Notes on groups and graphs

There are several ways to associate a graph with a group (or other algebraic structure such as a ring). There are many open questions.

Why would you want to do that? One possible reason is that the way people think about groups and about graphs are typically very different, and one approach might help you to see things which are not obvious in another.

These notes are meant as a brief introduction. I have not given references to the results cited. You can probably find these yourself, or I can supply them in most cases.

Also, it hardly needs saying, I haven't included everything known!

1 The commuting graph

Let G be a group. We make a graph called the *commuting graph* as follows: the vertices are the elements of G; two vertices x and y are joined if and only if $xy = yx$.

If G is an abelian group, then any two vertices are joined, and we have a complete graph – not very interesting!

The *centre* $Z(G)$ of G is the set

$$\{x \in G : xy = yx \text{ for all } y \in G\}$$

of elements which commute with everything in G. Group-theoretically, it is a normal subgroup of G; graph-theoretically, it consists of the vertices which are joined to all other vertices. It is common to delete the elements of the centre, and denote the resulting graph by $\Gamma(G)$.

Gruenberg and Kegel gave a (slightly complicated) characterisation of groups whose commuting graph is not connected. Here is an example. Let G be the symmetric group on a prime number p of points, where $p > 3$. Then an element of G which is a p-cycle commutes only with its powers, and so the graph has many components which are complete graphs of size $p - 1$. [HOW MANY?] It can be shown that all the other vertices lie in a single, rather complicated, connnected component of $\Gamma(G)$.

Quite a bit has been discovered recently about the diameter of the commuting graph. It is known that, if $Z(G) = \{1\}$, then any connected component of $\Gamma(G)$ has diameter at most 10 (Morgan and Parker). However, Giudici and Parker found examples of groups (whose order is a power of 2) in which the commuting graph has arbitrarily large diameter.

Other things about the commuting graph are interesting. For this, we go back to using the whole group, not throwing away the centre. A maximal complete subgraph of the commuting graph must be an abelian subgroup of G. (For let S be a complete subgraph. Then all the elements of S commute, and so all the elements of the subgroup they generate commute; if S is maximal, it must be this subgroup.)

Ramsey's Theorem states that an infinite graph contains either an infinite complete subgraph or an infinite subgraph with no edges. So an infinite group has either an infinite abelian subgroup or an infinite set of elements of which no two commute. Bernhard Neumann showed that if there is no infinite abelian subgroup, then there is a bound on how large a finite abelian subgroup can be.

A graph is said to be *perfect* if, for the graph and all of its subgraphs, the clique number (size of the largest complete subgraph) is equal to the chromatic number (smallest number of colours needed in a colouring of the graph in which no two adjacent vertices get the same colour). Britnell and Gill have classified the *quasisimple groups* for which the commuting graph is perfect, but the general classification is not known.

To what extent does the commuting graph of a group determine the group? It is conjectured that $\Gamma(G)$ determines the order of G, and in the case of simple groups determines G up to isomorphism, but this is not known in general. (The reason the first part is not trivial is that we have to throw away the elements of $Z(G)$. So we have to figure out the order of $Z(G)$ from the structure of $\Gamma(G)$.

There are also interesting open questions about the automorphism group of the commuting graph of a group.

2 The power graph

I will define the power graph of a group G in two steps. First I define the *directed power graph* $\vec{P}(G)$: the vertices are the elements of G, and there is a directed edge from x to y if y is a power of x (that is, $y = x^m$ for some m). Now the *power graph* $P(G)$ is the same graph but ignoring directions: x is joined to y if one of x and y is a power of the other.

It seems clear that $\vec{P}(G)$ contains more information than $P(G)$. However, it has been proved that if G and H are groups for which $P(G)$ and $P(H)$ are isomorphic, then $\vec{P}(G)$ and $\vec{P}(H)$ are isomorphic. However, this does not mean that G and H are isomorphic. For example, there are non-abelian groups G in which every non-identity element has order 3, having order 3^n for any $n \geq 3$; of course there are also abelian groups with this property (direct products of cyclic groups of order 3). For any such group, the power graph is a bunch of triangles with a common vertex.

Not so much is known about the power graph. But if y is a power of x, then x and y commute; so the power graph is a subgraph of the commuting graph (that is, it consists of all the vertices and some of the edges). Which groups have the property that these two graphs are equal? I don't know!

3 The generation graph

This graph only makes sense for groups which can be generated by two elements. The *generation graph* of G is the graph whose vertex set is G, and in which x and y are joined if and only if x and y generate G. A complete subgraph of the generation graph is thus a set of elements with the property that any two of them generate the group.

Any finite simple group can be generated by two elements. [We only know this because we have a classification of the finite simple groups, but this is an enormously complicated theorem.] So most research on the generation graph has focussed on groups which are simple, or almost simple.

But I think there are other interesting questions which could be asked. For example, let G be a non-abelian group which can be generated by two elements. Then two elements which generate the group cannot commute [else the whole group would be abelian!] So the generation graph is a subgraph of the *complement* of the commuting graph. For which groups does equality hold?

