

Permutation groups and transformation semigroups:

1. Permutation groups

Peter J. Cameron, University of St Andrews



Theoretical and Computational Algebra
July 2023

I'll know my song well before I start singing

Bob Dylan



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Higman's Theorem

Primitivity is possibly the most important concept in permutation group theory, and there are a number of conditions equivalent to it; for example, a transitive group is primitive if the **point stabiliser** is a maximal proper subgroup of G . Probably the most important of these is the theorem of Donald Higman:

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Note that we can construct G -invariant graphs by taking orbits of G on pairs of elements of Ω as edges. These are the **orbital (di)graphs**.

Multiple transitivity

Let t be a positive integer not exceeding n . We say G is **t -transitive** if its induced action on t -tuples of distinct elements of Ω is transitive; and G is **t -homogeneous** if the induced action on t -element subsets of Ω is transitive.

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Clearly t -transitivity implies t -homogeneity. If $5 \leq t \leq n/2$, a beautiful theorem of Livingstone and Wagner asserts that the converse is true. All t -homogeneous but not t -transitive groups for $t = 2, 3, 4$ were found by Kantor (before CFSG).

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The classification of t -transitive groups for $t \geq 2$ had to wait for CFSG (the **Classification of Finite Simple Groups** before it could be completed; but now we have a complete list of such groups.

A general scheme

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A virtue of this definition is that, for any class \mathcal{C} , the class of \mathcal{C} -free permutation groups is closed upwards.

Playing the game

We will see several examples. However, the way to play this game is not to think up an arbitrary class \mathcal{C} and examine the \mathcal{C} -free or \mathcal{C} -closed permutation groups. Rather, we have a property of permutation groups we want to study; understanding the \mathcal{C} -free or \mathcal{C} -closed structures for an appropriate class is likely to help the investigation. Even better are cases when we can build arbitrary permutation groups from the \mathcal{C} -free groups.

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Note that if G is not \mathcal{C} -free then it preserves a non-trivial \mathcal{C} -structure. The nicest cases are those where we can use this to get a reduction for G , and understand it in terms of smaller permutation groups. This is the case for transitivity and primitivity, for example.

How it works

Let \mathcal{C} be the class of “subsets”: a \mathcal{C} -object is a subset of Ω . The only subsets invariant under the symmetric group are the empty set and Ω ; so G is \mathcal{C} -free if and only if it is transitive.

How it works

Let \mathcal{C} be the class of “subsets”: a \mathcal{C} -object is a subset of Ω . The only subsets invariant under the symmetric group are the empty set and Ω ; so G is \mathcal{C} -free if and only if it is transitive. Again, let \mathcal{C} be the class of “directed graphs”. A directed graph invariant under the symmetric group is either null or complete; so, if G is \mathcal{C} -free, then any pair of distinct points can be mapped to any other pair by an element of G (otherwise an orbit of G would be a digraph which is neither complete nor null); in other words, G is 2-transitive.

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Partitions	Primitive
Graphs	2-homogeneous
Digraphs	2-transitive
Hamming graphs	Basic
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Another class \mathcal{C} we have just begun to study consists of **poset block structures**, where the \mathcal{C} -closed groups are the **generalised wreath products**.

A challenge for AI/ML

There are zillions of interesting classes of structures on sets.
Which ones give rise to interesting classes of permutation groups? Where should we look for them?

Reductions

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If G is transitive but imprimitive, it preserves a partition, and is embedded in the **wreath product** $H \wr K$, where H is the group induced on a block of the partition by its setwise stabiliser, and K the group induced on the set of parts of the partition. This is the **imprimitive action** of the wreath product.

Hamming graphs and basic groups

Let m, q be integers greater than 1. The **Hamming graph** $H(m, q)$ is the graph whose vertices are all words of length m over an alphabet of size q (so it has q^m vertices). A primitive group which preserves a Hamming graph is contained in the wreath product of the group (of degree q) induced on the symbols occurring in a given position by the stabiliser of that position in G and the group of permutations on the set of coordinate positions induced by G (of degree m).

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A primitive group is **basic** if it preserves no Hamming graph with $m, q > 1$. Thus, a group which is primitive but not basic is embeddable in a wreath product (in its **product action**).

