The Hanging Curtains conjecture

For a graph Γ, let $a(\Gamma)$ denote the number of acyclic orientations of Γ. Students of MT5821 will remember that this is the evaluation of the chromatic polynomial of Γ at -1 (a theorem of Richard Stanley).

We are interested in the distribution of $a(\Gamma)$ as Γ runs over all graphs with n vertices and m edges. The average and minimum values of this quantity are known, but much less is known about the maximum. The problem is to investigate

$$A_n(m) = \max\{a(\Gamma) : \Gamma \text{ has } n \text{ vertices and } m \text{ edges}\}.$$

In particular, the Hanging Curtains conjecture makes some predictions about the shape of this, as function of m for fixed n.

The *Turán graph* $T_k(n)$ is the graph on n vertices with the largest number of edges subject to containing no complete graph of size $k+1$. Turán showed that there is a unique such graph for any k, n with $2 \leq k \leq n$. It is obtained as follows: divide the n verties into k parts, with sizes as nearly equal as possible (that is, differing by at most one); put all possible edges between the parts, and none within any part. Thus $T_2(n)$ is the complete bipartite graph $K_{m,m}$ if $n = 2m$ is even, or $K_{m,m+1}$ if $n = 2m+1$ is odd.

Conjecture

(a) If m is the number of edges of a Turán graph on n vertices, say $T_{n,k}$, then $T_{n,k}$ is the unique graph with $A_n(m)$ acyclic orientations.

(b) The function $m \mapsto A_n(m)$ is convex between successive points corresponing to Turán graphs as in (a), but concave at these points.

In other words, the curve resembles the top of curtains suspended from points corresponding to Turán graphs.