Some problems in model theory

This is mostly restricted to the particular part of model theory that I find most interesting. I begin with some notes about $\aleph_0$-categorical structures and their automorphism groups. Note that real logicians usually talk about $\aleph_0$-categorial theories rather than structures, but my focus is a bit different.

1 Notes

1.1 First-order theories and structures

We always assume that we are working over a countable first-order language L. Thus, there are finitely or countably many symbols representing relations, functions, or constants in the language. A *structure* $\mathcal{M}$ over L consists of a set X together with actual relations, functions and constants on X interpreting the symbols on the language. For any first-order sentence in the language, it will either be true or false in the structure.

The *theory* of $\mathcal{M}$, written $\mathrm{Th}(\mathcal{M})$, is the set of sentences true in $\mathcal{M}$. Conversely, if T is a set of sentences, we say that $\mathcal{M}$ is a *model* for T if every sentence of T is true in $\mathcal{M}$, that is, $T \subseteq \mathrm{Th}(\mathcal{M})$. The theory of a structure $\mathcal{M}$ is a *complete* theory, that is, for every sentence, either it or its negation is a consequence of the theory.

1.2 Categoricity

According to the upward Löwenheim–Skolem theorem, if a theory has an infinite model, then it has models of arbitrarily large cardinality. Thus, first-order sentences can never characterise an infinite structure uniquely up to isomorphism. (An *isomorphism* between structures is a bijection between their underlying sets which preserves the interpretations of all the relations, functions and constants of the language.) So the best we can do is, for any cardinal number α, to call a theory *α-categorical* if it has a unique model of cardinality α (up to isomorphism).

Now Morley's Theorem asserts that if α and β are uncountable cardinals, then a theory is α-categorical if and only if it is β-caategorical. Thus there are only two kinds of categoricity: a theory may be $\aleph_0$-categorical (or *countably categorical*); it may be α-categorical for all uncountable α (or *uncountably categorical*). Of course, it may be both, or it may be neither.

Uncountably categorical theories have received a lot of attention from model theorists. Indeed, their study has given a big impetus to the study of *stability theory* and other areas of model theory. However, my focus here is on countably categorical theories and their countable models (which are called *countably categorical structures*).

An *automorphism* of a structure $\mathcal{M}$ is a bijective map from the domain of $\mathcal{M}$ to itself which is an isomorphism from $\mathcal{M}$ to $\mathcal{M}$. The automorphisms of $\mathcal{M}$ form a permutation group on the domain of $\mathcal{M}$.

Given a group G acting on a set X, we define a relation $\equiv$ on X by the rule that $x \equiv y$ if there exists $g \in G$ with $xg = y$. (I write group actions on the right and compose left-to-right.) From the group axioms, it follows easily that $\equiv$ is an equivalence relation. Its equivalence classes are the *orbits* of G. We say that G is *transitive* (more precisely, the action of G on X is transitive) if there is just one orbit.

Now, given a permutation group G on a set X, there is an induced action of G on the set X^n of all n-tuples of elements of X, for all natural numbers n. We say that the group G (or, more precisely, the action) is *oligomorphic* if the number of orbits of G on X^n is finite for all natural numbers n.

Now the following key theorem, proved independently by Engeler, Ryll-Nardzewski and Svenonius in 1959, is the basis of this area.

Theorem 1 *The countable first-order structure $\mathcal{M}$ is countably categorical if and only if the action of* $\mathrm{Aut}(\mathcal{M})$ *on the domain of $\mathcal{M}$ is oligomorphic.*

In my opinion, this is a remarkable equivalence between axiomatisability (countably categorical structures can be completely described by first-order axioms together with the collection of countability) and symmetry (having oligomorphic automorphism group is a very strong symmetry condition).

Example: The rational numbers as ordered set. Cantor proved a famous theorem according to which a countable totally ordered set which is dense and without least or greatest element is isomorphic to $(\mathbb{Q}, <)$ (the rational numbers with the usual order). This says that $(\mathbb{Q}, <)$ is the only countable model for the first-order sentences (in the language with a single binary relation $<$) asserting that $<$ is a total order, is dense, and has no least or greatest element. For example, an order is dense if and only if $(\forall x)(\forall y)((x < y) \Rightarrow (\exists z)(x < z < y))$ holds. So this structure is countably categorical.

Also, its automorphism group is oligomorphic. Suppose we are given two n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$; suppose first that each n-tuple is in increasing order. Then the unique order-preserving map carrying a_i to b_i for all i can be extended to a piecewise linear map on the entire set $\mathbb{Q}$ (see Figure 1).

