# Graphs defined on groups

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**Abstract**

These notes concern aspects of various graphs whose vertex set is a group $G$ and whose edges reflect group structure in some way (so that, in particular, they are invariant under the action of the automorphism group of $G$). The particular graphs I will discuss are the power graph, enhanced power graph, deep commuting graph, commuting graph, and non-generating graph, though I make some remarks on more general graphs. Particular aspects to be discussed include induced subgraphs, forbidden subgraphs, connectedness, and automorphism groups. Also, since these graphs form a hierarchy, we can ask about the graphs formed by the edges in one graph but not in an earlier graph in the hierarchy. I have also included some results on intersection graphs of subgroups of various types, which are often in a "dual" relation to one of the other graphs considered.

The graphs I mainly consider all have the property that they contain twins, pairs of vertices with the same neighbours (save possibly one another). For some purposes, we can merge twin vertices and get a smaller graph. Continuing until no further twins occur, the result is unique independent of the reduction, and is the trivial 1-vertex graph if and only if the original graph is a cograph. So I devote a section to cographs and twin reduction, and another to the consequences for automorphism groups (these exist because twins can be interchanged by an automorphism fixing the other vertices). In addition, I discuss the important question of deciding for each type of graph, for which groups is it a cograph.

These notes make no pretence to be a complete survey of the subject; the main interest lies in the relationships between the different graphs considered. However, a number of references to the speficic

types are included. By and large I have not included results about specific groups except where they provide examples or occur in conclusions to general theorems.

I'm grateful to several people, especially Saul Freedman, Michael Giudici, Bojan Kuzma and Natalia Maslova, for helpful comments on a previous version.

At present these notes are not designed for publication, but are being distributed in the hope of encouraging further research on the topic. Please contact me with comments.

# 1 Introduction

There are a number of graphs whose vertex set is a group $G$ and whose edges reflect the structure of $G$ in some way, so that the automorphism group of $G$ acts as automorphisms of the graph. These include the commuting graph (first defined in 1955), the generating graph (from 1996), the power graph (from 2000), and the enhanced power graph (from 2017), all of which have a considerable and growing literature. A relative newcomer, not published yet, is the deep commuting graph.

This paper does not aim to be a survey of all these areas, which would be far too ambitious a task. Rather, I am interested in comparisons among the different graphs. In particular, there is a hierarchy containing the null graph, power graph, enhanced power graph, deep commuting graph, commuting graph, non-generating graph (if the group is non-abelian), and complete graph: the edge set of each is contained in that of the next.

These graphs have some similarities: for example, the enhanced power graphs, commuting graphs, deep commuting graphs, and generating graphs of finite groups all form universal families (that is, every finite graph is embeddable in one of these graphs for some group $G$). However, the proofs of this require rather different techniques for the different graphs.

Another question, about which relatively little is currently known, concerns the differences between graphs in the hierarchy. Even rather basic questions such as connectedness are unstudied for most of these, although Saul Freedman and coauthors have results on the difference between the non-generating graph and the commuting graph (and, at top and bottom, the difference between the complete graph and the non-generating graph is the generating graph, while the difference between the power graph and the null graph is the power graph, both of which have an extensive literature).

Another curious feature is the appearence of the Gruenberg–Kegel graph, which determines (or almost determines) various features of the commuting graph and the power graph. The vertex set of this graph is not the group, but the much smaller set of prime divisors of the group order. For example, if $G$ has trivial centre, its reduced commuting graph (with the identity removed) is connected if and only if its Gruenberg–Kegel graph is. Conversely, for all graph types in the hierarchy except possibly the non-generating graph, the corresponding graph on $G$ determines the Gruenberg–Kegel graph of $G$.

Authors who have studied these have used a variety of notations for them. I have tried to use a consistent and helpful notation, for example, $\mathrm{Pow}(G)$ and $\mathrm{Com}(G)$ for the power graph and commuting graph, respectively, of $G$.

The final section concerns more general graphs defined on groups and invariant under group automorphisms.

Since much is not known, I have tried to emphasise open problems throughout.

Please be aware, however, that the current paper is just a rough draft ...

Computations reported here were performed using GAP [37], with the package GRAPE [70] for handling graphs.

## 1.1 Cayley graphs

One topic I will not consider, except in this section, concerns Cayley graphs. A *Cayley graph* for the group $G$ is a graph on the vertex set $G$ which is invariant under right translation by elements of $G$. (Some authors use left translation; the two concepts are equivalent, and the inversion map on $G$ converts one into the other.) Equivalently, if $S$ is an inverse-closed subset of $G \setminus \{1\}$, then the Cayley graph $\mathrm{Cay}(G, S)$ is the graph with vertex set $G$ in which $g$ and $h$ are adjacent whenever $gh^{-1} \in S$.

One reason for not considering these is that they have a huge literature, far more than I can survey here. I have heard the view expressed that algebraic graph theory is the study of Cayley graphs of finite groups (in fact, it is broader than this, but Cayley graphs are an important topic); while it is certainly arguable that geometric group theory is the study of Cayley graphs of finitely generated infinite groups.

The other is that Cayley graphs are not in general preserved by the automorphism group of $G$. I will say a few words about this.

Suppose that the set $S$ is a *normal subset* of $G$, that is, closed under conjugation. Then $\mathrm{Cay}(G,S)$ is invariant under both left and right translation. Such a graph is sometimes called a *normal Cayley graph*, see for example [48, 56]. However, the reader is warned that more recently this term has been used in a completely different sense: a Cayley graph $\Gamma = \mathrm{Cay}(G,S)$ is normal if the group of right translations of $\Gamma$ is a normal subgroup of $\mathrm{Aut}(\Gamma)$, see for example [78].

To avoid confusion, I propose to call a normal Cayley graph in the first sense above an *inner-automorphic* Cayley graph. Note that this condition is equivalent to saying that the graph is invariant under both left and right translation. (For the composition of right translation by $g$ and left translation by $g^{-1}$ is conjugation by $g$, and so a graph invariant under two of these maps is invariant under the third also.)

**Proposition 1.1** *The Cayley graph* $\mathrm{Cay}(G,S)$ *is inner-automorphic if and only if $S$ is a union of conjugacy classes in $G$.*

Note also that the minimal (non-null) inner-automorphic Cayley graphs for $G$ are the relations of the *conjugacy class association scheme* on $G$, see [23, 40, 77] (the last of these references calls this structure the *group scheme* of $G$).

I shall call the Cayley graph $\mathrm{Cay}(G,S)$ *automorphic* if it is invariant under the whole of $\mathrm{Aut}(G)$, that is, if $S$ is a union of orbits of $\mathrm{Aut}(G)$ acting on $G$. (Thus, if $\mathrm{Cay}(G,S)$ is automorphic, then its automorphism group contains the *holomorph* of $G$.) These graphs would fall under the rubric considered here, although I shall not be discussing them further.

# 2 Dramatis personae

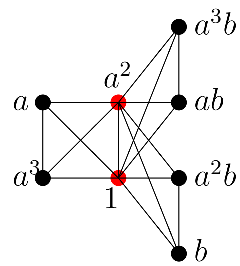
This section introduces the specific graphs on a group that I will be mainly concerned with.

## 2.1 The commuting graph

Let $G$ be a finite group. The *commuting graph* of $G$ is the graph with vertex set $G$ in which two vertices $x$ and $y$ are joined if $xy = yx$. This graph was introduced by Brauer and Fowler in their seminal paper [19] and has had a number of important applications in group theory. Vertices in $Z(G)$ are joined to everything, and for investigating questions such as connectedness these are often removed; this makes no difference here.

Also the definition puts a loop at every vertex. There is good reason for doing this; it follows from results of Jerrum [49] on the "Burnside process" that the limiting distribution of the random walk on the commuting graph with loops is uniform on conjugacy classes – that is, the limiting probability of being at a vertex is inversely proportional to the size of its conjugacy class. This is useful in finding representatives of very small conjugacy classes in large groups. But for my purposes here, I will imagine that the loops have been silently removed.

As with all of these graphs, we can ask: Which groups are characterised by their commuting graphs? For example, abelian groups of the same order have isomorphic commuting graphs, as do the dihedral and quaternion groups of order 8. Here are the commuting graphs of the two groups $D_8 = \langle a, b : a^4 = 1, b^2 = 1, b^{-1}ab = a^{-1}\rangle$ and $Q_8 = \langle a, b : a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1}\rangle$.




It was conjectured in [3] and proved in [4, 45, 71] that any non-abelian finite simple group is characterised by its commuting graph.

I note in passing an application of the commuting graphs of finite groups to the structure of finite quotients of the multiplicative group of a finite-dimensional division algebra by Segev [67].

## 2.2 The deep commuting graph

The *deep commuting graph* of a finite group $G$ was introduced very recently [28]. Two elements of $G$ are joined in the deep commuting graph if and only if their preimages in every central extension of $G$ (that is, every group $H$ with a central subgroup $Z$ such that $H/Z \cong G$) commute. More specifically, take the commuting graph of a *Schur cover* [66] of $G$ (this is a central extension $H$ of largest order such that $Z$ is contained in the derived group of $H$), and take the induced subgraph of the commuting graph of $H$ on a transversal to $Z$. It can be shown that the resulting graph is independent of the choice of Schur cover.

For example, $D_8$ and $Q_8$ are Schur covers of the Klein group $V_4$; so the deep commuting graph of the Klein group is the star $K_{1,3}$, though its commuting graph is the complete graph $K_4$.

We note in particular that the deep commuting graph is equal to the commuting graph if the *Schur multiplier* of $G$ (the central subgroup $Z$ in a Schur cover) is trivial. The converse is false; see [28].

**Question 1** Is it true that a non-abelian finite simple group is characterised by its deep commuting graph?

## 2.3 The power graph

The *directed power graph* of $G$ is the directed graph with vertex set $G$, with an arc $x \to y$ if $y = x^m$ for some integer $m$. The *power graph* of $G$ is the graph obtained by ignoring directions and double arcs; in other words, $x$ is joined to $y$ if one of $x$ and $y$ is a power of the other. It is clearly a spanning subgraph of the commuting graph. The power graph was introduced by Kelarev and Quinn [52].

The directed power graph is a *partial preorder*, that is, a reflexive and transitive relation on $G$; and the power graph is its *comparability graph* (two vertices joined if and only if they are related in the preorder). We will see later that the comparability graphs of partial preorders and *partial orders* form the same class.

The power graph does not uniquely determine the directed power graph; for example, if $G$ is the cyclic group of order 6, then the identity and the two generators are indistinguishable in the power graph (they are joined to all other vertices), but one is a sink and the other two are sources in the directed power graph. However, the following is shown in [24]:

**Theorem 2.1** *If two finite groups have isomorphic power graphs, then they have isomorphic directed power graphs.*

This is false for infinite groups; see [26].

## 2.4 The enhanced power graph

The *enhanced power graph* [1], of a group $G$ has vertex set $G$, with $x$ and $y$ joined if and only if $\langle x, y\rangle$ is cyclic. Equivalently, $x$ and $y$ are joined if there is an element $z \in G$ such that each is a power of the other.

The enhanced power graph can be obtained from the directed power graph by joining two vertices if both lie in the closed out-neighbourhood of some vertex. Thus, if two groups have isomorphic power graphs, then they have isomorphic enhanced power graphs. The converse is also true, see [79]:

**Theorem 2.2** *For a pair of finite groups, the following are equivalent:*

*(a) the power graphs are isomorphic;*

*(b) the directed power graphs are isomorphic;*

*(c) the enhanced power graphs are isomorphic.*

**Question 2** Is there a simple algorithm for constructing the directed power graph or the enhanced power graph from the power graph, or the directed power graph from the enhanced power graph?

Note that the enhanced power graph of $G$ is a union of complete subgraphs on the maximal cyclic subgroups of $G$. Similarly, the commuting graph is a union of complete subgraphs on the maximal abelian subgroups.

## 2.5 The generating graph

The *generating graph* of a finite group $G$ has vertex set $G$, with $x$ and $y$ joined if and only if $\langle x, y\rangle = G$. If the minimum number of generators of $G$ is greater than 2, then the generating graph is the null graph. If $G$ is cyclic, then its generating graph has loops; we will not be too much interested in this case. Note that, by the Classification of Finite Simple Groups, every non-abelian finite simple group is 2-generated.

