Machine Learning

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University of Illinois at Urbana-Champaign, 2018

L12: Structured Prediction (ILP, LP relaxation, message passing, graph cut)

Goals of this lecture

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- Getting to know structured inference algorithms

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Reading material:

Goals of this lecture

- Getting to know structured inference algorithms

Reading material:

- D. Koller and N. Friedman; Probabilistic Graphical Models: Principles and Techniques;

Recap: Inference Program

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$$\boldsymbol{y}^* = \arg\max_{\hat{\boldsymbol{y}}} \sum_r f_r(\hat{\boldsymbol{y}}_r)$$

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- Exhaustive search
- Dynamic programming

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Algorithms:

- Exhaustive search
- Dynamic programming
- Integer linear program
- Linear programming relaxation
- Message passing
- Graph-cut

Integer Linear Program

Example:

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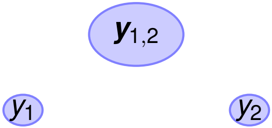

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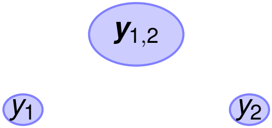

Integer Linear Program (LP) equivalence: variables $b_r(\boldsymbol{y}_r)$

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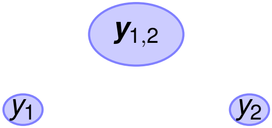

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$$\max_{b_1,b_2,b_{12}} \begin{bmatrix} b_1(1) \\ b_1(2) \\ b_2(1) \\ b_2(2) \\ b_{12}(1,1) \\ b_{12}(2,1) \\ b_{12}(1,2) \\ b_{12}(2,2) \end{bmatrix}^\top \begin{bmatrix} f_1(1) \\ f_1(2) \\ f_2(1) \\ f_2(2) \\ f_{12}(1,1) \\ f_{12}(2,1) \\ f_{12}(1,2) \\ f_{12}(2,2) \end{bmatrix}$$

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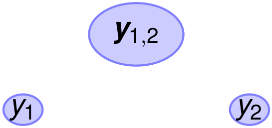

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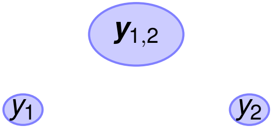

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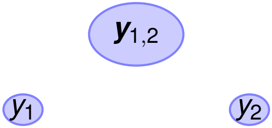

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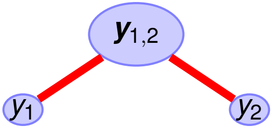

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Marginalization

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- **Advantage:** global optimum, very good solvers available
- **Disadvantage:** very slow for larger problems

Linear Programming Relaxation

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LP relaxation:

$$\cancel{b_r(\boldsymbol{y}_r) \in \{0,1\} \quad \forall r, \boldsymbol{y}_r}$$

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- **Advantage:** global optimum for LP, very good solvers available
- **Disadvantage:** no global optimum for ILP, slow for larger problems

Message Passing ([Loopy] Belief Propagation)

Exploit:

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Exploit: Graph structure defined via marginalization constraints

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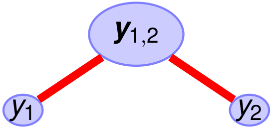

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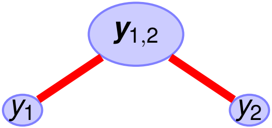

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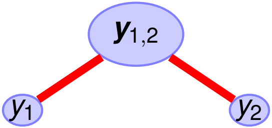

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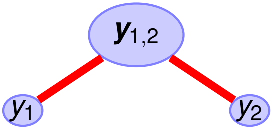

How: Compute the dual function

Message passing solvers:

- **Advantage:** Efficient due to analytically computable sub-problems
- **Disadvantage:** Special care required to find LP relaxation optimum

