Permutation groups and transformation semigroups: 1. Permutation groups

Peter J. Cameron, University of St Andrews

Theoretical and Computational Algebra
July 2023

I'll know my song well before I start singing

Bob Dylan

About 15 years ago, João tempted me to put a toe in the ocean of semigroup theory with some very attractive problems.

About 15 years ago, João tempted me to put a toe in the ocean of semigroup theory with some very attractive problems.
He claimed that, in the past, semigroup theorists believed their job was done if they reduced the question to one in group theory, and they could just hand it over to the group theorists; but it would be much more productive if the two subjects could have a dialogue.

About 15 years ago, João tempted me to put a toe in the ocean of semigroup theory with some very attractive problems.
He claimed that, in the past, semigroup theorists believed their job was done if they reduced the question to one in group theory, and they could just hand it over to the group theorists; but it would be much more productive if the two subjects could have a dialogue.
I am going to tell you about some aspects of this.

Permutation groups

I begin with a few basic definitions about permutation groups.

Permutation groups

I begin with a few basic definitions about permutation groups. Let G be a permutation group on a set Ω, that is, a subgroup of the symmetric group $\mathrm{Sym}(\Omega)$. We denote the image of a point $\alpha \in \Omega$ by a permutation $g \in G$ by αg. (Note that a more general concept is that of an action of G on Ω, a homomorphism from G to $\mathrm{Sym}(\Omega)$; since the image of an action is a permutation group, all these definitions immediately transfer to group actions.)

Permutation groups

I begin with a few basic definitions about permutation groups.
Let G be a permutation group on a set Ω, that is, a subgroup of the symmetric group $\mathrm{Sym}(\Omega)$. We denote the image of a point $\alpha \in \Omega$ by a permutation $g \in G$ by αg. (Note that a more general concept is that of an action of G on Ω, a homomorphism from G to $\mathrm{Sym}(\Omega)$; since the image of an action is a permutation group, all these definitions immediately transfer to group actions.)
We say G is transitive if, for all $\alpha, \beta \in \Omega$, there is an element $g \in G$ satisfying $\alpha g = \beta$. If G is transitive, then we say it is primitive if the only G-invariant partitions of Ω are the trivial ones (the partition into singletons, and the partition with a single part).

Higman's Theorem

Primitivity is possibly the most important concept in permutation group theory, and there are a number of conditions equivalent to it; for example, a transitive group is primitive if the point stabiliser is a maximal proper subgroup of G. Probably the most important of these is the theorem of Donald Higman:

Higman's Theorem

Primitivity is possibly the most important concept in permutation group theory, and there are a number of conditions equivalent to it; for example, a transitive group is primitive if the point stabiliser is a maximal proper subgroup of G. Probably the most important of these is the theorem of Donald Higman:

Theorem (Higman's Theorem)

The transitive permutation group G on Ω is primitive if and only if everyxnon-trivial graph (or digraph) on Ω which is G-invariant is connected.

Higman's Theorem

Primitivity is possibly the most important concept in permutation group theory, and there are a number of conditions equivalent to it; for example, a transitive group is primitive if the point stabiliser is a maximal proper subgroup of G. Probably the most important of these is the theorem of Donald Higman:

Theorem (Higman's Theorem)

The transitive permutation group G on Ω is primitive if and only if everyxnon-trivial graph (or digraph) on Ω which is G-invariant is connected.

Note that we can construct G-invariant graphs by taking orbits of G on pairs of elements of Ω as edges. These are the orbital (di)graphs.

Multiple transitivity

Let t be a positive integer not exceeding n. We say G is t-transitive if its induced action on t-tuples of distinct elements of Ω is transitive; and G is t-homogeneous if the induced action on t-element subsets of Ω is transitive.

Multiple transitivity

Let t be a positive integer not exceeding n. We say G is t-transitive if its induced action on t-tuples of distinct elements of Ω is transitive; and G is t-homogeneous if the induced action on t-element subsets of Ω is transitive.

Clearly t-transitivity implies t-homogeneity. If $5 \leq t \leq n/2$, a beautiful theorem of Livingstone and Wagner asserts that the converse is true. All t-homogeneous but not t-transitive groups for $t = 2, 3, 4$ were found by Kantor (before CFSG).

