Computer Graphics (CS 4731)
Transformations

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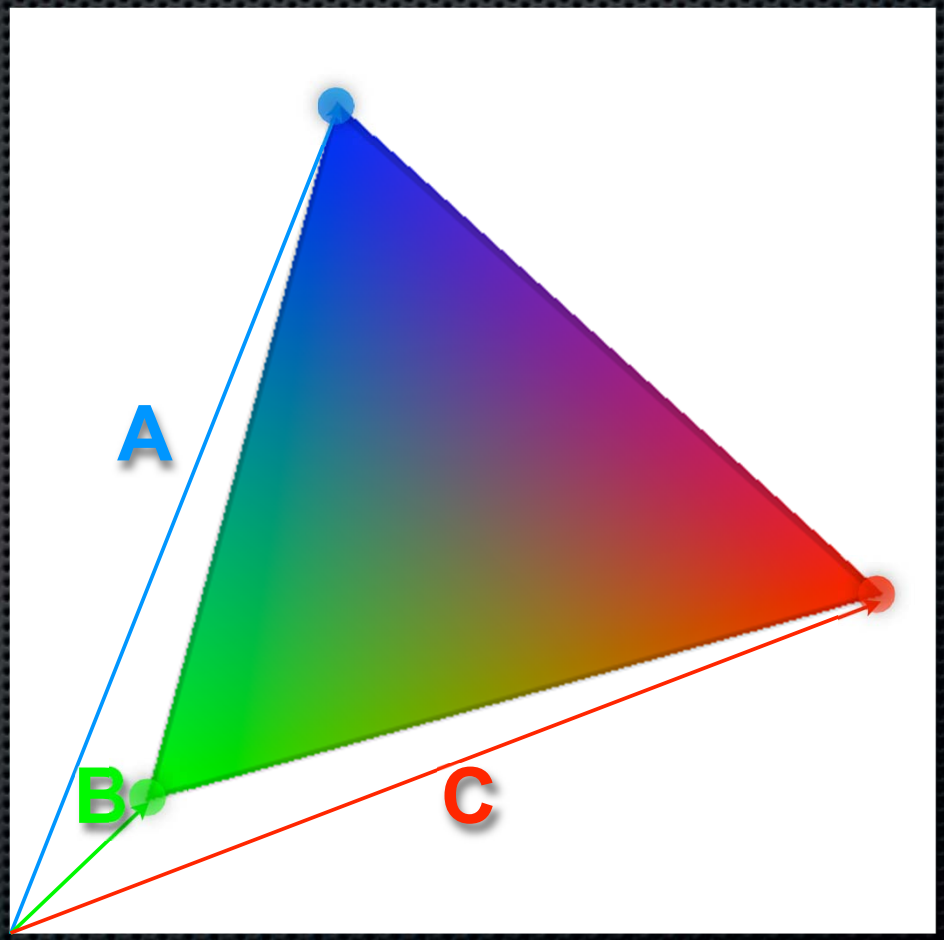
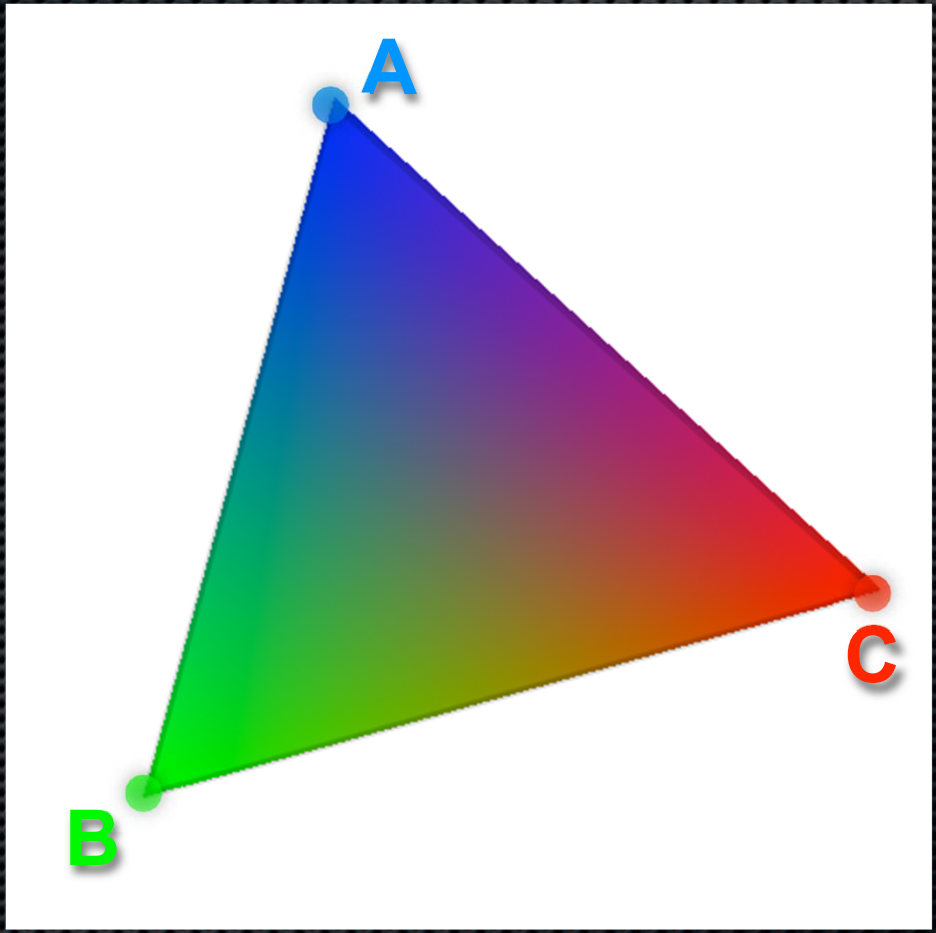
Slides Courtesy of Emmanuel Agu (WPI) and Matt Stoker (University of Utah)

What Can We Do So Far?

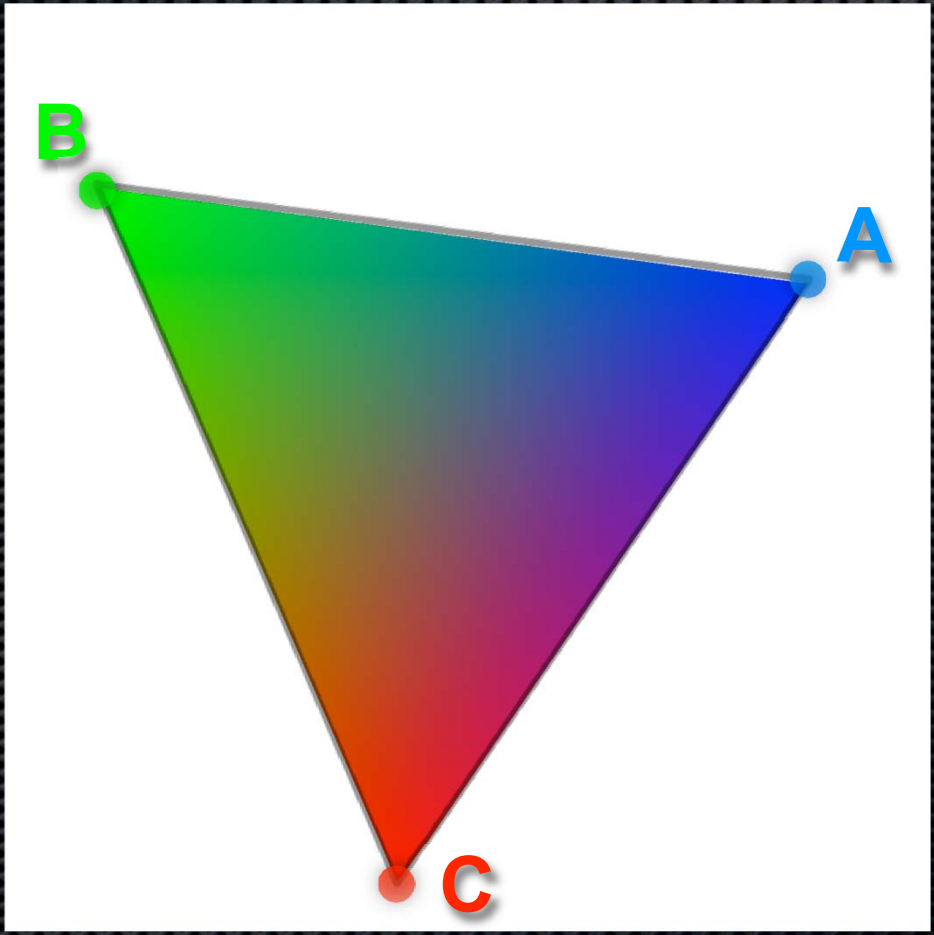
- Create and save rasters
- Define colors, vectors, and triangles
- Draw triangles using interpolated colors



