This document contains a mixture of problems which might be suitable for a summer research project. Of course, it is difficult to tell how hard a problem is until you try it! If any of these problems look interesting to you, come and discuss them; I probably know more about it than is said here.

1 Graphs and groups

There are many problems in this area; I have chosen a few of my favourites.

1.1 Power graph and commuting graph

We start with some definitions. Each of the following graph has vertex set a given group G, which may be finite or infinite.

- The *commuting graph* has x joined to y if $xy = yx$.
- The *directed power graph* has an arc from x to y if y is a power of x.
- The *power graph* is the directed power graph ignoring directions; that is, x is joined to y if one of them is a power of the other.
- The *enhanced power graph* has x joined to y if the group generated by x and y is cyclic (that is, both are powers of some element z).

A question which has been looked at a lot is:

Question 1.1.1 If two groups G and H have isomorphic directed power graphs, do they have isomorphic power graphs?

It is known that different groups can have isomorphic power graphs. This question has an affirmative answer for finite groups, and for some classes of infinite torsion-free groups: see the paper by Cameron, Guerra and Jurina in the references (which began as a summer research project). It would probably be hard to settle this question in general, but perhaps some more cases are within reach.

Note that the power graph is a subgraph of the enhanced power graph, which is itself a subgraph of the commuting graph. It is known exactly which finite groups have the property that any two of these graphs are equal. This is in the paper by Aalipour *et al.* in the references.

Question 1.1.2 What can be said about connectedness properties of the *differences*? That is, consider edges of the commuting graph not in the power graph, or one of the other cases. In a finite group, the identity is an isolated vertex; but can we classify the finite groups in which the remaining vertices are connected in this graph?

As said above, different finite groups can have the same power graph. For example, the group $C_3 \times C_3 \times C_3$ and the non-abelian group of order 27 and exponent 3 have power graph which is 13 triangles with a common vertex. But are there classes of groups in which the power graph determines the group up to isomorphism? And what about the enhanced power graph or the commuting graph?

1.2 A combinatorial reciprocity question

This question is taken from the paper by Cameron and Semeraro in the list.

The *cycle polynomial* of a permutation group G is the normalised generating function for the number of cycles of elements (including cycles of length 1): that is,

$$CP_G(x) = \frac{1}{|G|} \sum_{g \in G} x^{c(g)},$$

where $c(g)$ is the number of cycles of g. If G is a group of automorphisms of a graph X, the *orbital chromatic polynomial* $OP_{X,G}(q)$ is the polynomial whose value at a positive integer q is the number of orbits of G on proper colourings of X with q colours. There are a surprisingly large number of permutation groups G for which there exists a graph X with

$$OP_{X,G}(x) = (-1)^n CP_G(-x),$$

where n is the number of points permuted. We say that G satisfies *reciprocity* if this holds.

Question 1.2.1 Find all examples of permutation groups which satisfy reciprocity.

1.3 The generation graph

If a group G can be generated by two elements, we can form a graph whose vertex set is G in which x is joined to y whenever x and y generate G.

This graph can have isolated vertices. Most research on the generation graph has dealt with the case when only the identity is isolated.

Question 1.3.1 What can be said about the set of isolated vertices? Must it be a subgroup of G? If not, can we characterise the groups in which this fails?

The *Frattini subgroup* $\Phi(G)$ of a group G is the set of elements which can be dropped from any generating set for G; in other words,

$$\Phi(G) = \{x \in G : G = \langle S, x \rangle \Rightarrow G = \langle S \rangle\}.$$

It is a subgroup of G (indeed, it is the intersection of all the maximal subgroups), and its elements are isolated vertices of the generation graph (provided that G is not cyclic).

However there may be more. In the symmetric group S_4, for example, the isolated vertices are the identity and the products of two transpositions (but note that these do form a subgroup).

1.4 Trees and cycles

Given a tree T on vertex set $\{1, \ldots, n\}$, the edges form a set of $n-1$ transpositions whose product (in any order) is an n-cycle. If T is a star, then the $(n-1)!$ orderings of the edges give rise to the $(n-1)!$ cycles, each exactly once; but for any other shape of tree, we do not have a bijection. For example, for the path on 4 vertices, of the six 4-cycles we obtain two twice, two once, and two not at all.

