Parking functions and trees

Problem A car park has n spaces, numbered from 1 to n, arranged in a row. n drivers each independently choose a favourite parking space in the park. As each driver enters the park, he drives to his favourite space. If it is empty, he parks there. Otherwise, he continues along the line and parks in the first free space. If all spaces between his favourite and n are taken, he leaves in disgust.

(a) Show that, out of the n^n choices of favourite spaces that the drivers can make, just $(n+1)^{n-1}$ lead to all drivers parking successfully.

(b) This is the number of labelled trees on $n+1$ vertices. What is the connection?

Solution to (a) The problem is solved by a beautiful trick. We consider instead a circular car park with $n+1$ spaces numbered from 1 to $n+1$. We have the same n drivers choosing their favourite space (which is allowed to be space $n+1$.) There are $(n+1)^n$ possible choices of favourite spaces. This time, instead of leaving in disgust, a driver simply continues around the circle to the first empty space (which is bound to exist); so the drivers succeed in parking in every case, and one space remains empty.

I *claim* that a choice leads to successful parking in the linear problem if and only if it leads to space $n+1$ remaining empty in the circular problem. The forward implication is clear. Conversely, if space $n+1$ remains empty, then no driver has passed over this space in his search for a parking spot, and so we could cut the circle there and park the drivers successfully in the linear car park.

By symmetry, each of the $n+1$ spaces in the circular park is equally likely to remain empty. So the number of choices leading to space $n+1$ being empty (and hence successful parking in the linear park) is $1/(n+1)$ of the total, namely $(n+1)^{n-1}$.

Comments on (b) Let us say that a function f from $\{1,\ldots,n\}$ to itself is a *parking function* if it represents drivers' choices which lead to successful parking in the linear car park. Now we have counted parking functions. So, if we could find a bijection between the set of parking functions and the set of labelled trees on $n+1$ vertices, we would have a new proof of Cayley's Theorem on trees, potentially simpler than the two proofs outlined on pages 38–39 (or indeed, any of the other proofs of this theorem).

Unfortunately, I don't know a simple description and proof. A bijection has been found: for details, see

> J. D. Gilbey and L. H. Kalikow, Parking functions, valet functions and priority queues, *Discrete Math.* **197/198** (1999), 351–373

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