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Math 4005
Assignment 03

1. Assume $X \sim U(a,b)$. Density is $f(x) = \int_a^b \frac{1}{b-a}\,dx$

1.a. Prove $\mathbb{E}[X]$, the expected value of X, is $\frac{a+b}{2}$

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} x\,\mathbb{P}(x)\,dx$$

$$= \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\int_a^b x\,dx$$

$$= \frac{1}{b-a} \cdot \frac{x^2}{2}\Big|_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right]$$

$$= \frac{1}{\cancel{b-a}}\left[\frac{\cancel{(b-a)}(b+a)}{2}\right]$$

$$\boxed{= \frac{b+a}{2}}$$

1b. Prove $\mathbb{V}[X]$, the variance of X, is $\frac{(b-a)^2}{12}$

$$\mathbb{V}[X] = \mathbb{E}[X^2] - [\mathbb{E}[X]]^2$$

$$\mathbb{E}[X^2] = \int_a^b x^2 \frac{1}{(b-a)}\,dx$$

$$= \frac{1}{b-a}\int_a^b x^2\,dx$$

$$= \frac{1}{b-a}\cdot\frac{x^3}{3}\Big|_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^3}{3} - \frac{a^3}{3}\right]$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}} = \frac{b^2+ab+a^2}{3} = \mathbb{E}[X^2]$$

$$\frac{b^2+ab+a^2}{3} - \left[\frac{a+b}{2}\right]^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4b^2+4ab+4a^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

2.a. Let $k=2, n=5, p=0.5$

$$P_k = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{5!}{2!\,3!} = \frac{5 \cdot 4}{2} = 10$$

$$= 10 \cdot (0.5)^2 (0.5)^3$$

$$= 10(0.25)(0.125)$$

$= 0.3125$ or 31.25%

2b. $\binom{5}{4} = \frac{5!}{4!\,1!} = 5$

$P(4) = 5 \cdot (0.5)^4 (0.5)^1$

$= 5 \cdot 0.0625 (0.5)$

$= 0.15625 \rightarrow$ getting exactly 4 Heads

$\binom{5}{5} = \frac{5!}{5!(1!)} = 1$

$P(5) = 1 \cdot (0.5)^5 (0.5)^0$

$= 0.03125 \rightarrow$ getting exactly 5 Heads

$0.15625 + 0.03125 = 0.1875$ or 18.75%

3.a. $P_i = \dfrac{e^{-\lambda} \lambda^{i}}{i!}$ $\quad i = 6, \lambda = 10$

$P_6 = \dfrac{e^{-10}(10)^6}{6!} \approx 0.0631$ or about 6.31%

3b. $P_0 + P_1 + P_2 + P_3$

$P_0 = \dfrac{e^{-10}\, 10^{0}}{0!} \approx 0.0000454$ (100), about [illegible]%

$P_1 = \dfrac{e^{-10}(10)^1}{1!} \approx 0.000454$ (100) about [illegible]%

$P_2 = \dfrac{e^{-10}(10)^2}{2!} \approx 0.00227$ (100) about [illegible]%

$P_3 = \dfrac{e^{-10}(10)^3}{3!} \approx 0.00757$ (100) about [illegible]

$P(\leq 3) \approx 0.01034$

4. $\Gamma(10) = 9! = 362880$

5. $\beta(8,5)$

$$\beta(p,q) = \frac{\Gamma(p)\ \Gamma(q)}{\Gamma(p+q)}$$

$$= \frac{\Gamma(8)\ \Gamma(5)}{\Gamma(13)}$$

$$\approx 0.00025$$