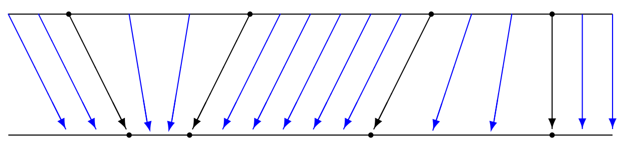

Figure 1: Extending an order-preserving map

Now a very similar argument shows that two n-tuples of distinct rationals belong to the same orbit if and only if they are in the same relative order, that is, $a_i < a_j$ if and only if $b_i < b_j$. So there are precisely $n!$ orbits on ordered n-tuples of distinct elements. From this it follows that the total number of orbits on all n-tuples is finite (indeed there is a formula for it in terms of *Stirling numbers of the second kind*, namely

$$\sum_{k=1}^{n} S(n,k)\,k!$$

(which, coincidentally, is the formula for the number of *preorders* on a set of n objects).

If a permutation group G is oligomorphic, then we can ask various counting questions related to the integer sequence (A_n), where A_n is the number of orbits on n-tuples of distinct elements.

Now the study of oligomorphic permutation groups reduces in a very precise way to the study of automorphism groups of countably categorical structures. I outline the steps.

- We can construct a first order theory of a group acting on a set with precisely A_n orbits on n-tuples of distinct elements. If such a group exists, then the theory is consistent, and so by the downward Löwenheim–Skolem theorem, has a countable model. So we may assume that the domain is countable.
- There is a natural topology on the symmetric group on a countable set X, the *topology of pointwise convergence*. Imagine that the set

being permuted is $\mathbb{N}$. Then two permutations are close together if they agree on a long initial sequence of $\mathbb{N}$ (and, for technical reasons, we also require that their inverses agree on a long initial sequence of $\mathbb{N}$). Now a subgroup of the symmetric group and its closure have the same orbits on n-tuples for all n; and a subgroup is closed if and only if it is the automorphism group of a first-order structure over some first-order language. Thus the closed oligomorphic groups acting on countable sets are the same as the automorphism groups of countably categorical first-order structures.

1.3 Homogeneity

There is an important method for constructing and proving uniqueness of certain nice structures. This is *Fraïssé's Theorem.* We need some terminology to state it.

The *age* of a relational structure $\mathcal{M}$ is the collection of all finite structures embeddable in $\mathcal{M}$. (Thus the age of the ordered set $(\mathbb{Q}, <)$ is the class of all finite totally ordered sets). A class $\mathcal{C}$ of finite relational structures has the *amalgamation property* if the following holds: if A, B_1, B_2 are structures in $\mathcal{C}$, and there are embeddings $f_i : A \to B_i$ for $i = 1, 2$, then there is a structure $C \in \mathcal{C}$ and embeddings $g_i : B_i \to C$ for $i = 1, 2$, so that $f_1 g_1 = f_2 g_2$. Informally, if B_1 and B_2 have isomorphic substructures (isomorphic to A), they can be "glued together" by identifying these two substructures inside a larger structure C.

Theorem 2

- *The class $\mathcal{C}$ is the age of a countable homogeneous relational structure if and only if $\mathcal{C}$ is non-empty, closed under isomorphism, closed under taking substructures, contains at most countably many non-isomorphic members, and has the amalgamation property.*
- *If these conditions hold, then the countable homogeneous structure is unique up to isomorphism.*

If $\mathcal{C}$ satisfies the above conditions, it is called a *Fraïssé class*; and the unique countable homogeneous structure $\mathcal{M}$ with age $\mathcal{C}$ is the *Fraïssé limit* of $\mathcal{C}$.

For example, it is easy to see that the class of all finite graphs is a Fraïssé class. Its Fraïssé limit (the unique countable homogeneous graph which is *universal*, in the sense of containing all finite graphs as induced subgraphs), is the *Erdős–Rényi random graph*, sometimes referred to as the *Rado graph.* (Erdős and Rényi showed that it is the unique graph obtained with probability 1 when edges are chosen from the 2-subsets of a countable set independently at random with probability $\frac{1}{2}$; they didn't bother to construct it since their argument was a non-constructive existence proof, but it had been constructed in a different context explicitly by Rado.)

1.4 Reducts

The final topic in the notes concerns *reducts*. The general definition is more complicated, so I will stick to the case of countably categorical structures. Let $\mathcal{M}$ and $\mathcal{N}$ be two relational structures on the same set X. We say that $\mathcal{N}$ is a *reduct* of $\mathcal{M}$ if $\mathrm{Aut}(\mathcal{M})$ is a subgroup of $\mathrm{Aut}(\mathcal{N})$.

For example, let $\mathcal{M}$ be $(\mathbb{Q}, <)$ (the rationals as ordered set). Define a ternary relation β, called *betweenness*, on $\mathbb{Q}$ by the rule that (a, b, c) satisfies β if and only if either $a < b < c$ or $c < b < a$ (in other words, b is between a and c. Then $\mathcal{N} = (\mathbb{Q}, \beta)$ is a reduct of $\mathcal{M}$. We see here a general fact: the relations in the reduct can be defined in terms of those in the original structure without extra parameters.

Trivially, every structure is a reduct of itself, and every structure is a reduct of the *trivial* structure (a set with no relations) whose automorphism group is the symmetric group.

A conjecture of Simon Thomas asserts:

Conjecture A countable homogeneous structure in a finite relational language has only finitely many reducts.

