Exchangeability and de Finetti's theorems

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Chapter 1

Introduction

This blueprint documents the formalization of **de Finetti's theorem** and the **de Finetti–Ryll-Nardzewski equivalence** for infinite sequences on *standard Borel spaces*.

The main result establishes a three-way equivalence between:

- **Contractable**: All strictly increasing subsequences of equal length have the same distribution
- **Exchangeable**: Distribution invariant under finite permutations
- **Conditionally i.i.d.**: There exists a probability kernel such that finite marginals equal mixtures of product measures

We formalize *all three proofs* from Kallenberg (2005):

1. **Koopman/Ergodic approach** using the Mean Ergodic Theorem
2. **$\mathbf{L}^2$ approach** using elementary contractability bounds
3. **Martingale approach** using reverse martingale convergence (after Aldous)

Chapter 2

Foundations

2.1 Core Definitions

Definition 1 (Exchangeable sequence)**.** A sequence $(X_n)_{n\in\mathbb{N}}$ of random variables is *exchangeable* with respect to measure μ if for every $n \in \mathbb{N}$ and every permutation σ of $\{0, \dots, n-1\}$, the joint distribution of $(X_{\sigma(0)}, \dots, X_{\sigma(n-1)})$ equals that of $(X_0, \dots, X_{n-1})$.

Definition 2 (Contractable sequence)**.** A sequence $(X_n)_{n\in\mathbb{N}}$ is *contractable* with respect to measure μ if for all $m \in \mathbb{N}$ and all strictly increasing functions $k, k' : \mathrm{Fin}(m) \to \mathbb{N}$, the joint distribution of $(X_{k(0)}, \dots, X_{k(m-1)})$ equals that of $(X_{k'(0)}, \dots, X_{k'(m-1)})$.

Definition 3 (Conditionally i.i.d. sequence)**.** A sequence $(X_n)_{n\in\mathbb{N}}$ is *conditionally i.i.d.* with respect to measure μ if there exists a probability kernel $\nu : \Omega \to \mathrm{Measure}(\alpha)$ such that for any strictly monotone $k : \mathrm{Fin}(m) \to \mathbb{N}$, the joint distribution of $(X_{k(0)}, \dots, X_{k(m-1)})$ equals the mixture $\mu.\mathrm{bind}(\omega \mapsto \nu(\omega)^{\otimes m})$.

Definition 4 (Tail σ-algebra)**.** The *tail σ-algebra* of a sequence (X_n) is $\mathcal{T} = \bigcap_{n=0}^{\infty} \sigma(X_n, X_{n+1}, \dots)$.

Chapter 3

Easy Directions

3.1 Exchangeable implies Contractable

Lemma 5 (Permutation extension)**.** *Any strictly increasing function* $k : \mathrm{Fin}(m) \to \mathbb{N}$ *with range contained in* $\{0, \ldots, n-1\}$ *extends to a permutation of* $\{0, \ldots, n-1\}$*.*

Theorem 6 (Exchangeable implies Contractable)**.** *If* (X_n) *is exchangeable, then it is contractable.*

3.2 Conditionally i.i.d. implies Exchangeable

Theorem 7 (Conditionally i.i.d. implies Exchangeable)**.** *If* (X_n) *is conditionally i.i.d., then it is exchangeable.*

Chapter 4

Main Implication: Contractable implies Conditionally i.i.d.

This is the deep direction of de Finetti's theorem. We formalize three independent proofs.

4.1 Via Martingale (Aldous' proof)

Definition 8 (Future filtration)**.** The *future filtration* at level m is $\mathcal{F}_m = \sigma(X_{m+1}, X_{m+2}, ...)$.