The generating graph was introduced in [58], and studied further in [20], where the authors showed that the generating graph of a non-abelian finite simple group has positive *spread*: every non-identity element of $G$ is contained in a 2-element generating set (so the identity is the only isolated vertex). A substantial strengthening has recently been proved by Burness *et al.* [22]:

**Theorem 2.3** *For a finite group $G$, the following three conditions are equivalent:*

*(a) any non-identity vertex has a neighbour in the generating graph;*

*(b) any two non-identity vertices have a common neighbour in the generating graph;*

*(c) any proper quotient of $G$ is cyclic.*

In particular, the conditions hold for a non-abelian finite simple group.

In the sequel, I will sometimes consider the *non-generating graph*, the complement of the generating graph.

## 2.6 The hierarchy

These graphs are given *ad hoc* names in the literature, but since I will be talking about all of them here, I prefer to give them names which help to distinguish them. Thus, the commuting graph of $G$ will be $\mathrm{Com}(G)$; the deep commuting graph $\mathrm{DCom}(G)$; the power graph $\mathrm{Pow}(G)$; the directed power graph $\mathrm{DPow}(G)$; the enhanced power graph $\mathrm{EPow}(G)$; the generating graph $\mathrm{Gen}(G)$; and the non-generating graph $\mathrm{NGen}(G)$. In each case, the vertex set is the group $G$. Sometimes I refer to the *reduced graph* of one of the above types, and denote it by a superscript $-$; this means that the identity element is deleted from the vertex set.

There are inclusions between these graphs, as follows. Here $E(\Gamma)$ denotes the edge set of a graph $\Gamma$; thus $E(\Gamma_1) \subseteq E(\Gamma_2)$ means that $\Gamma_1$ is a *spanning subgraph* of $\Gamma_2$ (a subgraph using all of the vertices and some of the edges).

**Proposition 2.4** *Let $G$ be a finite group.*

*(a)* $E(\mathrm{Pow}(G)) \subseteq E(\mathrm{EPow}(G)) \subseteq E(\mathrm{DCom}(G)) \subseteq E(\mathrm{Com}(G))$.

*(b) If $G$ is non-abelian or not 2-generated, then* $E(\mathrm{Com}(G) \subseteq E(\mathrm{NGen}(G))$.

**Proof** (a) All is obvious except possibly the inclusion of $E(\mathrm{EPow}(G))$ in $E(\mathrm{DCom}(G))$. So suppose that $\langle x, y\rangle$ is a cyclic subgroup of $G$, and let $H$ be a central extension of $G$, with $H/Z \cong G$. The lift of $\langle x, y\rangle$ is the extension of a central subgroup $Z$ by a cyclic group, and hence is abelian; so the lifts of $x$ and $y$ commute in $H$.

(b) If $G$ is not 2-generated, then $\mathrm{NGen}(G)$ is the complete graph, and the result is clear. If $G$ is non-abelian, it cannot be generated by two commuting elements. $\square$

Because of this, I will refer to the null graph, power graph, enhanced power graph, deep commuting graph, commuting graph, non-generating graph (in the case that $G$ is non-abelian) and complete graph on $G$ as the *graph hierarchy*, or just *hierarchy*, of $G$.

## 2.7 The Gruenberg–Kegel graph

A related graph will play a role in the investigation in several places. The *Gruenberg–Kegel graph*, also known as the *prime graph*, of a finite group $G$ has vertex set the set of prime divisors of the order of $G$; vertices $p$ and $q$ are joined by an edge if and only if $G$ contains an element of order $pq$.

The graph was introduced in an unpublished manuscript by Gruenberg and Kegel to study the integral group ring of a finite group, and in particular the decomposability of the augmentation ideal: see [43]. The main structural result was published by Williams (a student of Gruenberg) in [75]. It asserts that groups whose Gruenberg–Kegel graph is disconnected have a very restricted structure.

**Theorem 2.5** *Let $G$ be a finite group whose Gruenberg–Kegel graph is disconnected. Then one of the following holds:*

*(a) $G$ is a Frobenius or 2-Frobenius group;*

*(b) $G$ is an extension of a nilpotent $\pi$-group by a simple group by a $\pi$-group, where $\pi$ is the set of primes in the connected component containing 2.*

A 2-*Frobenius group* is a group $G$ with normal subgroups $H$ and $K$ with $H \le K$ such that

- $K$ is a Frobenius group with Frobenius kernel $H$;
- $G/H$ is a Frobenius group with Frobenius kernel $K/H$.

A typical example is the group $G = S_4$, with $K = A_4$, $H = V_4$ (the Klein group), and $G/K \cong S_3$.

Williams went on to examine the known finite simple groups to determine which ones could occur in conclusion (b) of the Theorem. He could not handle the groups of Lie type in characteristic 2; this was completed by Kondrat'ev in 1989, and some errors corrected by Kondrat'ev and Mazurov in 2000.

The next result indicates that the Gruenberg–Kegel graph is closely connected with our hierarchy of graphs.

**Theorem 2.6** *Let $G_1$ and $G_2$ be groups whose power graphs, or enhanced power graphs, or deep commuting graphs, or commuting graphs, are isomorphic. Then the Gruenberg–Kegel graphs of $G_1$ and $G_2$ are equal.*

**Proof** The four possible hypotheses each imply that $G_1$ and $G_2$ have the same order, so their GK graphs have the same set of vertices.

We show that in all cases except the power graph, primes $p$ and $q$ are adjacent in the GK graph of $G$ if and only if there is a maximal clique in the graph on $G$ with size divisible by $pq$. This is clear in the cases of the enhanced power graph and the commuting graph; for, as we observed earlier, the maximal cliques in these are maximal cyclic subgroups or maximal abelian subgroups of $G$ respectively, and if their order is divisible by $pq$ then they contain elements of order $pq$. Conversely an element of order $pq$ is contained in a maximal cyclic (or abelian) subgroup.

Consider the deep commuting graph of a group $G$. Let $H$ be a Schur cover of $G$, with $H/Z \cong G$. A maximal clique has the form $A = B/Z$, where $B$ is a maximal abelian subgroup of $H$ (containing $Z$). So $A$ is an abelian subgroup of $G$, and if $pq$ divides $|A|$ then $A$ contains an element of order $pq$. Conversely, suppose that $p$ and $q$ are joined in the GK graph, and let $x$ and $y$ be commuting elements of orders $p$ and $q$ in $G$, and $a$ and $b$ their lifts in $H$. Then $a$ and $b$ are contained in $\langle Z, ab\rangle$, which is an extension of a central subgroup by a cyclic group and hence is abelian; so $a$ and $b$ commute. Choosing a maximal abelian subgroup of $H$ containing $a$ and $b$ and projecting onto $G$ gives a maximal clique in $\mathrm{DCom}(G)$ with order divisible by $pq$.

This fails for the power graph. Instead we use the fact that groups with isomorphic power graphs also have isomorphic enhanced power graphs, and so have equal GK graphs, by what has already been proved. $\square$

I do not know whether the analogous result holds for the non-generating graph of a non-abelian 2-generated group.

The Gruenberg–Kegel graph is still an active topic of research; see [31] for a survey and some recent results. Some of the research concerns the question of whether a group is determined (perhaps up to finitely many possibilities) by its GK graph. There are two versions of this: two GK graphs could be equal (as graphs whose vertex set is a finite set of primes) or merely isomorphic as graphs but with possibly different labels for the vertices. For an interesting example, the GK graphs of the groups $A_{10}$ and $\mathrm{Aut}(\mathrm{J}_2)$ are isomorphic, and both have vertex sets $\{2,3,5,7\}$, but are not equal: the labels 2 and 3 are swapped.

### 2.8 Intersection graphs

Let $G$ be a finite group, not trivial and not a cyclic group of prime order. The *intersection graph* of $G$ is the graph whose vertices are the non-trivial proper subgroups of $G$, with two vertices $H_1$ and $H_2$ adjacent if $H_1 \cap H_2 \neq \{1\}$.

There are various other intersection graphs: we can restrict to subgroups in a particular class, or to maximal subgroups.

We will see a connection bewtween some intersection graphs and some of the graphs in our hierarchy.

## 3 Equality and differences

For a non-abelian finite group $G$, there are seven graphs in the hierarchy, and a natural question is: When can two of them be equal? If they are not equal, what can be said about their difference?

### 3.1 Equality

At the two ends, things are easy:

**Proposition 3.1** *(a)* $\mathrm{Pow}(G)$ *is equal to the null graph if and only if* $G$ *is the trivial group.*

*(b)* $\mathrm{NGen}(G)$ *is equal to the complete graph if and only if* $G$ *is not* 2*-generated.*

*(c)* $\mathrm{NGen}(G) = \mathrm{Com}(G)$ *if and only if* $G$ *is either abelian and not* 2*-generated, or a minimal non-abelian group.*

**Proof** Parts (a) and (b) are clear. So suppose that $\mathrm{NGen}(G) = \mathrm{Com}(G)$ and this is not the complete graph. Then $G$ is non-abelian, but if $x$ and

$y$ do not generate $G$ then they commute; so every proper subgroup of $G$ is abelian. Thus $G$ is minimal non-abelian. $\square$

The minimal non-abelian groups were determined by Miller and Moreno [61] in 1903. There are two types: the first consists of groups of prime power order; the second are extensions of an elementary abelian $p$-group by a cyclic $q$-group, where $p$ and $q$ are primes.

Leaving aside the deep commuting graph from the present, the following was shown in [1]:

**Proposition 3.2** *Let $G$ be a finite group.*

*(a) The power graph of $G$ is equal to the enhanced power graph if and only if $G$ contains no subgroup isomorphic to $C_p \times C_q$, where $p$ and $q$ are distinct primes; equivalently, the Gruenberg–Kegel graph of $G$ is a null graph.*

*(b) The enhanced power graph of $G$ is equal to the commuting graph if and only if $G$ contains no subgroup isomorphic to $C_p \times C_p$, where $p$ is prime; equivalently, the Sylow $p$-subgroups of $G$ are cyclic or generalized quaternion groups.*

*(c) The power graph of $G$ is equal to the commuting graph if and only if $G$ contains no subgroup isomorphic to $C_p \times C_q$, where $p$ and $q$ are primes (equal or distinct).*

**Proof** (a) If $G$ contains commuting elements of orders $p$ and $q$, they are adjacent in $\mathrm{EPow}(G)$ but not in $\mathrm{Pow}(G)$. Conversely, suppose that $x$ and $y$ are adjacent in $\mathrm{EPow}(G)$ but not in $\mathrm{Pow}(G)$. Then $x$ and $y$ are contained in a cyclic group $C$ but neither is a power of each other; $C$ must then have order divisible by two distinct primes.

(b) If $G$ contains commuting elements of the same prime order $p$ but not in a cyclic subgroup of order $p$, they are joined in the commuting graph but not in the enhanced power graph. Conversely, suppose that $x$ and $y$ are adjacent in $\mathrm{Com}(G)$ but not in $\mathrm{EPow}(G)$. The orders of $x$ and $y$ must have a common factor (otherwise they generate a cyclic group); so some powers of them have prime order $p$ and generate $C_p \times C_p$.

Now a theorem of Burnside (see [44, Theorem 12.5.2]) shows that a $p$-group containing no subgroup $C_p \times C_p$ is cyclic or generalized quaternion.

(c) The third part is immediate from the first two. $\square$

Using these results it is possible to classify the groups involved.

(a) Any group of prime power order has Gruenberg–Kegel graph consisting of a single vertex, so has power graph equal to enhanced power graph. Any other group with this property has disconnected Gruenberg–Kegel graph, and so satisfies the conclusion of Theorem 2.5. Cameron and Maslova [31] have worked out all these groups and hope to publish the list shortly.

(b) A group with all Sylow subgroups cyclic is metacyclic; indeed, if the primes dividing its order are $p_1, p_2, \ldots, p_r$ in increasing order, then it has a normal Hall subgroup corresponding to the last $i$ primes in this list, for $1 \leq i \leq r-1$.

   By Glauberman's $Z^*$-theorem [39], if $G$ has generalized quaternion Sylow 2-subgroup, and $O(G)$ is the largest normal subgroup of odd order in $G$, then $G/O(G)$ has a unique involution; the quotient $\bar{G}$ by the subgroup generated by this involution has dihedral Sylow 2-subgroup, so falls into the classification by Gorenstein and Walter [41]. Of the groups in their theorem, we retain only those with cyclic Sylow subgroups for odd primes, that is, $\bar{G}$ is isomorphic to $\mathrm{PSL}(2,p)$ or $\mathrm{PGL}(2,p)$ or two a dihedral 2-group. Conversely, each such group can be lifted to a unique group with a unique involution. The normal subgroup $O(G)$ has all its subgroups cyclic, so is metacyclic, as above.