For example, it is known that there are exactly five reducts of $(\mathbb{Q}, <)$: these are $(\mathbb{Q}, <)$ (the structure itself), the betweenness relation defined above, the circular order γ (in which (a, b, c) satisfies γ if and only if $a < b < c$ or $b < c < a$ or $c < a < b$), the separation relation (a 4-place relation whose definition I leave to your imagination), and the trivial structure.

2 Problems

2.1 Homogeneous structures and reducts

There are two obvious problems which would occur to you.

Problem 1: Describe all the countable homogeneous structures of some particular type (such as graphs). Equivalently, describe all the Fraïssé classes of finite structures of this type.

Problem 2: Find all the reducts of a particular countably categorical (or homogeneous) structure.

Problem 1 has been solved for graphs (Lachlan and Woodrow), tournaments (Cherlin and Lachlan), directed graphs (Cherlin – a major piece of work), partially ordered sets (Schmerl), and structures consisting of a set with at most three total orders on it (two orders by Cameron, three by Braunfeld very recently).

Problem 2 has been solved for the Erdős–Rényi random graph (Thomas), various hypergraph analogues of this graph (also by Thomas), the order $(\mathbb{Q},<)$ (Cameron), the ordered random graph (Bodirsky, Pinsker and Pongracz), and some other classes. A vector space of countable dimension over a fixed finite field is countably categorical and has finitely many reducts (Bodor and Szabó), but it is not homogeneous over a finite relational language – and Bodor, Cameron and Szabó showed that the vector space with a distinguished vector has infinitely many reducts.

For both problems, the arguments are rather difficult. However, vector spaces with additional structure look interesting from the viewpoint of mapping the boundaries of Thomas' conjecture.

Another interesting class consists of sets carrying m total orders, for $m > 3$, where not too much is known.

There is another important connection not touched on above. A set of finite structures is a *Ramsey class* if, given structures A, B in the class with A contained in B, there is a structure C such that, in any colouring of the embeddings of A into C with finitely many colours, there is a substructure of C isomorphic to B, all of whose A-substructures have the same colour.

It is known that a non-trivial Ramsey class must include a total order in its structure. Nešetřil showed that a Ramsey class must have the amalgamation property. A remarkable result of Kechris, Pestov and Todorcevic connects this with topological dynamics: the age of a countable homogeneous structure $\mathcal{M}$ is a Ramsey class if and only if the topological group $\mathrm{Aut}(\mathcal{M})$ is *extremely amenable* (this means that any continuous action of this group on a compact topological space has a fixed point).

In view of this it is very interesting for several reasons to decide which Fraïssé classes are Ramsey classes (equivalently, which countable homogeneous structures have automorphism groups which are extremely amenable).

2.2 Counting

As we saw, a structure is countably categorical if its automorphism group has only finitely many orbits on n-tuples of elements of the domain for all n (equivalently, on n-tuples of distinct elements, or on n-element subsets). This gives us three interesting sequences of positive integers associated with such a structure:

- F_n^*, the number of orbits on X^n (to a logician, this is the number of *n-types* over the first-order theory);
- F_n, the number of orbits on n-tuples of distinct elements;
- f_n, the number of orbits on n-sets.

The first two sequences are connected by a *Stirling transformation*:

$$F_n^* = \sum_{k=1}^{n} S(n,k)F_k$$

(we saw an example of this earlier). The second and third are related by

$$f_n \leq F_n \leq n!f_n.$$

It is known that all three sequences are non-decreasing. Moreover, if the structure is homogeneous over a relational language, F_n and f_n count respectively the numbers of n-element labelled and unlabelled structures in the corresponding Fraïssé class.

For example, for the structure $(\mathbb{Q},<)$, we have

$$f_n = F_n = 1, \quad F_n^* = B_n \text{ (the } n\text{th } \textit{Bell number}\text{)}.$$

Evidence from many examples suggests that the growth of these sequences is *rapid* (in general) and *smooth*. For example, in the case where $F_n = 1$ (that is, the automorphism group is transitive), it is known that either f_n is bounded by a polynomial, or its growth is at least *fractional exponential*. Recently, Falque and Thiéry showed that if the growth rate is bounded by a polynomial, then it is polynomial (asymptotic to cn^k for some natural number k and some $c \neq 0$).

A problem which might be within reach is to extend some of this to growth rates below $n^{1/2-\epsilon}$ for $\epsilon > 0$.

Associated with such a structure or permutation group is a *graded algebra*, that is an algebra whose vector space structure is

$$A = \bigoplus_{n \geq 0} V_n,$$

where the product of elements in V_n and V_m lies in V_{n+m}. We have $\dim(V_n) = f_n$, so the formal power series $\sum f_n x^n$ is the *Hilbert series* of the algebra. Pouzet showed that, if the group has no finite orbits, then this algebra is an integral domain. The algebra also played an important role in the work of Falque and Thiéry, where they showed that it is a *Cohen–Macauley algebra* in the polynomial growth case.

Again, it would be interesting to investigate this algebra in cases of slowish growth.