# Introduction to probability theory and stochastic processes

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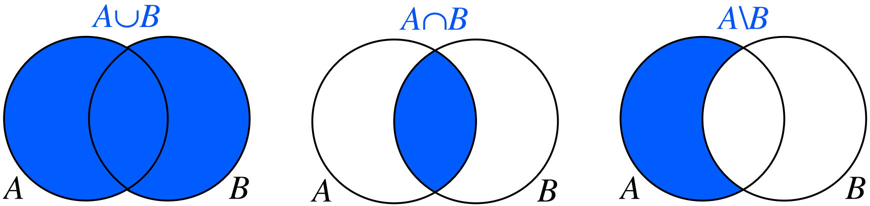
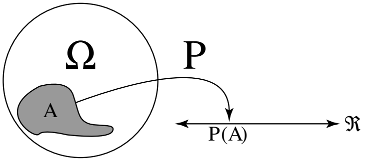
# Outline

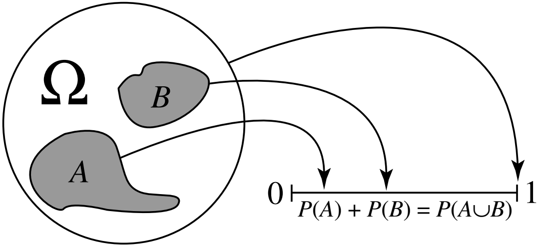
# sets



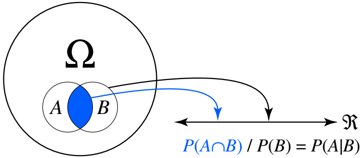
# probability measure



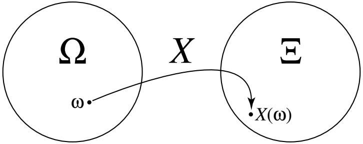
# probability axioms



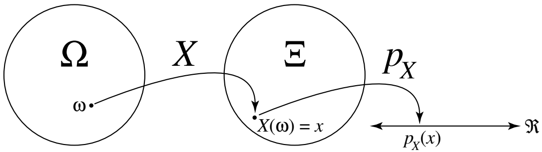
# conditional probability



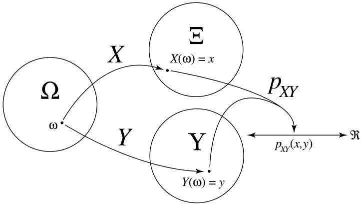
# random variables



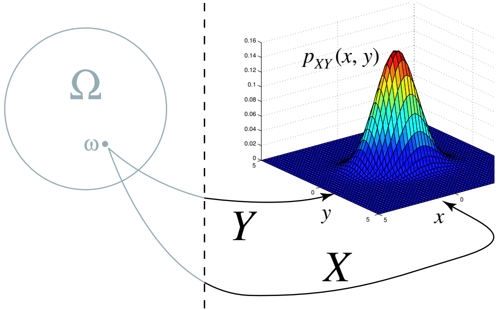
# probability densities



# joint probability densities



# the reality



- For the general theory of measure spaces, we first need a *measurable space* $(X, \Sigma)$, that is a set equipped with a collection $\Sigma$ of **measurable sets** complete under certain operations. Then this becomes a measure space $(X, \Sigma, \mu)$ by throwing in a function $\mu$ from $\Sigma$ to a space of values (such as the real line) that gets along with the set-theoretic operations that $\Sigma$ has. If $E$ is a measurable set, then $\mu(E)$ is called the **measure** of $E$ with respect to $\mu$. [1]

1. Given a set $X$,
2. a $\sigma$-**algebra** is a collection of subsets of $X$ that is closed under complementation, countable unions, and countable intersections.
3. A **measurable space**, by the usual modern definition, is a set $X$ equipped with a $\sigma$-algebra $\Sigma$.
4. The elements of $\Sigma$ are called the **measurable sets** of $X$ (or more properly, the measurable subsets of $(X, \Sigma)$). [2]

A **measure space** is a **measurable space** equipped with a **measure**. There are many different types of measures parametrized by the type of their codomains. Let $(X, \Sigma)$ be a measurable space. A **probability measure** on $X$ (due to Kolmogorov) is a function $\mu$ from the collection $\Sigma$ of measurable sets to the unit interval $[0, 1]$ such that:

1. The measure of the empty set is zero: $\mu(\emptyset) = 0$;
2. The measure of the entire space is one: $\mu(X) = 1$;
3. Countable additivity: $\mu(\bigcup_{i=1}^{\infty} S_i) = \sum_{i=1}^{\infty} \mu(S_i)$ whenever the $S_i$ are mutually disjoint sets—disjoint. (Part of the latter condition is the requirement that the sum on the right-hand side must converge.)

It is sometimes stated (but in fact follows from the previous) that:

- Finitary additivity: $\mu(S \cup T) = \mu(S) + \mu(T)$ whenever $S$ and $T$ are disjoint.
- $\mu$ is increasing: $\mu(A) \leq \mu(B)$ if $A \subseteq B$.

# Outline

Intuitively, **stochastic processes** are $\infty$-dimensional probability distributions.

- In most applications, stochastic processes model systems that evolve randomly in time, which is likely the origin of the word *process* in the name.
- The order of this evolution can be described through the use of an index and an index set respectively $t \in T$.

Definition

*Consider a random experiment with sample space $X$, a $\sigma$-algebra $\Sigma$, a base probability measure $\mu : \Sigma \to [0, 1]$, and a collection of random variables $S_t$ indexed by a set $T$. A* **stochastic process** *is then defined by the set $\{S_t, t \in T\}$.*

- This definition can be specialized to the case of discrete or continuous stochastic processes by taking the index set to be $T \in \mathbb{N}$ or $T \in \mathbb{R}_+$ respectively.

# Outline

Theoretical background

Stochastic processes

References

[1] David Houle, Christophe Pélabon, Günter P Wagner, and Thomas F Hansen.
Measurement and meaning in biology.
*The Quarterly review of biology*, 86(1):3–34, March 2011.

[2] Yee Whye Teh, Michael I Jordan, Matthew J Beal, and David M Blei.
Hierarchical Dirichlet Processes.
*Journal of the American Statistical Association*, 101(476):1566–1581, December 2006.