Finally, the deep commuting graph lies between the enhanced power graph and the commuting graph. In order to investigate equality here, we need another construction. Recall that the Schur multiplier of $G$ is the largest kernel $Z$ in a stem extension $H$ of $G$ (with $Z \leq Z(H) \cap H'$ and $H/Z \cong G$). An extension is said to be *commutation-preserving*, or CP, if whenever two elements $x, y \in G$ commute, their preimages in $H$ also commute. Now there is a well-defined largest kernel of a CP stem extension of $G$; this is the *Bogomolov multiplier* of $G$, see [16, 50].

**Proposition 3.3** *Let $G$ be a finite group.*

*(a)* $\mathrm{DCom}(G) = \mathrm{EPow}(G)$ *if and only if $G$ has the following property: let $H$ be a Schur cover of $G$, with $H/Z = G$. Then for any subgroup $A$ of $G$, with $B$ the corresponding subgroup of $H$ (so $Z \leqslant B$ and $B/Z = A$), if $B$ is abelian, then $A$ is cyclic.*

(b) $\mathrm{DCom}(G) = \mathrm{Com}(G)$ *if and only if the Bogomolov multiplier of $G$ is equal to the Schur multiplier.*

I refer to [28] for the proofs.

A precise characterisation of the groups attaining either equality is not known; but examples exist where one bound but not the other is met, or where neither bound is met (see [28]):

- If $G$ is the symmetric or alternating group of degree at least 8, then $E(\mathrm{EPow}(G)) \subset E(\mathrm{DCom}(G)) \subset E(\mathrm{Com}(G))$.
- If $G$ is a dihedral group of order $2^n$ with $n \geq 3$, then $E(\mathrm{EPow}(G) = E(\mathrm{DCom}(G) \subset E(\mathrm{Com}(G))$.
- If $G$ is a certain group of order 64 (number 182 in the GAP library), then $E(\mathrm{EPow}(G)) \subset E(\mathrm{DCom}(G)) = E(\mathrm{Com}(G))$.

Note that

- if the Schur multiplier of $G$ is trivial, then $\mathrm{DCom}(G) = \mathrm{Com}(G)$;
- in general, the Bogomolov multiplier is much smaller than the Schur multiplier; for example, if $G$ is a non-abelian finite simple group, then its Bogomolov multiplier is trivial [55].

**Question 3** (a) What can be said about groups $G$ for which $\mathrm{DCom}(G) = \mathrm{EPow}(G)$?

(b) What can be said about groups $G$ for which $\mathrm{DCom}(G) = \mathrm{Com}(G)$?

## 3.2 Differences

For any pair of graphs in the hierarchy, if $G$ is a group such that these two graphs are unequal, we could ask about the graph whose edge set is the difference. We could denote these by using, for example, $(\mathrm{Com} - \mathrm{Pow})(G)$ for the graph whose edges are those belonging to the commuting graph but not the power graph, with similar notation in other cases.

At the top, the difference between the complete graph and the non-generating graph is just the generating graph, which has been extensively studied. At the next level, the difference between the generating graph and the commuting graph (the graph $(\mathrm{NGen} - \mathrm{Com})(G)$) has been studied by

Saul Freedman. The most complete results are for nilpotent groups, and are reported in [25]. In particular, if $G$ is nilpotent and the non-commuting non-generating graph is not null, then after deletion of the identity it is connected, with diameter 2 or 3.

Other differences (apart from the difference between the power graph and the null graph) have not been studied.

**Question 4** For each pair of graph types in the hierarchy, what can be said about groups for which the difference is connected (after removing isolated vertices and vertices joined to all others)?

### 3.3 Further problems

We saw the result of [79] that two groups have isomorphic power graphs if and only if they have isomorphic enhanced power graphs.

**Question 5** Are there any other implications of this kind between pairs of graphs in the hierarchy?

For a simple negative example, the groups $C_{p^2}$ and $C_p \times C_p$ have isomorphic commuting graphs but nonisomorphic power graphs, while the group $C_p \times C_p \times C_p$ and the non-abelian group of order $p^3$ and exponent $p$ have isomorphic power graphs but nonisomorphic commuting graphs.

Do there exist groups $G_1$ and $G_2$ such that, for example, $\mathrm{Pow}(G_1)$ is isomorphic to $\mathrm{Com}(G_2)$? This will be true if $\mathrm{Pow}(G_1) = \mathrm{Com}(G_1)$ and $G_1$ and $G_2$ have isomorphic commuting graphs, or if $\mathrm{Pow}(G_2) = \mathrm{Com}(G_2)$ and $G_1$ and $G_2$ have isomorphic power graphs.

**Question 6** Can $\mathrm{Pow}(G_1)$ and $\mathrm{Com}(G_2)$ be isomorphic for groups $G_1$ and $G_2$ which both have power graph not equal to commuting graph? Similar questions for other pairs of graphs in the hierarchy.

## 4 Induced subgraphs

In this section I will consider the question, for each of the graphs in our hierarchy: For which finite graphs $\Gamma$ does there exist a finite group $G$ such that $\Gamma$ is isomorphic to an induced subgraph of the group of that type defined on $G$? (An *induced subgraph* of $\Gamma$ on a subset $A$ of the vertex set consists of the vertices of $A$ and all edges of $\Gamma$ which are contained in $A$.)

To summarise the results:

- A finite graph $\Gamma$ is isomorphic to an induced subgraph of the power graph of some finite group $G$ if and only if $\Gamma$ is the comparability graph of a partial order.
- For each of the other graphs in the hierarchy, every finite graph is isomorphic to an induced subgraph of that graph defined on some finite group.

Three related questions are:

**Question 7** (a) What is the smallest group for which a given graph is embeddable in the enhanced power graph/deep commuting graph/commuting graph/non-generating graph?

(b) What is the smallest group in which every graph on $n$ vertices can be embedded in one of these graphs?

(c) Which graphs occur if we restrict the group to have a particular property such as nilpotence or simplicity?

On the first question, here is a very rough lower bound for the order of a group which embeds an $n$-vertex graph. Suppose $N$ is such that every $n$-vertex graph can be embedded in the enhanced power graph, deep commuting graph, commuting graph, or non-generating graph of some group of order at most $N$. For our rough calculation, we need only consider groups of order $N$. It is known that there are at most $2^{c(\log N)^3}$ such groups (see [15]); each has at most $N^n$ subsets of size $n$. But there are at least $2^{n(n-1)/2}/n!$ graphs on $n$ vertices up to isomorphism. So we require

$$2^{c(\log N)^3} \cdot N^n \geq 2^{n(n-1)/2}/n!,$$

which implies that $N \geq 2^{n^{2/3-\epsilon}}$. So the exponential bound we find in some cases is not too far from the truth.

For the second question, we note that every $n$-vertex graph is embeddable in a Paley graph of order $q$, where $q$ is a prime power congruent to 1 (mod 4) and $q > n^2 2^{2n-2}$ (see [17]); so we only need to embed this graph.

## 4.1 The commuting graph

**Theorem 4.1** *Every finite graph is isomorphic to an induced subgraph of the commuting graph of a finite group. This group can be taken to be nilpotent of class* 2 *and exponent* 4*.*

**Proof** Let $F$ be the two-element field, $V$ a vector space over $F$, and $B$ a bilinear form on $V$. Define an operation $\circ$ on $V \times F$ by the rule

$$(v_1, a_1) \circ (v_2, a_2) = (v_1 + v_2, a_1 + a_2 + B(v_1, v_2)).$$

It is a straightforward exercise to show that this operation makes $V \times F$ a group. This group is nilpotent of class 2 and exponent (dividing) 4, since $\{0\} \times F$ is a central subgroup with elementary abelian quotient. Moreover, $(v_1, a_1)$ and $(v_2, a_2)$ commute if and only if $B(v_1, v_2) = B(v_2, v_1)$.

Now a bilinear form is uniquely determined by its values on pairs of vectors taken from a basis for $V$; these values can be assigned arbitrarily. So let $\Gamma$ be a graph with vertex set $\{1, \ldots, n\}$, and let $v_1, \ldots, v_n$ be a basis. Assign the values $B(v_i, v_j) = 0$ if $i \leq j$; for $i > j$, put $B(v_i, v_j) = 0$ if vertices $i$ and $j$ are adjacent, 1 if not. Then it is clear that the induced subgraph of the commuting graph on the set $\{v_1, \ldots, v_n\}$ is isomorphic to $\Gamma$. $\square$

The construction above shows that the smallest group whose commuting graph contains a given $n$-vertex graph has order at most $2^{n+1}$ if $\Gamma$ has $n$ vertices. However, it may be very much smaller; for the complete graph $K_n$, the answer is clearly $n$.

## 4.2 The deep commuting graph

**Theorem 4.2** *Every finite graph is isomorphic to an induced subgraph of the deep commuting graph of a finite group.*

**Proof** Let $\Gamma$ be a finite graph. As we have seen, $\Gamma$ is isomorphic to an induced subgraph of the commuting graph of some group. So it is enough to show that this group can be chosen to have trivial Schur multiplier. Since the induced subgraph on a subgroup $H$ of the commuting graph of $G$ is the commuting graph of $H$, it suffices to show that every finite group can be embedded in a finite group with trivial Schur multiplier.

By Cayley's Theorem, every finite group of order $n$ can be embedded in the symmetric group $S_n$. Unfortunately the symmetric group has Schur multiplier $C_2$ if $n \geq 8$. So we embed $S_n$ into the general linear group $\mathrm{GL}(n, 2)$ by permutation matrices of order $n$. Now the Schur multiplier of $\mathrm{GL}(n, 2)$ is trivial except for $n = 3$ or $n = 4$ [42]. $\square$

## 4.3 The power graph

A *partial order* is a partial preorder satisfying *antisymmetry*, that is, if $x \to y$ and $y \to x$ then $x = y$. Comparability graphs of partial orders and of partial preorders form the same class. For clearly every partial order is a partial preorder. Conversely, suppose that $\to$ is a partial preorder on $X$. Define a relation $\equiv$ on $X$ by the rule that $x \equiv y$ if $x \to y$ and $y \to x$. This is an equivalence relation on $X$. Now extend the partial preorder by imposing a total order on each equivalence class of $\equiv$. The result is a partial order with the same comparability graph as the partial preorder.

**Theorem 4.3** *A finite graph is isomorphic to an induced subgraph of the power graph of a finite group if and only if it is the comparability graph of a partial order. The group can be taken to be cyclic of squarefree order.*

**Proof** One way round follows from our preliminary remarks: the power graph is the comparability graph of a partial order, and the class of such graphs is closed under taking induced subgraphs.

So suppose that we have a partial order $\leqslant$ on $X$. For each $x \in X$, let $[x] = \{y \in x : y \leqslant x\}$. A routine check shows that

- $[y] \subseteq [x]$ if and only if $y \leqslant x$;
- $[x] = [y]$ if and only if $x = y$.

So the given partial order is isomorphic to the set of subsets of $X$ of the form $[x]$, ordered by inclusion.

Now chose distinct prime numbers $p_x$ for $x \in X$. Let $G$ be the direct product of cyclic groups $C_{p_x} = \langle a_x \rangle$ of order $p_x$ for $x \in X$. Now map the subset $Y$ of $X$ to the element $g_X = (g_x : x \in X)$ of the direct product, where

$$g_x = \begin{cases} a_x & \text{if } x \in Y, \\ 1 & \text{otherwise.} \end{cases}$$

It is readily checked that $g_X$ and $g_Y$ are adjacent in the power graph if and only if $X$ and $Y$ are adjacent in the comparability graph of the inclusion order on $X$.

To conclude, we simply note that $G$ is a cyclic group of squarefree order.
$\square$

Note that a graph is the comparability graph of a partial order if and only if there is a transitive orientation of the edges. There is a list of forbidden induced subgraphs for comparability graphs [36], but it is not straightforward to state.

## 4.4 The enhanced power graph

**Theorem 4.4** *Every finite graph is isomorphic to an induced subgraph of the enhanced power graph of some group (which can be taken to be abelian).*

**Proof** The proof is by induction. For a graph with a single vertex, there is no problem. So let $\Gamma$ be a graph with vertex set $\{1,\ldots,n\}$, and suppose that $i \mapsto x_i$ (for $i = 1,\ldots,n-1$) is an isomorphism to an induced subgraph of $G$.