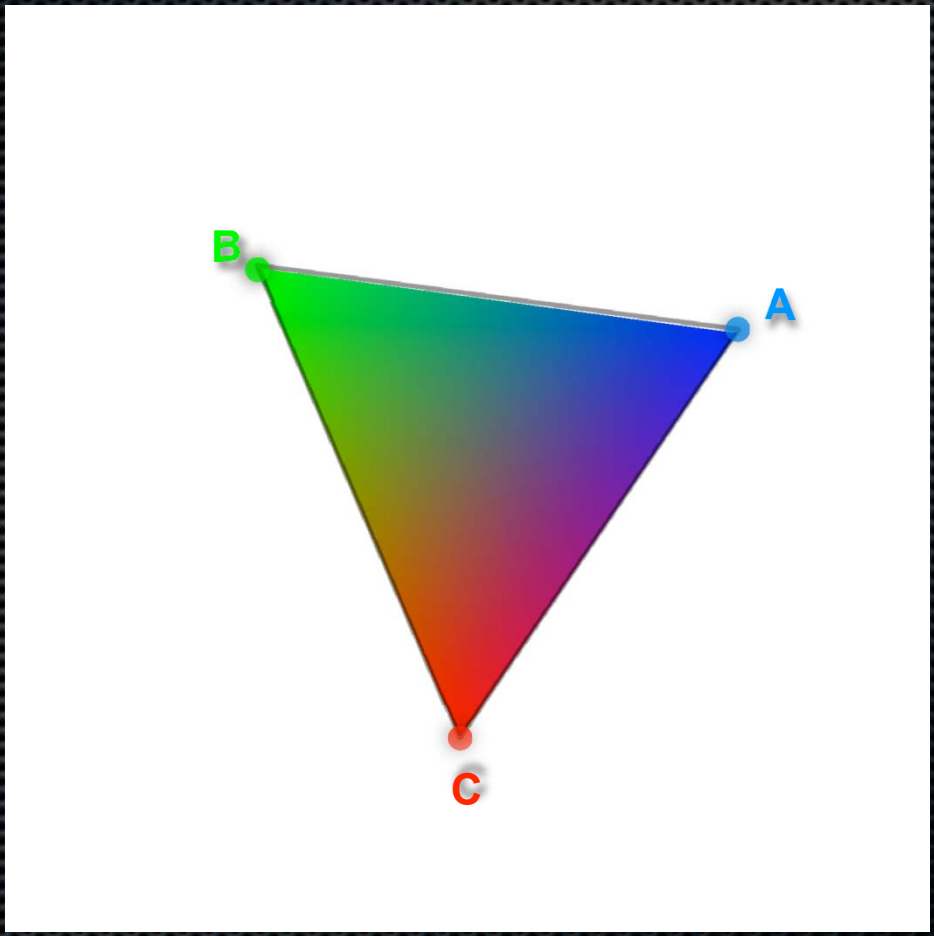
Transformations



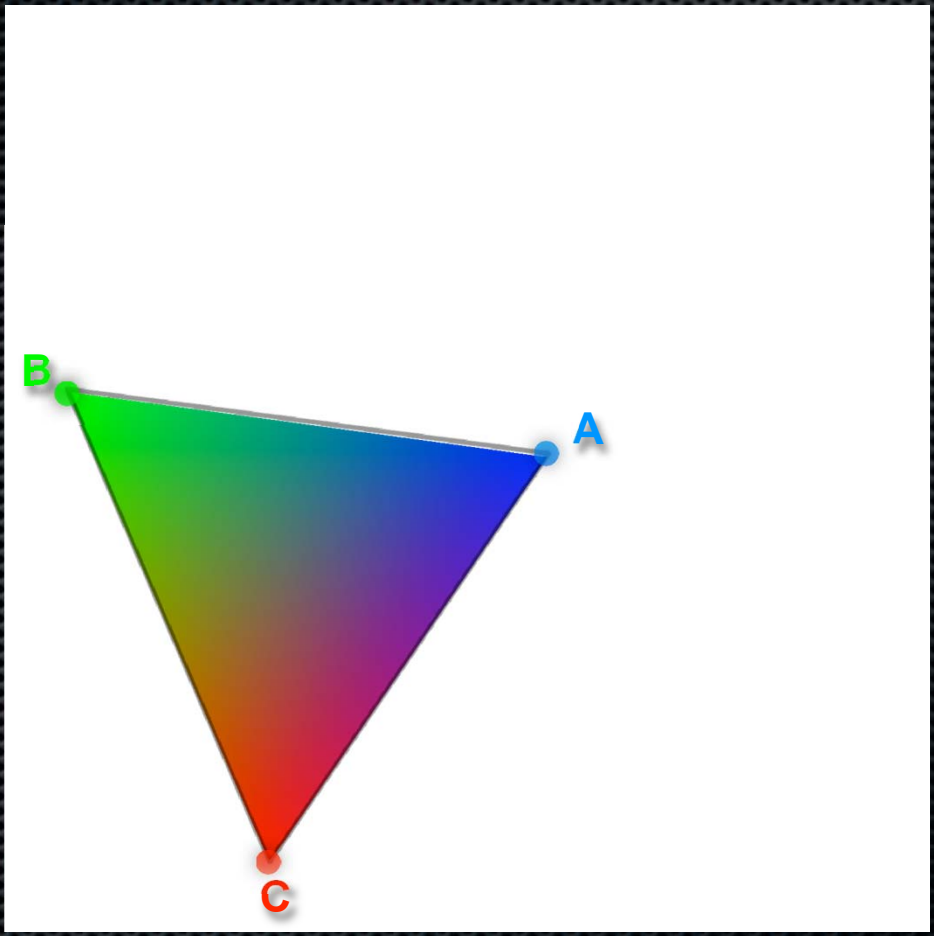
Rotate



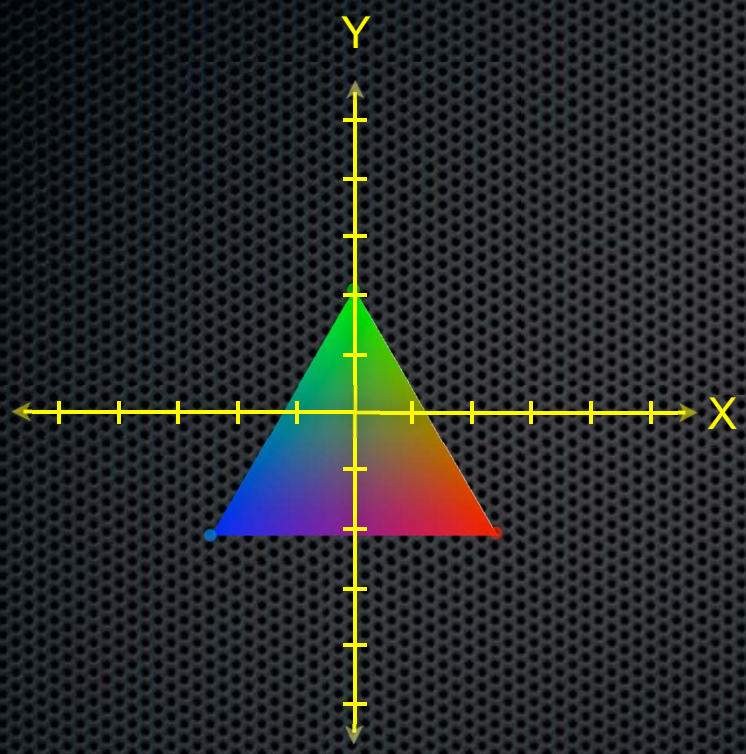
Scale

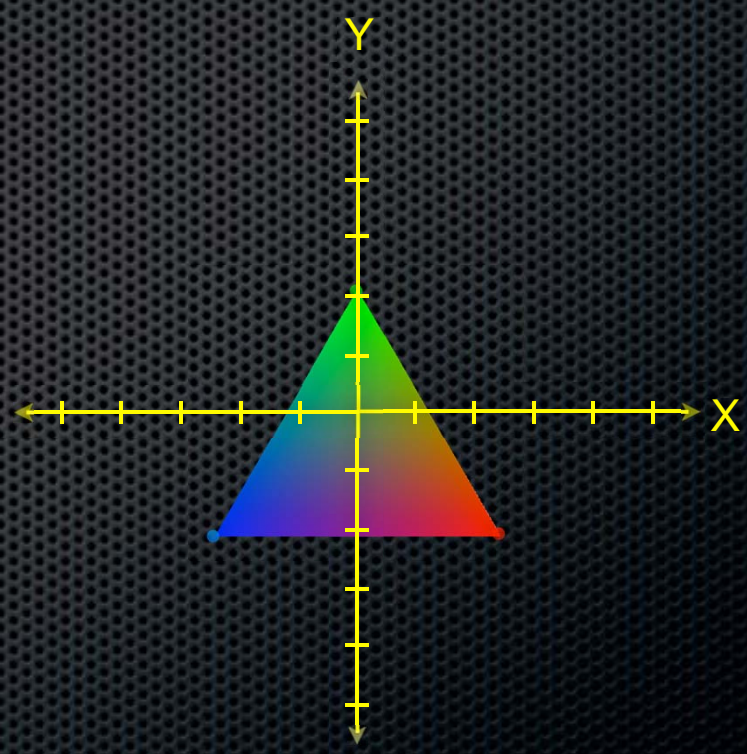


Translate

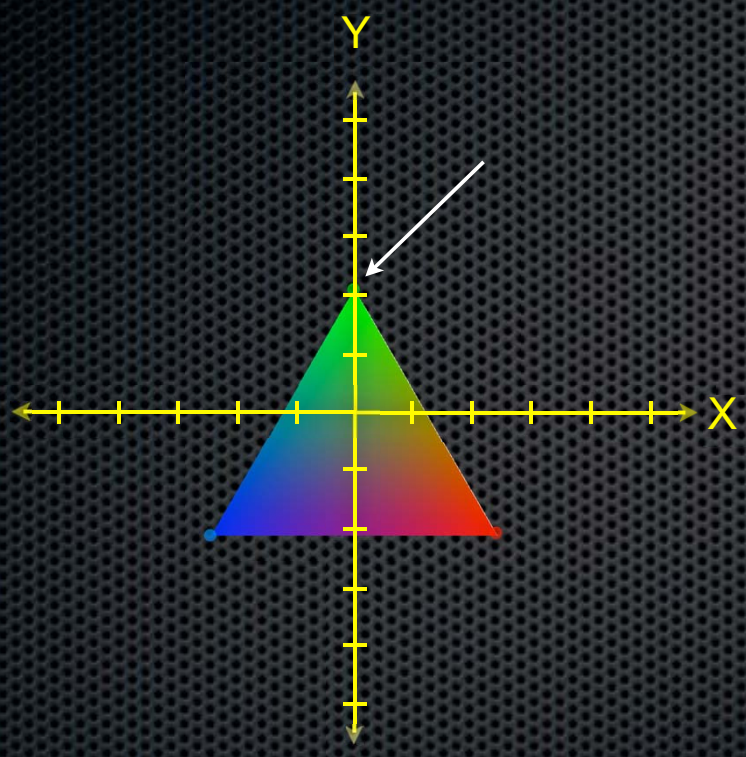


Translate Mathematics

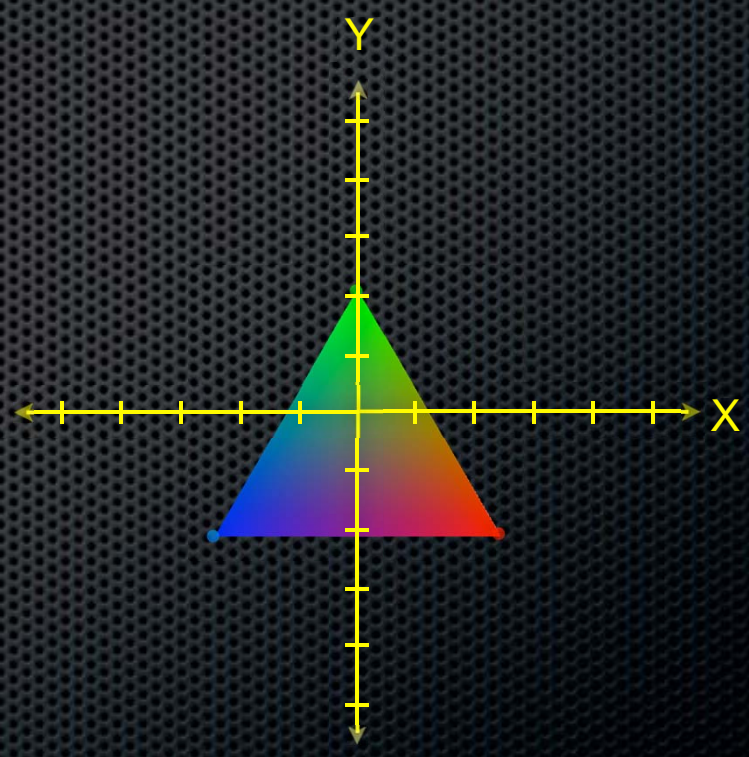




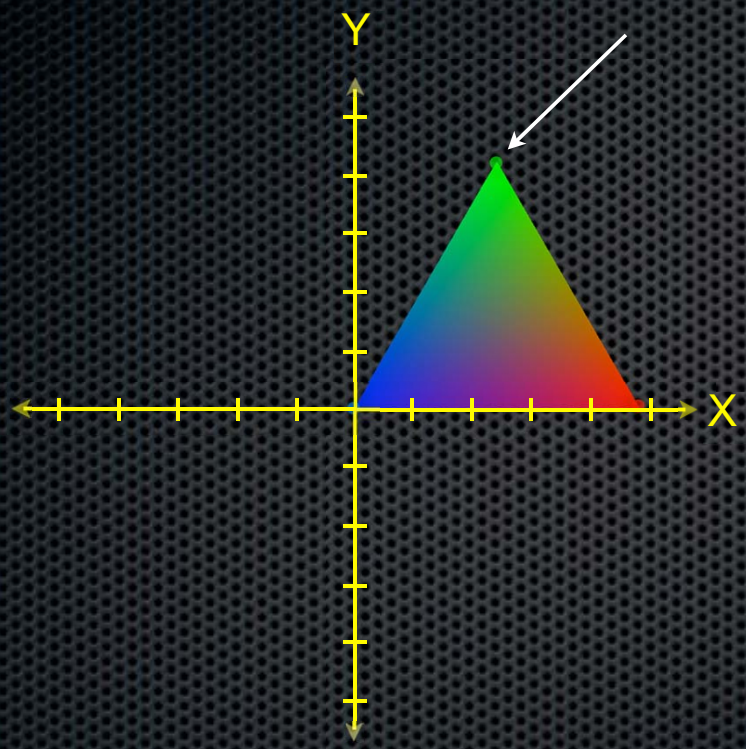
Translate Mathematics

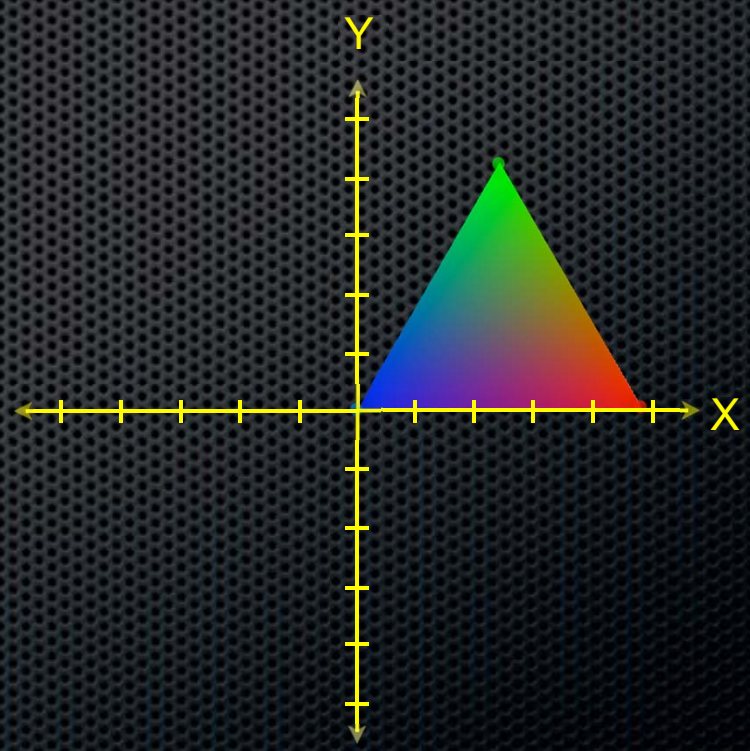


0
2



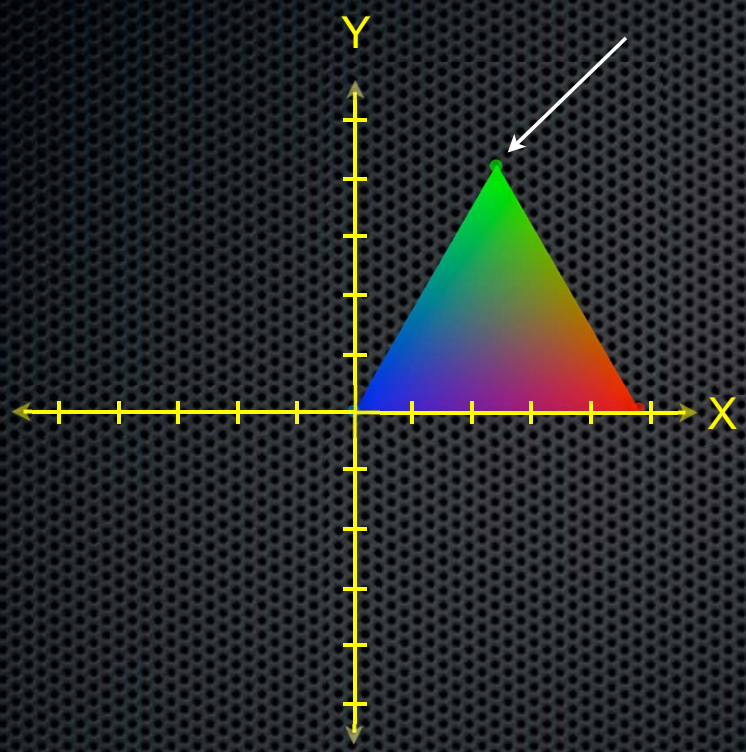
Translate Mathematics

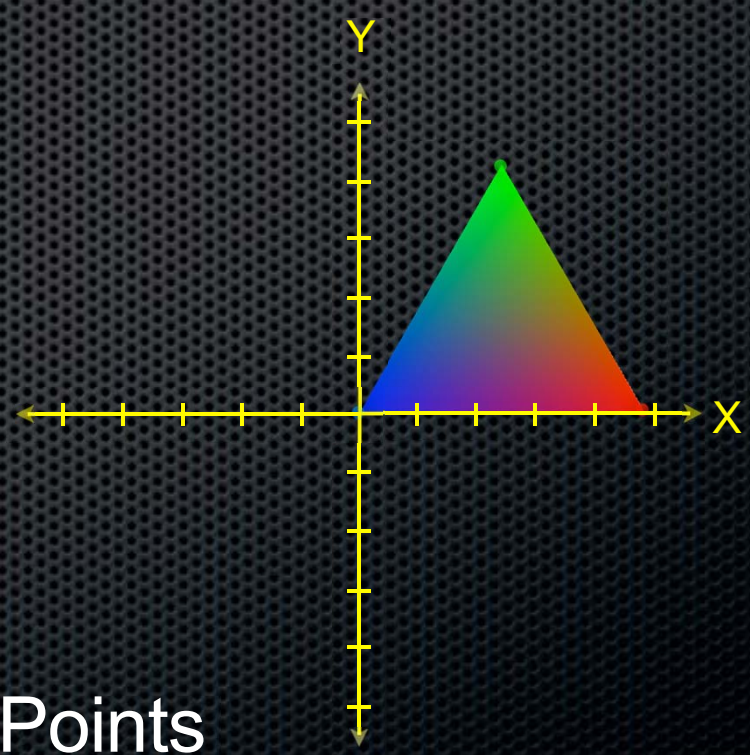




2.5 = 0 + 2.5
4.3 = 2 + 2.3

Translate Mathematics



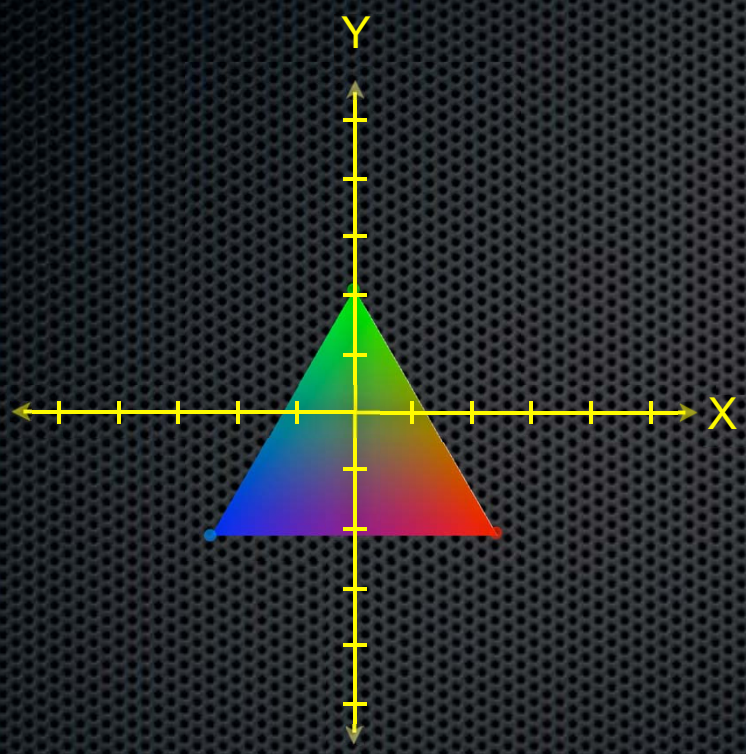