Lemma 9 (Pair law equality)**.** *For a contractable sequence, the joint distribution of* $(X_k, \theta_{m+1}X)$ *equals that of* $(X_m, \theta_{m+1}X)$ *for all* $k \leq m$*, where* θ_n *is the shift operator.*

Lemma 10 (Kallenberg chain lemma)**.** *Conditional expectations of indicators given the reverse filtration converge to conditional expectations given the tail* σ*-algebra.*

Lemma 11 (Conditional expectation convergence)**.** *For* $k \leq m$ *and measurable* B*:* $\mathbb{E}[\mathbf{1}_{X_m \in B} \mid \mathcal{F}_m] = \mathbb{E}[\mathbf{1}_{X_k \in B} \mid \mathcal{F}_m]$ *a.s.*

Lemma 12 (Extreme members equal on tail)**.** *For any measurable* B*:* $\mathbb{E}[\mathbf{1}_{X_m \in B} \mid \mathcal{T}] = \mathbb{E}[\mathbf{1}_{X_0 \in B} \mid \mathcal{T}]$ *a.s.*

Theorem 13 (Contractable implies Conditionally i.i.d. (via Martingale))**.** *If* (X_n) *is contractable, then it is conditionally i.i.d. The directing kernel* $\nu(\omega)(B) = \mathbb{E}[\mathbf{1}_{X_0 \in B} \mid \mathcal{T}](\omega)$ *is constructed from the tail* σ*-algebra.*

4.2 Via L^2 (Elementary proof)

The L^2 approach uses elementary contractability bounds on block averages. This is Kallenberg's "second proof" and has the lightest dependencies.

Note: This proof applies to *real-valued* sequences ($X : \mathbb{N} \to \Omega \to \mathbb{R}$) with L^2 integrability (i.e., $\mathbb{E}[X_i^2] < \infty$ for all i).

Lemma 14 (L^2 contractability bound)**.** *For contractable sequences, certain* L^2 *norms of block averages are bounded.*

Theorem 15 (Contractable implies Conditionally i.i.d. (via L^2))**.** *If (X_n) is contractable, then it is conditionally i.i.d.*

4.3 Via Koopman (Mean Ergodic Theorem)

The Koopman approach uses the Mean Ergodic Theorem via the shift operator on L^2. This is Kallenberg's "first proof" and uses disjoint-block averaging.

Definition 16 (Block average)**.** The *block average* $A_{m,n,k}(f)(\omega) = \frac{1}{n}\sum_{j=0}^{n-1} f(\omega_{k \cdot n+j})$ averages f over the k-th block of size n (indices $[kn, kn+n)$). For $n = 0$, the block average is defined as 0.

Lemma 17 (Block averages converge in L^1)**.** *For a shift-invariant measure, block averages converge in L^1 to the conditional expectation given the shift-invariant σ-algebra: $\int |\bar{A}_{m,n,k}(f) - \mathbb{E}[f \circ \pi_0 \mid \mathcal{I}]| \, d\mu \to 0$ as $n \to \infty$.*

Lemma 18 (Integral product equals block average product)**.** *For contractable sequences, integrals of products factor through block averages.*

Lemma 19 (Bridge from contractability)**.** *For contractable sequences, indicator products satisfy the bridge condition.*

Theorem 20 (Contractable implies Conditionally i.i.d. (via Koopman))**.** *If (X_n) is contractable, then it is conditionally i.i.d. This proof uses the Mean Ergodic Theorem via the Koopman operator on L^2.*

Chapter 5

Common Ending

All three proofs converge to the same final step: extending from indicators to general sets via a monotone class argument.

Lemma 21 (π-system uniqueness)**.** *Measures on product spaces are determined by their finite-dimensional marginals.*

Theorem 22 (Conditional independence extension)**.** *Conditional independence on indicators extends to the full product σ-algebra.*

Chapter 6

Main Theorem

Theorem 23 (de Finetti–Ryll-Nardzewski equivalence)**.** *For an infinite sequence* $(X_n)_{n\in\mathbb{N}}$ *of random variables taking values in a* standard Borel space α *(with* α *nonempty), the following are equivalent:*

1. (X_n) *is contractable*
2. (X_n) *is exchangeable*
3. (X_n) *is conditionally i.i.d. (i.e., there exists a directing kernel* ν*)*

Remark: *The martingale proof constructs* ν *from the tail* σ*-algebra* $\mathcal{T}$ *via* $\nu(\omega)(B) = \mathbb{E}[\mathbf{1}_{X_0\in B} \mid \mathcal{T}](\omega)$.