SICP Exercise 1.13

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Fibonacci Definition:

$$\mathrm{Fib}(n) = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ \mathrm{Fib}(n-2) + \mathrm{Fib}(n-1), & n > 1 \end{cases}$$

Assertion:

$$\mathrm{Fib}(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}, \text{ where } \phi = \frac{1+\sqrt{5}}{2} \text{ and } \psi = \frac{1-\sqrt{5}}{2}$$

Proof:

$$\phi \cdot \psi = \frac{1-5}{4} = -1$$

$$\phi + \psi = \frac{2}{2} = 1$$

$$\mathrm{Fib}(0) = \frac{\phi^0 - \psi^0}{\sqrt{5}} = 0$$

$$\mathrm{Fib}(1) = \frac{\phi^1 - \psi^1}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$$

$$\begin{aligned}
\mathrm{Fib}(n) &= \frac{\phi^n - \psi^n}{\sqrt{5}} \\
&= \frac{\phi^{n-1}\phi - \psi^{n-1}\psi}{\sqrt{5}} \\
&= \frac{\phi^{n-1}(1-\psi) - \psi^{n-1}(1-\phi)}{\sqrt{5}} \\
&= \frac{\phi^{n-1} - \phi^{n-1}\psi - \psi^{n-1} + \psi^{n-1}\phi}{\sqrt{5}} \\
&= \frac{\phi^{n-1} + \phi^{n-2} - \psi^{n-1} - \psi^{n-2}}{\sqrt{5}} \\
&= \frac{\phi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\phi^{n-2} - \psi^{n-2}}{\sqrt{5}} \\
&= \mathrm{Fib}(n-2) + \mathrm{Fib}(n-1)
\end{aligned}$$

Note

Because $\forall n > 0, |\psi^n| < \frac{\sqrt{5}}{2}$, the closest integer to $\frac{\phi^n}{\sqrt{5}}$ is equal to $\frac{\phi^n - \psi^n}{\sqrt{5}}$ which is equal to $\mathrm{Fib}(n)$.