Multiple transitivity

Let t be a positive integer not exceeding n. We say G is t-transitive if its induced action on t-tuples of distinct elements of Ω is transitive; and G is t-homogeneous if the induced action on t-element subsets of Ω is transitive.

Clearly t-transitivity implies t-homogeneity. If $5 \leq t \leq n/2$, a beautiful theorem of Livingstone and Wagner asserts that the converse is true. All t-homogeneous but not t-transitive groups for $t = 2, 3, 4$ were found by Kantor (before CFSG).

The classification of t-transitive groups for $t \geq 2$ had to wait for CFSG (the Classification of Finite Simple Groups before it could be completed; but now we have a complete list of such groups.

A general scheme

We will need to define several further properties of permutation groups. Here is a general scheme for such definitions, which handles most cases we need.

A general scheme

We will need to define several further properties of permutation groups. Here is a general scheme for such definitions, which handles most cases we need.

Let $\mathcal{C}$ be a class of relational or combinatorial objects. A member of $\mathcal{C}$ is said to be trivial if it is invariant under all permutations of its domain.

A general scheme

We will need to define several further properties of permutation groups. Here is a general scheme for such definitions, which handles most cases we need.

Let $\mathcal{C}$ be a class of relational or combinatorial objects. A member of $\mathcal{C}$ is said to be trivial if it is invariant under all permutations of its domain.

Now let G be a permutation group on a set Ω. We say that G is

- $\mathcal{C}$-free if the only $\mathcal{C}$-structures on Ω invariant under G are the trivial ones;

A general scheme

We will need to define several further properties of permutation groups. Here is a general scheme for such definitions, which handles most cases we need.

Let $\mathcal{C}$ be a class of relational or combinatorial objects. A member of $\mathcal{C}$ is said to be trivial if it is invariant under all permutations of its domain.

Now let G be a permutation group on a set Ω. We say that G is

- $\mathcal{C}$-free if the only $\mathcal{C}$-structures on Ω invariant under G are the trivial ones;
- $\mathcal{C}$-closed if any permutation of Ω which preserves all G-invariant $\mathcal{C}$-structures belongs to G,

A general scheme

We will need to define several further properties of permutation groups. Here is a general scheme for such definitions, which handles most cases we need.

Let $\mathcal{C}$ be a class of relational or combinatorial objects. A member of $\mathcal{C}$ is said to be trivial if it is invariant under all permutations of its domain.

Now let G be a permutation group on a set Ω. We say that G is

- $\mathcal{C}$-free if the only $\mathcal{C}$-structures on Ω invariant under G are the trivial ones;
- $\mathcal{C}$-closed if any permutation of Ω which preserves all G-invariant $\mathcal{C}$-structures belongs to G,

A virtue of this definition is that, for any class $\mathcal{C}$, the class of $\mathcal{C}$-free permutation groups is closed upwards.

Playing the game

We will see several examples. However, the way to play this game is not to think up an arbitrary class $\mathcal{C}$ and examine the $\mathcal{C}$-free or $\mathcal{C}$-closed permutation groups. Rather, we have a property of permutation groups we want to study; understanding the $\mathcal{C}$-free or $\mathcal{C}$-closed structures for an appropriate class is likely to help the investigation. Even better are cases when we can build arbitrary permutation groups from the $\mathcal{C}$-free groups.

Playing the game

We will see several examples. However, the way to play this game is not to think up an arbitrary class $\mathcal{C}$ and examine the $\mathcal{C}$-free or $\mathcal{C}$-closed permutation groups. Rather, we have a property of permutation groups we want to study; understanding the $\mathcal{C}$-free or $\mathcal{C}$-closed structures for an appropriate class is likely to help the investigation. Even better are cases when we can build arbitrary permutation groups from the $\mathcal{C}$-free groups.

Note that if G is not $\mathcal{C}$-free then it preserves a non-trivial $\mathcal{C}$-structure. The nicest cases are those where we can use this to get a reduction for G, and understand it in terms of smaller permutation groups. This is the case for transitivity and primitivity, for example.