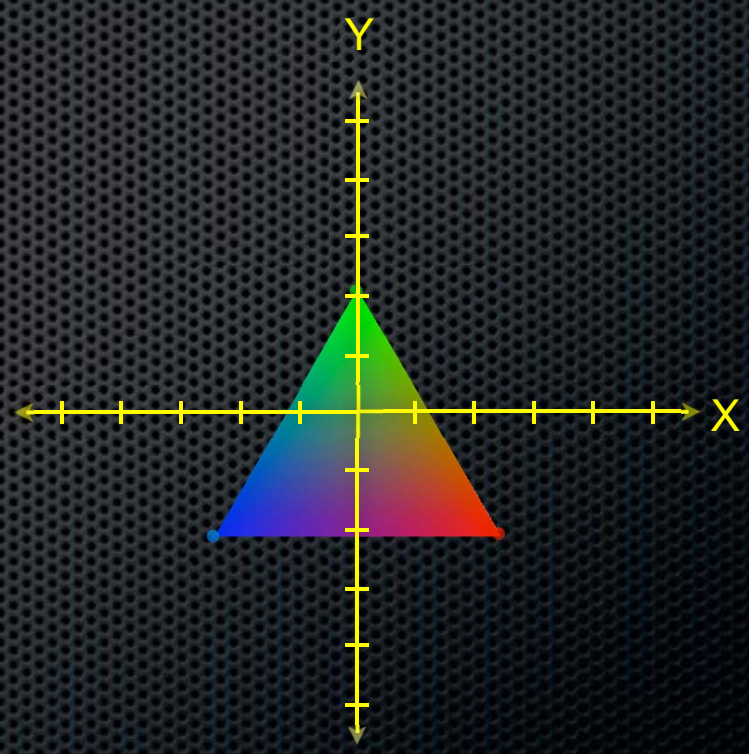
$x_f = x_o + t_x$

$y_f = y_o + t_y$

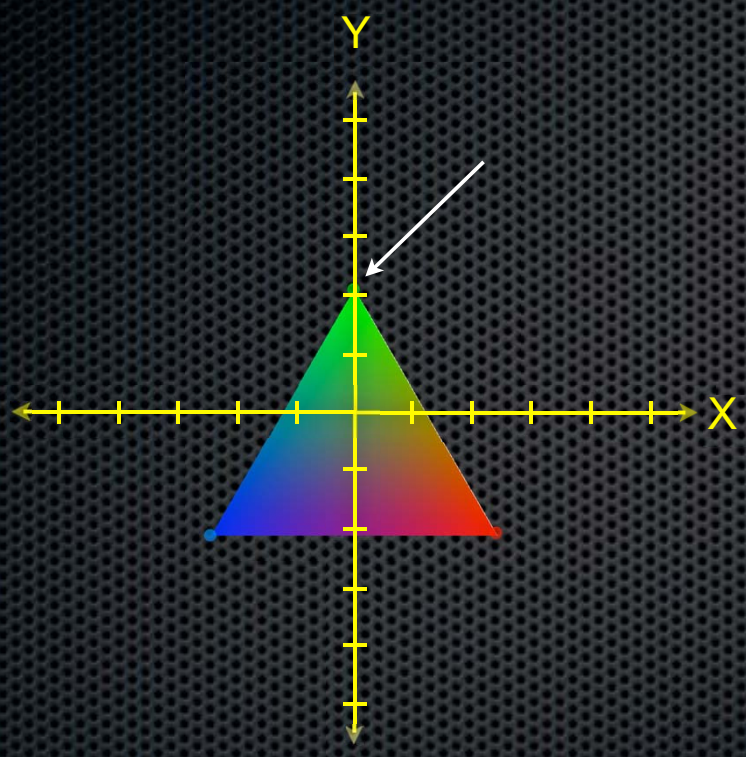
All Points

Scale Mathematics

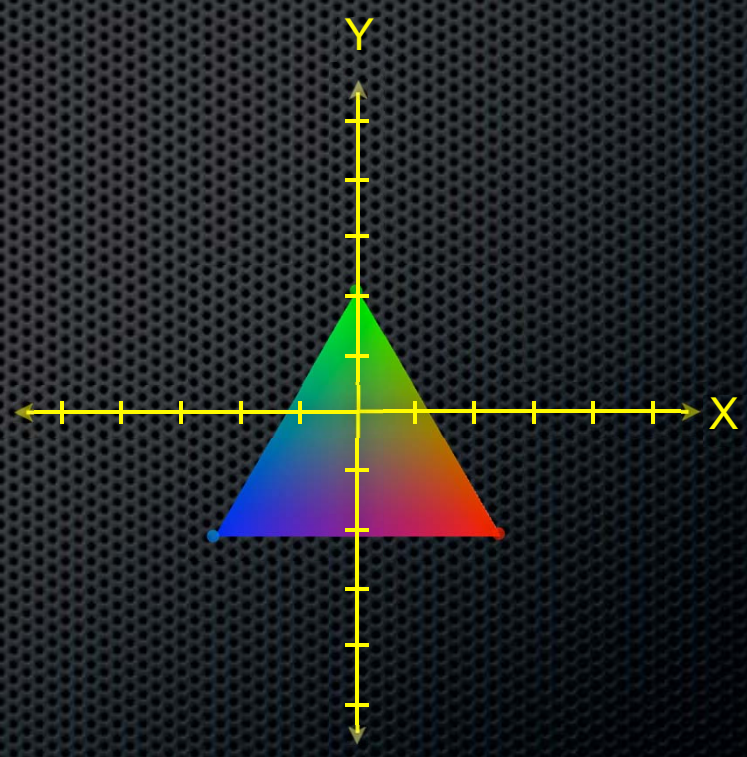




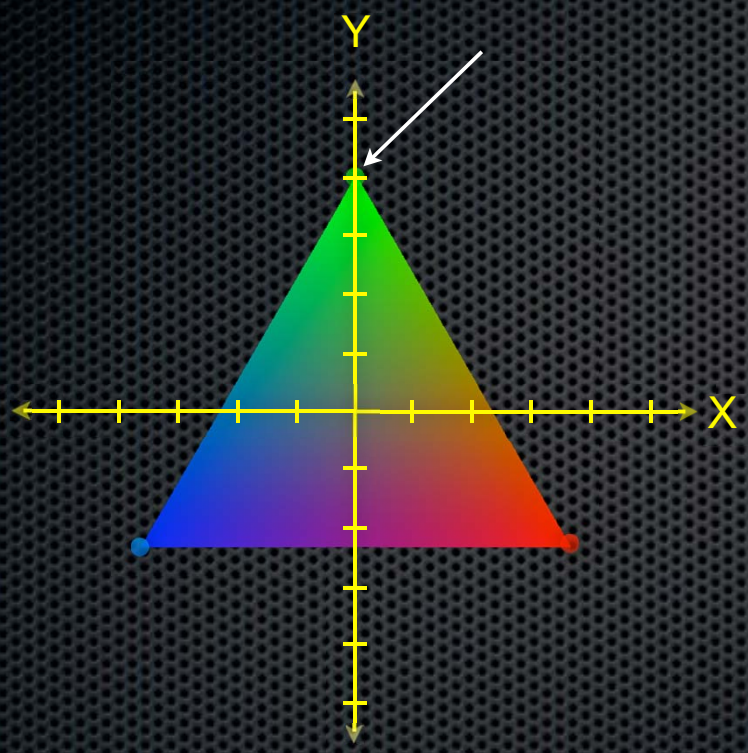
Scale Mathematics

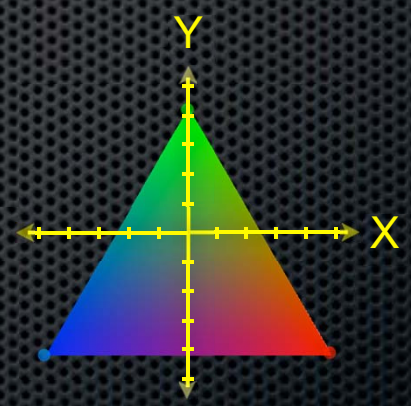


0
2



Scale Mathematics

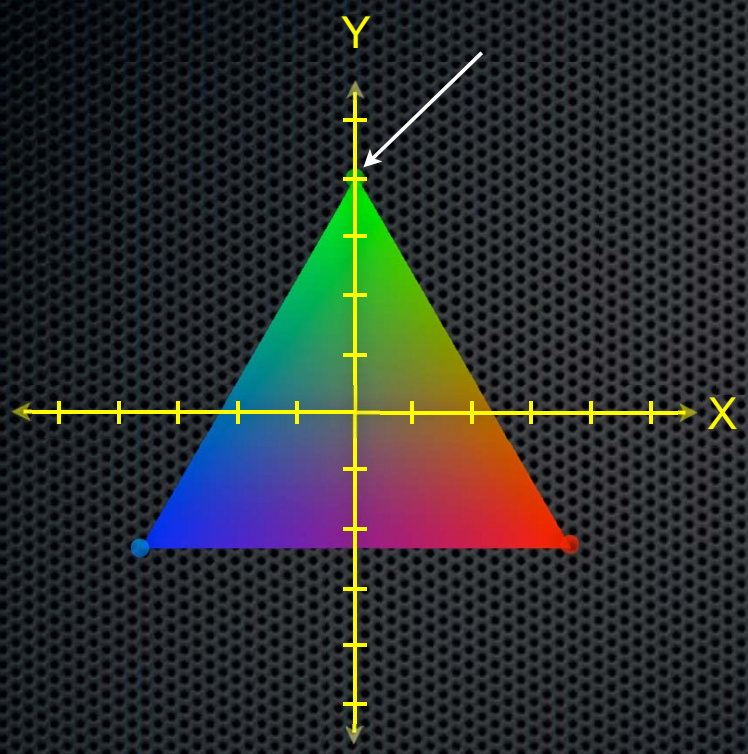


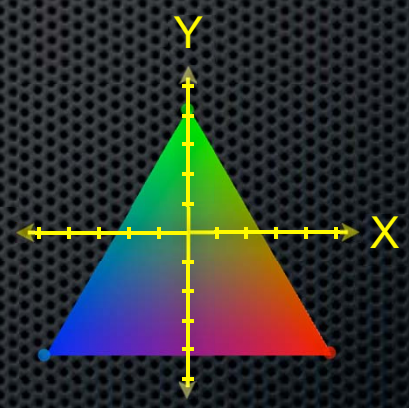


$0 = 0 \cdot 2$

$4 = 2 \cdot 2$

Scale Mathematics





$x_f = x_o \bullet s_x$

$y_f = y_o \bullet s_y$

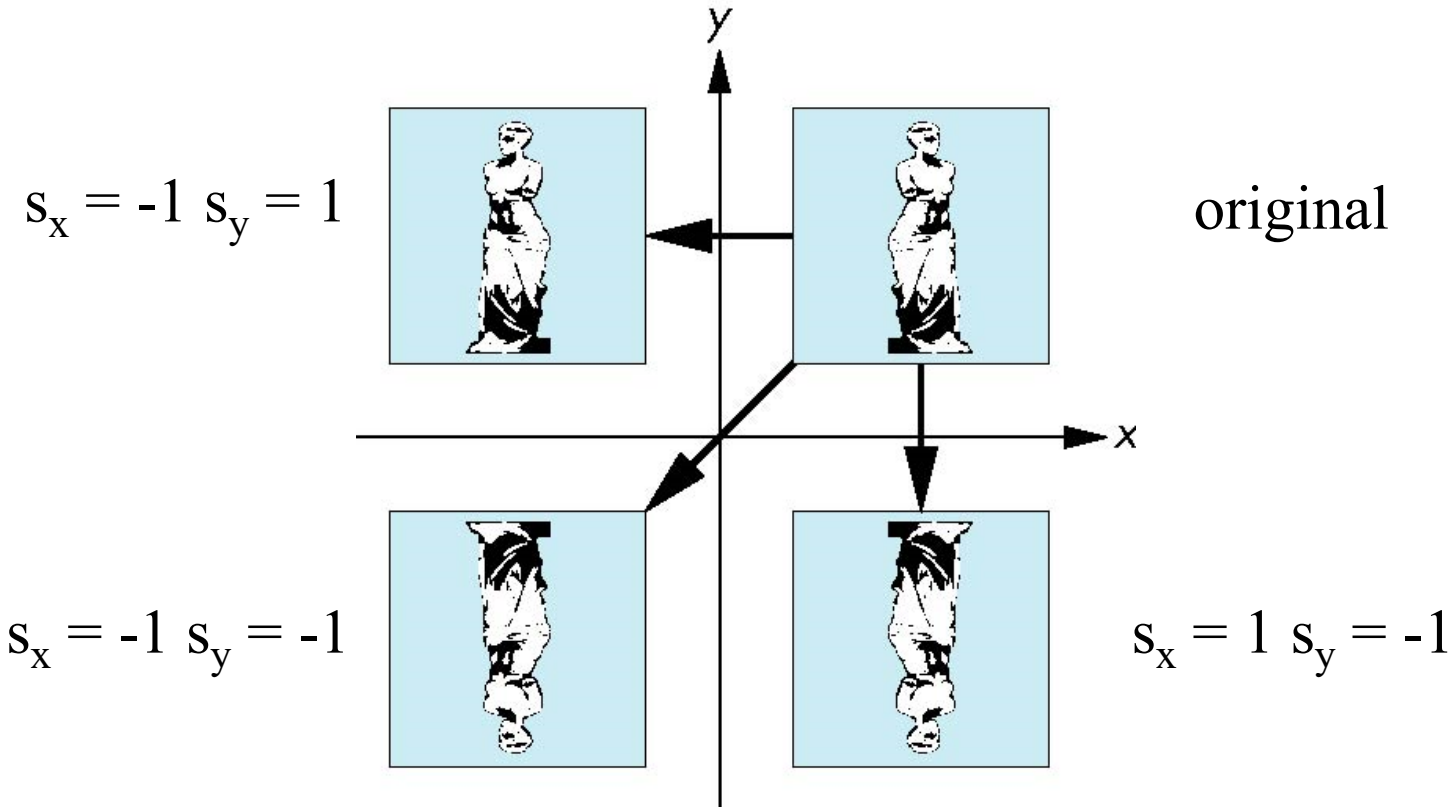
All Points

Think-Pair-Share

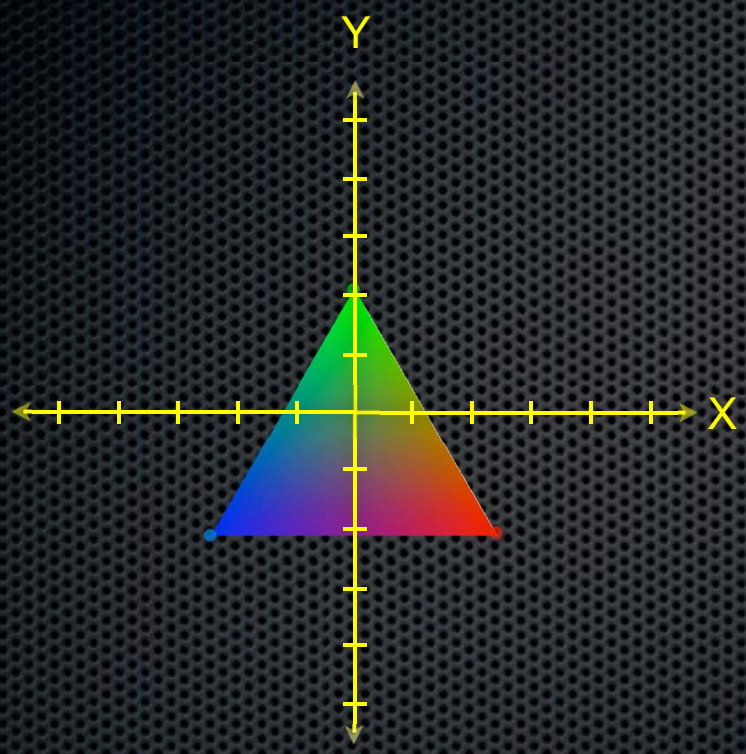
What happens if our scale factor is < 0?