Choose a prime $p$ not dividing the order of $G$, and let $H = \langle a, b\rangle$ be an elementary abelian group of order $p^2$. Now in the group $G \times H$, replace $x_i$ by $x_i a$ if $i$ is not joined to $n$ in $\Gamma$, and leave it as is if $i$ is joined to $n$. Then map $n$ to $x_n = b$.

Since $p \nmid |G|$, for any $z \in G$ we have $\langle z, a\rangle = \langle z\rangle \times \langle a\rangle$, which is cyclic. So the embedding of $\{1,\ldots,n-1\}$ is still an isomorphism to an induced subgraph. Moreover, $\langle x_i, b\rangle$ is cyclic while $\langle x_i a, b\rangle$ is not, so we have the correct edges from $b$ to the other vertices, and the result is proved.

The resulting group is the product of $n$ copies of $C_p \times C_p$ for distinct primes $p$. $\square$

## 4.5 The generating graph

**Theorem 4.5** *Every finite graph is isomorphic to an induced subgraph of the generating graph of a finite group.*

**Proof** Let $\Gamma$ be a finite graph. We proceed in a number of steps.

**Step 1** Replace $\Gamma$ by its complement.

**Step 2** Every graph can be represented as the intersection graph of a *linear hypergraph*, a family of sets which intersect in at most one point (where intersection 1 corresponds to adjacency). The ground set $E$ is the set of edges of the graph; the vertex $v$ is represented by the set $S(v)$ of edges incident with $v$. Then for distinct vertices $v, w$,

$$S(v) \cap S(w) = \begin{cases} e, & \text{if } \{v, w\} \text{ is an edge } e, \\ \emptyset & \text{if } v \text{ and } w \text{ are nonadjacent.} \end{cases}$$

**Step 3** Add some dummy points, each lying in just one of the sets, so that they all have the same cardinality $k$, with $k \geq 3$. Now add some dummy points in none of the sets so that the cardinality $n$ of the set $\Omega$ of points satisfies the conditions that $n > 2k$ and $n - k$ is prime.

**Step 4** Now replace each set by its complement. The complements of two subsets of $\Omega$ have union $\Omega$ if and only if the two sets are disjoint. Thus, each original vertex is now represented by an $(n-k)$-set where two such sets have union $\Omega$ if and only if the corresponding vertices are adjacent in $\Gamma$.

**Step 5** Replace each set by a cyclic permutation on that set, fixing the remaining point. Each of these cycles has odd prime length, so each is an even permutation, and so lies in the alternating group $A_n$. Let $g_v$ be the permutation corresponding to the vertex $v$ of $\Gamma$.

- If $v$ and $w$ are nonadjacent, then the supports of $g_v$ and $g_w$ have union strictly smaller than $\Omega$, so $\langle g_v, g_w \rangle \neq A_n$.
- Suppose $v$ and $w$ are adjacent. Then the supports of $g_v$ and $g_w$ have union $\Omega$, so $H = \langle g_v, g_w \rangle$ is transitive on $\Omega$. It is primitive: for each of $g_v$ and $g_w$ is a cycle of prime length $n - k > n/2$, and a block of imprimitivity either contains the cycle (and so has length greater than $n/2$, hence $n$) or meets it in one point (and so there are more than $n/2$ blocks, hence $n$ blocks). Hence $H$ is a primitive group of degree $n$ containing a cycle of prime length $p$ with $n/2 < p < n - 2$, By Jordan's theorem [74, Theorem 13.9], $H$ contains the alternating group $A_n$. Since it is generated by even permutations, $H = A_n$.

Thus we have embedded $\Gamma$ as an induced subgraph in the generating graph of $A_n$, as required. □

## 4.6 Differences

We can also ask which graphs can be embedded in the graph whose edge set is the difference of the edge sets of two graphs in the hierarchy.

The proof that enhanced power graphs are universal uses abelian groups for the embedding. So, by embedding the complement, it shows:

**Corollary 4.6** *Let $\Gamma$ be a finite graph. Then there is a group $G$ such that $\Gamma$ is isomorphic to an induced subgraph of* $\mathrm{Com} - \mathrm{EPow}(G)$.

However, a much stronger result is true:

**Theorem 4.7** *Let $\Gamma$ be a finite complete graph, whose edges are coloured red, green and blue in any manner. Then there is an embedding of $\Gamma$ into a finite group $G$ so that*

*(a) vertices joined by red edges are adjacent in the enhanced power graph;*

*(b) vertices joined by green edges are adjacent in the commuting graph but not in the enhanced power graph;*

*(c) vertices joined by blue edges are non-adjacent in the commuting graph.*

**Proof** We begin with two observations. First, the direct product of cyclic (resp. abelian) groups of coprime orders is cyclic (resp. abelian).

Second, consider the non-abelian group of order $p^3$ and exponent $p^2$, where $p$ is an odd prime:

$$P = \langle a, b \mid a^{p^2} = b^p = 1, [a, b] = a^p \rangle.$$

Any two elements of $\langle a \rangle$ generate a cyclic group; and the group generated by $b$ and $x$ is cyclic if $x = 1$, abelian but not cyclic if $x = a^p$, and non-abelian if $x = a$.

The proof is by induction on the number $n$ of vertices. The result is clearly true if $n = 1$. So let $\{v_1, \ldots, v_n\}$ be the vertex set of $\Gamma$, and suppose that we have an embedding of $\{v_1, \ldots, v_{n-1}\}$ into a group $G$ satisfying (a)–(c).

Choose an odd prime $p$ not dividing $|G|$, and consider the group $P \times G$, where $P$ is as above. Modify the embedding of the first $n - 1$ vertices by replacing $v_i$ by $(1, v_i)$ if $\{v_i, v_n\}$ is red, by $(a^p, v_i)$ if $\{v_i, v_n\}$ is green, and by $(a, v_i)$ if $\{v_i, v_n\}$ is blue. It is easily checked that we still have an embedding of $\{v_1, \ldots, v_n\}$ satisfying (a)–(c). Moreover, if we now embed $v_n$ as $(b, 1)$, we find that the conditions hold for the remaining pairs as well. $\square$

Clearly there are plenty of problems along similar lines to investigate here.

# 5 Producs

There are a number of graph products. Here I will be chiefly concerned with the strong product, defined as follows.

Let $\Gamma$ and $\Delta$ be graphs with vertex sets $V$ and $W$ respectively. The *strong product* $\Gamma \boxtimes \Delta$ has vertex set the Cartesian product $V \times W$; vertices $(v_1, w_1)$ and $(v_2, w_2)$ are joined whenever $v_1$ is equal or adjacent to $v_2$ and $w_1$ is equal or adjacent to $w_2$, but not equality in both places. (All of the graphs in the hierarchy naturally have loops at each vertex, which we have discarded; the strong product is the natural categorical product in the category of graphs with a loop at each vertex.)

I note that the strong product, along with the Cartesian and categorical products, is denoted by a symbol representing the corresponding product of two edges: the Cartesian product is $\Gamma \mathbin{\square} \Delta$, while the categorical product is $\Gamma \times \Delta$.

The only group product that concerns us here is the direct product.

**Proposition 5.1** *Let $G$ and $H$ be finite groups.*

*(a)* $\mathrm{Com}(G \times H) = \mathrm{Com}(G) \boxtimes \mathrm{Com}(H)$.

*(b) If $G$ and $H$ have coprime orders, then* $\mathrm{EPow}(G \times H) = \mathrm{EPow}(G)$ *boxtimes* $\mathrm{EPow}(H)$.

*(c) If $G$ and $H$ are perfect (that is, equal to their derived groups), then* $\mathrm{DCom}(G \times H) = \mathrm{DCom}(G) \boxtimes \mathrm{DCom}(H)$.

**Proof** (a) Distinct elements $(g_1, h_1)$ and $(g_2, h_2)$ in $G \times H$ commute if and only if $g_1$ and $g_2$ are equal or commute, and $h_1, h_2$ are equal or commute.

(b) Suppose that $|G|$ and $H$ are coprime. If $\langle g_1, g_2 \rangle$ and $\langle h_1, h_2 \rangle$ are cyclic, then (as their orders are coprime) their direct product is also cyclic and contains $(g_1, h_1)$ and $(g_2, h_2)$. Conversely, again using coprimeness, if $\langle (g_1, h_1), (g_2, h_2) \rangle$ is cyclic, then it contains $(g_1, 1)$, $(g_2, 1)$, $(1, h_1)$ and $(1, h_2)$.

(c) A formula of Schur gives the Schur multiplier of $G$ and $H$ to be $M(G) \times M(H) \times (G \otimes H)$, where $M(G)$ is the Schur multiplier of $G$. If $G$ and $H$ are perfect, the third term is absent. It follows that a Schur cover of $G \times H$ is the direct product of Schur covers of $G$ and $H$. The result follows. $\square$

Thus, questions about the commuting graph or enhanced power graph of a nilpotent group can be reduced to questions about the corresponding graphs for their Sylow subgroups.

The corresponding result fails for the power graph and the non-generating graph. The power graphs of $C_2$ and $C_3$ are complete but the power graph of $C_2 \times C_3$ are not. For the non-generating graph, we note that for any non-abelian finite simple group $G$, there is an integer $m$ such that $G^n$ fails to be 2-generated if $n > m$.

# 6 Cographs and twin reduction

Two vertices in a graph are called twins if they have the same neighbours (possibly excluding one another). Equivalently, $v$ and $w$ are twins if the transposition $(v, w)$ (fixing the other vertices) is an automorphism of $\Gamma$. If $G$ is a non-trivial group, then any of the graphs in our hierarchy based on $G$ will contain many pairs of twins. Thus, twins will play an important part when we come to look at automorphism groups.

If a graph has twins, then we can make a new graph by merging the twins to a single vertex. The process can be continued until no pairs of twins remain. If the resulting graph has just a single vertex, the original graph is called a cograph.

Cographs also play an important part in the story, and make another link with the Gruenberg–Kegel graph. So we make a detour to look at twin reduction and cographs.

A graph $\Gamma$ is a *cograph* if either of the following equivalent conditions holds for it:

- $\Gamma$ does not contain the four-vertex path $P_4$ as an induced subgraph;
- $\Gamma$ can be constructed from the 1-vertex graph by the operations of complement and disjoint union.

In particular, a cograph is connected if and only if its complement is disconnected. This leads to a tree representation of cographs and to very efficient algorithms for determining their properties.

Cographs have been rediscovered a number of times, and as a result appear in the literature with very different names, such as "complement-reducible graphs", "hereditary Dacey graphs" and "N-free graphs". See [68, 72, 51] for information about cographs.

In a graph $\Gamma$, we can define two kinds of "twin relations" on vertices. The *open neighbourhood* $\Gamma(v)$ of $v$ in $\Gamma$ is the set of vertices in $\Gamma$ joined to $v$; the *closed neighbourhood* is $\Gamma(v) \cup \{v\}$. Two vertices $v, w$ are *open twins* if they have the same open neighbourhoods; they are *closed twins* if they have the same closed neighbourhoods. Both of these relations are obviously equivalence relations. Two vertices are open twins in $\Gamma$ if and only if they are closed twins in the complement of $\Gamma$. Note that open twins are not joined, while closed twins are joined.

For either of these relations, we can define a *reduced graph* by collapsing each equivalence class to a single vertex. The original graph can be reconstructed uniquely from the reduced graph and the partition into equivalence classes. We call the two reductions *open twin reduction* and *closed twin reduction* respectively. In the graph obtained by open twin reduction, the open twin relation is the relation of equality, and similarly for the closed relation. So, given any graph, we can apply the two reductions alternately until the resulting graph has both twin relations equal to the relation of identity. We call such a graph the *cokernel* of the original graph, for reasons which are made clear by the following result. Note that, in the most general form of twin reduction, each move simply identifies one pair of twins, and these can be open or closed twins arbitrarily.

**Theorem 6.1** *Given a graph $\Gamma$, the result of performing a sequence of twin reductions until the graph is twin-free is unique up to isomorphism, independent of the chosen sequence.*

**Proof** Open and closed twin classes of sizes greater than 1 are disjoint. For suppose that $x$ and $y$ are open twins and $y$ and $z$ are closed twins. Then $xy$ is a non-edge while $yz$ is an edge. Since $y$ and $z$ are twins, $x$ is not joined to $z$; but since $y$ and $x$ are twins, $z$ is joined to $x$. These conclusions are contradictory. (Alternatively, since $y$ and $z$ are twins, $(y, z)$ is an automorphism, and so $x$ and $z$ are open twins; and similarly using the automorphism $(x, y)$, $x$ and $z$ are closed twins.)