How it works

Let $\mathcal{C}$ be the class of "subsets": a $\mathcal{C}$-object is a subset of Ω. The only subsets invariant under the symmetric group are the empty set and Ω; so G is $\mathcal{C}$-free if and only if it is transitive.

How it works

Let $\mathcal{C}$ be the class of "subsets": a $\mathcal{C}$-object is a subset of Ω. The only subsets invariant under the symmetric group are the empty set and Ω; so G is $\mathcal{C}$-free if and only if it is transitive.

Again, let $\mathcal{C}$ be the class of "directed graphs". A directed graph invariant under the symmetric group is either null or complete; so, if G is $\mathcal{C}$-free, then any pair of distinct points can be mapped to any other pair by an element of G (otherwise an orbit of G would be a digraph which is neither complete nor null); in other words, G is 2-transitive.

Examples

Here are some examples. We have met the first few, and will see the basic property next, and the synchronizing property in the next lecture.

Examples

Here are some examples. We have met the first few, and will see the basic property next, and the synchronizing property in the next lecture.

$\mathcal{C}$	$\mathcal{C}$-free
Subsets	Transitive
Partitions	Primitive
Graphs	2-homogeneous
Digraphs	2-transitive
Hamming graphs	Basic
Association schemes	AS-free
Weakly perfect graphs	Synchronizing

Examples

Here are some examples. We have met the first few, and will see the basic property next, and the synchronizing property in the next lecture.

$\mathcal{C}$	$\mathcal{C}$-free
Subsets	Transitive
Partitions	Primitive
Graphs	2-homogeneous
Digraphs	2-transitive
Hamming graphs	Basic
Association schemes	AS-free
Weakly perfect graphs	Synchronizing

Another class $\mathcal{C}$ we have just begun to study consists of poset block structures, where the $\mathcal{C}$-closed groups are the generalised wreath products.

Two challenges

For AI/ML specialists:

Question

There are zillions of interesting classes of structures on sets. Which ones give rise to interesting classes of permutation groups? Where should we look for them?

Two challenges

For AI/ML specialists:

Question

There are zillions of interesting classes of structures on sets. Which ones give rise to interesting classes of permutation groups? Where should we look for them?

For semigroup theorists:

Question

Can we define interesting classes of (partial) transformation monoids in this way?

Reductions

I mentioned some reductions earlier: here is a bit more detail. These often reduce the study of a permutation group to groups of smaller degree, or those with a very specific structure, and eventually pave the way to applications of CFSG.

Reductions

I mentioned some reductions earlier: here is a bit more detail. These often reduce the study of a permutation group to groups of smaller degree, or those with a very specific structure, and eventually pave the way to applications of CFSG.

If G is intransitive, then it has more than one orbit, and induces a transitive group on each orbit. So G is embedded in a direct product of transitive groups.

Reductions

I mentioned some reductions earlier: here is a bit more detail. These often reduce the study of a permutation group to groups of smaller degree, or those with a very specific structure, and eventually pave the way to applications of CFSG.

If G is intransitive, then it has more than one orbit, and induces a transitive group on each orbit. So G is embedded in a direct product of transitive groups.

If G is transitive but imprimitive, it preserves a partition, and is embedded in the wreath product $H \wr K$, where H is the group induced on a block of the partition by its setwise stabiliser, and K the group induced on the set of parts of the partition. This is the imprimitive action of the wreath product.

Hamming graphs and basic groups

Let m, q be integers greater than 1. The Hamming graph $H(m, q)$ is the graph whose vertices are all words of length m over an alphabet of size q (so it has q^m vertices. A primitive group which preserves a Hamming graph is contained in the wreath product of the group (of degree q) induced on the symbols occurring in a given position by the stabiliser of that position in G and the group of permutations on the set of coordinate positions induced by G (of degree m).

Hamming graphs and basic groups

Let m, q be integers greater than 1. The Hamming graph $H(m, q)$ is the graph whose vertices are all words of length m over an alphabet of size q (so it has q^m vertices. A primitive group which preserves a Hamming graph is contained in the wreath product of the group (of degree q) induced on the symbols occurring in a given position by the stabiliser of that position in G and the group of permutations on the set of coordinate positions induced by G (of degree m).
A primitive group is basic if it preserves no Hamming graph with $m, q > 1$. Thus, a group which is primitive but not basic is embeddable in a wreath product (in its product action).

