Metrics-Math Bootcamp Day 1

Cameron Taylor

August 12, 2019

Stanford GSB

Overview

1. Intro to Metrics
2. Intro to Stats
3. Intro to Linear Algebra
4. Matrices and Stats in R

Metrics

What is Metrics?

- "Econometrics is the application of statistical methods to economic data in order to give empirical content to economic relationships."
- Economics research broadly consists of three things: theory, data, and econometrics
- Econometrics is the thing that allows you to say something about the theory with the data you have
  - Note that data is an important input! "Garbage-in-garbage-out"
- Common empirical idea/questions: What is causal effect of X on Y? What will be the impact of changing from policy A to policy B?

Brief General Ideas

- Empirical research generally consists of specifying a model of the data X and some parameters θ, $M(X, \theta)$, and then optimizing some criteria of the model to back out which parameters seem "best" given the data
- OLS will choose parameters θ linear in model, and then minimize the sum of the squared deviations between model predictions and data
- MLE and GMM will also do something similar
- Key ideas in each come from statistics and require linear algebra - we will go over both very briefly (MGTECON 603 does this in more depth - a good class that many RF's before took)

Stats

Basic Stats in Econometrics

- 5/7 responses: have seen probability and stats with calculus, so mostly refresh
- As stated, metrics combines stats and economics, so stats important input
- We will go over: random variable basics, expectation, covariance/correlation, conditional expectation

Random variables

- A random variable X takes values from some sample space Ω into a measurable space, which for now will use $\mathbb{R}$ (so it is a function)
- Ω also specifies probability of different events happening
- $Pr(X = x) = Pr\Big(\{\omega \in \Omega | X(\omega) = x\}\Big)$
- Examples
    - Number on die role: $\Omega = \{1, 2, 3, 4, 5, 6\}$, uniform probability distribution, $X(\omega) = \omega$.
    - Coin flip outcome is heads: $\Omega = \{H, T\}$, $Pr(\omega = H) = Pr(\omega = T) = 1/2$. $X(\omega) = 1$ when $\omega = H$.
- Gives us flexible way to express economic quantities that are random
- Can be continuous and/or discrete - usually work with continuous stuff

Distribution and Expectation

- How to describe a random variable?
- A "full" way to do it: CDF $Pr(X \leq x) = F(x)$ and PDF $Pr(X = x) = f(x)$
    - Why full? Any info we want about the random variable comes from this
- Don't usually take this approach unless doing some descriptive stats because: (1) hard to report and (2) hard to estimate the whole thing
- Instead, usually report other summaries of data
- A particularly useful one: **expectation** $E(X) = \int xf(x)dx$ ("average"); can generalize to functions $E(g(X)) = \int g(x)f(x)dx$
- Also care about "spread"- variance $V(X) = E(X^2) - E(X)^2$
- There are many nice properties about expectations that one can derive: the best one is probably linearity, can look up online

Joint Dist

- While summarizing a single variable interesting, economics more concerned with relationship between multiple variables
- Consider joint distribution of (Y, X), two random variables or a random vector
- These also have a joint dist which can be described by $f(x, y) \approx Pr(X = x \text{ and } Y = y)$
- Similarly though, want different ways to summarize

Covariance, Correlation and Conditional Expectation

- $Cov(X, Y) = E(XY) - E(X)E(Y)$ a way to express how X and Y move together
- $Cor(X, Y)$ normalizes the covariance to be in $[-1, 1]$
- A particular object of interest is $E[Y|X]$, the conditional expectation function (CEF). This is a random variable since X is a random variable
- $E[Y|X = x]$ can be seen as "best guess of Y given $X = x$" (See Mostly Harmless Thm 3.1.2)
- Note that we can always write $Y = E[Y|X] + \epsilon$ where $E[\epsilon|X] = 0$ (Why? HW!)
- A useful tool: Law of Iterated Expectations: $E(Y) = E(E(Y|X))$

Population to Sample

- So far everything has been abstract modeling - at population level
- What do we actually do with data? Example: how to calculate or approximate $E(X)$ with data $(x_1, \ldots, x_n)$
- General idea: sample analog of population: $\hat{E}(X) = \bar{x} = \frac{1}{n}\sum_i x_i$ (this is like a numerical integral)
- Then can say things about how close this will be to true mean as get more or less data (this is inference)
- Example inference: note that $E(\bar{x}) = E(X)$ and $V(\bar{x}) = \frac{1}{n}V(X)$ so as $n \to \infty$, $V(\bar{x}) \to 0$ if $V(X) < \infty$ so we can be more confident in the estimate