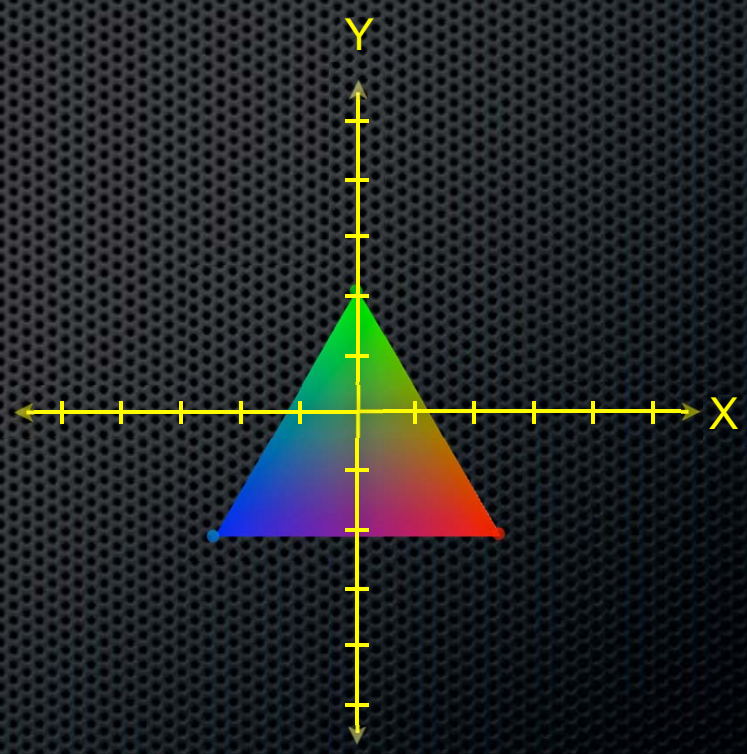
Reflection

- corresponds to negative scale factors

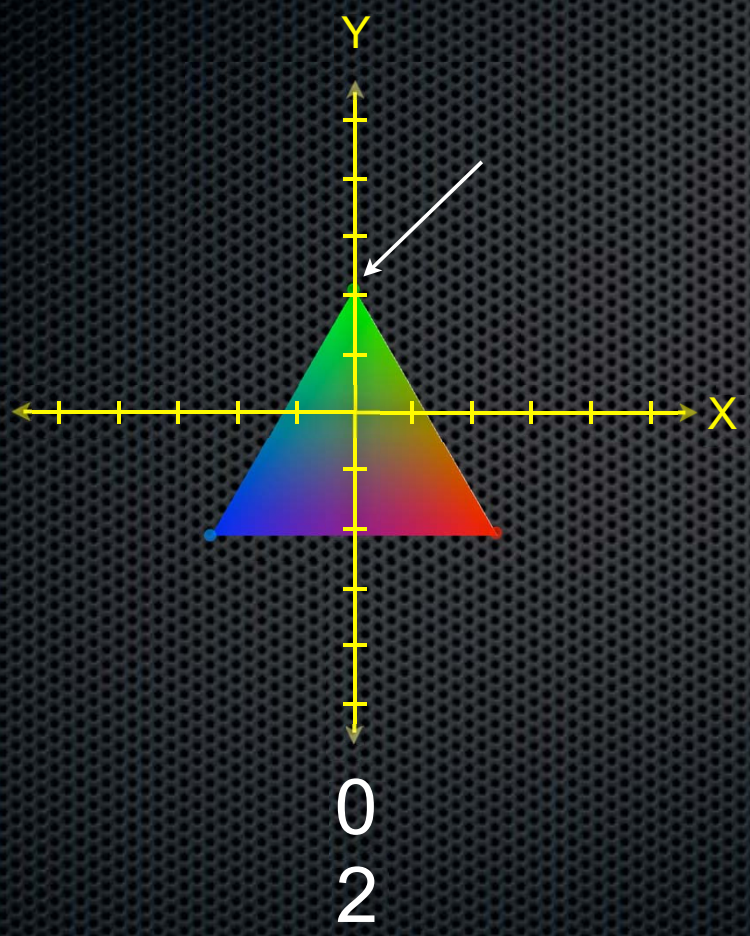


Rotate Mathematics

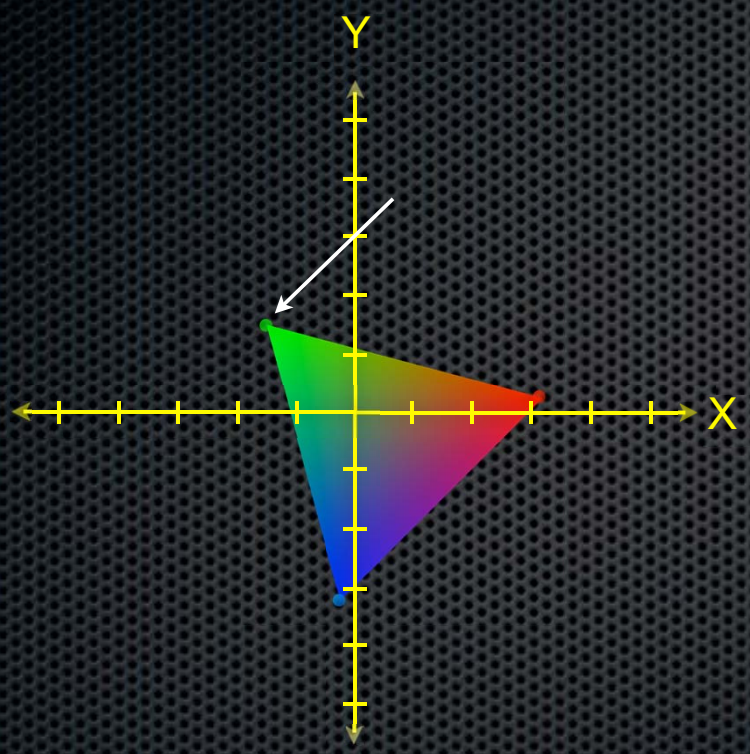




Rotate Mathematics



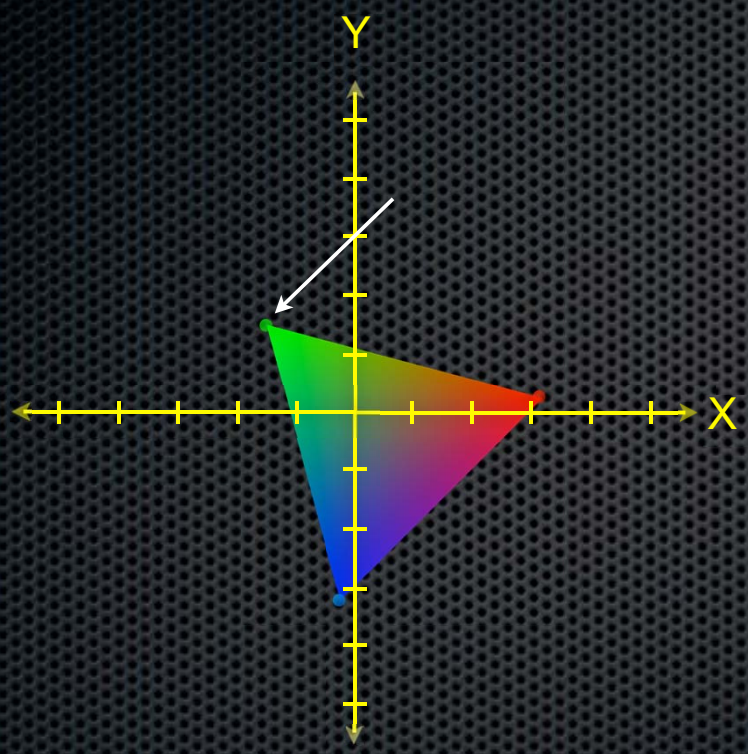
Rotate Mathematics

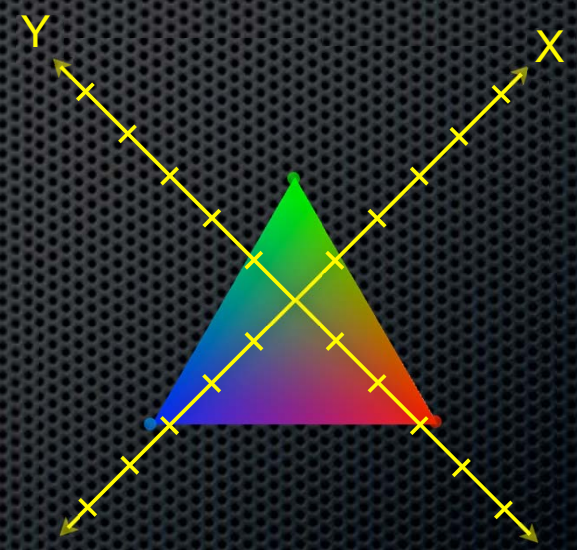


$$-1.41 = 0 \cdot \cos(45) - 2 \cdot \sin(45)$$
$$1.41 = 0 \cdot \sin(45) + 2 \cdot \cos(45)$$

All Points

Rotate Mathematics

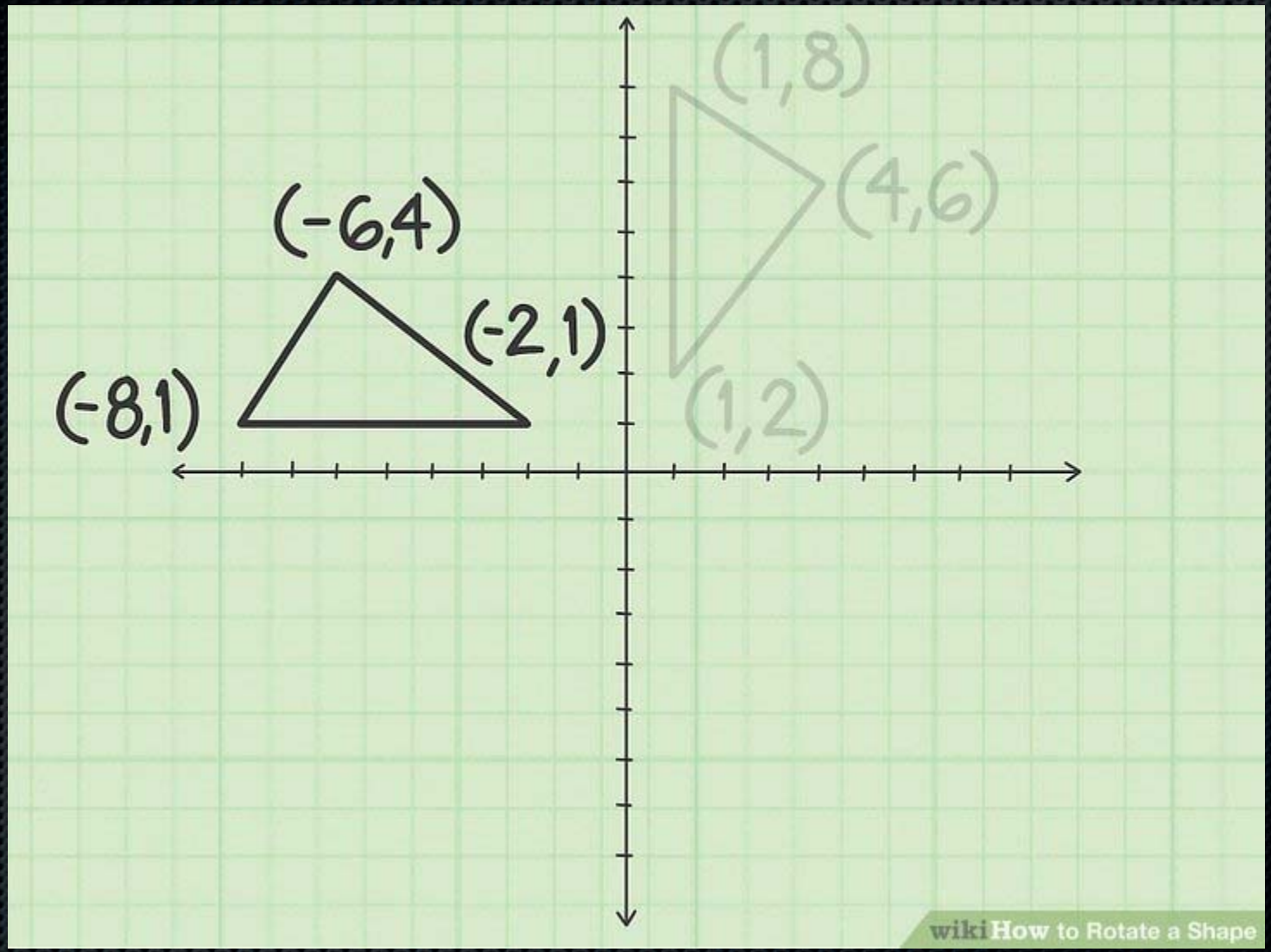




$$x_f = x_i \bullet \cos\theta - y_i \bullet \sin\theta$$
$$y_f = x_i \bullet \sin\theta + y_i \bullet \cos\theta$$

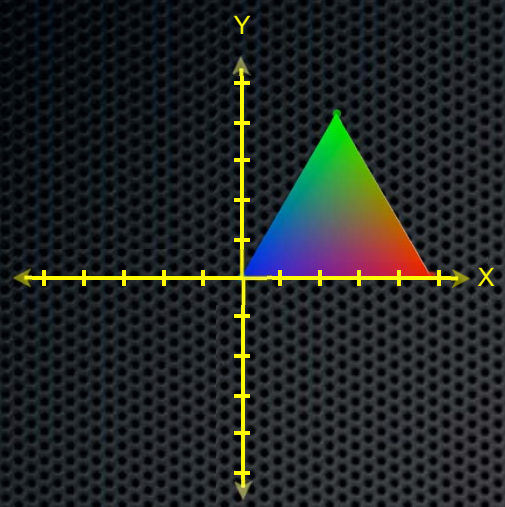
Think-Pair-Share

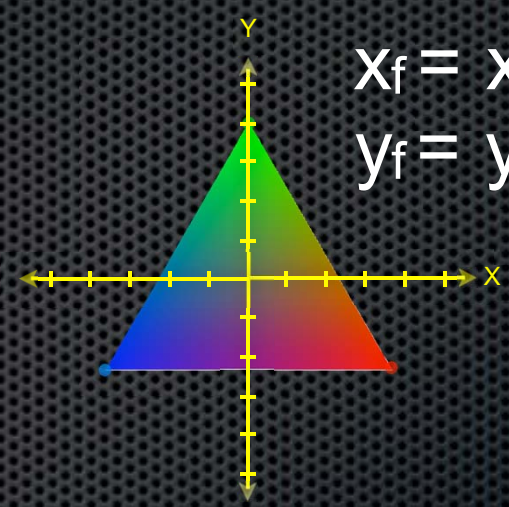
What would a rotation look like if our object was not centered at the origin?


(1,8)
(4,6)
(-6,4)
(-2,1)
(-8,1)
(1,2)
wikiHow to Rotate a Shape

Transformations

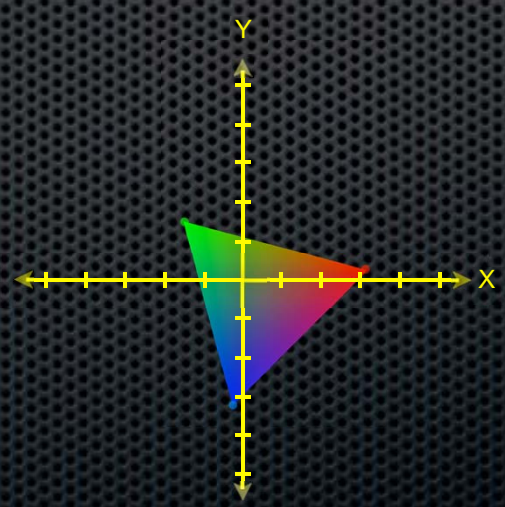
All Points





$$x_f = x_o \cdot s_x$$
$$y_f = y_o \cdot s_y$$

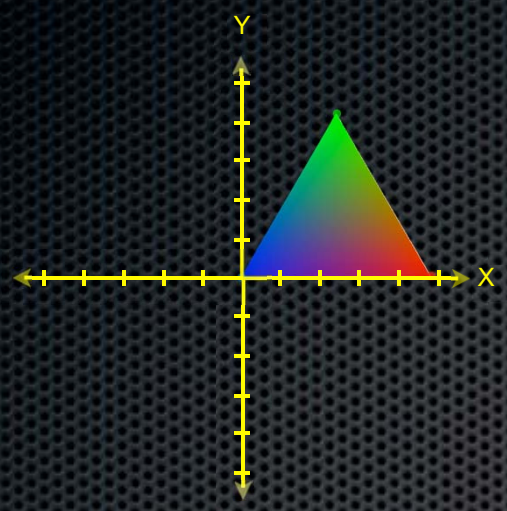
$$x_f = x_o + t_x$$
$$y_f = y_o + t_y$$



$$x_f = x_i \cdot \cos\theta - y_i \cdot \sin\theta$$
$$y_f = x_i \cdot \sin\theta + y_i \cdot \cos\theta$$

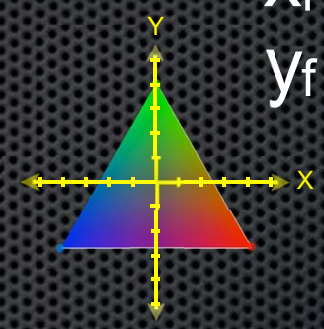
Transformations

All Points



$x_f = x_o \cdot s_x$

$y_f = y_o \cdot s_y$



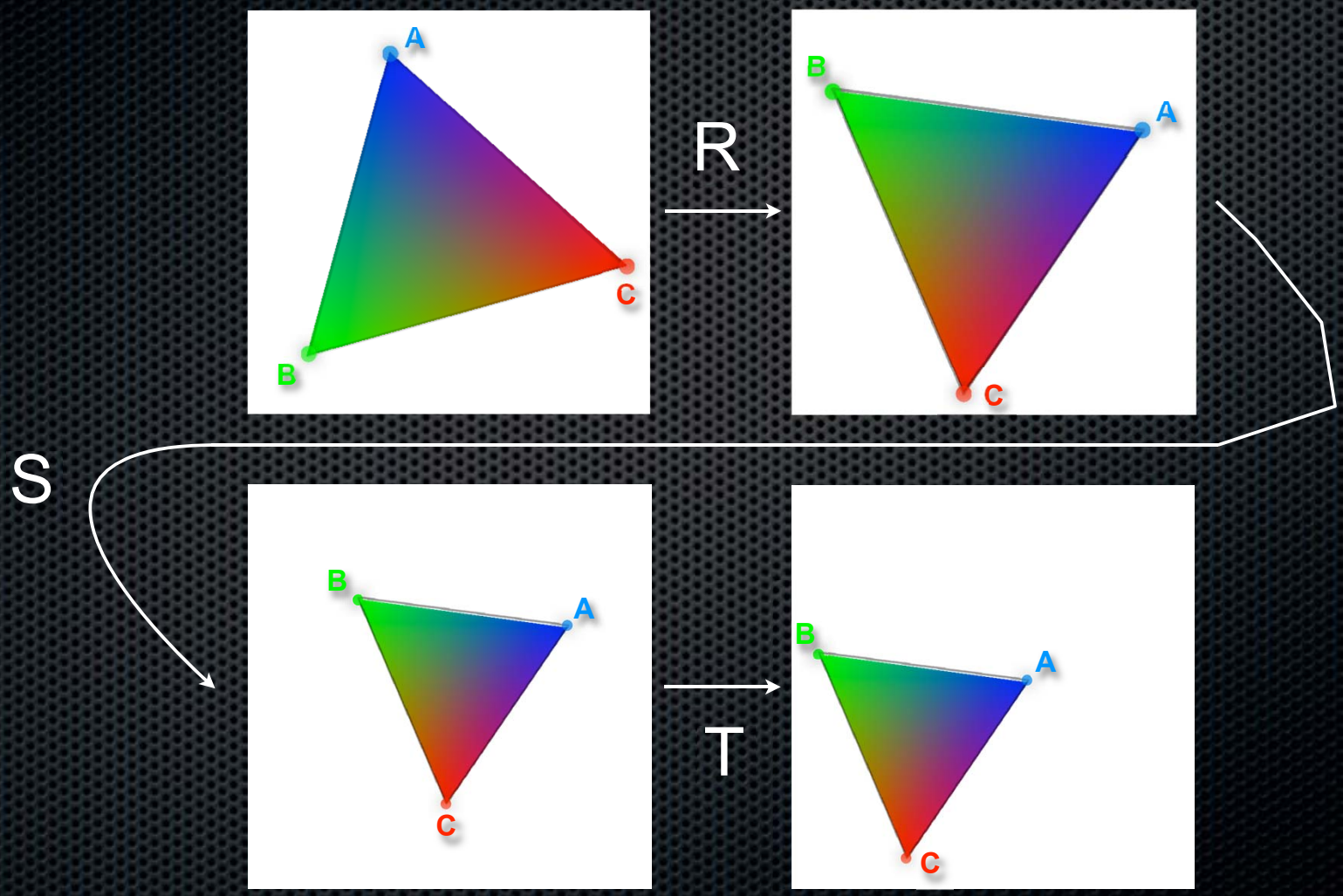
$x_f = x_o + t_x$

$y_f = y_o + t_y$

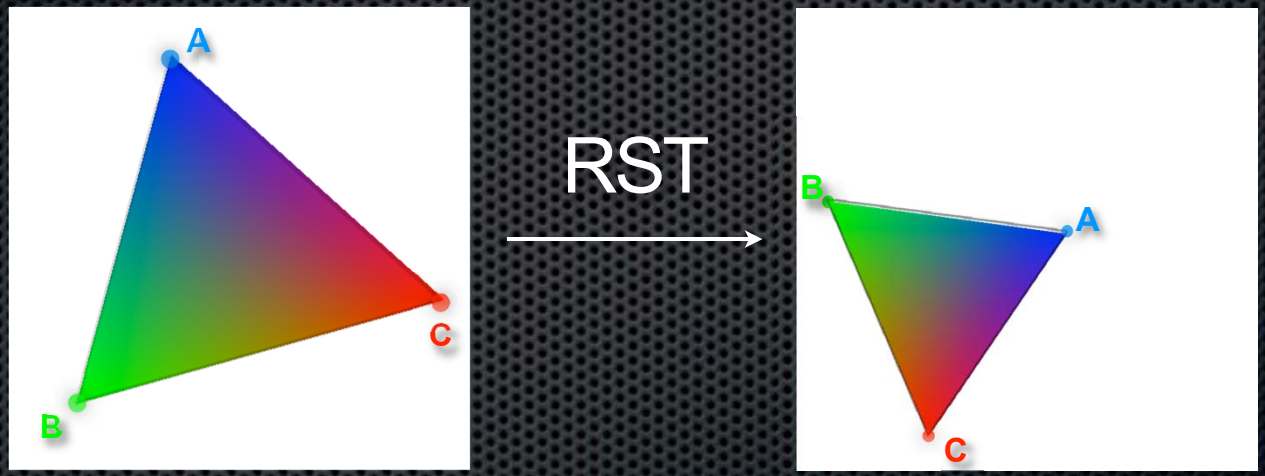
$x_f = x_i \cdot \cos\theta - y_i \cdot \sin\theta$