Two special types of group

Let V be a finite vector space. The affine group $\mathrm{AGL}(V)$ is the group of permutations of V generated by translations and invertible linear maps. (It is the semidirect product of the abelian translation group T and the general linear group $\mathrm{GL}(V)$.)

Two special types of group

Let V be a finite vector space. The affine group $\mathrm{AGL}(V)$ is the group of permutations of V generated by translations and invertible linear maps. (It is the semidirect product of the abelian translation group T and the general linear group $\mathrm{GL}(V)$.)

A subgroup of $\mathrm{AGL}(V)$ containing T is the semidirect product of T with a subgroup H of $\mathrm{GL}(V)$. It is necessarily transitive, since T is; it is primitive if and only if H is an irreducible linear group; and it is basic if and only if H is a primitive linear group, one which preserves no non-trivial direct sum decomposition of V.

Two special types of group

Let V be a finite vector space. The affine group $\mathrm{AGL}(V)$ is the group of permutations of V generated by translations and invertible linear maps. (It is the semidirect product of the abelian translation group T and the general linear group $\mathrm{GL}(V)$.)

A subgroup of $\mathrm{AGL}(V)$ containing T is the semidirect product of T with a subgroup H of $\mathrm{GL}(V)$. It is necessarily transitive, since T is; it is primitive if and only if H is an irreducible linear group; and it is basic if and only if H is a primitive linear group, one which preserves no non-trivial direct sum decomposition of V.

I will not give the rather involved definition of a diagonal group here; suffice to say that the diagonal group $D(H, m)$ depends on a group H and a positive integer m; it has degree $|H|^m$ and has a normal subgroup H^{m+1} acting on the cosets of a diagonal subgroup, the quotient contained in the group generated by $\mathrm{Aut}(H)$ and the symmetric group S_{m+1}.

The O'Nan–Scott Theorem

This theorem was proved by O'Nan and Scott (independently) in 1979, and improved by Aschbacher, Kovács, and others. What I state here is only a part of the theorem, but will be adequate for our needs.

The O'Nan–Scott Theorem

This theorem was proved by O'Nan and Scott (independently) in 1979, and improved by Aschbacher, Kovács, and others. What I state here is only a part of the theorem, but will be adequate for our needs.

A group G is almost simple if $S \leq G \leq \mathrm{Aut}(S)$ for some non-abelian finite simple group S. (Note that S is embedded in $\mathrm{Aut}(S)$ as the group of inner automorphisms.)

The O'Nan–Scott Theorem

This theorem was proved by O'Nan and Scott (independently) in 1979, and improved by Aschbacher, Kovács, and others. What I state here is only a part of the theorem, but will be adequate for our needs.

A group G is almost simple if $S \leq G \leq \mathrm{Aut}(S)$ for some non-abelian finite simple group S. (Note that S is embedded in $\mathrm{Aut}(S)$ as the group of inner automorphisms.)

Theorem (O'Nan–Scott Theorem)

A basic primitive group is affine, diagonal or almost simple.

The O'Nan–Scott Theorem

This theorem was proved by O'Nan and Scott (independently) in 1979, and improved by Aschbacher, Kovács, and others. What I state here is only a part of the theorem, but will be adequate for our needs.

A group G is almost simple if $S \leq G \leq \mathrm{Aut}(S)$ for some non-abelian finite simple group S. (Note that S is embedded in $\mathrm{Aut}(S)$ as the group of inner automorphisms.)

Theorem (O'Nan–Scott Theorem)

A basic primitive group is affine, diagonal or almost simple.

Since affine groups preserve affine spaces, and diagonal groups preserve structures called diagonal semilattices, we can say that a permutation group which preserves no non-trivial subset, partition, Hamming graph, affine space, or diagonal semilattice is almost simple.