We are going to prove the theorem by induction on the number of vertices. There is nothing to do for graphs with a single vertex, so let $\Gamma$ have $n$ vertices, with $n > 1$, and assume that the result is true for any graph with fewer than $n$ vertices. Take two twin reduction sequences on $\Gamma$. Suppose that the first begins by identifying $x$ and $y$, and the second by identifying $u$ and $v$.

If $\{x, y\} = \{u, v\}$, then the two sequences result in the same graph, and induction finishes the job.

If $|\{x, y\} \cap \{u, v\}| = 1$, then our initial remark shows that the two pairs of twins have the same type, so the graphs obtained after one step are isomorphic, and again induction finishes the job.

Suppose that $\{x, y\} \cap \{u, v\} = \emptyset$. Then the two reductions commute. Let $\Delta$ be the graph obtained by applying the two reductions. Then $\Delta$ occurs after two steps in reduction sequences for $\Gamma$ beginning by identifying $x$ and $y$, or by identifying $u$ and $v$. By induction the end result of either given sequence is the same as the result of reducing $\Delta$ (up to isomorphism).

The theorem is proved. $\square$

The next result gives the connection between cographs and twin reduction.

**Proposition 6.2** *A graph $\Gamma$ is a cograph if and only if the cokernel of $\Gamma$ is the graph with a single vertex.*

**Proof** For the necessity, we show by induction that a cograph with more than one vertex contains twins. Let $\Gamma$ be a cograph with more than one vertex, and suppose that any smaller cograph with more than one vertex contains twins. If $\Gamma$ is disconnected, and has a component with more than one vertex, then this component contains twins; otherwise $\Gamma$ is a null graph and all pairs of vertices are open twins. If $\Gamma$ is connected, then its complement is disconnected, and we argue in the complement instead.

Now the result of twin reduction is an induced subgraph of $\Gamma$, and so also a cograph; so so the reduction continues until only one vertex remains.

Conversely, suppose that $\Gamma$ is not a cograph. Then $\Gamma$ contains a 4-vertex path, say $(w, x, y, z)$. Then any pair of these vertices are not twins, and so are not identified in any twin reduction; so the result of the reduction still contains a 4-vertex path. So no sequence of reductions can terminate in a single vertex. $\square$

This gives another test for a cograph: apply twin reductions until the process terminates, and see whether just one vertex remains.

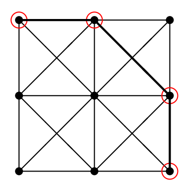
The relevance of this to our problem is:

**Proposition 6.3** *Let $\Gamma$ be the power graph, enhanced power graph, deep commuting graph, commuting graph, or non-generating graph of a non-trivial group $G$. Then the closed twin relation on $\Gamma$ is not the relation of equality.*

**Proof** Suppose that $G$ contains an element $g$ of order greater than 2. Let $h$ be an element such that $g \neq h$ but $\langle g \rangle = \langle h \rangle$. Then any element joined to one of $g$ and $h$ in one of the graphs listed is also joined to the other. The arguments are all easy; let us look at the least trivial, the deep commuting graph. Let $H$ be a Schur cover of $G$ with kernel $Z$, and $x$ and $y$ elements of $H$ covering $g$ and $h$ respectively. Then $\langle Z, x \rangle$ is abelian and contains $y$, so $x$ and $y$ commute.

The groups not covered by this are elementary abelian groups. In these cases, everything is clear: the power graph, enhanced power graph and deep commuting graph are stars; the commuting graph is complete; the non-generating graph is complete if the group has order greater than 4. All these graphs are cographs. $\square$

Finally for this section, I note that the class of cographs is not preserved by strong product. The picture shows $P_4$ as an induced subgraph of $P_3 \boxtimes P_3$.



# 7 Forbidden subgraphs

In this section we consider various classes of graphs defined by forbidden induced subgraphs, and ask: for which finite groups $G$ can one of the graphs in the hierarchy defined on $G$ belong to this class? Not much is known.

One of the most important graph classes is the class of perfect graphs. A graph $\Gamma$ is perfect if every induced subgraph has clique number equal to chromatic number. According to the Strong Perfect Graph Theorem [32], (confirming a conjecture by Claude Berge), a graph is perfect if and only if it has no induced subgraph which is isomorphic to an odd cycle of length greater than 3 or the complement of one. In particular, a cograph (which contains no induced 4-vertex path) is perfect.

We have seen that power graphs are comparability graphs of partial orders. Now every graph which is the comparability graph of a partial order is perfect, by (the easy direction of) Dilworth's Theorem. However, as we saw in the preceding section, for our other types of graph, no proper subclass defined by forbidden induced subgraphs contains them all; so we have to ask a different question: which groups have the property that one of these graphs belongs to a graph class defined in this way?

## 7.1 Power graphs

We begin with the question: When is the power graph of the group $G$ a cograph? This question was considered in the paper [30]; here the Gruenberg–Kegel graph makes another appearance. The question is answered completely for nilpotent groups, but a necessary and a sufficient condition are known for general groups in terms of the GK graph; these conditions do not coincide, and we will see that no condition just in terms of the GK graph can be both necessary and sufficient.

**Theorem 7.1** *Let $G$ be a finite nilpotent group. Then $\mathrm{Pow}(G)$ is a cograph if and only if either*

*(a) $G$ has prime power order; or*

*(b) $G$ is cyclic of order $pq$, where $p$ and $q$ are distinct primes.*

**Theorem 7.2** *(a) Let $G$ be a finite group whose Gruenberg–Kegel graph is a null graph. Then the power graph of $G$ is a cograph.*

*(b) Let $G$ be a finite group whose power graph is a cograph. Then, with possibly one exception, a connected component of the Gruenberg–Kegel graph has at most two vertices, the exception being the component containing the prime 2. If $\{p, q\}$ is a connected component of the GK graph, with $p$ and $q$ odd primes, then $p$ and $q$ divide $|G|$ to the first power only.*

For the final claim, note that the groups $\mathrm{PSL}(2, 11)$ and $M_{11}$ have the same GK-graph (an edge $\{2, 3\}$ and isolated vertices $\{5\}$ and $\{11\}$), but the power graph of the first is a cograph, that of the second is not.

**Question 8** Classify the groups whose power graph is a cograph.

The remainder of this subsection is based on the paper [30].

We saw that the power graph of a finite group is a comparability graph of a partial order, and so in particular is a perfect graph. Various interesting subclasses of the perfect graphs are defined by forbidding certain induced subgraphs. Cographs form an example: they forbid the 4-vertex path $P_4$. Here are some other graph classes.

- A graph is *chordal* if it contains no induced cycles of length greater than 3 (that is, every cycle of length greater than 3 has a chord).
- A graph is *split* if the vertex set is the disjoint union of two subsets, one inducing a complete graph and the other a null graph (with possibly some edges between them). A graph is split if it contains no induced subgraph isomorphic to $C_4$, $C_5$ or $2K_2$.
- A graph is *threshold* if it can be constructed from the 1-vertex graph by adding vertices joined either to all or to no existing vertices. A graph is threshold if and only if it contains no induced subgraph isomorphic to $P_4$, $C_4$, or $2K_2$.

**Theorem 7.3** *For a finite nilpotent group $G$, the power graph of $G$ is chordal if and only if one of the following conditions holds:*

*(a) $G$ has prime power order;*

*(b) $G$ is the direct product of a cyclic group of $p$-power order and a group of exponent $q$, where $p$ and $q$ are distinct primes.*

**Question 9** Which non-nilpotent groups have the property that the power graph is chordal?

**Theorem 7.4** *The following conditions for a finite group are equivalent:*

*(a)* $\mathrm{Pow}(G)$ *is a threshold graph;*

*(b)* $\mathrm{Pow}(G)$ *is a split graph;*

*(c)* $\mathrm{Pow}(G)$ *contains no induced subgraph isomorphic to $2K_2$;*

*(d) $G$ is cyclic of prime power order, or an elementary abelian or dihedral 2-group, or cyclic or order $2p$, or dihedral of order $2p^n$ or $4p$, where $p$ is an odd prime.*

Note that this theorem does not assume that $G$ is nilpotent.

## 7.2 Other graphs

For our other classes of graphs, the problem of deciding for which groups the graph in question forbids a certain induced subgraph has not been much worked on, and a number of interesting questions are open. One difference is that, as we have seen, every finite graph occurs as an induced subgraph of each type of graph on some finite group.

In line with the preceding subsection we could ask a multipart question:

**Question 10** For which finite groups is the enhanced power graph/deep commuting graph/commuting graph/nongenerating graph a perfect graph, or a cograph, or a chordal graph, or a split graph, or a threshold graph?

We can give a partial answer for groups of prime power order.

**Theorem 7.5** *Let $G$ be a group of prime power order.*

*(a) The power graph of $G$ is equal to the enhanced power graph, and contains no induced $P_4$ or $C_4$.*

*(b) If $G$ is 2-generated of order $p^n$, then the non-generating graph of $G$ consists of $p+1$ complete subgraphs of order $p^{n-1}$, any two intersecting in the same subset of size $p^{n-2}$.*

**Proof** (a) In a group of prime power order, if two elements generate a cyclic subgroup, then one is a power of the other; this shows that the power graph and enhanced power graph coincide. Suppose that $(x, y, z)$ is an induced path of length 2. If $x, z \in \langle y \rangle$, or if $y \in \langle x \rangle$ and $z \in \langle y \rangle$, or if $x \in \langle y \rangle$, $y \in \langle z \rangle$, then $x$ and $z$ are joined, a contradiction. So $y \in \langle x \rangle \cap \langle z \rangle$. Now if $w$ is joined to $x$ but not to $y$, then $(w, x, y)$ is an induced path of length 2, so $x \in \langle y \rangle$ by the same argument. But then $y \in \langle z \rangle$, so $x \in \langle z \rangle$, a contradiction.

(b) By the Burnside Basis Theorem, $\Phi(G)$ consists of vertices lying in no generating pair and has index $p^2$ in $G$; moreover, the generating pairs are all pairs of elements lying in distinct non-trivial cosets of $\Phi(G)$. $\square$

No such result holds for the commuting graph; Theorem 4.1 shows that the commuting graphs of 2-groups form a universal class. Indeed, computation shows that the smallest group whose commuting graph is not a cograph is the symmetric group $S_4$: the elements $(1, 2, 3, 4)$, $(1, 3)(2, 4)$, $(1, 2)(3, 4)$

and $(1, 3, 2, 4)$ induce a 4-vertex path. In fact, seven groups of order 32 have commuting graphs which are not cographs.

Question 10 has been considered for the commuting graph by Britnell and Gill [21], who obtained a partial description of groups for which the commuting graph is a perfect graph. Assuming that $G$ has a component (a subnormal quasisimple subgroup), they determine all possible components of such groups.

**Question 11** What about other graph classes, for example planar graphs?

For planarity, this may not be too hard for graphs in the hierarchy. Since the complete graph on 5 vertices is not planar, it follows that $G$ has no elements of order greater than 4. (For all cases except the power graph, a cyclic subgroup induces a complete graph, so the claim is clear. In the power graph, the power graph of a cyclic group of prime power order is complete, and the power graph of $C_6$ contains a $K_5$.) So $G$ is solvable.

Indeed, bounding the genus of a graph bounds its clique number, and so (for graphs in the hierarchy) bounds the orders of elements.

See [5] for the commuting graph and its complement.

## 7.3 Results for simple groups

**Proposition 7.6** *Let $G = \mathrm{PSL}(2, q)$, with $q$ a prime power and $q \geq 4$.*

*(a) If $q$ is even, then $\mathrm{EPow}(G)$, $\mathrm{DCom}(G)$ and $\mathrm{Com}(G)$ are cographs; $\mathrm{Pow}(G)$ is a cograph if and only if $q - 1$ and $q + 1$ are either prime powers or products of two distinct primes.*

*(b) If $q$ is odd, then $\mathrm{EPow}(G)$ and $\mathrm{DCom}(G)$ are cographs; $\mathrm{Pow}(G)$ is a cograph if and only if $(q - 1)/2$ and $(q + 1)/2$ are either prime powers or products of two distinct primes.*

**Proof** We consider the graphs obtained by removing the identity. We note that a graph $\Gamma$ is a cograph if and only if the graph obtained by adding a vertex joined to all others is a cograph.