In any finite group G, the set of elements which can always be omitted from any generating set forms a subgroup, called the *Frattini subgroup* and denoted by $\Phi(G)$. (Notes about this in an appendix.) If G can be generated by two elements but is not cyclic, then the elements of the Frattini subgroup are *isolated vertices*, not joined to anything else [WHY?] But there can be more isolated vertices. For example, if G is the symmetric group S_4, then the Frattini subgroup of G is the trivial group $\{1\}$, but the "double transpositions" are all isolated vertices of the generation graph. Note that they and the identity form a normal subgroup of G. For which 2-generator groups do the isolated vertices form a normal subgroup?

What can one say about the automorphism group of the generation graph, or about groups which are determined up to isomorphism by knowledge of their generation graph?

4 Cayley graphs

The subject of Cayley graphs is a big research area.

Suppose that G is a group with a subset G, not containing the identity. Form a directed graph $\vec{X}(G,S)$ as follows: the vertices are the elements of G; and there is a directed edge from x to sx for every element $s \in S$. The undirected Cayley graph $X(G,S)$ is obtained by forgetting the directions; or, we can enlarge S by including the inverses of all its elements. (Note that $y = sx$ if and only if $x = s^{-1}y$.)

One feature of the Cayley graph is that we can see G acting on it by right multiplication. If there is an edge from x to y, then $sx = y$ for some $s \in S$; so $s(xg) = (sx)g = yg$, using the associative law, and so there is an edge from xg to yg. This says that the map $\rho_g : x \mapsto xg$ is an automorphism of the graph. The set of all these maps ρ_g for $g \in G$ is a group isomorphic to G. Indeed, this is the action of G that is used in the proof of *Cayley's theorem*, stating that every group is isomorphic to a subgroup of some symmetric group. (Here we have G as a subgroup of the symmetric group on the elements of G.)

An important fact about Cayley graphs is:

Proposition 1 *The Cayley graph $X(G,S)$ is connected if and only if G is generated by the set S.*

For consider the vertices that we can reach by starting at the identity. We can assume that the set S contains the inverses of its elements. So starting at the identity element 1, we can move to $s_1^{\pm 1}$, then to $s_2^{\pm 1}s_1^{\pm 1}$, then to $s_3^{\pm 1}s_2^{\pm 1}s_1^{\pm 1}$, and so on; we only reach elements of the subgroup generated by S. $\square$

Another property is based on this fact. We call a directed graph *strongly connected* if there is a directed path from x to y for any choice of vertices x and y: in other words, we are required to follow the directions of the edges. It is *connected* if there is a path from x to y which is allowed to ignore directions. (This is the difference between driving and walking in a town with many one-way streets.)

Proposition 2 *If a finite graph has a group which acts* transitively *on its vertices, then it is connected if and only if it is strongly connected.*

One way round is clear: strongly connected implies connected. I will show you why the converse holds. For any vertex x, let $A(x)$ be the set of all vertices that can be reached by following a directed path starting at x. Then we have:

- $x \in A(x)$. (Follow a path of length 0.)
- if $y \in A(x)$, then $A(y) \subseteq A(x)$. (For if $z \in A(y)$, there is a directed path from y to z; prefixing this with a path from x to y gives a path from x to z, so $z \in A(x)$.
- If a group of automorphisms acts transitively on the vertices, then all the sets $A(x)$ have the same size. (For an automorphism which maps x to y must map $A(x)$ to $A(y)$.)

But this means that, if $y \in A(x)$, then $A(y) \subseteq A(x)$ and $|A(y)| = |A(x)|$; so it must be that $A(y) = A(x)$, whence $x \in A(y)$. So, if you can get from x to y, then you can get from y back to x; thus connectedness and strong connectedness agree. $\square$

In particular, if the Cayley graph is connected (that is, if S generates G), then it is strongly connected.

Now we see that the generation graph and the Cayley graphs for 2-element generating sets should have something to do with one another. Can we use this fact for anything?

Appendix: the Frattini subgroup

Proposition 3 *The following properties of an element x of a finite group G are equivalent:*

(a) x lies in every maximal subgroup of G;

(b) x can be deleted from any generating set for G.

For suppose that x satisfies (a), and let $\{x, y_1, \ldots, y_r\}$ generate G. Suppose taht $\{y_1, \ldots, y_r\}$ does not generate G. Then it does generate a proper subgroup of G, which is therefore contained in some maximal subgroup M.

By assumption, x is also contained in M, and so the whole generating set of G is contained in M, which is impossible. So (b) holds.

Suppose conversely that (b) holds. To prove (a), we argue by contradiction, and suppose that there is a maximal subgroup M of G which does not contain x. Let $\{y_1, \ldots, y_r\}$ be a set of elements which generates M. Then as $x \notin M$, the set $\{x, y_1, \ldots, y_r\}$ must generate a subgroup larger than M, which must be all of G; and clearly x cannot be dropped from this generating set, contradicting (b). So (a) must be true. $\square$

Peter J. Cameron
June 2015