Finite simple groups

The Classification of Finite Simple Groups is probably the biggest theorem ever proved, involving thousands of pages by hundreds of mathematicians. However, the result is easy to state:

Finite simple groups

The Classification of Finite Simple Groups is probably the biggest theorem ever proved, involving thousands of pages by hundreds of mathematicians. However, the result is easy to state:

Theorem (CFSG)

A non-abelian finite simple group is one of the following:

Finite simple groups

The Classification of Finite Simple Groups is probably the biggest theorem ever proved, involving thousands of pages by hundreds of mathematicians. However, the result is easy to state:

Theorem (CFSG)

A non-abelian finite simple group is one of the following:

- *an alternating group* A_n*, for* $n \geq 5$*;*

Finite simple groups

The Classification of Finite Simple Groups is probably the biggest theorem ever proved, involving thousands of pages by hundreds of mathematicians. However, the result is easy to state:

Theorem (CFSG)

A non-abelian finite simple group is one of the following:

- *an alternating group* A_n*, for* $n \geq 5$*;*
- *a group of Lie type (these are central quotients of specific linear groups over finite fields);*

Finite simple groups

The Classification of Finite Simple Groups is probably the biggest theorem ever proved, involving thousands of pages by hundreds of mathematicians. However, the result is easy to state:

Theorem (CFSG)

A non-abelian finite simple group is one of the following:

- *an alternating group A_n, for $n \geq 5$;*
- *a group of Lie type (these are central quotients of specific linear groups over finite fields);*
- *one of the 26 sporadic groups.*

It follows from CFSG that, if S is one of these groups, then $\mathrm{Aut}(S)/S$ is very small (and in any case soluble). The combined efforts of many mathematicians has led to a good understanding of simple (and almost simple) groups, such as knowledge of their maximal subgroups and linear representations.

Applications

The classification of 2-transitive groups follows from this. A 2-transitive group is clearly primitive and basic, and it is not hard to show that diagonal groups cannot be 2-transitive. So these groups are affine or almost simple; and using knowledge of the almost simple groups and their representations, a complete list can be found. (In fact, much less than the full strength of O'Nan–Scott is needed here; the reduction is due to Burnside.)

Applications

The classification of 2-transitive groups follows from this. A 2-transitive group is clearly primitive and basic, and it is not hard to show that diagonal groups cannot be 2-transitive. So these groups are affine or almost simple; and using knowledge of the almost simple groups and their representations, a complete list can be found. (In fact, much less than the full strength of O'Nan–Scott is needed here; the reduction is due to Burnside.)

More generally, Wielandt intruduced the class of $\frac{3}{2}$-transitive groups, those which are transitive and the stabiliser of a point α has all remaining orbits of the same size. (This class is not upward-closed so cannot be included in our general scheme.)

Wielandt showed that a $\frac{3}{2}$-transitive group is either primitive or a Frobenius group, a group in which all 2-point stabilisers are trivial. Any Frobenius group is $\frac{3}{2}$-transitive; the primitive ones have been classified, using CFSG.

Low in the hierarchy

The properties we have examined so far are almost all at least as strong as primitivity. I want to conclude with several properties which are weaker, which I have investigated with Marina Anagnostopoulou-Merkouri and, in part, with Enoch Suleiman and Rosemary Bailey.

Low in the hierarchy

The properties we have examined so far are almost all at least as strong as primitivity. I want to conclude with several properties which are weaker, which I have investigated with Marina Anagnostopoulou-Merkouri and, in part, with Enoch Suleiman and Rosemary Bailey.

I begin with a property which has been studied extensively by Cheryl Praeger and others. A permutation group G on Ω is quasiprimitive if every non-trivial normal subgroup of G is transitive. Since the orbit partition of a normal subgroup is G-invariant, a primitive group is quasi-primitive.

Low in the hierarchy

The properties we have examined so far are almost all at least as strong as primitivity. I want to conclude with several properties which are weaker, which I have investigated with Marina Anagnostopoulou-Merkouri and, in part, with Enoch Suleiman and Rosemary Bailey.

I begin with a property which has been studied extensively by Cheryl Praeger and others. A permutation group G on Ω is quasiprimitive if every non-trivial normal subgroup of G is transitive. Since the orbit partition of a normal subgroup is G-invariant, a primitive group is quasi-primitive.

Many results, including the O'Nan–Scott theorem, have been extended from primitive to quasiprimitive groups.