Two special types of group

Let V be a finite vector space. The **affine group** $\text{AGL}(V)$ is the group of permutations of V generated by translations and invertible linear maps. (It is the semidirect product of the abelian translation group T and the **general linear group** $\text{GL}(V)$.)

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A subgroup of $\text{AGL}(V)$ containing T is the semidirect product of T with a subgroup H of $\text{GL}(V)$. It is necessarily transitive, since T is; it is primitive if and only if H is an **irreducible** linear group; and it is basic if and only if H is a **primitive** linear group, one which preserves no non-trivial direct sum decomposition of V .

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I will not give the rather involved definition of a **diagonal group** here; suffice to say that the diagonal group $D(H, m)$ depends on a group H and a positive integer m ; it has degree $|H|^m$ and has a normal subgroup H^{m+1} acting on the cosets of a diagonal subgroup, the quotient contained in the group generated by $\text{Aut}(H)$ and the symmetric group S_{m+1} .

The O'Nan–Scott Theorem

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Since affine groups preserve affine spaces, and diagonal groups preserve structures called **diagonal semilattices**, we can say that a permutation group which preserves no non-trivial subset, partition, Hamming graph, affine space, or diagonal semilattice is almost simple.

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- ▶ one of the 26 **sporadic groups**.

It follows from CFSG that, if S is one of these groups, then $\text{Aut}(S)/S$ is very small (and in any case soluble). The combined efforts of many mathematicians has led to a good understanding of simple (and almost simple) groups, such as knowledge of their maximal subgroups and linear representations.

Applications

The classification of 2-transitive groups follows from this. A 2-transitive group is clearly primitive and basic, and it is not hard to show that diagonal groups cannot be 2-transitive. So these groups are affine or almost simple; and using knowledge of the almost simple groups and their representations, a complete list can be found. (In fact, much less than the full strength of O'Nan–Scott is needed here; the reduction is due to Burnside.)

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More generally, Wielandt introduced the class of $\frac{3}{2}$ -transitive groups, those which are transitive and the stabiliser of a point α has all remaining orbits of the same size. (This class is not upward-closed so cannot be included in our general scheme.) Wielandt showed that a $\frac{3}{2}$ -transitive group is either primitive or a Frobenius group, a group in which all 2-point stabilisers are trivial. Any Frobenius group is $\frac{3}{2}$ -transitive; the primitive ones have been classified, using CFSG.

Low in the hierarchy

The properties we have examined so far are almost all at least as strong as primitivity. I want to conclude with several properties which are weaker, which I have investigated with Marina Anagnostopoulou-Merkouri and, in part, with Enoch Suleiman and Rosemary Bailey.

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Peter Neumann pointed out that in the Second Memoir, Galois sometimes confused the notions of primitivity and quasiprimitivity.

Pre-primitivity

Suppose that P and Q are permutation group properties such that P implies Q . The philosophy of what follows is to define a property “pre- P ” such that it is independent of Q but together with Q it is equivalent to P . (Note that this is not well-defined!)

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We have various results about such groups, including the fact that a wreath product of transitive groups is pre-primitive if and only if the factors are.

Invariant partitions

The set of partitions of Ω forms a lattice: the meet of two partitions P and Q is the partition whose parts are all non-empty intersections of parts of P and Q , and the join is the partition into connected components of the graph in which two points are adjacent if and only if they are in the same part of either P or Q .

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In addition, a partition is **uniform** if all parts have the same size, and two partitions **commute** if the corresponding equivalence relations do.

Statisticians define an **orthogonal block structure** to be a sublattice of the partition lattice consisting of commuting orthogonal partitions. Any OBS is a modular lattice; a **poset block structure** is a distributive OBS.

OB and PB permutation groups

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Generalized wreath product

PB groups are related to another concept, which I cannot describe in detail. It is well-known that a finite distributive lattice is the lattice of down-sets in a finite poset. There is a concept of **generalized wreath product** defined by a poset with a permutation group at each element.

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For example, there are two 2-element posets. Suppose that groups H and K are given at the two points. If the poset is an antichain, the GWP is the direct product; if it is a chain, with H above K , the GWP is the wreath product $K \wr H$.

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A transitive group with the PB property is naturally embedded in a generalized wreath product of symmetric groups.

Indeed, we expect to be able to replace the symmetric group by appropriate subgroups induced by the action of G ; but this is work in progress.

References

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