A block design problem

The project

Statisticians use block designs in the design of experiments on inhomogeneous material which is partitioned into blocks in some natural way. They have introduced various notions of optimality for block designs. A design is *optimal* on some criterion if it scores better than any competing design with the same numbers of points, blocks, and points per block. The various criteria are typically functions of the eigenvalues of the *information matrix* of the block design, which depends on concurrences between pairs of points.

Suppose that a block design has v points, b blocks, and k points per block. If the design is *balanced* (that is, any two points lie in exactly λ blocks), then it is optimal on all standard criteria. However, for most values of the parameters (v, b, k), no balanced design will exist, and we have to do the best we can.

This project concerns a family of designs related to an important graph called the *Sylvester graph*. This is a graph on 36 vertices. All these designs have $v = 36$ and $k = 6$. Two of them have $b = 42$. These values of (v, b, k) would coincide with those of a hypothetical *affine plane* of order 6, which would be a balanced design with these parameters having $\lambda = 1$. However, it is known that an affine plane of order 6 does not exist. So we suspect that one of the designs from the Sylvester graph may be optimal.

The project would consist of examining these designs, calculating the values of various optimality criteria for them, and if possible proving that they beat all competing designs. This is with a view to a publication drawing the attention of statisticians to the new designs. The project would be suitable for you if you like computing with discrete structures, and might well involve some GAP programming.

If you are interested in this project, please talk to either either Peter Cameron or Rosemary Bailey in room 317. Peter will be away for most of the first week of summer research, arriving back at teatime on Thursday, 15 June.

The Sylvester graph and its designs

The *Sylvester graph* is a graph with 36 vertices and valency 5, with a large automorphism group. The 36 vertices can be identified with the points of a

6×6 grid. For each vertex v, we take a block $B(v)$ consisting of v and its five neighbours. Now it follows from properties of the graph that two points v and w lie in at most two blocks:

- If v and w lie in the same row or column, then no block contains both.
- If v and w are adjacent in the graph, then they lie in the two blocks $B(v)$ and $B(w)$ (and no others, since there are no triangles in the graph).
- In all other cases, just one block contains v and w.

Twelve further blocks consist of the six rows and six columns of the grid. Adding these in gives 48 blocks; any two points now lie in either 1 or 2 blocks.

The blocks $B(v)$ as v runs through a row (or a column) of the grid form what is called a *parallel class*; they are pairwise disjoint and cover all the points. The same is true for the six rows, or for the six columns. The remaining designs are obtained by removing some parallel classes. It is clear that there are only a few different ways of doing this. In particular, our approximation to an affine plane is obtained by removing all the columns, leaving the rows and all blocks $B(v)$.

The Sylvester graph can be found at

`http://www.distanceregular.org/graphs/sylvester.html`