Low in the hierarchy

The properties we have examined so far are almost all at least as strong as primitivity. I want to conclude with several properties which are weaker, which I have investigated with Marina Anagnostopoulou-Merkouri and, in part, with Enoch Suleiman and Rosemary Bailey.

I begin with a property which has been studied extensively by Cheryl Praeger and others. A permutation group G on Ω is quasiprimitive if every non-trivial normal subgroup of G is transitive. Since the orbit partition of a normal subgroup is G-invariant, a primitive group is quasi-primitive.

Many results, including the O'Nan–Scott theorem, have been extended from primitive to quasiprimitive groups.

Peter Neumann pointed out that in the Second Memoir, Galois sometimes confused the notions of primitivity and quasiprimitivity.

Pre-primitivity

Suppose that P and Q are permutation group properties such that P implies Q. The philosophy of what follows is to define a property "pre-P" such that it is independent of Q but together with Q it is equivalent to P. (Note that this is not well-defined!)

Pre-primitivity

Suppose that P and Q are permutation group properties such that P implies Q. The philosophy of what follows is to define a property "pre-P" such that it is independent of Q but together with Q it is equivalent to P. (Note that this is not well-defined!) We say that the transitive group G is pre-primitive if every G-invariant partition is the orbit partition of a subgroup of G.

Pre-primitivity

Suppose that P and Q are permutation group properties such that P implies Q. The philosophy of what follows is to define a property “pre-P” such that it is independent of Q but together with Q it is equivalent to P. (Note that this is not well-defined!)

We say that the transitive group G is pre-primitive if every G-invariant partition is the orbit partition of a subgroup of G.

This does what is required: Neither of pre-primitivity and quasi-primitivity implies the other, but together they are equivalent to primitivity.

Pre-primitivity

Suppose that P and Q are permutation group properties such that P implies Q. The philosophy of what follows is to define a property "pre-P" such that it is independent of Q but together with Q it is equivalent to P. (Note that this is not well-defined!)

We say that the transitive group G is pre-primitive if every G-invariant partition is the orbit partition of a subgroup of G.

This does what is required: Neither of pre-primitivity and quasi-primitivity implies the other, but together they are equivalent to primitivity.

We have various results about such groups, including the fact that a wreath product of transitive groups is pre-primitive if and only if the factors are.

Invariant partitions

The set of partitions of Ω forms a lattice: the meet of two partitions P and Q is the partition whose parts are all non-empty intersections of parts of P and Q, and the join is the partition into connected components of the graph in which two points are adjacent if and only if they are in the same part of either P or Q.

Invariant partitions

The set of partitions of Ω forms a lattice: the meet of two partitions P and Q is the partition whose parts are all non-empty intersections of parts of P and Q, and the join is the partition into connected components of the graph in which two points are adjacent if and only if they are in the same part of either P or Q.

The greatest element is the partition with a single part Ω, and the least is the partition into singletons.

Invariant partitions

The set of partitions of Ω forms a lattice: the meet of two partitions P and Q is the partition whose parts are all non-empty intersections of parts of P and Q, and the join is the partition into connected components of the graph in which two points are adjacent if and only if they are in the same part of either P or Q.

The greatest element is the partition with a single part Ω, and the least is the partition into singletons.

In addition, a partition is uniform if all parts have the same size, and two partitions commute if the corresponding equivalence relations do.

Invariant partitions

The set of partitions of Ω forms a lattice: the meet of two partitions P and Q is the partition whose parts are all non-empty intersections of parts of P and Q, and the join is the partition into connected components of the graph in which two points are adjacent if and only if they are in the same part of either P or Q.
The greatest element is the partition with a single part Ω, and the least is the partition into singletons.
In addition, a partition is uniform if all parts have the same size, and two partitions commute if the corresponding equivalence relations do.
Statisticians define an orthogonal block structure to be a sublattice of the partition lattice consisting of commuting orthogonal partitions. Any OBS is a modular lattice; a poset block structure is a distributive OBS.

OB and PB permutation groups

A transitive permutation group G has the OB property (resp., the PB property) if the lattice of G-invariant partitions is an OBS (resp. a PBS). Note that the G-invariant partitions always form a lattice, and if G is transitive then they are all uniform.