We begin with the case when $q$ is a power of 2, noting that in this case $\mathrm{PSL}(2, q)$ has trivial Schur multiplier except when $q = 4$. The elements of the group have order 2 or prime divisors of $q - 1$ or $q + 1$. The centralisers of involutions are elementary abelian of order $q$, while the centralisers of other elements are cyclic of order $q-1$ and $q+1$. In other words, centralisers are abelian and intersect only in the identity. So the commuting graph (which is the deep commuting graph if $q \neq 4$) is a disjoint union of complete graphs; the enhanced power graph is a disjoint union of complete graphs and isolated vertices (corresponding to the involutions); and the power graph is a disjoint union of the power graphs of cyclic groups of orders $q \pm 1$ and isolated vertices. Using the fact that the power graph of the cyclic group $C_m$ is a cograph if and only if $m$ is either a prime power or the product of two primes [30, Theorem 3.2], the result follows. The result for the deep commuting graph for $\mathrm{PSL}(2,4)$ can be proved by computation, or by identifying this group with $\mathrm{PSL}(2,5)$ (which is dealt with in the next paragraph).

Now consider the case when $q$ is a power of an odd prime $p$. The centralisers of elements of order $p$ are elementary abelian of order $q$; centralisers of other elements are cyclic of orders $(q \pm 1)/2$ except for involutions, which are centralised by dihedral groups of order $q \pm 1$, whichever is divisible by 4. So, although centralisers may not be disjoint, the maximal cyclic subgroups are, so the statements about the enhanced power graph and the commuting graph are true.

If $q$ is odd and $q \neq 9$, the Schur multiplier of $\mathrm{PSL}(2,q)$ is cyclic of order 2, and so a Schur cover of this group is $\mathrm{SL}(2,q)$. The unique involution in this group is $-I$, the kernel of the extension; so involutions in $\mathrm{PSL}(2,q)$ lift to elements of order 4, which lie in unique maximal abelian subgroups.

The group $\mathrm{PSL}(2,9)$ has Schur multiplier of order 6. The deep commuting graph of $\mathrm{PSL}(2,9)$ was analysed using GAP, with generators for the Schur cover from the on-line Atlas of Finite Group Representations [76]. $\square$

**Question 12** Are there infinitely many prime powers $q$ for which the power graph of $\mathrm{PSL}(2,q)$ is a cograph?

Here is a preliminary analysis of this question.

**Case $q$ even** Then $q = 2^d$, say, and each of $q+1$ and $q-1$ is either a prime power or the product of two primes. By Catalan's conjecture (now Mihăilescu's Theorem: see [33, Section 6.11]), the only two proper powers differing by 1 are 8 and 9. So, unless $q = 8$, we conclude that each of $q+1$ and $q-1$ is either prime or the product of two primes. Moreover, one of these numbers is divisible by 3, so we can say further that one of $q+1$ and $q-1$ is three times a prime, while the other is a prime or the product of two primes. For example, $2^{11} - 1 = 23 \cdot 89$ while $2^{11} + 1 = 3 \cdot 683$. The values of $d$ up to 200 for which $q = 2^d$ satisfies the condition are 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 31, 61, 101, 127, 167, 199.

**Case $q$ odd** If $q$ is congruent to $\pm 1 \pmod 8$, then one of $(q+1)/2$ and $(q-1)/2$ is divisible by 4, and so must be a power of 2; so either $q = 9$, or $q$ is a Fermat or Mersenne prime. So, apart from this case, $q$ is congruent to $\pm 3 \pmod 8$. Now either $q$ is an odd power of 3, or one of $(q+1)/2$ and $(q-1)/2$ is twice a prime, while the other is three times a prime or a power of 3 (unless $q = 11$ or $q = 13$). The odd prime powers up to 500 satisfying the condition are 3, 5, 7, 9, 11, 13, 17, 19, 27, 29, 31, 43, 53, 67, 163, 173, 243, 257, 283, 317.

Table 1 gives the numbers of vertices in cokernels for small finite simple groups. Note that a graph is a cograph if and only if its cokernel has one vertex.

| $G$ | $\lvert G\rvert$ | $\mathrm{Pow}(G)$ | $\mathrm{EPow}(G)$ | $\mathrm{DCom}(G)$ | $\mathrm{Com}(G)$ | $\mathrm{NGen}(G)$ |
|---|---|---|---|---|---|---|
| $A_5$ | 60 | 1 | 1 | 1 | 1 | 32 |
| $\mathrm{PSL}(2,7)$ | 168 | 1 | 1 | 1 | 44 | 79 |
| $A_6$ | 360 | 1 | 1 | 1 | 92 | 167 |
| $\mathrm{PSL}(2,8)$ | 504 | 1 | 1 | 1 | 1 | 128 |
| $\mathrm{PSL}(2,11)$ | 660 | 1 | 1 | 1 | 112 | 244 |
| $\mathrm{PSL}(2,13)$ | 1092 | 1 | 1 | 1 | 184 | 366 |
| $\mathrm{PSL}(2,17)$ | 2448 | 1 | 1 | 1 | 308 | 750 |
| $A_7$ | 2520 | 352 | 352 | 352 | 352 | 842 |
| $\mathrm{PSL}(2,19)$ | 3420 | 1 | 1 | 1 | 344 | 914 |
| $\mathrm{PSL}(2,16)$ | 4080 | 1 | 1 | 1 | 1 | 784 |
| $\mathrm{PSL}(3,3)$ | 5616 | 756 | 756 | 808 | 808 | 1562 |
| $\mathrm{PSU}(3,3)$ | 6048 | 786 | 534 | 499 | 499 | 1346 |
| $\mathrm{PSL}(2,23)$ | 6072 | 1267 | 1 | 1 | 508 | 1313 |
| $\mathrm{PSL}(2,25)$ | 7800 | 1627 | 1 | 1 | 652 | 1757 |
| $M_{11}$ | 7920 | 1212 | 1212 | 1212 | 1212 | 2444 |

Table 1: Sizes of cokernels of graphs on small simple groups

The table suggests various conjectures, some of which can be proved. For example:

**Theorem 7.7** *Let $G$ be a non-abelian finite simple group. Then* $\mathrm{NGen}(G)$ *is not a cograph.*

**Proof** We consider the reduced graph obtained by deleting the identity vertex. The reduced non-generating graph of a simple group is connected, and has diameter at most 6 (this follows from the results of Ma, Herzog *et al.*, Shen and Freedman discussed in Section 11 below). Also, the generating graph is connected (see [20]), and indeed has diameter 2 (see [22]). But the complement of a connected cograph is disconnected.

**Question 13** Find, or estimate, the number of vertices in the cokernel of the non-generating graph of a finite simple group.

The hypothesis of simplicity is essential here. For example, the non-generating graph of any 2-generator $p$-group is a cograph (it consists of a star $K_{1,p+1}$ with the central vertex blown up to a clique of size $p^{n-2}$ and the remaining vertices to cliques of size $p^{n-2}(p-1)$, where the group has order $p^n$). This graph is a cograph: its complement is complete multipartite with some isolated vertices.

# 8 Connectedness

In this section, we examine the question of connectedness of these graphs. All those in the hierarchy (except the null graph) are connected, since the identity is joined to all other vertices. The question is non-trivial, however, if we remove vertices joined to all others. The first job is to characterise these vertices.

## 8.1 Centres

In the commuting graph of $G$, the set of vertices joined to all others is simply the centre $Z(G)$ of $G$. So we adapt the terminology by defining analogues of the centre for other graphs in the list. So, if X denotes Pow, EPow, DCom, Com or NGen, we define the $X$-centre of $G$, denoted $Z_{\mathrm{X}}(G)$, to be the set of vertices joined to all others in $\mathrm{X}(G)$. It turns out that (aside from the non-commuting graph), in almost all cases, $Z_{\mathrm{X}}(G)$ is a normal subgroup of $G$; the only exception is for the power graph of a cyclic group of non-prime-power order.

**Theorem 8.1** *(a)* $Z_{\mathrm{Pow}}(G)$ *is equal to* $G$ *if* $G$ *is cyclic of prime power order; or the set consisting of the identity and the generators if* $G$ *is cyclic of non-prime-power order; or* $Z(G)$ *if* $G$ *is a generalized quaternion group; or* $\{1\}$ *otherwise.*

*(b)* $Z_{\mathrm{EPow}}(G)$ *is the product of the Sylow* $p$*-subgroups of* $Z(G)$ *for* $p \in \pi$*, where* $\pi$ *is the set of primes* $p$ *for which the Sylow* $p$*-subgroup of* $G$ *is cyclic or generalized quaternion; in particular,* $Z_{\mathrm{EPow}}(G)$ *is cyclic.*

*(c)* $Z_{\mathrm{DCom}}(G)$ *is the projection into* $G$ *of* $Z(H)$*, where* $H$ *is a Schur cover of* $G$*.*

*(d)* $Z_{\mathrm{Com}(G)} = Z(G)$*.*

**Proof** (a) This is [24, Proposition 4].

(b) Suppose that $x$ is joined to all other vertices in $\mathrm{EPow}(G)$. Then $\langle x, y\rangle$ is cyclic for all $y \in G$; so certainly $x \in Z(G)$.

If three elements of a group have the property that any two of them generate a cyclic group, then all three generate a cyclic group: see [1, Lemma 32]. So $Z_{\mathrm{EPow}}(G)$ is a subgroup, since if $x, y \in Z_{\mathrm{EPow}}(G)$ then, for all $w \in G$, $\langle x, w\rangle$ and $\langle y, w\rangle$ are cyclic, and so if $\langle x, y\rangle = \langle z\rangle$ then $\langle z, w\rangle$ is cyclic for all $w \in G$, so that $z \in Z_{\mathrm{EPow}}(G)$.

Now let $x$ be an element of prime order $p$ in $Z_{\mathrm{EPow}}(G)$. If $G$ contains a subgroup $C_p \times C_p$ then there is an element of order $p$ not in $\langle x\rangle$, so not adjacent to $x$, a contradiction. So the Sylow $p$ subgroup of $G$ is cyclic or generalised quaternion, by Burnside's theorem. But now $x$ lies in every Sylow $p$-subgroup of $G$, so is joined to every element of $p$-power order, and hence to every element of $G$.

(c) and (d) Part (d) is clear, and (c) follows since $\mathrm{DCom}(G)$ is a projection of the commuting graph of a Schur cover of $G$. $\square$

By contrast, $Z_{\mathrm{NGen}}(G)$ is not necessarily a subgroup of the 2-generated group $G$. If $G$ is non-abelian, then $Z_{\mathrm{NGen}}(G)$ must contain $Z(G)$, since the non-generating graph contains the commuting graph. Also, it contains the Frattini subgroup $\Phi(G)$ of $G$, since 2-element generating sets are minimal and so their elements do not lie in the Frattini subgroup (which consists of the elements which can be dropped from any generating set).

If the order of $G$ is a prime power, then the Burnside basis theorem shows that $Z_{\mathrm{NGen}}(G) = \Phi(G)$, since a set of elements generates $G$ if and only if its projection onto $G/\Phi(G)$ generates this quotient.

Also, by the result of [20] (see Theorem 2.3), if all proper quotients of $G$ are cyclic, then $Z_{\mathrm{NGen}}(G) = \{1\}$.

In general, however, $Z_{\mathrm{NGen}}(G)$ may be a subgroup different from both $Z(G)$ and $\Phi(G)$. For example, let $G$ be the symmetric group $S_4$. Then both $Z(G)$ and $\Phi(G)$ are trivial, but $Z_{\mathrm{NGen}}(G)$ is the Klein group $V_4$ (the minimal normal subgroup of $G$).

Moreover, it may not be a subgroup at all. For example, if $G = C_6 \times C_6$, then $Z_{\mathrm{NGen}}(G)$ consists of the elements not of order 6, since both elements in any generating pair must have order 6.

**Question 14** Characterise the 2-generated groups in which $Z_{\mathrm{NGen}}(G)$ is a subgroup of $G$.

## 8.2 Connectedness

Each of our types of graph is connected, since the corresponding "centre" is non-empty and its vertices are joined to all others. So the question becomes interesting if we ask whether the induced subgraph on the elements outside this centre is connected.