Key Statistical Concepts for Metrics: Small Sample

- Bias
    - An estimator is unbiased if its expectation equals the population object
    - Key tools: Properties of expectations, law of iterated expectations
    - Ex: We will see that OLS is unbiased under some assumptions
- Variance
    - An estimator's variance measures how much it deviates between different samples
    - Key tools: Properties of expectation, variance
- Note: Estimators/statistics will be random because they are functions of data which are random!
- Note: An important idea in stats/econometrics is the bias/variance trade-off. More present in non-parametric methods where need to "tune"

Key Statistical Concepts for Metrics: Asymptotics

- Consistency
    - An estimator is consistent if it "converges in probability" to the true object (can make as close in probability space with more data)
    - Key tool: Law of Large Numbers
    - Idea: The average is consistent in probability under general conditions
- Asymptotic Distribution
    - Since estimators random, will have an asymptotic distribution that they approach as we use more data
    - This is useful because then we can do "asymptotic inference" with estimators
    - Key tool: Central Limit Theorem
    - Idea: The sum of independent random variables converges to a normal distribution

Linear Algebra

Overview of LA

- 6/7 have seen LA so I will cruise through stuff like intro to matrices, etc. and focus on more advanced stuff important to metrics

Linear Algebra

- Linear Algebra is an area of mathematics studying linear mappings between vector spaces
- The most common vector space is Euclidean space $\mathbb{R}^n$, the space of n-dimensional vectors of real numbers. Linear mappings between Euclidean spaces are represented by matrices.
- A solid understanding of matrices and vectors is essential for studying econometrics

Matrices

- A *matrix* is a two dimensional grid

$$\mathcal{M} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

- Each *row* is read left to right: the first row of the above matrix is $\begin{bmatrix} a & b & c \end{bmatrix}$
- Each *column* is read top to bottom: the first column of the above matrix is $\begin{bmatrix} a \\ d \end{bmatrix}$

Matrix Transpose

- The transpose of a matrix $\mathcal{M}$ is denoted $\mathcal{M}'$ - you rotate the entries

$$\mathcal{M} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

$$\mathcal{M}' = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

- The transpose of an $N \times K$ matrix is thus a $K \times N$ matrix.
- Importantly: $(AB)' = B'A'$ and $A^{-1'} = A^{'-1}$

Matrix Multiplication

- Let A be an $N \times K$ matrix, and let B be a $K \times M$ matrix. The *matrix product* AB is a $N \times M$ matrix with $[AB]_{ij} = \sum_l A_{il} B_{lj}$
- For this to be well defined A must the same number of columns as B has rows.

$$(N \times K) \times (K \times M) = (N \times M)$$

$$(N \times K) \times (J \times M) = \text{Not possible!}$$

Identity Matrix

- A *square* matrix is an $N \times N$ matrix, i.e., a matrix with the same number of rows and columns
- A *diagonal matrix* is a square matrix whose nondiagonal entries equal 0.
- The N dimensional *Identity matrix*, often denoted I_N or just I, is a square matrix whose diagonal entries are all 1:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Can verify that for any $M \times N$ matrix A, $AI_N = A$ and $I_N A' = A'$, hence identity

Matrix Inverse

- Some square matrices are *invertible*. An $N \times N$ matrix $\mathcal{M}$ is invertible if and only if there exists an $N \times N$ matrix $\mathcal{M}^{-1}$ such that

$$\mathcal{M}^{-1}\mathcal{M} = \mathcal{M}\mathcal{M}^{-1} = I_N$$

- And we call $\mathcal{M}^{-1}$ the inverse of $\mathcal{M}$. It is not hard to prove that if an inverse exists, it is unique.
- There is a deep result that almost all square matrices are invertible, more on this later. The inverse is not well defined for nonsquare matrices.

Matrix Addition

- Matrix addition works exactly the way you want it to. Let A and B be $N \times K$ matrices:
$$[A + B]_{ij} = A_{ij} + B_{ij}$$
- Matrix addition is only defined if A and B have the same dimensions
- Componentwise multiplication or addition by a scalar (number) is also straightforward: If A is a matrix and c is a number, cA the matrix A with each element multiplied by c. Informally, $c + A$ is the matrix whose ij-th element is $c + A_{ij}$.

