Machine Learning

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University of Illinois at Urbana-Champaign, 2018

L5: Optimization Dual

Goals of this lecture

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- Constrained optimization

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- Understanding duality for optimization

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Reading Material

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Reading Material

- S. Boyd and L. Vandenberghe; Convex Optimization; Chapter 5

Optimization problems that we have seen so far:

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- Linear Regression

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Analytically computable optimum vs. gradient descent

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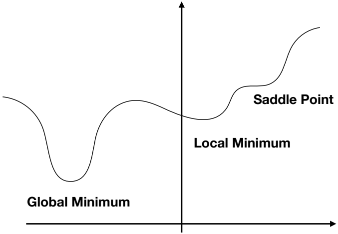

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How to optimize this?

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- General optimization problems are very difficult to solve
- Often compromise between accuracy and computation time

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- Convex program when all f_i convex (generalizes the above)

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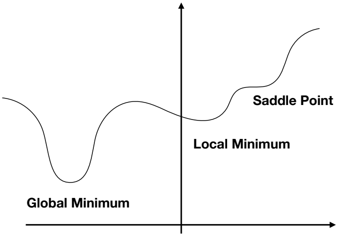

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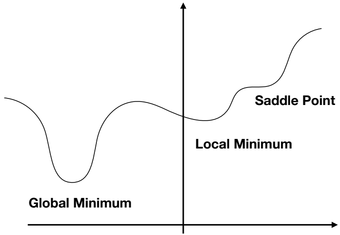

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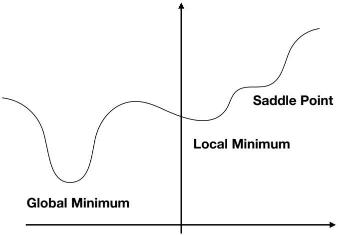

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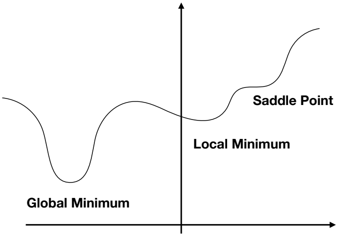

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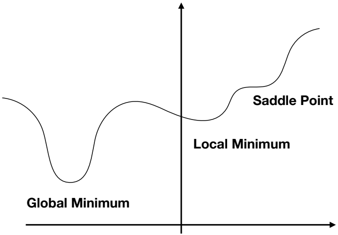

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This makes convex optimization special!

Algorithms to search for the optimum?

Descent methods

$$\min_{\boldsymbol{w}} f(\boldsymbol{w})$$

Intuition

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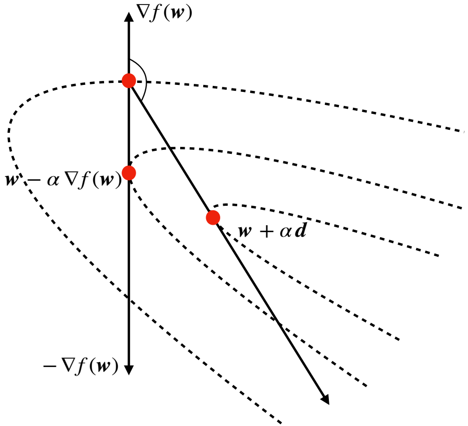

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Descent direction d_k satisfies $\nabla f(\boldsymbol{w})^\top \boldsymbol{d}_k < 0$

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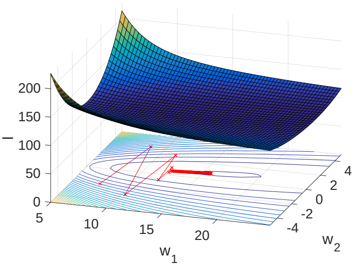

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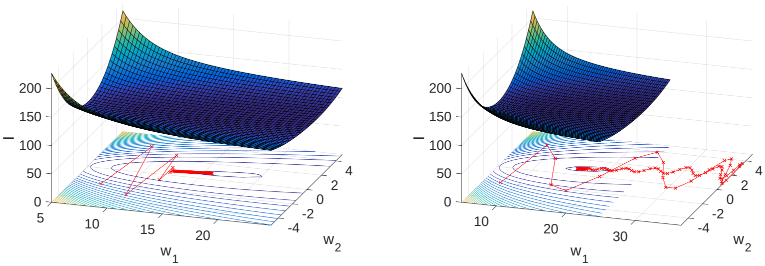