Computing the dual of

$$\max_b \sum_{r,\boldsymbol{y}_r} b_r(\boldsymbol{y}_r) f_r(\boldsymbol{y}_r) \quad \text{s.t.} \quad \left\{ \begin{array}{ll} b_r(\boldsymbol{y}_r) \geq 0 & \forall r, \boldsymbol{y}_r \\ \sum_{\boldsymbol{y}_r} b_r(\boldsymbol{y}_r) = 1 & \forall r \\ \sum_{\boldsymbol{y}_p \setminus \boldsymbol{y}_r} b_p(\boldsymbol{y}_p) = b_r(\boldsymbol{y}_r) & \forall r, p \in P(r), \boldsymbol{y}_r \end{array} \right.$$

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Lagrangian:

$$\begin{aligned} L() &= \sum_{r,\boldsymbol{y}_r} b_r(\boldsymbol{y}_r) f_r(\boldsymbol{y}_r) + \sum_{r,p \in P(r),\boldsymbol{y}_r} \lambda_{r\to p}(\boldsymbol{y}_r) \left(\sum_{\boldsymbol{y}_p \backslash \boldsymbol{y}_r} b_p(\boldsymbol{y}_p) - b_r(\boldsymbol{y}_r) \right) \\ &= \end{aligned}$$

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Lagrange multipliers are messages defined on edges of the graph. They shift 'energy' such that local maximization (dual) is identical to global maximization (primal).

Graph-cut Solvers

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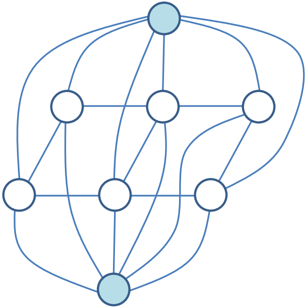
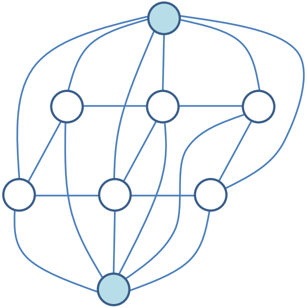


Graph-cut Solvers

For binary problems $y_d \in \{1, 2\}$:

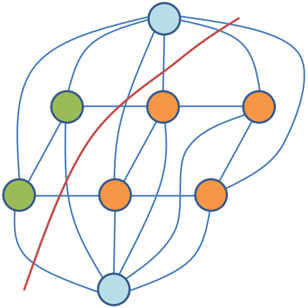
- Convert scoring function F into auxiliary graph (not the same graph as before!)



Graph-cut Solvers

For binary problems $y_d \in \{1, 2\}$:

- Convert scoring function F into auxiliary graph (not the same graph as before!)
- Compute a weighted cut cost corresponding to the labeling score



Graph-cut Solvers

For binary problems $y_d \in \{1, 2\}$:

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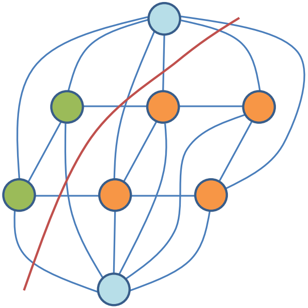
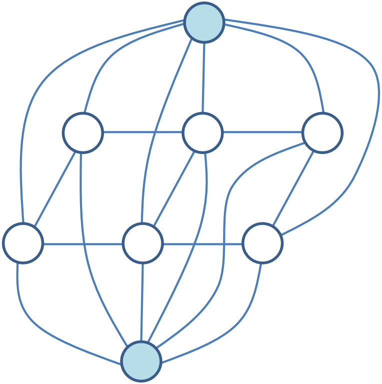


What are the nodes and what are the weights on the edges in this auxiliary graph?

Graph-cut Solvers

What are the nodes in the auxiliary graph?