$y_f = x_i \cdot \sin\theta + y_i \cdot \cos\theta$

Transformations

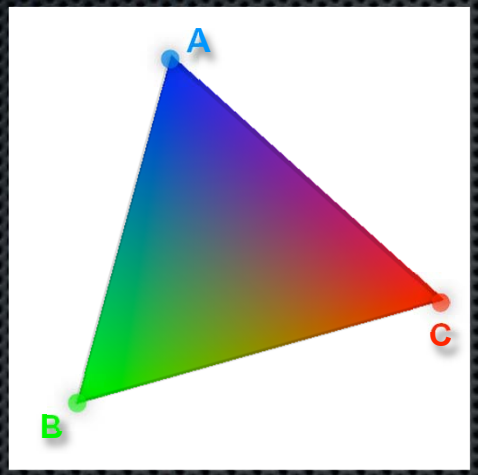


Transformations

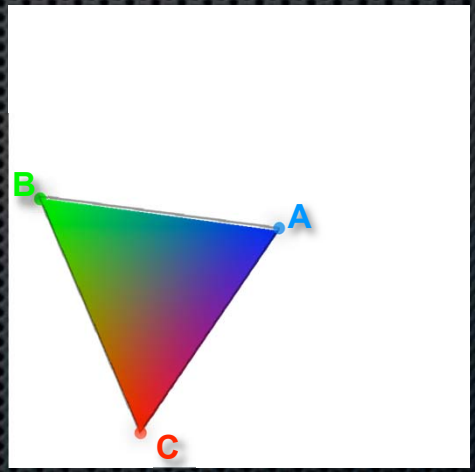


Transformations

Coordinate System 1



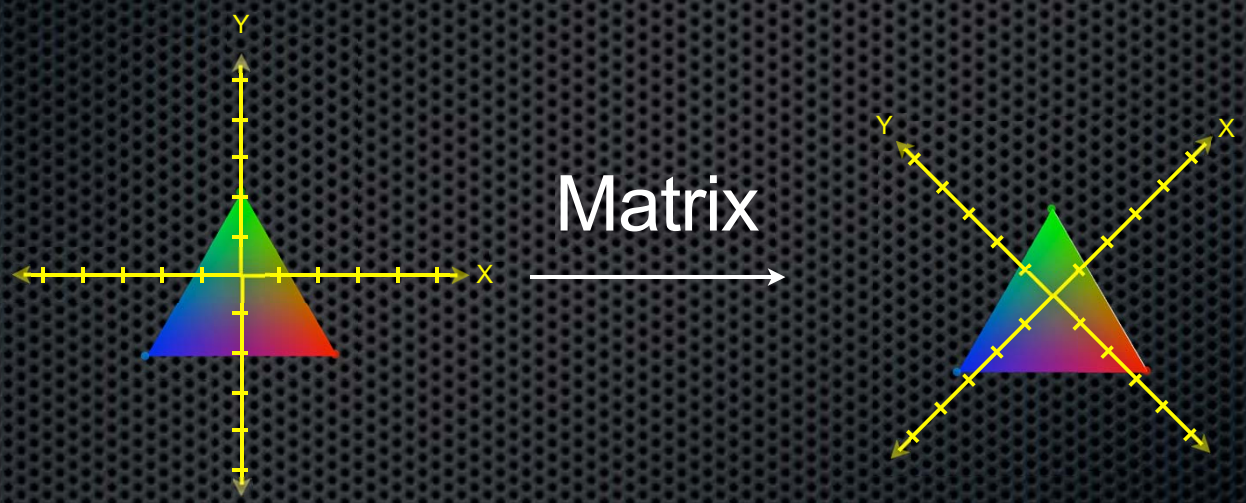
Matrix



Coordinate System 2

Transformations

Coordinate System 1

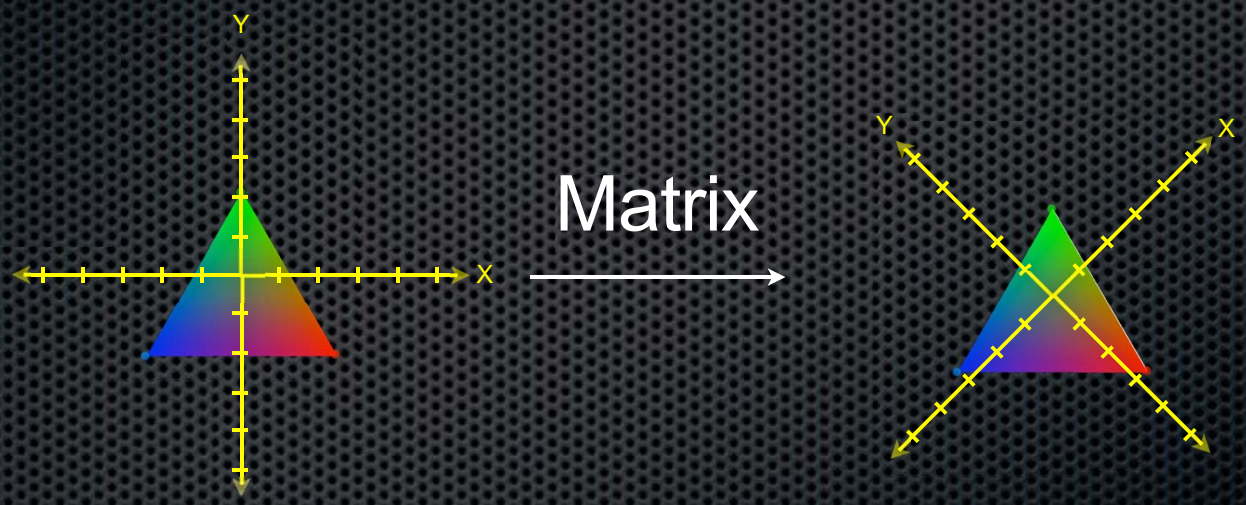


Coordinate System 2

Transformations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Coordinate System 1



Coordinate System 2

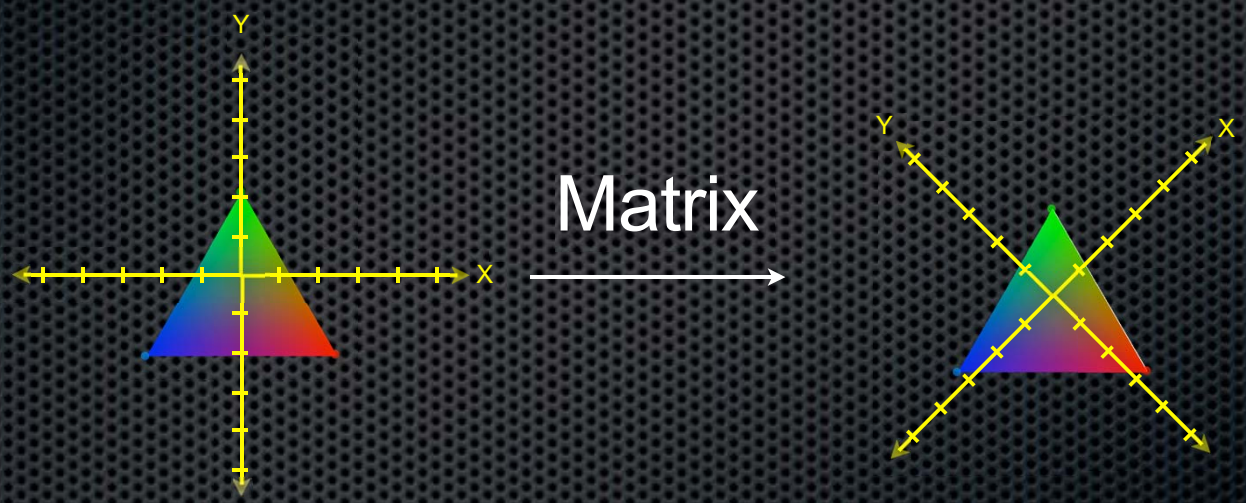
$x_f = x_i$???

$y_f = x_i$???

Rotate Matrix

$$\begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Coordinate System 1



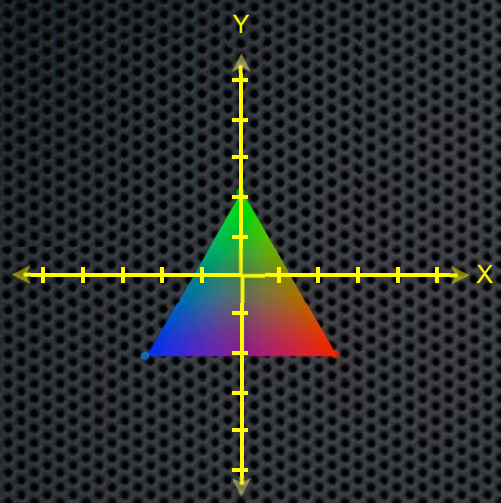
Coordinate System 2

$$x_f = x_i \cdot \cos\theta - y_i \cdot \sin\theta$$
$$y_f = x_i \cdot \sin\theta + y_i \cdot \cos\theta$$

Rotate Matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Coordinate System 1



Matrix

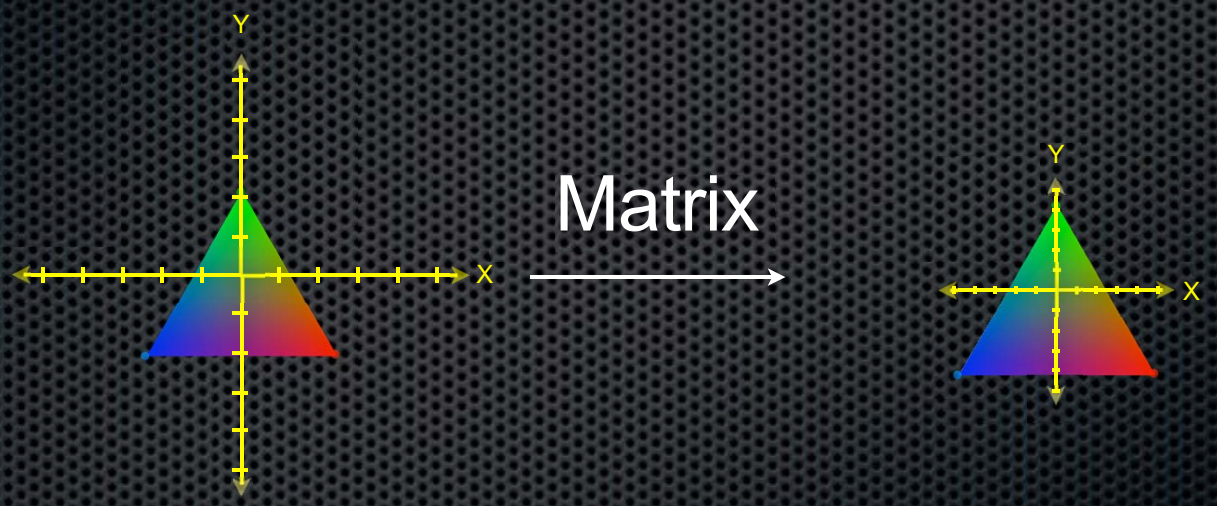
Coordinate System 2

$$x_f = x_i \bullet \cos\theta - y_i \bullet \sin\theta$$
$$y_f = x_i \bullet \sin\theta + y_i \bullet \cos\theta$$