OB and PB permutation groups

A transitive permutation group G has the OB property (resp., the PB property) if the lattice of G-invariant partitions is an OBS (resp. a PBS). Note that the G-invariant partitions always form a lattice, and if G is transitive then they are all uniform. Suppose that G is pre-primitive: that is, any invariant permutation is the orbit partition of a subgroup. Without loss, the subgroup is normal. Hence the partitions commute, and so form an OBS. Thus, pre-primitivity implies the OB property.

OB and PB permutation groups

A transitive permutation group G has the OB property (resp., the PB property) if the lattice of G-invariant partitions is an OBS (resp. a PBS). Note that the G-invariant partitions always form a lattice, and if G is transitive then they are all uniform. Suppose that G is pre-primitive: that is, any invariant permutation is the orbit partition of a subgroup. Without loss, the subgroup is normal. Hence the partitions commute, and so form an OBS. Thus, pre-primitivity implies the OB property. Of course, the PB property also implies the OB property; but we don't know a relation between PB and pre-primitivity.

Generalized wreath product

PB groups are related to another concept, which I cannot describe in detail. It is well-known that a finite distributive lattice is the lattice of down-sets in a finite poset. There is a concept of generalized wreath product defined by a poset with a permutation group at each element.

Generalized wreath product

PB groups are related to another concept, which I cannot describe in detail. It is well-known that a finite distributive lattice is the lattice of down-sets in a finite poset. There is a concept of generalized wreath product defined by a poset with a permutation group at each element.

For example, there are two 2-element posets. Suppose that groups H and K are given at the two points. If the poset is an antichain, the GWP is the direct product; if it is a chain, with H above K, the GWP is the wreath product $K \wr H$.

Generalized wreath product

PB groups are related to another concept, which I cannot describe in detail. It is well-known that a finite distributive lattice is the lattice of down-sets in a finite poset. There is a concept of generalized wreath product defined by a poset with a permutation group at each element.

For example, there are two 2-element posets. Suppose that groups H and K are given at the two points. If the poset is an antichain, the GWP is the direct product; if it is a chain, with H above K, the GWP is the wreath product $K \wr H$.

The following extends well-known results about direct and wreath products:

Generalized wreath product

PB groups are related to another concept, which I cannot describe in detail. It is well-known that a finite distributive lattice is the lattice of down-sets in a finite poset. There is a concept of generalized wreath product defined by a poset with a permutation group at each element.

For example, there are two 2-element posets. Suppose that groups H and K are given at the two points. If the poset is an antichain, the GWP is the direct product; if it is a chain, with H above K, the GWP is the wreath product $K \wr H$.

The following extends well-known results about direct and wreath products:

Theorem

A transitive group with the PB property is naturally embedded in a generalized wreath product of symmetric groups.

Generalized wreath product

PB groups are related to another concept, which I cannot describe in detail. It is well-known that a finite distributive lattice is the lattice of down-sets in a finite poset. There is a concept of generalized wreath product defined by a poset with a permutation group at each element.

For example, there are two 2-element posets. Suppose that groups H and K are given at the two points. If the poset is an antichain, the GWP is the direct product; if it is a chain, with H above K, the GWP is the wreath product $K \wr H$.

The following extends well-known results about direct and wreath products:

Theorem

A transitive group with the PB property is naturally embedded in a generalized wreath product of symmetric groups.

Indeed, we expect to be able to replace the symmetric group by appropriate subgroups induced by the action of G; but this is work in progress.

References

- Marina Anagnostopoulou-Merkouri, Peter J. Cameron and Enoch Suleiman, A new property of permutation groups, arXiv `2302.13703`
- R. A. Bailey, *Association Schemes: Designed Experiments, Algebra and Combinatorics*, Cambridge Univ. Press, 2004.
- R. A. Bailey, Cheryl E. Praeger, C. A. Rowley and T. P. Speed, Generalized wreath products of permutation groups, *Proc. London Math. Soc.* (3) **47** (1983), 69–82.
- Peter J. Cameron, *Permutation Groups*, Cambridge Univ. Press, 1990.
- John D. Dixon and Brian Mortimer, *Permutation Groups*, Springer, 1996.
- Cheryl E. Praeger, An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs, *J. London Math. Soc.* (2) **47** (1993), 227–239.