This document contains some problems where we hope that further progress will be possible duing the summer research session. They are roughly classified by subject.

1 Groups and graphs

This is a very big area. We will concentrate on a few specific topics.

1.1 Power graph and commuting graph

Let G be a group. We define three graphs on the vertex set of G as follows:

> The *power graph* $P(G)$ of G has an (undirected) edge from x to y if one of them is a power of the other. There is also a directed version $\vec{P}(G)$, which has an arc $x \to y$ if y is a power of x.
>
> The *enhanced power graph* $P_e(G)$ of G has an edge from x to y if there exists $z \in G$ such that both x and y are powers of z.
>
> The *commuting graph* of G has an edge from x to y if $xy = yx$.

It is easy to see that the power graph is a subgraph of the enhanced power graph, which is itself a subgraph of the commuting graph. In [1], the finite groups are determined in which the power graph is equal to the enhanced power graph, or the enhanced power graph is equal to the commuting graph.

Problem 1.1 *What about the infinite groups satisfying one or other of these equalities?*

Problem 1.2 *Let G be a finite group such that $P(G) \neq P_e(G)$. Is it true that the graph whose edges are those in $P_e(G)$ but not in $P(G)$ is connected? What about other pairs from the list above?*

One of the main questions is: for which groups G, H is it the case that, if $P(G) \cong P(H)$ implies $\vec{P}(G) \cong \vec{P}(H)$? This is true for finite groups [2]. It is false in general for groups, but in [3] it is proved for some classes of torsion-free groups.

Problem 1.3 *Find further classes of torsion-free groups for which power graph isomorphism implies directed power graph isomorphism. What happens for mixed groups?*

1.2 Cayley graphs

This is a well-worked area, but there is probably still room for some experimentation and possibly some theorems.

Let G be a group, and S a subset of G satisfying $S = S^{-1}$ and $1 \notin S$. Form a graph $\mathrm{Cay}(G, S)$ by the rule that, for every $g \in G$ and $s \in S$, there is an edge from g to sg. Our assumptions on S imply that this graph has no loops and that its edges are undirected. It is easy to see that $\mathrm{Cay}(G, S)$ is connected if and only if S generates G.

The main property of Cayley graphs is that G, acting on itself by right multiplication, is a group of automorphisms of $\mathrm{Cay}(G, S)$.

Many of the open problems concern *random Cayley graphs* for a group G: that is, we form S by choosing inverse pairs of non-identity elements of G independently with fixed probability p, where $0 < p < 1$.

Problem 1.4 *What can be said about the* spectrum *(the eigenvalues of the adjacency matrix) of a random Cayley graph for the finite group G? Related to this are various properties measuring how good a graph is as a communications network, such as* expansion *properties. How do these properties depend on the structure of G?*

There is a wide class of countably infinite groups G with the property that a random Cayley graph for G is the famous *countable random graph* or *Rado graph*. However, not every countable group has this property.

Problem 1.5 *Which countable groups G have the property that a random Cayley graph for G is not almost surely isomorphic to the countable random graph? An example of such a graph is the group*

$$G = \langle a, b \mid b^4 = 1, b^{-1}ab = a^{-1} \rangle;$$

what does a random Cayley graph for this group look like?

1.3 Other algebraic structures

The notions of commuting graph and power graph can be extended to other algebraic structures; sometimes care is required, since if the associative law fails then powers are not uniquely defined. However, there are non-associative structures such as *Moufang loops* for which all 2-generated subloops are associative.

Problem 1.6 *Is it true that, if the power graphs of two Moufang loops are isomorphic, then their directed power graphs are isomorphic? Does this extend to wider classes of loops?*

Problem 1.7 *What about semigroups? (Note that the power graph was originally defined for a semigroup.)*

2 Permutation group polynomials

Let G be a *permutation group* of degree n (a subgroup of the symmetric group S_n). Any element of G has an essentially unique *cycle decomposition* into disjoint cycles (we always include cycles of length 1, that is, fixed points, in this decomposition).

The *cycle index* of G is a polynomial Z_G in indeterminates $s_1, \ldots, s_n$ defined by

$$Z_G(s_1, \ldots, s_n) = \frac{1}{|G|} \sum_{g \in G} s_1^{c_1(g)} s_2^{c_2(g)} \cdots s_n^{c_n(G)},$$

where $c_i(g)$ is the number of cycles of length i in the cycle decomposition of G. (The factor $1/|G|$ is just a normalising factor, which is convenient in applications of the cycle index to orbit counting.)

Problem 2.1 *For which permutation groups G is the cycle index polynomial* reducible*, that is, have a factorisation into polynomials of smaller degree?*

For example, if $G = G_1 \times G_2$, where the groups G_1 and G_2 act on disjoint sets whose union is $\{1, \ldots, n\}$, then

$$Z_G = Z_{G_1} Z_{G_2}.$$

Apart from this, very few examples are known!

It is easier to think about polynomials in a single indeterminate. Two examples have been considered:

> The *fixed point polynomial* [6], the generating polynomial for the numbers of fixed points, which is
>
> $$\frac{1}{|G|} \sum_{g \in G} x^{c_1(g)}.$$

> The *cycle polynomial* [5], the generating function for the total number of cycles, which is
> $$\frac{1}{|G|}\sum_{g\in G} x^{c(g)},$$
> where $c(g) = c_1(g) + c_2(g) + \cdots + c_n(g)$.

These can be obtained from the cycle index by the substitutions

> $s_1 = x$, $s_2 = \cdots = s_n = 1$, and
>
> $s_1 = s_2 = \cdots = s_n = x$,

respectively.

We can ask many questions about reducibility, location of roots, and so on, for these polynomials.

Here is a specific question about the cycle polynomial. It relates it to the *chromatic polynomial* P_Γ of a graph Γ, the polynomial whose value at a positive integer q is equal to the number of proper q-colourings of the vertices of Γ ("proper" means that vertices joined by an edge must get different colours). In [4], an *orbital chromatic polynomial* $OP_{\Gamma,G}$ was associated with a graph Γ and a group G of automorphisms of Γ; its value at a positive integer q is equal to the number of orbits of G on the set of proper q-colourings of Γ.

The authors of [5] point out that, in a number of interesting cases, given a permutation group G, there is a graph Γ containing G as a group of automorphisms, such that

$$(-1)^n C_G(-x) = OP_{\Gamma,G}(x),$$

where $C_G(x)$ is the cycle polynomial of G. This relation between combinatorial polynomials was invented by Richard Stanley in the 1960s and is referred to as *reciprocity.*

Problem 2.2 *For which permutation groups G does there exist a graph Γ such that the above reciprocity holds between the cycle polynomial of G and the orbital chromatic polynomial of Γ and G? In cases where this does not hold, is there a combinatorial interpretation of the polynomial $(-1)^n C_G(-x)$?*

References

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