Question 1.4.1 What is the distribution of frequencies of occurrence of cycles arising from orders of a given tree T?

2 Number theory

Here are a mixed bag of number theory problems.

2.1 Sums of powers of the same prime

Apart from 2, there are just six positive integers known which can be represented as the sum of two powers of the same prime in three different ways: the largest is

$$132 = 131^1 + 131^0 = 2^7 + 2^2 = 11^2 + 11^1.$$

Question 2.1.1 Are there any more? Can one bound the number of different such representations that a positive integer can have?

2.2 A universal sequence from the primes

A sequence of zeros and ones is *universal* if every finite sequence occurs in it somewhere as a consecutive subsequence. For example, if you choose the terms in the sequence independently at random (by tossing a coin), almost surely the result will be universal.

Now define a sequence s whose ith term is 0 if the ith odd prime is congruent to 1 mod 4, and 1 if the ith odd prime is congruent to -1 mod 4.

Question 2.2.1 Is s universal?

2.3 A curious recursion

Given positive integers k, a, b, with $a > b$, define a function $f(n) = f_{k,a,b}(n)$ by the recursion

$$f(0) = k, \qquad f(n) = \lceil f(n-1)(a+n)/(b+n) \rceil \text{ for } n > 0.$$

Question 2.3.1 How does this function behave for large n?

If it were not for the ceilings, it would grow like $(k/((b+1)\cdots a))n^{a-b}$. So we might guess that $f(n)/n^{a-b}$ tends to a limit. For $a = 4$, $b = 2$, and $k = 1, \ldots, 12$, the limit appears to be 1/6, 1/4, 1/3, 1/2, 1/2, 1/2, 2/3, 3/4, 5/6, 1, 1, 1; then adding 12 to k appears to add 1 to the limit.

Can this example be generalised?

2.4 Sums of consecutive primes

It is known that the average number of ways in which a positive integer in the range $[1, \ldots, n]$ can be written as a sum of consecutive primes tends to the limit $\log_e 2$ as n tends to infinity.

Question 2.4.1 Is it true that the limiting distribution of the number of representations of this form is Poisson with parameter $\log_e 2$?

This would mean, in particular, that in the limit half of all positive integers have no such representation.

3 Combinatorial questions

3.1 Maximising the number of acyclic orientations

An *acyclic orientation* of a graph is an assignment of directions to the edges of the graph so that there are no directed cycles

$$v_0 \to v_1 \to \cdots \to v_{n-1} \to v_0.$$

Note that we can assume without loss that the graph is simple: if there is a loop, then there are no acyclic orientations; if there are multiple edges, they must all be directed the same way to avoid a directed 2-cycle, so we only need one of them.

Let $a(G)$ be the number of acyclic orientations of the graph G. Richard Stanley showed that, if G has n vertices, then $a(G) = (-1)^n P_G(-1)$, where $P_G(x)$ is the chromatic polynomial of G.

A *Turán graph* $T(n, k)$ is the complete k-partite graph on n vertices for which the parts are as nearly equal as possible. For example, $T(8, 3)$ has parts of sizes 3, 3 and 2, with all edges between parts and none within. The name comes from *Turán's Theorem*, which asserts that the unique graph on n vertices containing no complete subgraph on $k+1$ vertices but subject to this having the maximum number of edges is the Turán graph $T(n, k)$.

This problem concerns the function $M(n, m)$, defined to be the maximum number of acyclic orientations of a graph with n vertices and m edges.

The *hanging curtain conjecture*, which first appeared in the PhD thesis of Robert Schumacher at City University London, says in rough terms that, for fixed n and varying m, the shape of the graph of $M(n, m)$ looks like a curtain suspended at the points corresponding to Turán graphs. In more detail,

- if m is such that there is a Turán graph with n vertices and m edges, then this Turán graph realises the maximum number of acyclic orientations for that n and m;
- between successive Turán graphs, the graph is convex.

It would be nice to get any results about this. For example, the first Turán graph is the complete bipartite graph with parts of size $n/2$ and $n/2$ (if n is even) and $(n + 1)/2$ and $(n - 1)/2$ (if n is odd). For these values, we have a formula for the number of acyclic orientations of the complete bipartite graphs. Can we prove that this is the maximum?