Scale Matrix

$$\begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Coordinate System 1



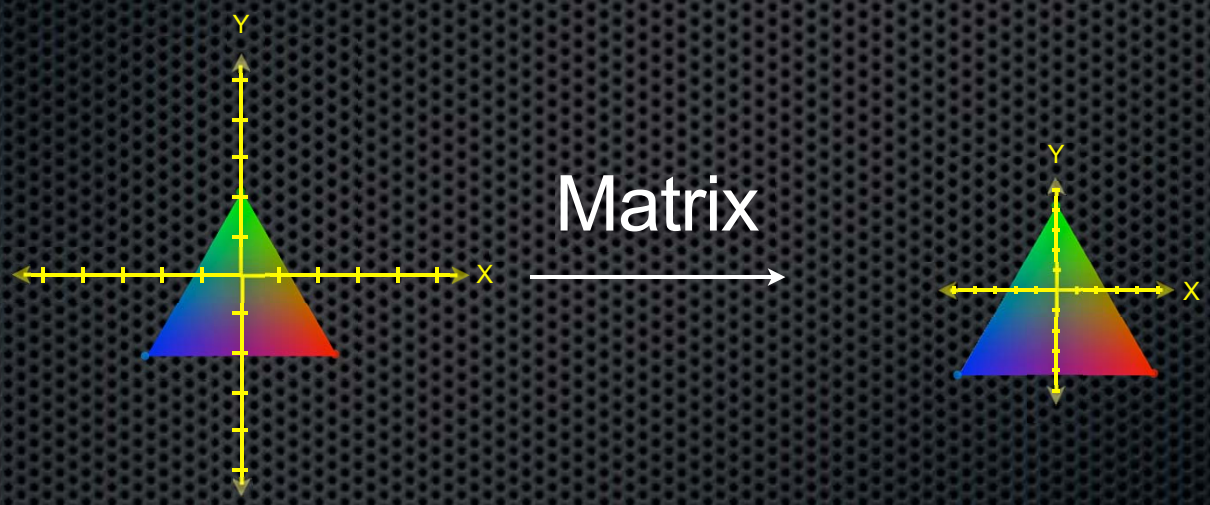
Coordinate System 2

$$x_f = x_o \cdot s_x$$
$$y_f = y_o \cdot s_y$$

Scale Matrix

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Coordinate System 1



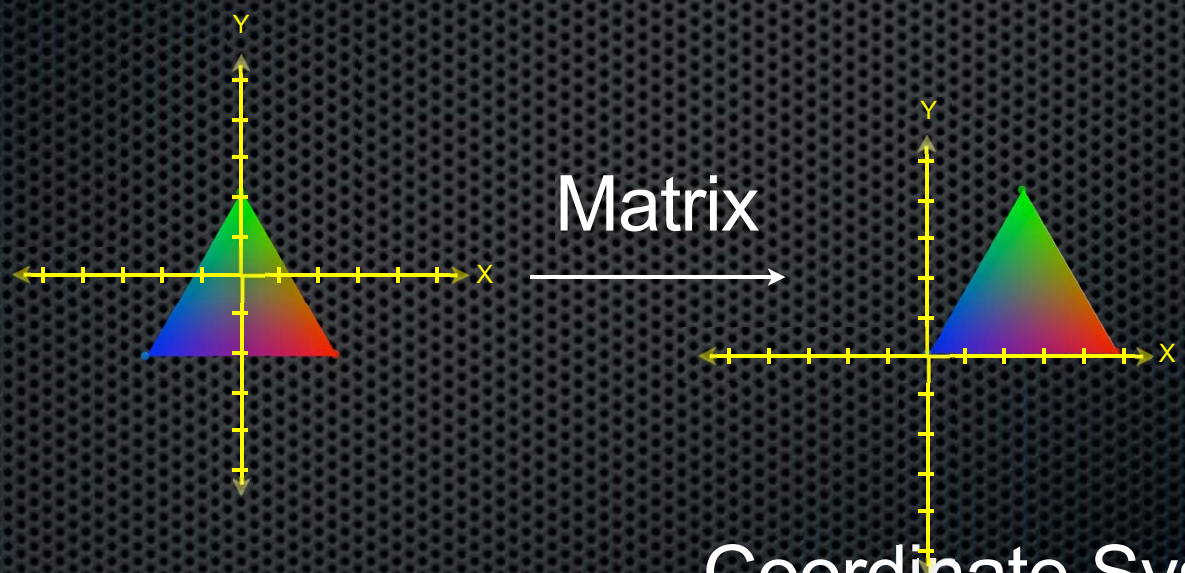
Coordinate System 2

$$x_f = x_o \cdot s_x$$
$$y_f = y_o \cdot s_y$$

Translate Matrix

$$\begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

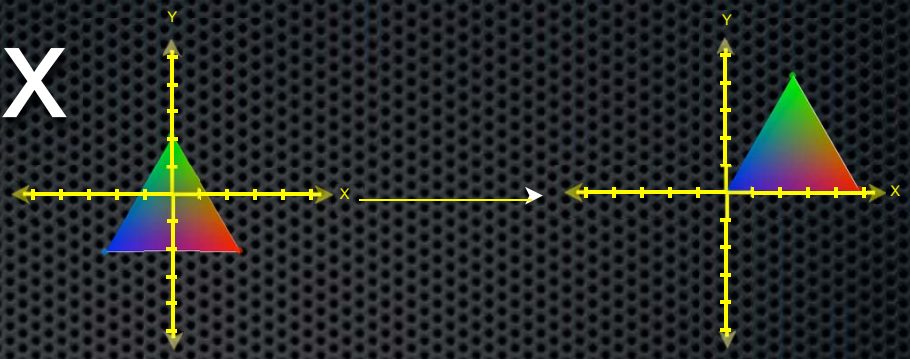
Coordinate System 1



Coordinate System 2

$$x_f = x_o + t_x$$
$$y_f = y_o + t_y$$

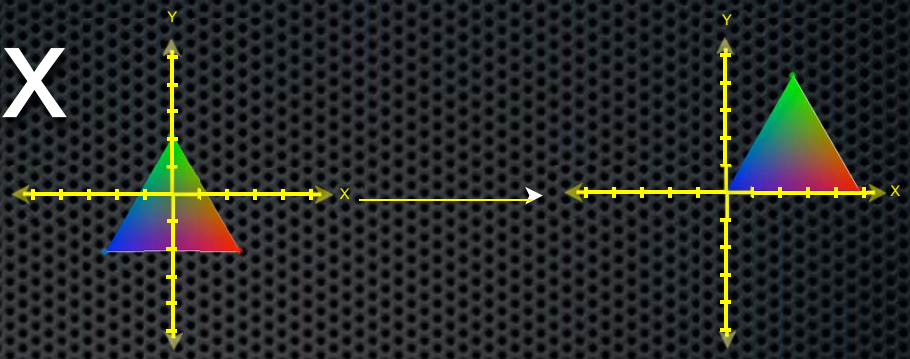
Translate Matrix



$$x_f = x_o + t_x$$
$$y_f = y_o + t_y$$

$$\begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

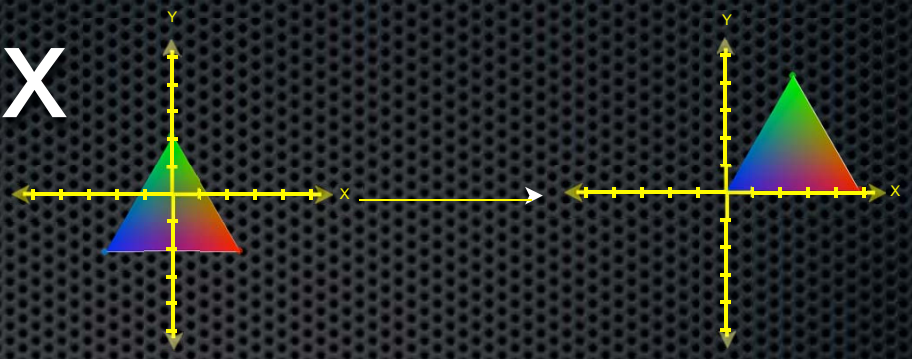
Translate Matrix



$$x_f = x_o + t_x$$
$$y_f = y_o + t_y$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

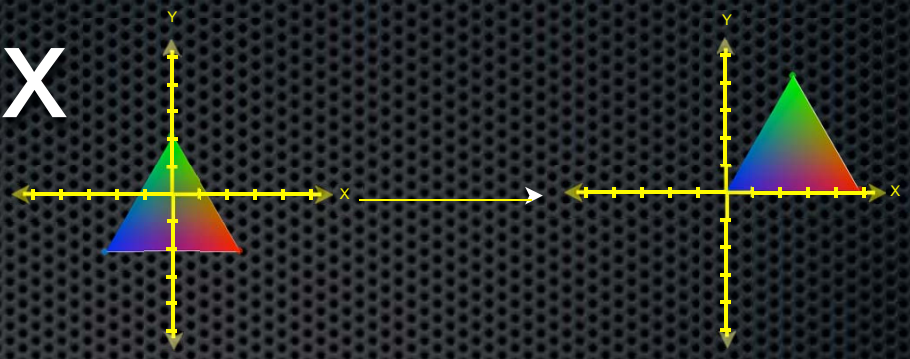
Translate Matrix



$$x_f = x_o + t_x$$
$$y_f = y_o + t_y$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Translate Matrix



$$x_f = x_o + t_x$$
$$y_f = y_o + t_y$$

$$\begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1x + by \\ cx + 1y \end{bmatrix}$$

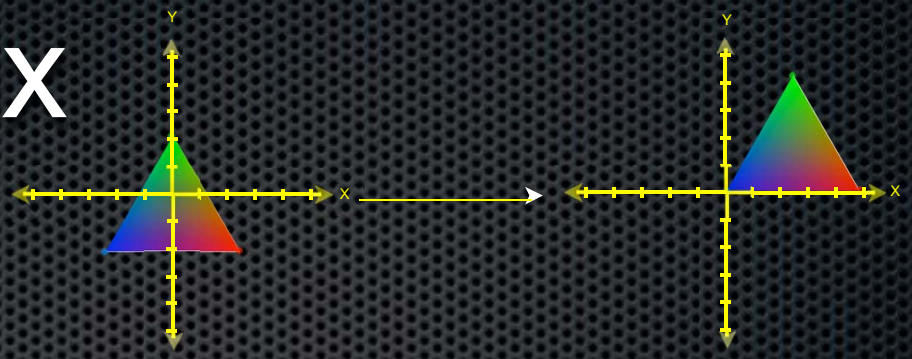
Think-Pair-Share

What are the values of b and c?

$x_f = x_o + t_x$
$y_f = y_o + t_y$

$$\begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1x + by \\ cx + 1y \end{bmatrix}$$

Translate Matrix



$x_f = x_o + t_x$

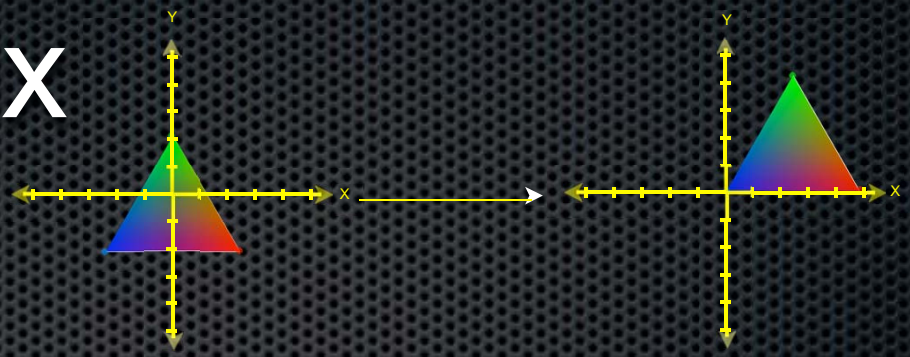
$y_f = y_o + t_y$

$$\begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1x + by \\ cx + 1y \end{bmatrix}$$

b = ?

c = ?

Translate Matrix



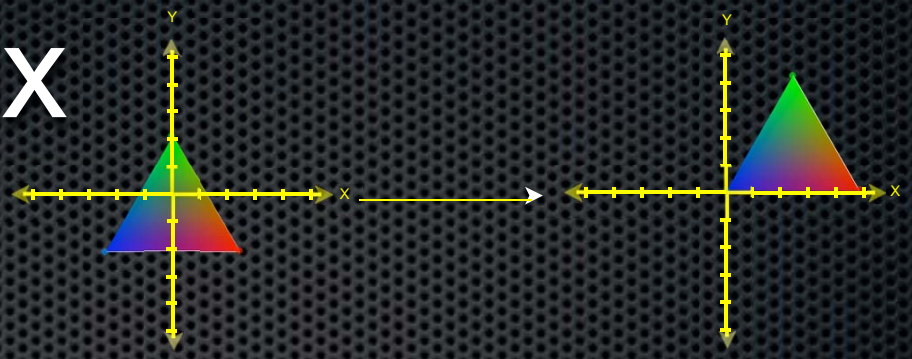
$x_f = x_o + t_x$

$y_f = y_o + t_y$

$$\begin{bmatrix} 1 & b & \\ c & 1 & \end{bmatrix} \quad + \begin{bmatrix} by \\ 1y \end{bmatrix}$$

c = ?

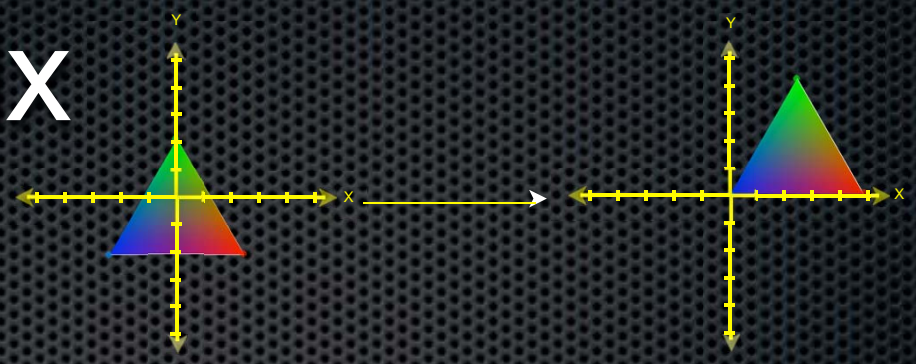
Translate Matrix



$$x_f = x_o + t_x$$
$$y_f = y_o + t_y$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate Matrix

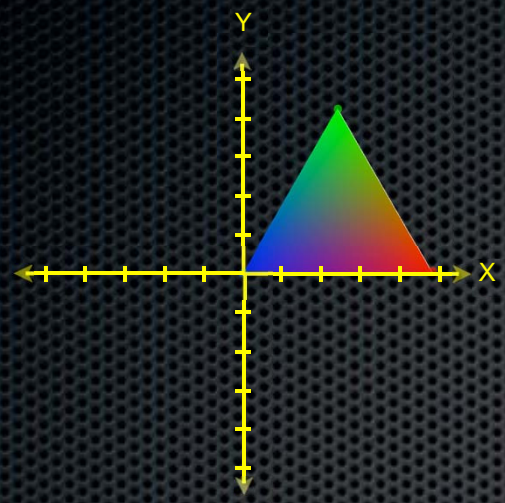


$x_f = x_o + t_x$

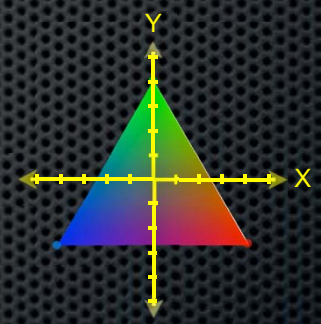
$y_f = y_o + t_y$

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Transformations



$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

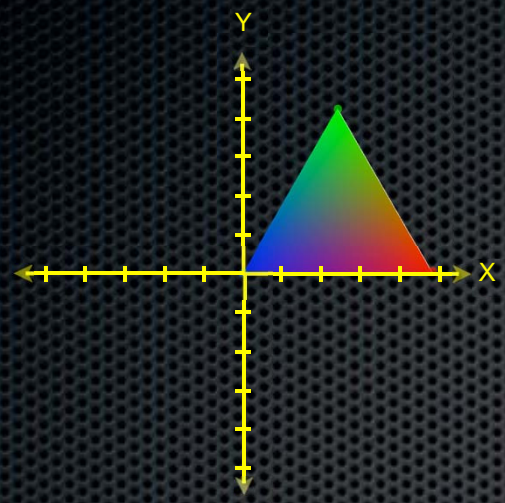


$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

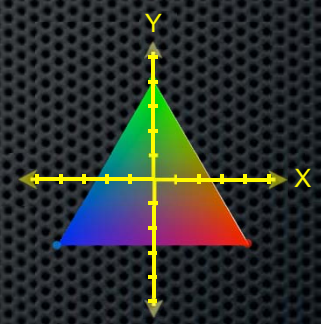
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Transformations

All Points



$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

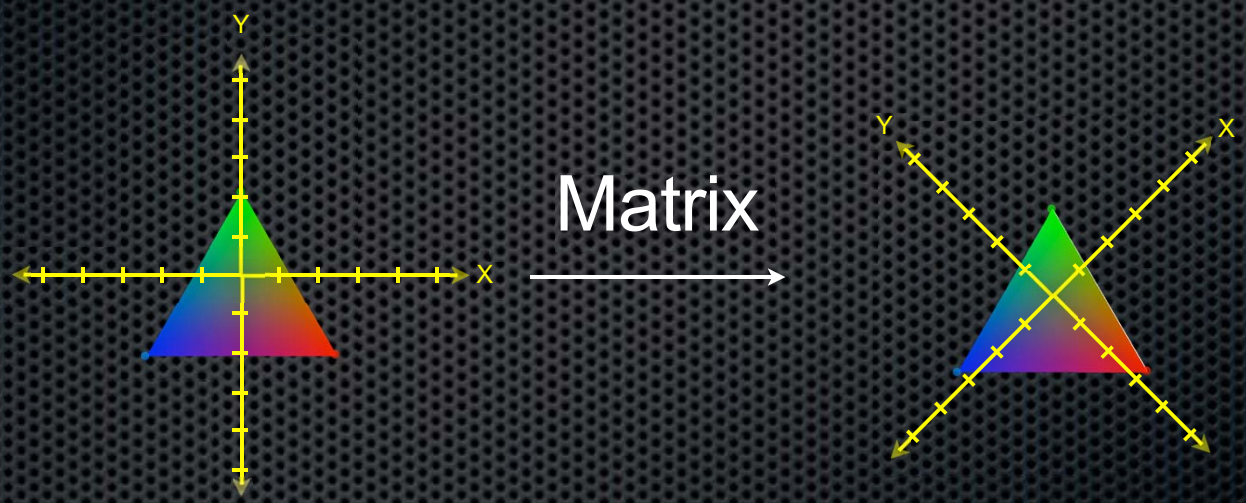


$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Transformations

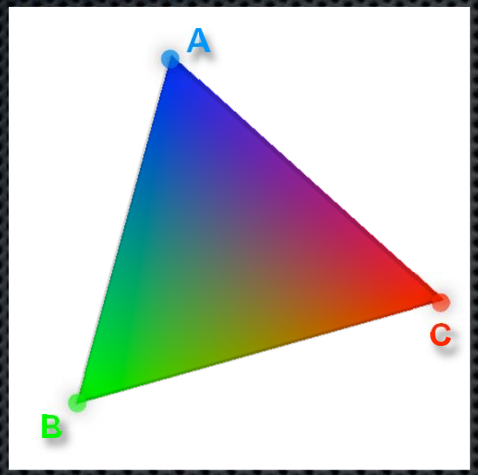
Coordinate System 1



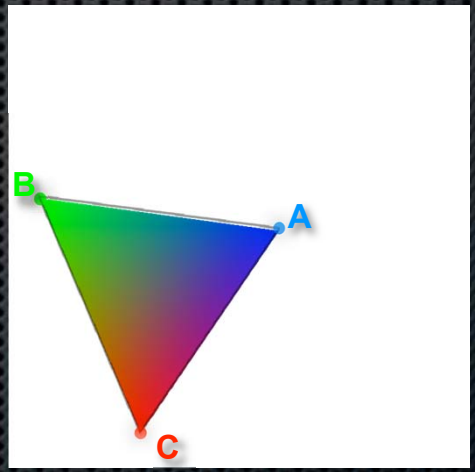
Coordinate System 2

Transformations

Coordinate System 1



RST



Coordinate System 2

Matrices

- Matrices
  - Concatenation & Persistence
  - Vector Multiplication

Matrix Composition

2.1 X
-3.6 Y
0.8 Z
Translate

PI/3 Rotate

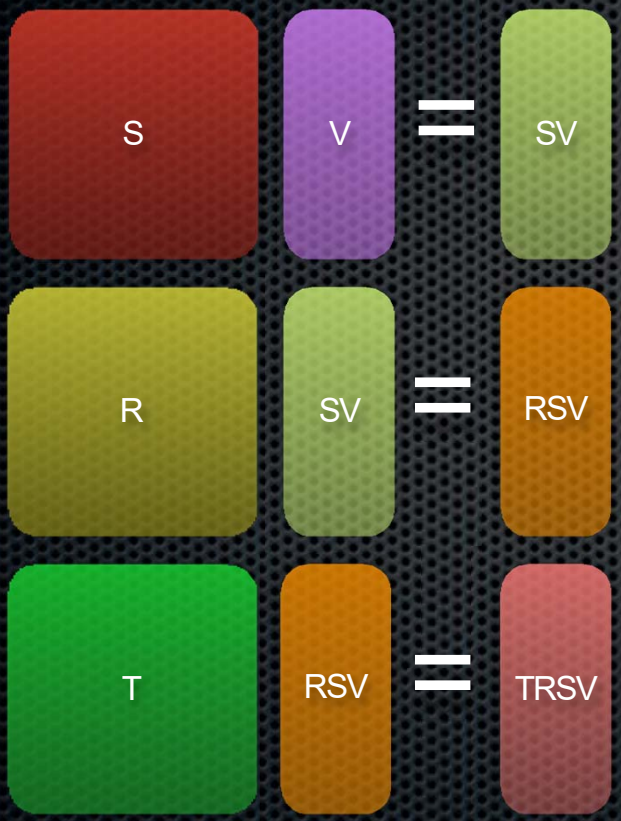
2x Scale

=

Combined Matrix That Does All 3 Ops!