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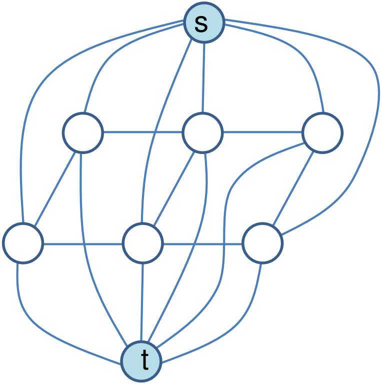

- Two special nodes called 'source' and 'terminal'

Graph-cut Solvers

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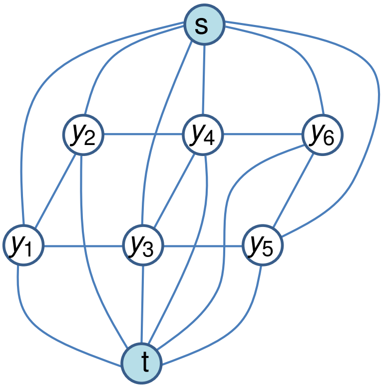

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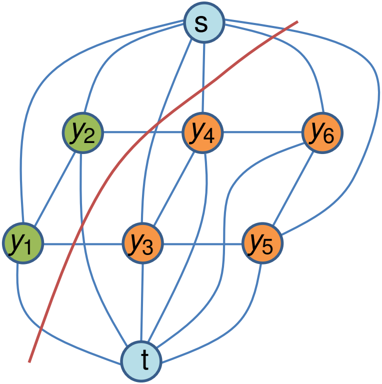

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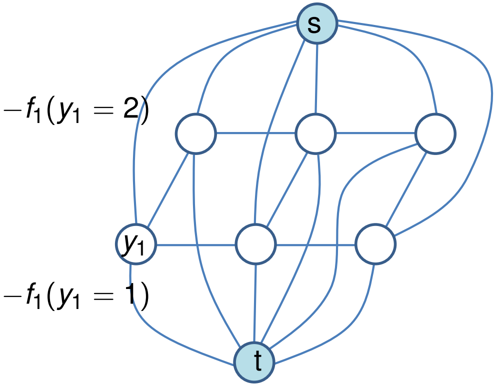

Graph-cut Solvers

What weights do we assign to edges?
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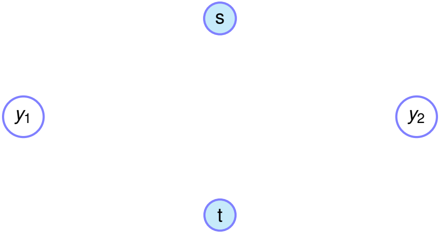

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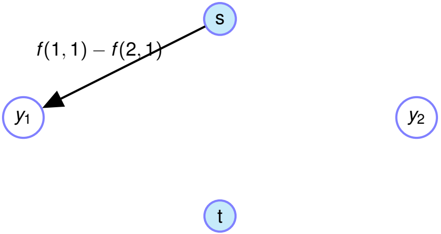

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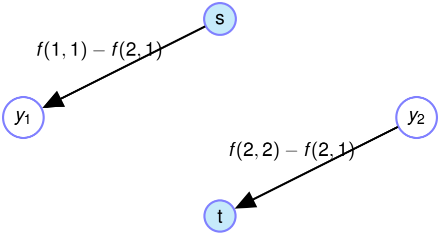

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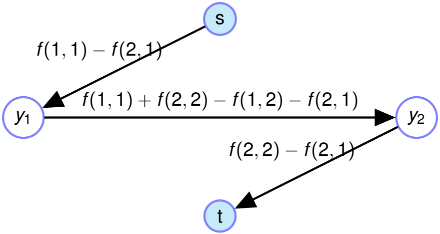
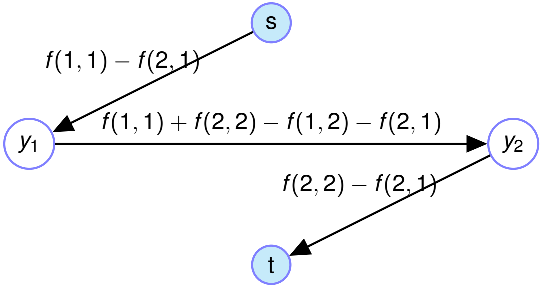
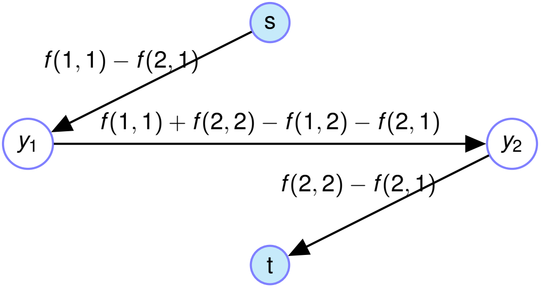