3.2 A generalisation of biplanes

A *biplane* is a collection of subsets (called blocks) of an n-element set of points with the properties

- two points lie in exactly two blocks;
- two blocks meet in exactly two points.

In a biplane, there is a number k such that any block contains k points and any point lies in k blocks; moreover, $n = (k^2 - k + 2)/2$. Only finitely many biplanes are known; these are described in detail in Marshall Hall's book *Combinatorial Theory.*

Question 3.1.1 What can be said about configurations of points and blocks (other than biplanes) satisfying the conditions

- two points lie in at most two blocks;
- two blocks meet in at least two points?

Examples do exist: the subsets $\{1, 2, 5, 8, 9\}$, $\{1, 2, 6, 7, 9\}$, $\{3, 4, 5, 7, 9\}$, $\{3, 4, 6, 8, 9\}$, $\{1, 3, 5, 6, 10\}$, $\{2, 4, 5, 6, 10\}$, $\{1, 4, 7, 8, 10\}$ and $\{2, 3, 7, 8, 10\}$ of $\{1, \ldots, 10\}$ form an example.

4 Miscellanea

A mixed bag to finish.

4.1 Endomorphisms and partial isomorphisms

Given a finite group G, let $\mathrm{End}(G)$ be the semigroup of endomorphisms of G, and $\mathrm{PIso}(G)$ the semigroup of partial isomorphisms of G (isomorphisms between subgroups of G).

If G is abelian, then $|\mathrm{End}(G)| = |\mathrm{PIso}(G)|$.

Question 4.1.1 Does the converse hold? What can be said about groups where the left hand side is larger (or smaller) than the right hand side?

4.2 A group theory question

Question 4.2.1 Which finite groups G have the property that the automorphism group of G is transitive on ordered pairs of non-commuting elements?

Pablo Spiga pointed out that such a group must be a p-group, for some prime p, and its Frattini subgroup must be elementary abelian.

4.3 Network flow where channels silt up

This is perhaps a model for a network where channels silt up if not used. It is rather open-ended, so feel free to interpret or change the hypotheses.

Consider a network in which all edges have capacity 1. At unit time steps, perform the following operation:

- Choose a random maximal flow f in the network. (The maximal flows form a compact subset of a finite-dimensional Euclidean space; use the measure induced from the uniform measure on a suitable box.)
- Choose a random edge e.
- Delete e with probability $1 - f(e)$.

Eventually the network will be reduced to c edge-disjoint paths from source to target, where c is the capacity.

Question 4.3.1 What can be said about the time required for this to happen?

For example, if the network is a cube and the source and target are antipodal vertices, the expected time required is 19.5.

4.4 Problems in $\mathbb{Z}/(p)$

Donald Preece and I have failed to solve the following problem.

Let p be an odd prime, and k an integer at least 3. A k-AP decomposition mod p is an arithmetic progression $(a_1, \ldots, a_k)$ consisting of integers not congruent to 0 or 1 mod p such that the group of units of $\mathbb{Z}/(p)$ is the direct product of the cyclic groups generated by the terms of the progression.

There are many examples of 3-AP decompositions, but we found no 4-AP decompositions; none exist for primes less than 20000.

Question 4.4.1 Do k-AP decompositions exist for $k > 3$?

(Note: we assume that each cyclic factor has order greater than 1. Without this assumption, the AP $(3528, 1148, 2381, 1)$ mod 3613 is an example.)

Question 4.4.2 Characterise the primes p which have the property that there exists an element c of the multiplicative group of $\mathbb{Z}/(p)$ with the property that $\{c, c-1, 1\}$ generates a proper subgroup of the multiplicative group.

Primes congruent to 1 mod 4 have this property since -1 is a square and there are two consecutive squares. Also primes congruent to 1 mod 3 have the property: take c to be a primitive 6th root of unity. There are others: the remaining primes less than 1000 with this property are 131, 191, 239, 251, 311, 419, 431 and 491.

This problem arose in my work with João Araújo on regular semigroups.

4.5 Ordering finite vector spaces

Question 4.5.1 How many total orderings of a finite vector space have the property that the unique order-preserving bijection between two subspaces of the same dimension is linear? Can they be described?

The lexicographic ordering of $\mathrm{GF}(q)^n$ has this property (if the field is ordered arbitrarily). There are other examples: how many?

References

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