Linear Combinations

- Recall a vector is a $N \times 1$ or $1 \times N$ dimensional matrix.
- Let $V^1, V^2, ..., V^K$ be a collection of K column vectors, or $N \times 1$ dimensional matrices
- We just learned that the sum $\sum_{i=1}^{K} V^i$ of the K vectors will also be a $N \times 1$ dimensional column vector. We also learned that if we multiply V^i by a scalar c_i, then $c_i V^i$ is an $N \times 1$ dimensional vector.
- A *linear combination* of the vectors $V^1, ..., V^K$ is the sum of the scalar multiples $\sum_{i=1}^{K} c_i V^i$ for any arbitrary numbers $c_1, c_2, ..., c_K$
- The same concept applies to a collection of $1 \times N$ dimensional row vectors

Linear Independence

- A collection of column vectors $V^1, V^2, ..., V^K$ is said to be *linearly dependent* if one of the vectors V^l can be written as a linear combination of the other $K-1$ vectors:

$$V^l = \sum_{i \neq l} c_i V^i \text{ for some } \{c_i\}_{i \neq l}$$

- A collection of column vectors $V^1, V^2, ..., V^K$ is *linearly independent* if they are NOT linearly dependent.
- For example, $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ is a linearly dependent collection, because

$$1 * \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1) * \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Rank

- For any $N \times K$ matrix A, the K columns of A are a collection of $N \times 1$ dimensional column vectors.
- The *column rank* of a $N \times K$ matrix A is the maximum number of columns that can form a linearly independent collection. For example, the 2×3 matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

  has a column rank of 2: we just saw that the three vectors are linearly dependent, but
  $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ are linearly independent.

Column Space

- For any $N \times K$ matrix A, the *column space* of A is the set of all column vectors that can be written as a linear combination of the columns of A.
- A linear combination of the columns of A can be written as $\underbrace{A}_{N\times K}\underbrace{b}_{K\times 1}$ where b_{i1} is the scalar applied to column i of A.
- For any collection of vectors V, we say the columns of A *span* V if for any vector $v \in V$, there exists a vector b such that $\underbrace{v}_{N\times 1} = \underbrace{A}_{N\times K}\underbrace{b}_{K\times 1}$.
- By definition, the columns of A span the column space of A. An equivalent definition of the rank of A is the smallest number r such that the columns of some $N \times r$ dimensional matrix M span the column space of A.

Full Rank and Invertibility

- For any $N \times K$ matrix A, we say the matrix A has *full column rank* if it has column rank K.
- It turns out that for any $N \times K$ matrix A, the $K \times K$ square matrix $A'A$ is invertible if and only if A has full column rank.
- The $N \times N$ square matrix AA' is invertible if and only if A has full row rank.
- In general, an $N \times N$ matrix B is invertible if and only if B has full column rank.

Positive Definite Matrix

- A square $N \times N$ matrix $\mathcal{M}$ is *positive definite* if for any column vector $\underbrace{v}_{N\times 1}$, we have $\underbrace{v'\mathcal{M}v}_{1\times 1} > 0$
- Suppose $x^1, ..., x^K$ are random variables. The *covariance matrix* of the collection $\{x^1, ..., x^K\}$ is the $K \times K$ matrix C whose diagonal entries C_{ii} equal the variance of x^i, and whose off diagonal entries C_{ij} equal $Cov(x^i, x^j)$
- Every covariance matrix is positive definite, and *symmetric* which means $C = C'$.

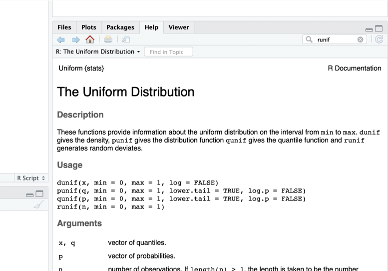
Stats and LA in R

Overview of Stats and LA in R

- Many of you are familiar with R or use something similar (like Python)
- For this bootcamp I don't care what language you use, but I think R is a good one (seems like many RFs used it in the past)
- More improtant that you get practice applying the concepts on real data than learning specific syntax
- Here: some basic stats and matrix algebra in R

Intro to R

- R has a nice editor (RStudio) and a great online community
- Some important basics
  - # symbol for comments
  - People use arrows to define variables (I don't know why) but you can use equals
  - The "Help" sidebar in RStudio is extremely useful



