QUEEN MARY, UNIVERSITY OF LONDON

MTHM A30 Permutation Groups

Assignment 4 Weeks 10-12

In these questions, a direct product of permutation groups acts on the disjoint union of the domains, while a wreath product acts on the cartesian product of domains (i.e. in the imprimitive action).

1 Let $S = \mathrm{Sym}(\Omega)$ be the symmetric group on an infinite set Ω. Show that the number of orbits of $S \times \cdots \times S$ (with k factors) on the set of n-sets is equal to the number of choices of k non-negative integers $x_1, \ldots, x_k$ with sum n. Hence show that this number is $\binom{n+k-1}{n}$.

2 Let A denote the group of order-preserving permutations of the rational numbers.

(a) Prove that the number of orbits of $A \,\mathrm{Wr}\, A$ on n-sets is equal to the number of expressions for n as an ordered sum of positive integers. Hence show that this number is 2^{n-1} for $n \geq 1$.

(b) More generally show that the number of orbits of $A \,\mathrm{Wr} \cdots \mathrm{Wr}\, A$ (with k factors) on n-sets is equal to k^{n-1} for $n \geq 1$.

3 Let $G = \mathrm{Sym}(\Delta)$ be the symmetric group on an infinite set Δ, and let Ω be the set of 2-element subsets of Δ. Then G acts as a permutation group on Ω. Show that G is oligomorphic, and prove that the number $f_n(G)$ of orbits of G on n-element subsets of Ω is equal to the number of simple undirected graphs with n edges and no isolated vertices.

4 Let $\mathcal{C}$ be the Fraïssé class consisting of all finite undirected graphs which have no multiple edges but may have loops (with the proviso that there is at most one loop at each vertex). Let X be the Fraïssé limit of $\mathcal{C}$, and let $G = \mathrm{Aut}(X)$. Prove that the number F_n of orbits of G on ordered n-tuples of distinct elements of X is equal to $2^{n(n+1)/2}$.

5 Let Ω be a countable set. A permutation of Ω is said to be *cofinitary* if it fixes only finitely many elements of Ω; a permutation group G on Ω is *cofinitary* if all its non-identity elements are cofinitary.

Let $N = \mathrm{FSym}(\Omega)$ be the group of finitary permutations of Ω, and let G be any cofinitary permutation group on Ω. Show that any permutation in the product NG is either finitary or cofinitary. Hence show that any complement for N in NG (that is, any subgroup H of NG satisfying $H \cap N = \{1\}$, $NH = NG$) is cofinitary.