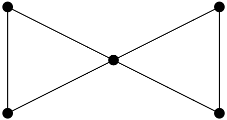

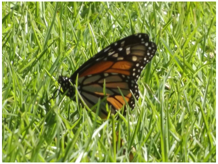

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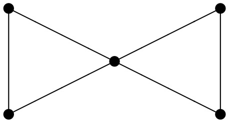

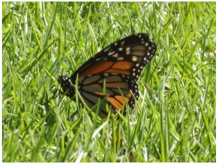

Suppose further that the graph has valency 4 and the closed neighbourhood of a vertex is itself a butterfly. Then it has a non-uniform endomorphism of rank 5.
Fortunately, group theorists have determined such graphs. There are two, with 45 and 153 vertices: the line graphs of the Tutte–Coxeter and Biggs–Smith graphs.

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If Γ has primitive automorphism group G, then $\Gamma \mathbin{\Box} \Gamma$ has primitive automorphism group $G \operatorname{wr} C_2$.

An example

The graph $K_k \square K_k$ is the square lattice graph $L_2(k)$, with two vertices joined if and only if they are in the same row or column. It has clique number k (rows and columns are maximum cliques) and chromatic number k (a k-colouring is a Latin square):

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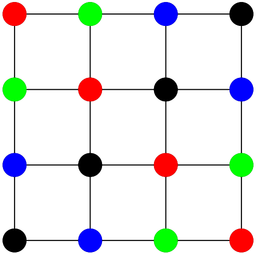

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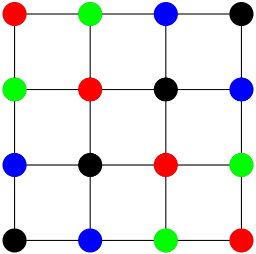

The complementary graph also has clique number and chromatic number k: a clique is a transversal, and the row numbers give a proper colouring.

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Each function must be a Latin square, since we require that $y \neq z$ implies $f(x, y) \neq f(y, z)$, and similarly for g; but there is no connection between the two squares.

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So we have a primitive group of degree k^4 which fails to synchronize maps of every possible rank between k and k^2 inclusive except for $k + 1$ and $k^2 - 1$.

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So we have a primitive group of degree k^4 which fails to synchronize maps of every possible rank between k and k^2 inclusive except for $k+1$ and k^2-1.
Many more examples can no doubt be found …

Another conjecture

We saw that, if G is imprimitive, preserving a relation with m equivalence classes of size k, then G preserves the disjoint union of m copies of K_k, and also its complete bipartite complement. These graphs have endomorphisms whose ranks are all multiples of k (for mK_k) and all integers between m and $mk = n$ (for the complete multipartite graph). This gives $(3/4 - o(1))n$ ranks of maps not synchronized by G.

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For our example, this number is about $\sqrt{n}$.

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A transducer is an automaton which writes as well as reading.

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We say that an automaton is k-determined if every word of length k is a reset word for it. In other words, when it reads k symbols, the state it is in depends only on the symbols read, and not on the state it was in before reading them.

Finitely-determined automata

A transducer representing an invertible transformation must have the property that the "inverse" (reversing the writing and reading operations) is also a transducer. Thus it is constructed from a pair of automata with the same underlying graph.

We say that an automaton is *k-determined* if every word of length k is a reset word for it. In other words, when it reads k symbols, the state it is in depends only on the symbols read, and not on the state it was in before reading them.

The automata involved in automorphisms of the Higman–Thompson groups turn out to be finitely determined. So we need to examine these further.

De Bruijn graphs

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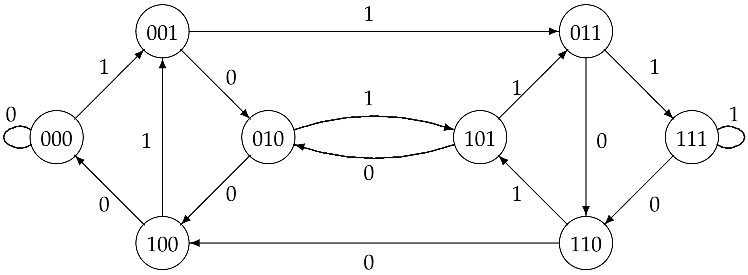
Here is the de Bruijn graph $\mathrm{DB}_{2,3}$.



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(An Eulerian cycle is a cycle passing once through each directed edge, in the correct direction; a directed graph has an Eulerian cycle if and only if it is strongly connected and has the in-degree of each vertex equal to its out-degree – an obvious necessary condition).
For us the crucial property (from which the strong connectedness follows) is that, regarded as an automaton over the alphabet A, the de Bruijn graph $\mathrm{DB}_{n,k}$ is k-determined.

Foldings of de Bruijn graphs

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So in order to study k-determined automata over A, we simply have to study foldings of $\mathrm{DB}_{n,k}$.

Counting foldings

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$n \setminus k$	1	2	3	4
2	2	5	30	1247
3	5	192	?	?
4	15	78721	?	?
5	52	519338423	?	?

Word length 2

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Theorem

Let

$$R(s,t) = \sum_{\pi} (-1)^{|\pi|-1} (|\pi|-1)! \prod_{i=1}^{|\pi|} B(a_i s),$$

where π runs over all partitions of $\{1, \ldots, t\}$, $|\pi|$ is the number of parts of π, and a_i is the size of the ith part. Then

$$F(n,2) = \sum_{\pi} \prod_{i=1}^{s} R(m, a_i),$$

where π runs over all partitions of the n-letter alphabet, m is the number of parts of π, and a_i is the cardinality of the ith part for $i = 1, \ldots, m$.

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If you stare at the formula you will see the technique: Möbius inversion over the lattice of partitions of a set. But we haven't made it work for longer words yet!
Möbius inversion is a general technique for arbitrary partially ordered sets which generalises the Inclusion-Exclusion principle (the case for the lattice of subsets of a set). I have known the form of the Möbius function for the partition lattice for many years, but never before now did I have a chance to use it seriously.

Counting automorphisms?

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- decide when two foldings give rise to isomorphic directed graphs, and count these (since a transducer consists of two automata with isomorphic graphs);
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So we still have plenty more to do!