The situation for the commuting graph is well-understood, thanks to the results of [38, 62]. But first I mention another link with the Gruenberg–Kegel graph. This has been known for some time, but the first mention I know in the literature is [62, Section 3].

**Theorem 8.2** *Let $G$ be a group with trivial centre. Then the induced subgraph of the commuting graph on $G \setminus \{1\}$ is connected if and only if the Gruenberg–Kegel graph is connected.*

**Proof** Suppose first that $Z(G) = 1$ and the commuting graph is connected. Let $p$ and $q$ be primes dividing $|G|$. Choose elements $g$ and $h$ of orders $p$ and $q$ respectively, and suppose their distance in the commuting graph is $d$. We show by induction on $d$ that there is a path from $p$ to $q$ in the GK graph.

If $d = 1$, then $g$ and $h$ commute, so $gh$ has order $pq$, and $p$ is joined to $q$. So assume the result for distances less than $d$, and let $g = g_0, \ldots, g_d = h$ be a path from $g$ to $h$.

Let $i$ be mimimal such that $p$ does not divide the order of $g_i$ (so $i > 0$). Now some power of $g_{i-1}$, say $g_{i-1}^a$, has order $p$, while a power $g_i^b$ of $g_i$ has prime order $r \neq p$.

The distance from $g_i^b$ to $g_d$ is at most $d - i < d$, so there is a path from $r$ to $q$ in the GK graph. But $g_{i-1}^a$ and $g_i^b$ commute, so $p$ is joined to $r$.

For the converse, assume that the GK graph is connected.

Note first that for every non-identity element $g$, some power of $g$ has prime order, so it suffices to show that all elements of prime order lie in the same connected component of the commuting graph. Also, since a non-trivial $p$-group has non-trivial centre, the non-identity elements of any Sylow subgroup lie in a single connected component.

Let $C$ be a connected component. Connectedness of the GK graph shows that $C$ contains a Sylow $p$-subgroup for every prime $p$ dividing $|G|$. Also, every element of $C$, acting by conjugation, fixes $C$. It follows that the normaliser of $C$ is $G$, and hence that $C$ contains every Sylow subgroup of $G$, and thus contains all elements of prime order, as required. $\square$

The main results are the following.

**Theorem 8.3 (Giudici and Parker)** *There is no upper bound for the diameter of the commuting graph of a finite group; for any given $d$ there is a 2-group whose commuting graph is connected with diameter greater than $d$.*

On the other hand:

**Theorem 8.4 (Morgan and Parker)** *Suppose that the finite group $G$ has trivial centre. Then every connected component of its commuting graph has diameter at most 10.*

For the power graph and enhanced power graph, we note that, if the group $G$ is not cyclic or generalized quaternion, then the corresponding "centre" is just the identity. So the natural question is: if $G$ is not cyclic or generalized quaternion, is the induced subgraph of the power graph on non-identity elements connected? This question has been considered in several papers, for example [27, 79].

The next result shows that we have only one rather than two problems to consider.

**Proposition 8.5** *Let $G$ be a group with $Z(G) = \{1\}$. Then the reduced power graph of $G$ is connected if and only if the reduced enhanced power graph of $G$ is connected.*

**Proof** If $g$ and $h$ are joined in the power graph, they are joined in the enhanced power graph; if they are joined in the enhanced power graph, then they lie at distance at most 2 in the power graph; both $g$ and $h$ are powers of the intermediate vertex, which is thus not the identity. $\square$

The argument shows that, if these graphs are connected, the diameter of the power graph is at least as great, and at most twice, the diameter of the enhanced power graph. Can these bounds be improved?

I have already quoted the result of Burness *et al.* on the generating graph. For the non-generating graph, the results of Freedman and others on the difference between the non-generating graph and the commuting graph (that is, the graph $(\mathrm{NGen}-\mathrm{Com})G$) have been mentioned also.

# 9 Automorphisms

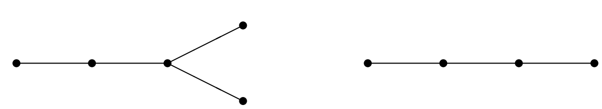
Each type of graph in the hierarchy on a group $G$ is preserved by the automorphism group of $G$. But in almost all cases, the automorphism group of the graph is much larger. This question has been considered in [8].

We saw in Proposition 6.3 that any of our hierarchy of graphs has non-trivial twin relation. So the first thing we need to do is to take a look at automorphisms of such graphs.

Let $\Gamma$ be a graph. It is clear that its automorphism group $\mathrm{Aut}(\Gamma)$ preserves twin relations on $\Gamma$, and that vertices in a twin equivalence class can be permuted arbitrarily. It follows by induction that the group induces an automorphism group on the cokernel $\Gamma^*$ of $\Gamma$, say $\mathrm{Aut}^-(\Gamma^*)$. We say that the twin reduction on $\Gamma$ is *faithful* if $\mathrm{Aut}^-(\Gamma^*) = \mathrm{Aut}(\Gamma^*)$.

Trivially, if $\Gamma$ is a cograph (so that its cokernel is the 1-vertex graph), the twin reduction is faithful; we ignore this case.

The reduction process is not always faithful. For a simple example, consider the following pair of graphs:



In the left-hand graph, the two leaves on the right are twins, and twin reduction gives the right-hand graph as the cokernel. But the cokernel has an automorphism (reflection in the vertical axis of symmetry) not induced from an automorphism of the original.

**Question 15** Given a finite group $G$ and one of our types of graph (say X),

(a) When is twin reduction on $\mathrm{X}(G)$ faithful?

(b) What is the automorphism group of the cokernel of $\mathrm{X}(G)$?

Very little seems to be known about this question. I first discuss cographs, then give a couple of examples.

**Proposition 9.1** *Let $\Gamma$ be a cograph. Then the automorphism group of $\Gamma$ can be built from the trivial group by the operations of direct product and wreath product with a symmetric group.*

**Proof** Recall that a cograph can be built from the 1-vertex graph by the operations of complement and disjoint union. Now complementation does not change the automorphism group. If $\Gamma$ is the disjoint union of $m_1$ copies of $\Delta_1$, ..., $m_r$ copies of $\Delta_r$, then

$$\mathrm{Aut}(\Gamma) = (\mathrm{Aut}(\Delta_1) \wr S_{m_1}) \times \cdots \times (\mathrm{Aut}(\Delta_r) \wr S_{m_r}).$$

Assuming inductively that each of $\mathrm{Aut}(\Delta_1)$, ..., $\mathrm{Aut}(\Delta_r)$ can be built by direct producs and wreath products with symmetric groups; then the same is true for $\mathrm{Aut}(\Gamma)$. $\square$

**Example** Let $G$ be the alternating group $A_5$, and consider the power graph of $G$. The identity is joined to all other vertices; after removing it we have six cliques of size 4 (corresponding to cyclic subgroups of order 5), ten of size 2 (corresponding to cyclic subgroups of order 3), and fifteen isolated vertices (corresponding to elements of order 2). This graph is easily seen to be a cograph, so its cokernel has a single vertex. In fact, closed twin reduction contracts the cliques of sizes 2 and 4 to single vertices, giving a star on 32 vertices; then open twin reduction produces a single edge, and closed twin reduction reduces this to a single vertex.

**Example** Let $G$ be the Mathieu group $M_{11}$. The power graph of $G$ has 7920 vertices. On removing the identity, we are left with a graph consisting of

- 144 complete graphs of size 10, corresponding to elements of order 11;

- 396 complete graphs of size 4, corresponding to elements of order 5;
- a single connected component $\Delta$ on the remaining 4895 vertices.

Two steps of twin reduction remove all the components which are complete. If we take $\Delta$, and first factor out the relation “same closed neighbourhood”, and then factor out from the result the relation “same open neighbourhood”, we obtain a connected graph on 1210 vertices whose automorphism group is $M_{11}$. (This is shown by a GAP computation.) This group is induced by the automorphism group of the original power graph; so the reduction is faithful.

**Exercise** Why is the number 1210 given above two less than the number of vertices of the cokernel of the power graph of $M_{11}$ given in Table 1?

**Question 16** For which non-abelian finite simple group $G$ is it the case that the twin reduction on the power graph/enhanced power graph/deep commuting graph/commuting graph/generating graph of $G$ is faithful?

**Example** A curious example showing that this is not true for all such groups is described in [29]. Let $G$ be the simple group $\mathrm{PSL}(2,16)$. The automorphism group of $G$ is the group $\mathrm{P\Gamma L}(2,16)$, four times as large as $G$; but the cokernel of the generating graph of $G$ has an extra automorphism of order 2, interchanging the sets of vertices coming from elements of orders 3 and 5 in $G$. Twin reduction of this graph is thus not faithful. The cokernel is a graph on 784 vertices with automorphism group $C_2 \times \mathrm{P\Gamma L}(2,16)$.

Non-faithfulness means, as in this example, that extra automorphisms are introduced by twin reduction.

# 10 Above the commuting graph

Given a subgroup-closed class of graphs $\mathcal{C}$, we can define a graph on $G$ in which $x$ and $y$ are joined if $\langle x, y\rangle$ belongs to $\mathcal{C}$.

For $\mathcal{C}$ the class of cyclic groups, we obtain the enhanced power graph; and for the class of abelian groups, we obtain the commuting graph.

The most natural classes to consider are those of nilpotent and solvable groups; let us denote the corresponding graphs by $\mathrm{Nilp}(G)$ and $\mathrm{Sol}(G)$ respectively.

A *Schmidt group* is a non-nilpotent group all of whose proper subgroups are nilpotent. These groups were characterised by Schmidt [65]; see [11] for an accessible account. All are 2-generated.

I do not know of a similar characterisation of the non-solvable groups all of whose proper subgroups are solvable. However, we can conclude that they are 2-generated, as follows. Let $G$ be such a group, and $S$ the solvable radical of $G$ (the largest solvable normal subgroup). Then $G/S$ is a non-abelian simple group. (For if $H/S$ is a minimal normal subgroup of $G/S$, then $H/S$ is a product of isomorphic simple groups, so $H$ is not solvable, and by minimality $H = G$.) Now every finite simple group is 2-generated. If we take two cosets $Sg, Sh$ which generate $G/S$, then $\langle g, h \rangle$ is a subgroup of $G$ which projects onto $G/S$, and so is non-solvable; by minimality it is equal to $G$. (In fact we do not need the Classification of Finite Simple Groups here. For clearly $G/S$ is a minimal simple group, and so is covered by Thompson's classification of N-groups [73].)

It follows that a group $G$ is nilpotent (resp. solvable) if and only if every 2-generated subgroup of $G$ is nilpotent (resp. solvable). For if every 2-generated subgroup of $G$ is nilpotent, then $G$ cannot contain a minimal non-nilpotent subgroup, and so $G$ is nilpotent; similarly for solvability.

**Proposition 10.1** *(a) For any finite group $G$, we have $E(\mathrm{Com}(G)) \subseteq E(\mathrm{Nilp}(G)) \subseteq E(\mathrm{Sol}(G))$.*

*(b) If $G$ is non-nilpotent, then $E(\mathrm{Nilp}(G)) \subseteq E(\mathrm{NGen}(G))$; equality holds if and only if $G$ is a Schmidt group.*

*(c) If $G$ is non-solvable, then $E(\mathrm{Sol}(G)) \subseteq E(\mathrm{NGen}(G))$; equality holds if and only if $G$ is a minimal non-solvable group.*

(The forward direction in the last two points uses the fact that these groups are 2-generated, as remarked above. For if $\mathrm{Nilp}(G)$ and $\mathrm{NGen}(G)$ have the same edges, then two elements which do not generate $G$ must generate a nilpotent group, and similarly for solvability.)

**Question 17** Examine the earlier results in this paper in the extended hierarchy of graphs containing $\mathrm{Nilp}(G)$ and $\mathrm{Sol}(G)$.

Other classes of groups for which the corresponding graphs could be studied, for which the minimal groups not in the class have been considered, include the supersolvable groups (those for which every composition factor is cyclic) and the $p$-nilpotent groups (groups with normal $p$-complements). The groups minimal with respect to not lying in these classes are considered in [10] and [11] respectively.

# 11 Intersection graphs

There turns out to be a close connection between certain intersection graphs defined on $G$, and some of the graphs in our hierarchy. First I look briefly at the connection in the abstract, then discuss some particular cases.