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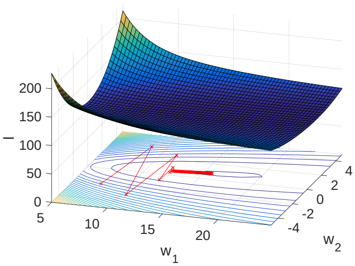

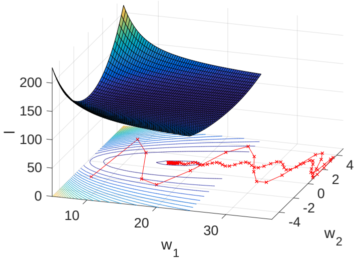

Video

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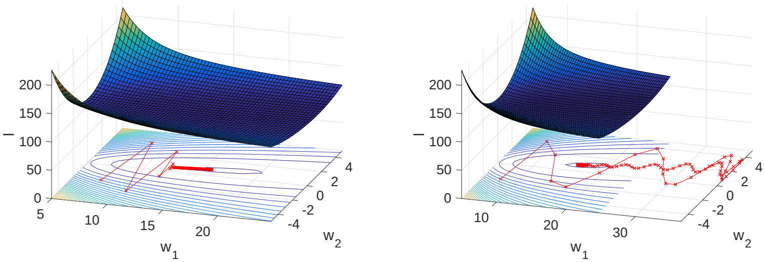

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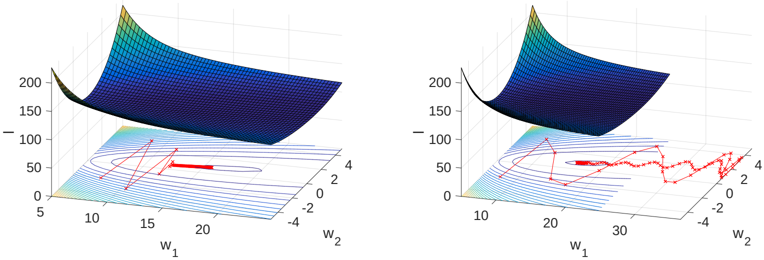

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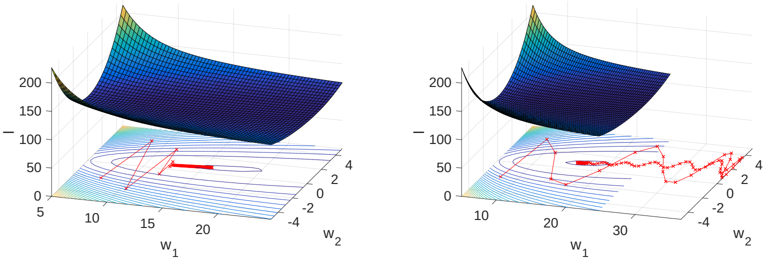

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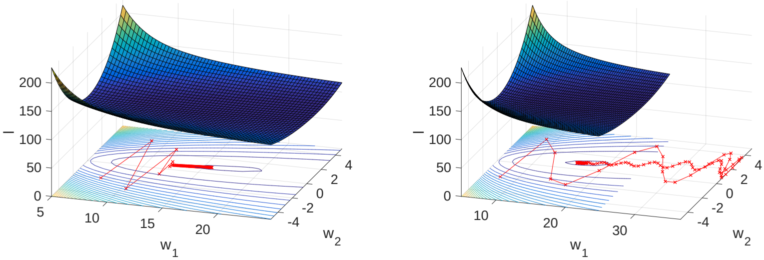

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- Armijo Rule

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How to deal with this?

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- Select a subset of samples $\mathcal{B}_k$
- Gradient update using approximation

$$\nabla f(\boldsymbol{w}) \approx \sum_{(x^{(i)},y^{(i)})\in\mathcal{B}_k} \nabla \ell(y^{(i)}, F(x^{(i)}, \boldsymbol{w}))$$

How about constraints?

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Dual Program:

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Up next:

- Support vector machines