Simulating in R

- I think simulating is extremely useful for understanding basic metrics problems and working through examples
- In R usually specify r-DISTRIBUTION NAME to simulate data. First argument is number of draws, and then require parameter specifications.
- R stores draws in a vector...

```r
# Simulating data
n_samp <- 10
set.seed(1)
data1 <- rnorm(n=n_samp, mean=0, sd=1)
data2 <- runif(n=n_samp, min=0, max=1)
data1
data2
```

```
> data1
 [1] -0.6264538  0.1836433 -0.8356286  1.5952808  0.3295078 -0.8204684  0.4874291  0.7383247
 [9]  0.5757814 -0.3053884
> data2
 [1] 0.93470523 0.21214252 0.65167377 0.12555510 0.26722067 0.38611409 0.01339033 0.38238796
 [9] 0.86969085 0.34034900
```

Figure 2: Simulating

Vectors in R

- The notation c() is used for defining vectors
- Adding vectors and multiplying works very intuitively: for example a*b will multiply each entry of the vector if the same length

```
> # Define a vector
> a <- c(1,2,3)
> b <- c(4,5,6)
>   # Multiply by constant and vector
> a*2
[1] 2 4 6
> a*b
[1]  4 10 18
```

Figure 3: Vectors

Matrices in R

- There are multiple ways to define matrices
  - Can use the matrix() command
  - I prefer using cbind() or rbind() which combines matrices
- %*% is used for matrix multiplication (including multiplying vectors by matrices) (make sure dimensions match! R will tell you)

```
> # Define a matrix
>   # Directly
> m1 <- matrix(1, nrow=2, ncol=2)
>   # From vectors
> m2 <- cbind(c(1,0), c(0,1))
> m1
     [,1] [,2]
[1,]    1    1
[2,]    1    1
> m2
     [,1] [,2]
[1,]    1    0
[2,]    0    1
```

```
>   # Multiply matrices
> m1%*%m2
     [,1] [,2]
[1,]    1    1
[2,]    1    1
```

Statistics in R

- Some useful basic functions
  - mean(x) takes average of vector x
  - sd(x) takes standard deviation of vector x
  - lm() for running OLS (see Help in R for more details)
  - Many packages in R will give you standard errors and other important stats objects when you estimate (ex: lm() for OLS)
- Combining these functions with simulation tools allows you to test your intuition and experiment

Wrap-Up

- Hopefully you have an idea of what metrics is, and how stats and linear algebra fit into it
- Also you have gotten familiar with R or a comparable language so you can practice

Some Extra Resources for Today's Content

- For intro to metrics: Mostly Harmless has a nice intro, Angrist and Pishcke have a nice JEP piece (2010)
  - For an alternate viewpoint: can look at Frank Wolak and Peter Reiss *Structural Econometric Modeling: Rationales and Examples from Industrial Organiziation* - talks about reduced form vs. structural
  - Also Deaton (2010) on randomized control trials
- For intro stats for metrics (and general tools related to metrics): MGTECON 603 is a good intro, even a bit of overkill
  - Goes over some stuff we did not have time to do from stat perspective: MLE, hypothesis testing, suff stats, more on probability theory, etc.
  - Accompanying book (which I like): Casella and Berger *Statistical Inference*
- For linear algebra: I really like Gilbert Strang *Intro to Linear Algebra*

HW

1. Prove the CEF property: we can always write $Y = E[Y|X] + \epsilon$ where $E[\epsilon|X] = 0$ and show that ϵ is uncorrelated with any function of X.
2. Calculate $V(\bar{x})$ where $\bar{x}$ is the mean of N i.i.d. draws $\{x_i\}_{i=1}^{N}$ where $V(x_i) = \sigma^2$. Calculate it when the draws are independent but not identical and $V(x_i) = \sigma_i^2$ (make sure you know what i.i.d. means!). Bonus: under what conditions will this go to 0 as $n \to \infty$?
3. Look up idempotent matrices. Prove that $I - X'(X'X)^{-1}X'$ is idempotent.
4. Plot the true and empirical CEF from simulated data with $N = 100$ for (Y, X) where $Y = X + 3 * X^2 + \epsilon$, $X \sim U\{0, \ldots, 10\}$ (uniform distribution over integers 0 to 10) and $\epsilon \sim N(0, 1)$. Also try it for $N = 1000$. Note: I do NOT want you to run a linear model, but to approximate the CEF using conditional means.