## 11.1 Dual pairs

Let B be a bipartite graph. If it is connected, it has a unique bipartition: take a vertex $v$; then the bipartite blocks are the sets of vertices at even (resp. odd) distance from $v$. If B is not connected, the bipartition is not unique; in fact, there are $2^{\kappa-1}$ bipartitions, where $\kappa$ is the number of connected components, since we can make a bipartite block by choosing a bipartite block in each component and taking their union. However, I will always assume that the bipartition of B is given, and is part of its structure.

The *halved graphs* arising from B are the graphs $\Gamma_1$ and $\Gamma_2$ whose vertex set is a bipartite block, two vertices adjacent in the relevant graph if and only if they lie at distance 2 in B.

We call a pair of graphs $\Gamma_1$ and $\Gamma_2$ a *dual pair* if there is a bipartite graph $\mathcal{B}$ without isolated vertices such that $\Gamma_1$ and $\Gamma_2$ are the halved graphs of B.

**Proposition 11.1** *Let $\Gamma_1$ and $\Gamma_2$ be a dual pair of graphs. Then $\Gamma_1$ is connected if and only if $\Gamma_2$ is connected. More generally, there is a natural bijection between connected components of $\Gamma_1$ and connected components of $\Gamma_2$ with the property that corresponding components have diameters which are either equal or differ by* 1.

**Proof** Any vertex of $\Gamma_1$ is joined (by an edge of B) to a vertex of $\Gamma_2$, and *vice versa*, since B has no isolated vertices. Now suppose that two vertices of $\Gamma_1$ are joined by a path of length $d$. Then there is a path of length $2d$ in B joining them. So a connected component of B is the union of corresponding connected components in $\Gamma_1$ and $\Gamma_2$. Suppose that a component of $\Gamma_1$ has diameter $d$. Take two vertices $v_1$, $v_2$ in the corresponding component of $\Gamma_2$.

Choose vertices $u_1$ and $u_2$ of $\Gamma_1$ joined in B to $v_1$ and $v_2$ respectively. These two vertices lie at distance $r \leq d$, say; so there is a path of length at most $2r$ in B joining them. Thus $v_1$ and $v_2$ have distance at most $2r+2$ in B, whence their distance in $\Gamma_2$ is at most $r+1$, hence at most $d+1$. So the diameter of a component of $\Gamma_2$ has diameter at most one more than the corresponding component of $\Gamma_1$. Interchanging the roles of the dual pair completes the proof. □

**Question 18** What other relations hold between properties of a dual pair of graphs?

## 11.2 Graphs on groups and intersection graphs

In order to apply this result, I give a general construction showing that certain graphs defined on the non-identity elements of a group form dual pairs with certain intersection graphs of families of subgroups.

**Proposition 11.2** *Let $G$ be a finite non-cyclic group, and let $\mathcal{F}$ be a family of non-trivial proper subgroups of $G$ with the property that its union is $G$. Let $\Gamma$ be the graph defined on the non-identity elements of $G$ by the rule that $x$ is joined to $y$ if and only if there is a subgroup $H \in \mathcal{F}$ with $x, y \in H$. Then $\Gamma$ and the intersection graph of $\mathcal{F}$ form a dual pair.*

**Proof** We form the bipartite graph B whose vertex set is $(G \setminus \{1\}) \cup \mathcal{F}$, where a group element $x \neq 1$ is joined to a subgroup $H \in \mathcal{F}$ if and only if $x \in H$. We verify the conditions for a dual pair.

First, B has no isolated vertices: for each subgroup in $\mathcal{F}$ is non-trivial, so contains an element of $G \setminus \{1\}$, and every such element is contained in a subgroup in $\mathcal{F}$, since the union of this family is $G$.

Next, two subgroups are joined in the intersection graph if and only if their intersection is non-trivial (that is, contains an element of $G \setminus \{1\}$; and, by assumption, two non-trivial elements are adjacent if and only if some element of $\mathcal{F}$ contains both.

Note that we have assumed that $G$ is non-cyclic; this in fact follows from the fact that it is a union of proper subgroups, since a generator would lie in no proper subgroup. □

## 11.3 Applications

I will consider several cases. I begin with the "classical" case, where the vertices are all the non-trivial proper subgroups of $G$, joined if two vertices are adjacent. These were first investigated by Csákány and Pollák, who considered non-simple groups; they determined the groups for which the intersection graph is connected and showed that, in these cases, its diameter is at most 4. For simple groups, Shen [69] showed that the graph is connected and asked for an upper bound; Herzog *et al.* [46] gave a bound of 64, which was improved to 28 by Ma [60], and to the best possible 5 by Freedman [35], who showed that the upper bound is attained only by the Baby Monster and some unitary groups (it is not currently known exactly which).

**Proposition 11.3** *Let $G$ be a non-cyclic finite group. Then the induced subgraph of the non-generating graph of $G$ on non-identity elements and the intersection graph of $G$ form a dual pair.*

**Proof** Take $\mathcal{F}$ to be the family of all non-trivial proper subgroups of $G$. $\square$

So the reduced non-generating graph of a non-abelian finite simple group has diameter at most 6.

Now we turn to the commuting graph.

**Proposition 11.4** *Let $G$ be a finite group with $Z(G) = 1$. Then the reduced commuting graph of $G$ (on the vertex set $G \setminus \{1\}$) and the intersection graph of non-trivial abelian subgroups of $G$ form a dual pair.*

**Proof** The condition $Z(G) = 1$ ensures that the reduced commuting graph does have vertex set $G \setminus \{1\}$, and also implies that $G$ is not cyclic. Take $\mathcal{F}$ to be the family of all non-trivial abelian subgroups of $G$ (all are proper subgroups since $G$ is not abelian). Two elements are joined in the reduced commuting graph if and only if the group they generate is abelian. $\square$

**Corollary 11.5** *For a finite group $G$ with $Z(G) = \{1\}$, the following four conditions are equivalent:*

*(a) the Gruenberg–Kegel graph of $G$ is connected;*

*(b) the reduced commuting graph of $G$ is connected;*

*(c) the intersection graph of non-trivial abelian subgroups of $G$ is connected;*

*(d) the intersection graph of maximal abelian subgroups of $G$ is connected.*

**Proof** The equivalence of (a) and (b) comes from Theorem 8.2, and that of (b) and (c) from Proposition 11.1. For the equivalence of (c) and (d), note that any non-trivial abelian subgroup is contained in a maximal abelian subgroup, to which it is joined, so (d) implies (c). The converse holds because any path in the intersection graph of non-trivial abelian subgroups can be lifted to a path in the intersection graph of maximal abelian subgroups. □

**Proposition 11.6** *Let $G$ be a group which is not cyclic or generalised quaternion. Then the induced subgraph of the enhanced power graph of $G$ on the set of non-identity elements and the intersection graph of non-trivial cyclic subgroups of $G$ form a dual pair.*

**Proof** We take $\mathcal{F}$ to be the family of non-trivial cyclic subgroups of $G$. □

# 12 More general graphs

When we think about graphs on groups, we want there to be some connection between the graph and the group. This connection is mostly exoressed in terms of invariance of the graph under something, either right translations or automorphisms of the group. The first gives rise to Cayley graphs; I will not discuss this, except to say that if a graph is invariant under both left and right translations then it is also invariant under inner automorphisms of the group. (This is naturally called a *normal Cayley graph* for the group, though unfortunately this term is also used for a completely different concept, namely a Cayley graph such that the right translations form a normal subgroup of the automorphism group of the graph.)

So the focus here is on graphs on a group $G$ invariant under the automorphism group $\mathrm{Aut}(G)$ of $G$. We have seen that all graphs in the hierarchy do satisfy this condition.

There are several ways we could approach the general case.

- Any graph invariant under $\mathrm{Aut}(G)$ is a union of orbital graphs for $\mathrm{Aut}(G)$.
- We could define the adjacency in the graph by a first-order formula with two free variables.

- We could define adjacency by some more recondite group-theoretic property.

We will see examples below.

However, it matters whether we are defining the graph on a single group, or defining it on the class of all groups.

## 12.1 On a specific group

If we are given a group $G$, and can compute $\mathrm{Aut}(G)$, then the first procedure (taking unions of orbital graphs) obviously gives all orbital (di)graphs for $G$.

**Theorem 12.1** *Given a group $G$, for every $\mathrm{Aut}(G)$-invariant graph, there is a formula $\phi$ in the first-order language of groups such that $x \sim y$ if and only if $G \models \phi(x,y)$.*

**Proof** By the so-called Ryll-Nardzewski Theorem, proved also by Engeler and by Svenonius (see [47]), $G$ is oligomorphic, so the $G$-orbits on $n$-tuples are $n$-types over $G$, that is, maximal sets of $n$-variable formulae consistent with the theory of $G$; but all types are principal, so each is given by a single formula. $\square$

**Question 19** Given $G$, is there a bound for the complexity of the formulae defining orbital graphs for $\mathrm{Aut}(G)$ acting on $G$ (for example, for the alternation of quantifiers)?

Clearly the commuting graph can be defined by the quantifier-free formula $xy = yx$. If $G$ is an elementary abelian 2-group, there are only three (non-diagonal) orbital graphs, defined by the formulae $(x = 1) \wedge (y \neq 1)$, $(x \neq 1) \wedge (y = 1)$, and $(x \neq 1) \wedge (y \neq 1) \wedge (x \neq y)$ respectively.

## 12.2 For classes of groups

As we have seen, the commuting graph is defined uniformly for all groups by the quantifier-free formula $xy = yx$.

It seems unlikely that the other graphs listed earlier have uniform first-order definitions. The statement $\langle x, y\rangle = G$ seems to require quantification either over words in $x, y$ or over subsets of $G$, and so to need some version of higher-order logic for its definition.

For example, suppose that there is a formula $\phi(x, y)$ which, in any finite group, specifies that $x$ and $y$ are joined in the power graph. Taking $C_n$ with $n$ even, with $x$ a generator and $y$ of order 2, the formula is always satisfied. So it should hold in an ultraproduct of such groups (see [13]). But in the ultraproduct, $x$ has infinite order and $y$ has order 2, so $y$ cannot be a power of $x$.

### 12.3 Applications

If we are given a specific group $G$ and know its automorphism group, then constructing all the orbital graphs is a simple polynomial-time procedure.

If we are given $G$ and don't know (and maybe are trying to find out about) its automorphism group, then clearly some indication of which first-order formulae need to be considered would be helpful. Maybe, given $g, h \in G$, the type of $(g, h)$ (the set of $\phi(x, y)$ such that $G \models \phi(g, h)$) could be described, and a formula generating the type found.

Another situation that might arise would be that we are given one or more graphs defined on general groups and are interested to know for which groups they have some property, e.g. two graphs equal. If we had first-order descriptions of the graphs, we would just be looking for models of some first-order sentence.

## 13 Beyond groups

The ideas behind some of these graphs can be extended to other algebraic structures. For example, the commuting graph and the power graph of a semigroup can be defined in the same way as for a group (and, indeed, the power graph was defined for semigroups early in their study, see [53]). The same argument that shows that the power graph of a group is the comparability graph of a partial order works without change for semigroups, and indeed for *power-associative magmas* (structures with a binary operation such that the associative law holds for powers of any element).

The commuting graphs of semigroups are considered by Araújo *et al.* [7], who pose a number of questions about them.

As for groups, the power graph of a semigroup is a spanning subgraph of its commuting graph. The enhanced power graph could be defined for any semigroup; to my knowledge this has not been studied. It is not clear whether the definition of the deep commuting graph could be adapted for semigroups.

Commuting graphs of semigroups are universal (and power graphs are universal for comparability graphs of partial orders), since thes statements hold for groups.

The intersection graph of the subsemigroups of a semigroup had been studied much earlier: Bosák [18] raised the question of its connectedness in 1963, and the question was soon resolved by Lin [59] and Ponděliček [63]: for any finite semigroup, this graph is connected with diameter at most 3. These results preceded the investigation of the intersection graph for groups, mentioned earlier. (The intersection graphs of semigroups are not comparable with the intersection graphs of groups, since any subgroup of a group contains the identity, so adjacency requires their intersection to be non-trivial.)

The *zero-divisor graph* of a ring was introduced by Beck [12] in 1988: the elements are the ring elements, with $a$ and $b$ joined whenever $ab = 0$. If the ring is commutative, the graph is undirected. Another graph associated with a ring is the *unit graph*, in which $a$ and $b$ are joined whenever $a + b$ is a unit [9].

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