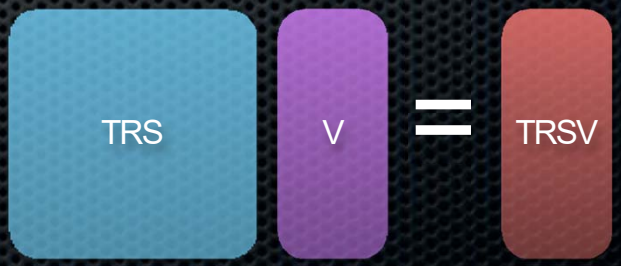
Matrix Composition

Either This 10,000 Times



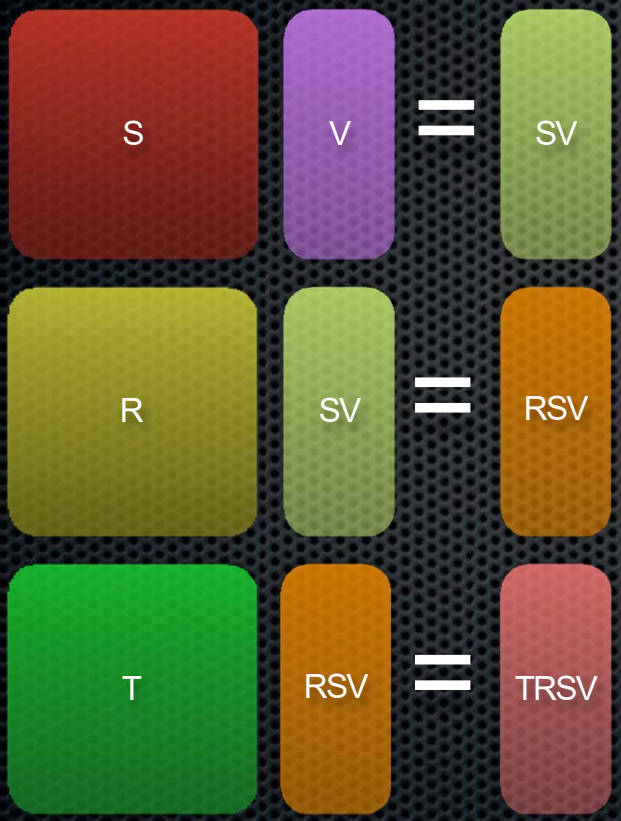
Or This 1 Time

And This 10,000 Times



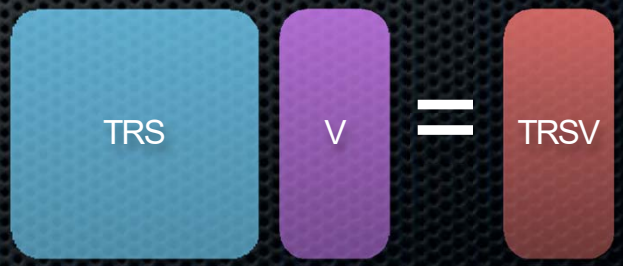
Either This
10,000 Times
S
V
=
SV
R
SV
=
RSV
T
RSV
=
TRSV
Or This
1 Time
T
R
S
=
TRS
And This
10,000 Times
TRS
V
=
TRSV

Either This
10,000 Times



Notice the ordering!

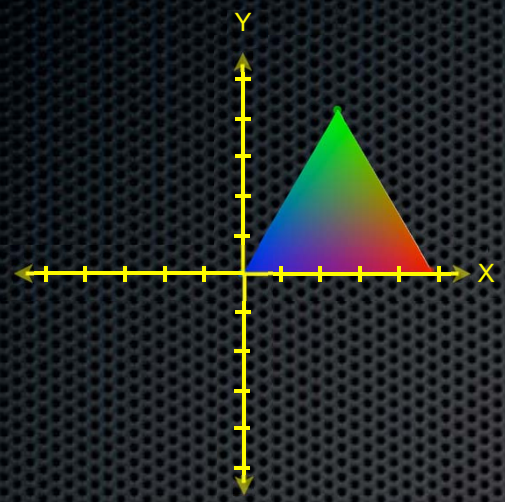
And This
10,000 Times

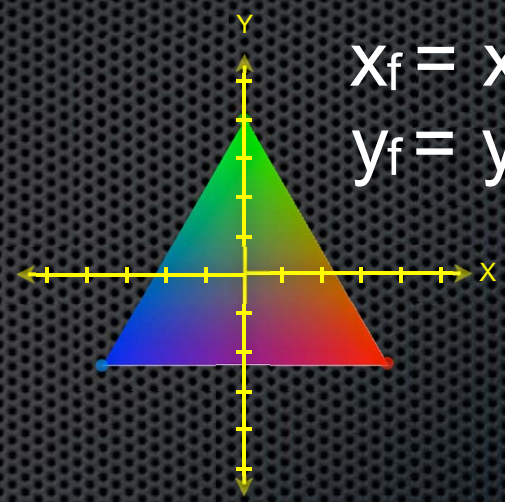


3D Transformations

Transformations

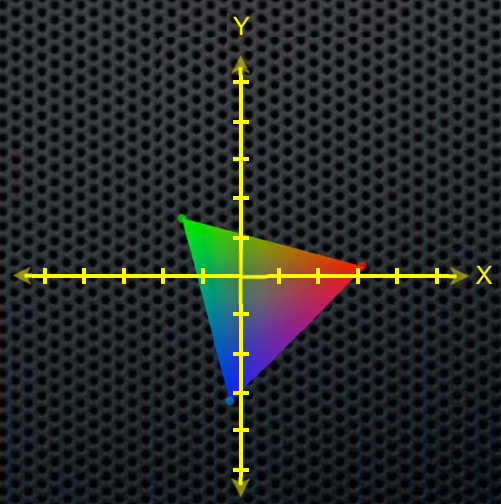
All Points





$$x_f = x_o \cdot s_x$$
$$y_f = y_o \cdot s_y$$

$$x_f = x_o + t_x$$
$$y_f = y_o + t_y$$



$$x_f = x_i \cdot \cos\theta - y_i \cdot \sin\theta$$
$$y_f = x_i \cdot \sin\theta + y_i \cdot \cos\theta$$

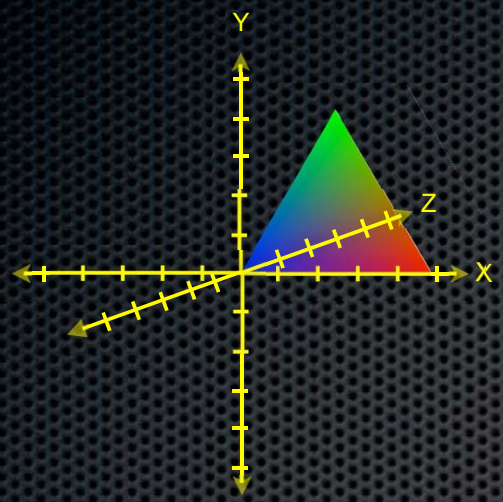
Transformations

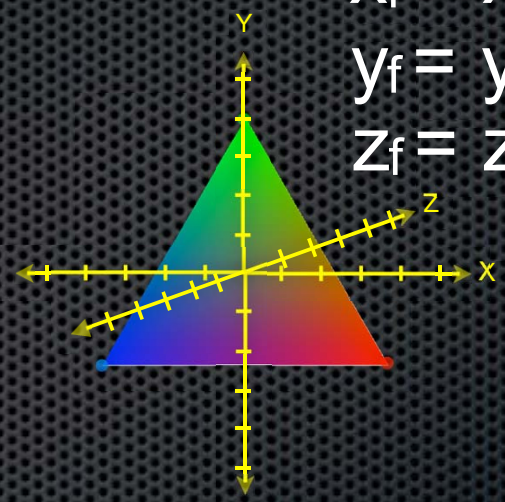
All Points

$x_f = x_o \cdot s_x$

$y_f = y_o \cdot s_y$

$z_f = z_o \cdot s_z$

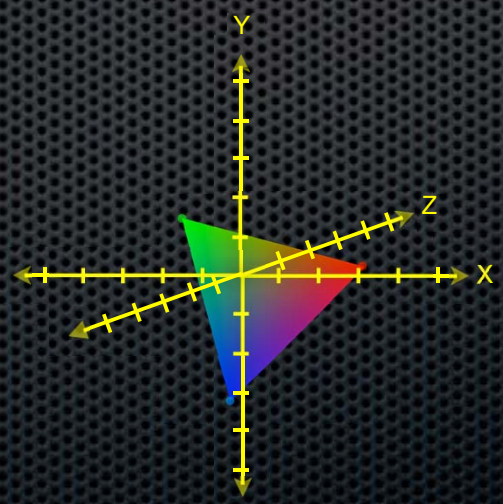




$x_f = x_o + t_x$

$y_f = y_o + t_y$

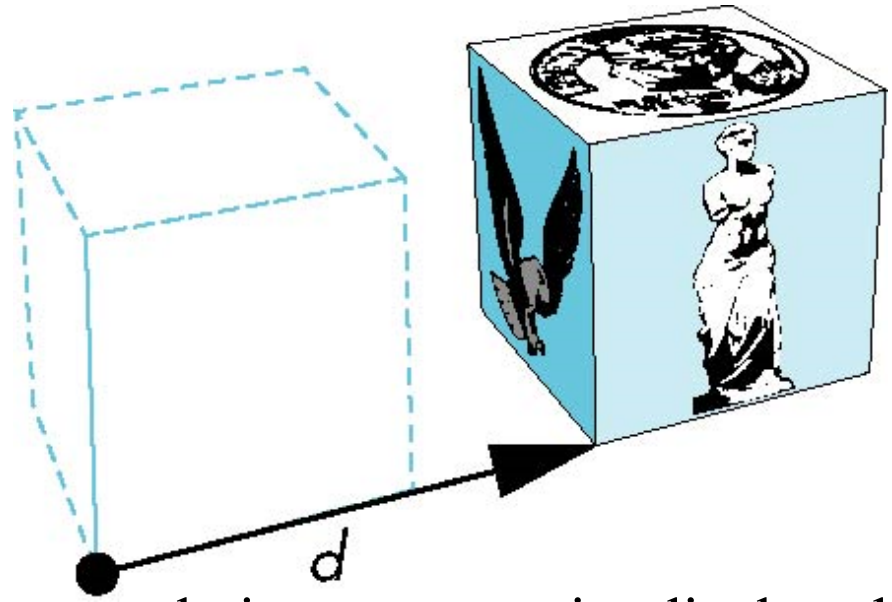
$z_f = z_o + t_z$



More Complex

Translation

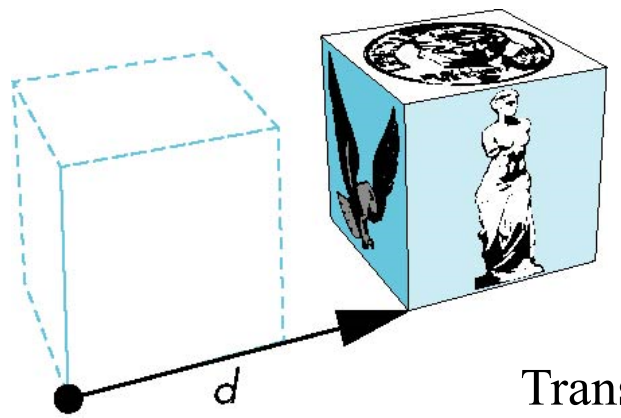
object



translation: every point displaced by same vector

3D Translation Example

object



Translation of object

- **Example:** If we translate a point (2,2,2) by displacement (2,4,6), new location of point is (4,6,8)

Using matrix multiplication for translation

Translate(2,4,6)

- Translate x: 2 + 2 = 4
- Translate y: 2 + 4 = 6
- Translate z: 2 + 6 = 8

$$\begin{pmatrix} 4 \\ 6 \\ 8 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

Translated point **Translation Matrix** **Original point**

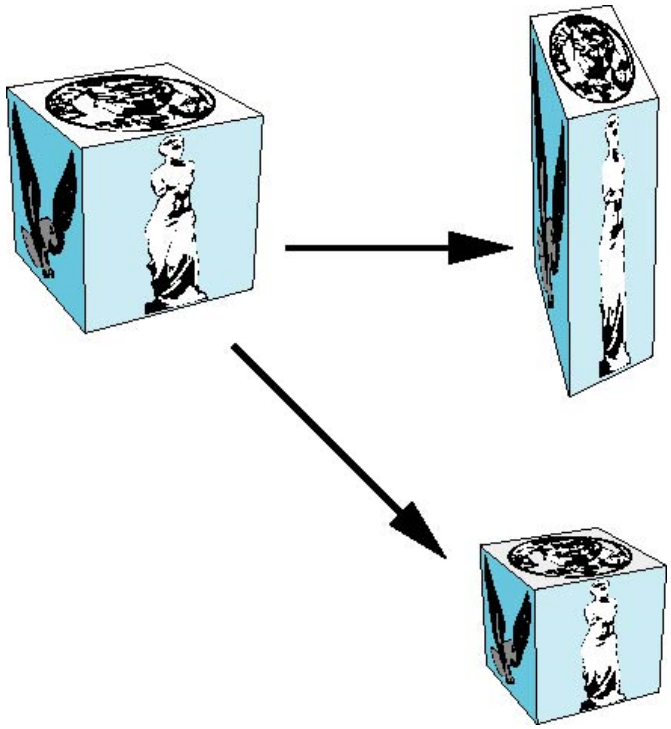
Scaling

$$x' = s_x x$$
$$y' = s_y y$$
$$z' = s_z z$$

$$\mathbf{p}' = \mathbf{S}\mathbf{p}$$

where

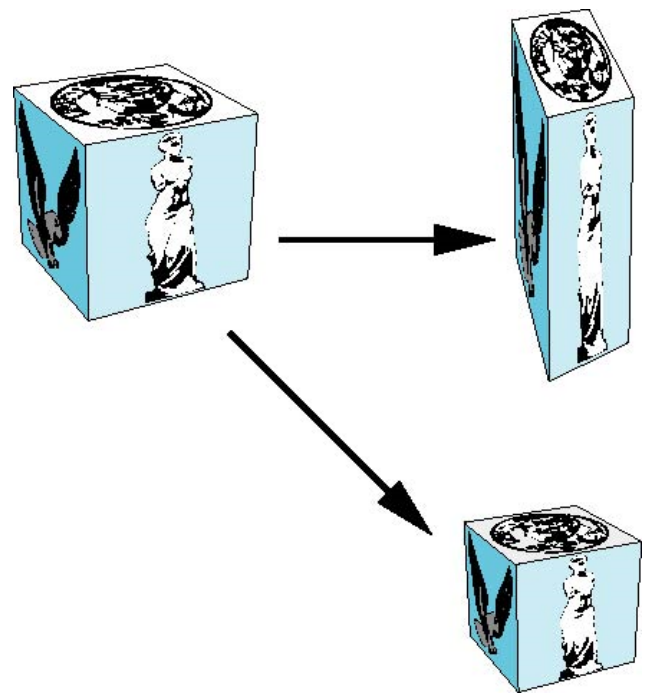
$$\mathbf{S} = \mathbf{S}(s_x, s_y, s_z)$$



Scaling Example

If we scale a point (2,4,6) by scaling factor (0.5,0.5,0.5)
Scaled point position = (1, 2, 3)