Graph-cut Solvers



Requirement for optimality:

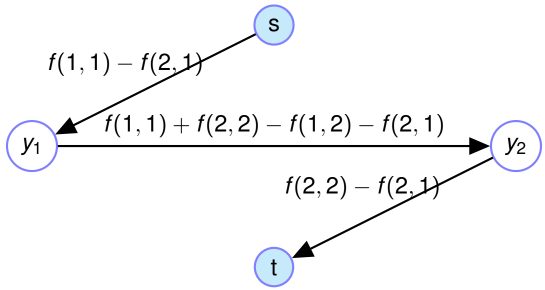
Graph-cut Solvers



Requirement for optimality: Pairwise edge weights are positive

$$f(1,1) + f(2,2) - f(1,2) - f(2,1) \geq 0 \quad \text{sub-modularity}$$

Graph-cut Solvers



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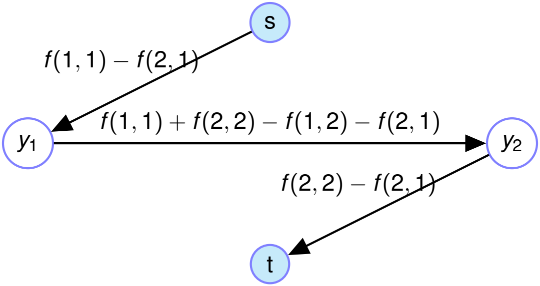
For higher order functions?

Graph-cut Solvers



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For higher order functions? More complicated graph constructions

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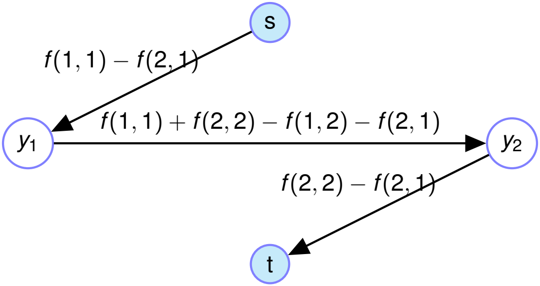

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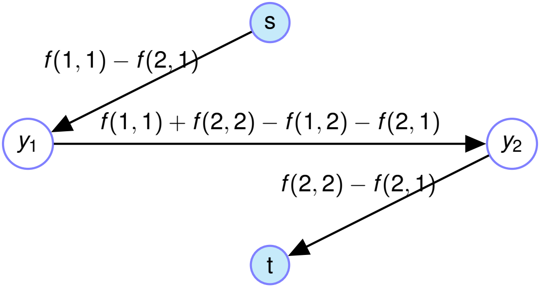

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For higher order functions? More complicated graph constructions
For more than two labels? Move making algorithms

Structured Prediction

Inference:

$$\boldsymbol{y}^* = \arg\max_{\hat{\boldsymbol{y}}} \sum_r f_r(\boldsymbol{w}, x, \hat{\boldsymbol{y}}_r)$$

Efficiency and accuracy of inference algorithms is problem dependent:

Structured Prediction

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- Graph-cut

Quiz:

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- When is a graph-cut algorithm optimal?

Important topics of this lecture

- More inference algorithms for structured spaces

Up next:

- Learning models for structured output spaces