- Scale x: 2 x 0.5 = 1
- Scale y: 4 x 0.5 = 2
- Scale z: 6 x 0.5 = 3

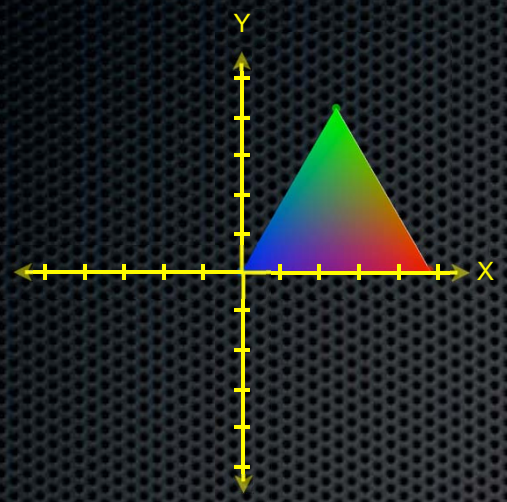


$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \\ 6 \\ 1 \end{pmatrix}$$

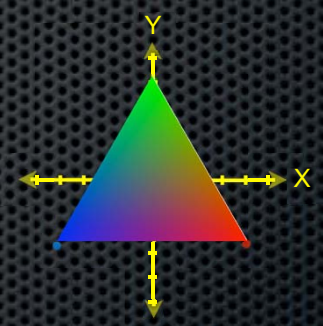
Scale Matrix for Scale(0.5, 0.5, 0.5)

Transformations

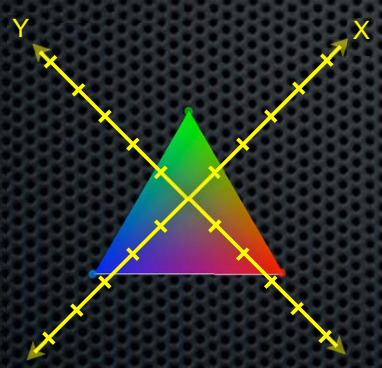
All Points



$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



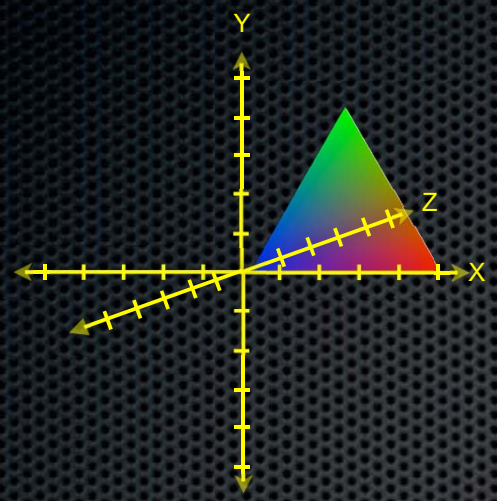
$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



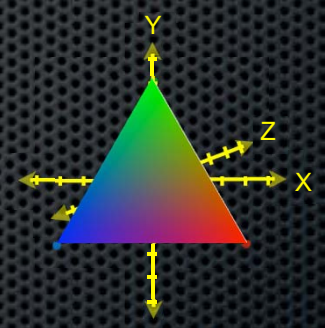
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Transformations

All Points

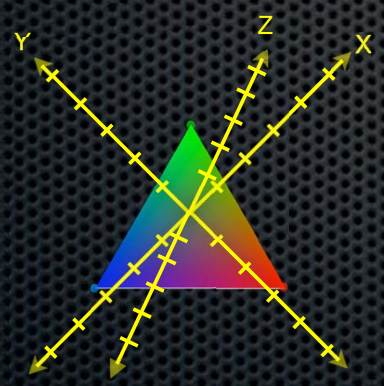


$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

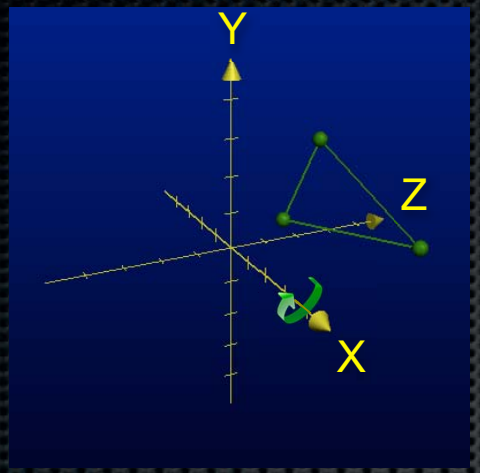
M = vector rotate matrix



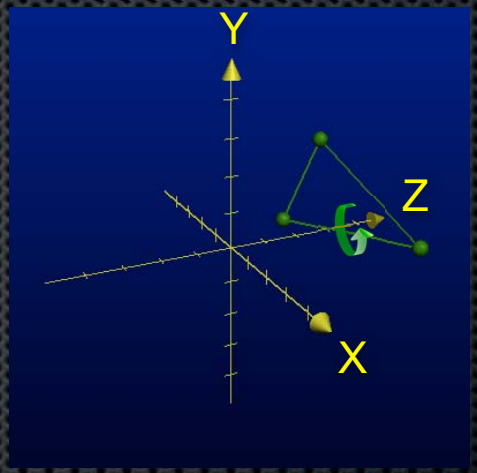
$$\begin{bmatrix} m & m & m & 0 \\ m & m & m & 0 \\ m & m & m & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation in 3D

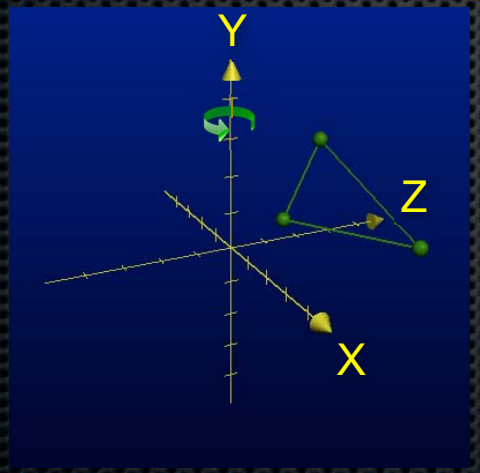
Rotation About X-Axis



Rotation About Z-Axis

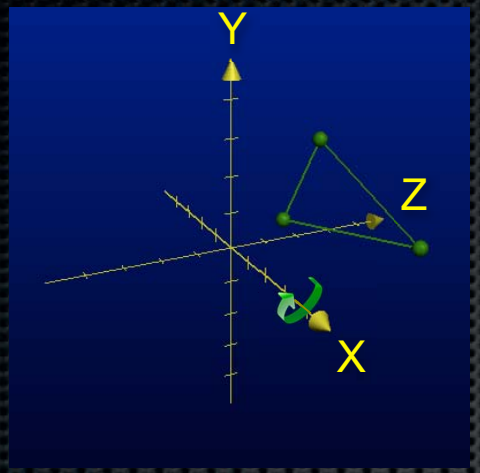


Rotation About Y-Axis

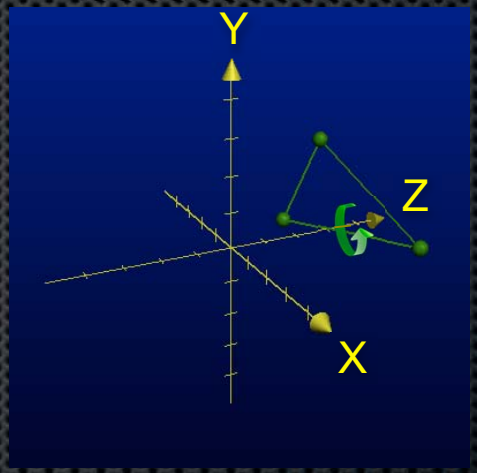


Rotation in 3D

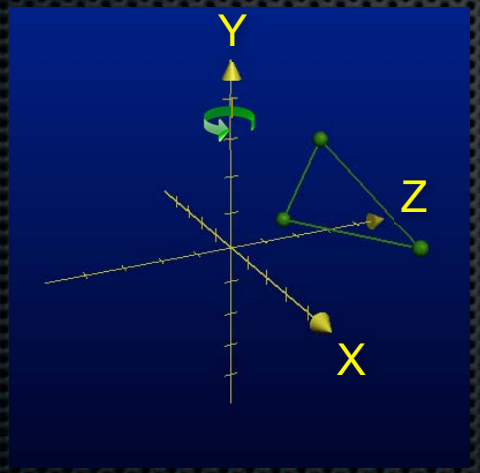
Rotation About X-Axis



Rotation About Z-Axis



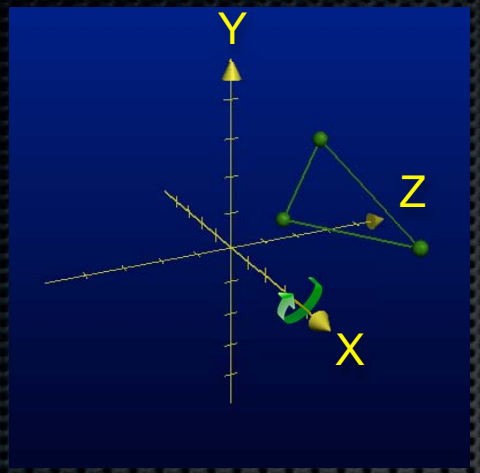
Rotation About Y-Axis



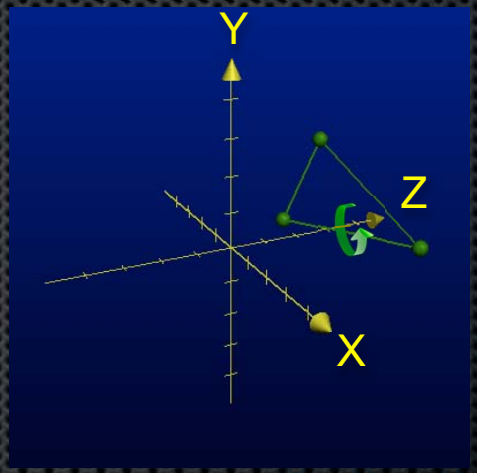
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation in 3D

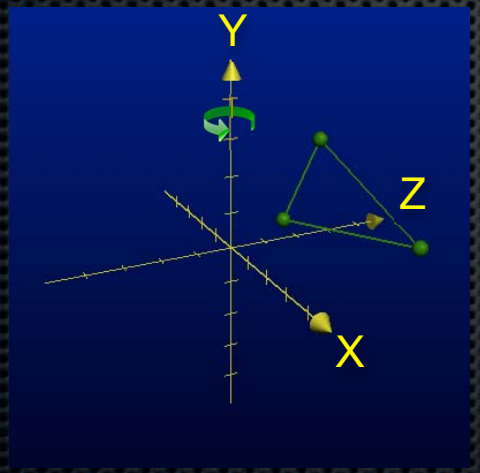
Rotation About X-Axis



Rotation About Z-Axis



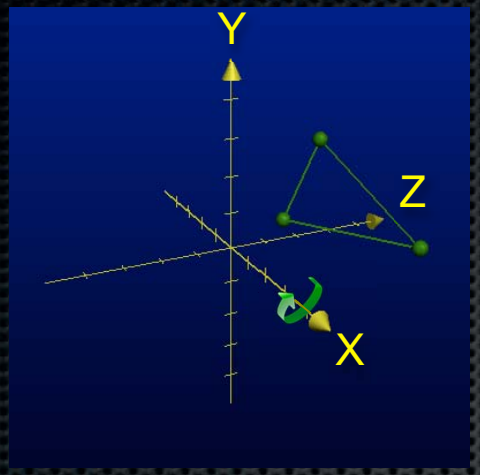
Rotation About Y-Axis



$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

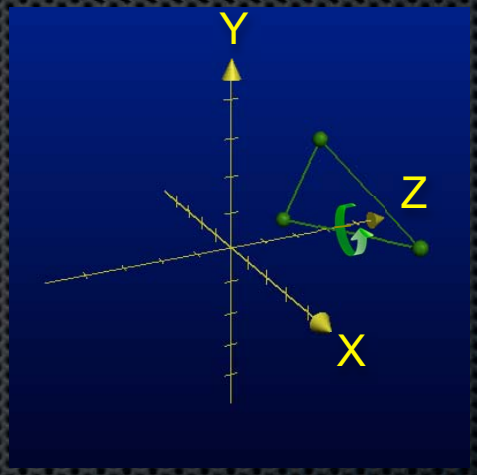
Rotation in 3D

Rotation About X-Axis

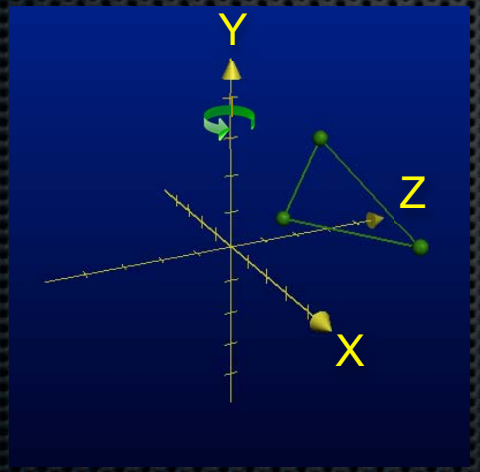


$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation About Z-Axis



Rotation About Y-Axis



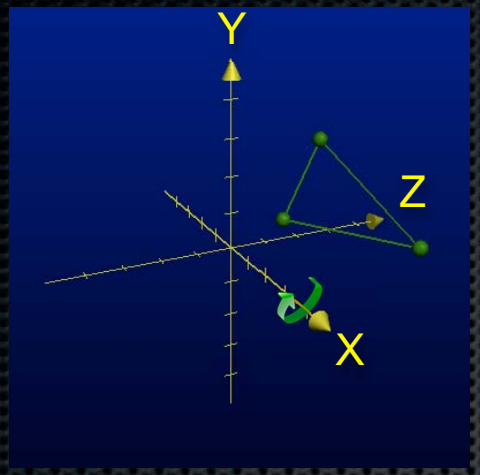
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Think-Pair-Share

What would the matrix for rotation around the y-axis look like?

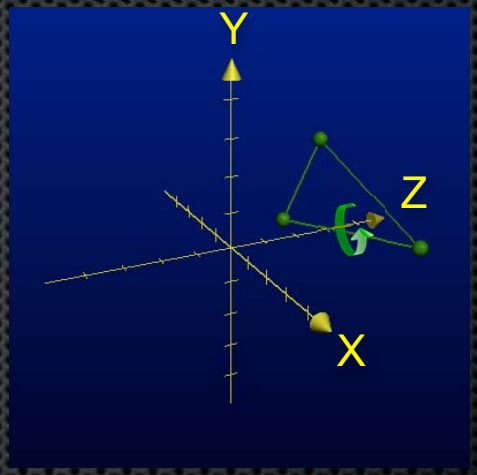
Rotation in 3D

Rotation About X-Axis

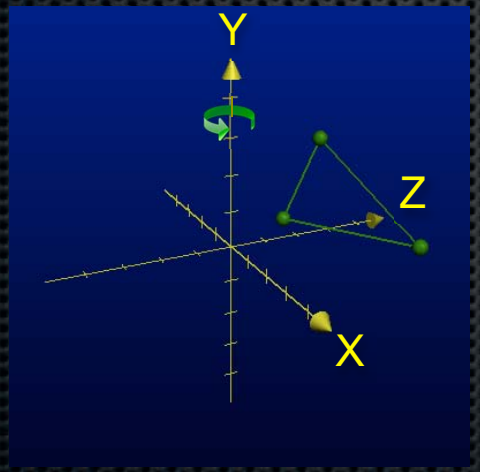


$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation About Z-Axis



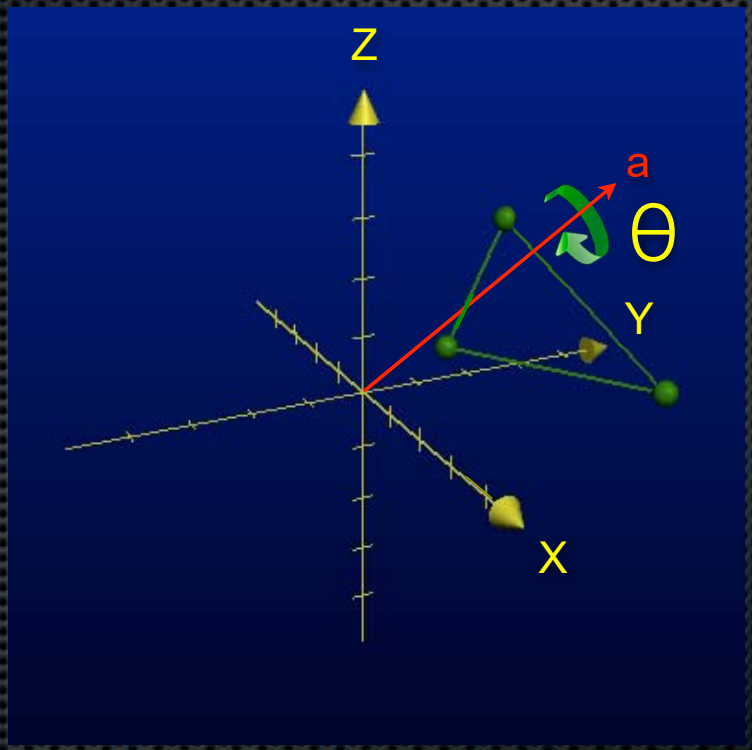
Rotation About Y-Axis



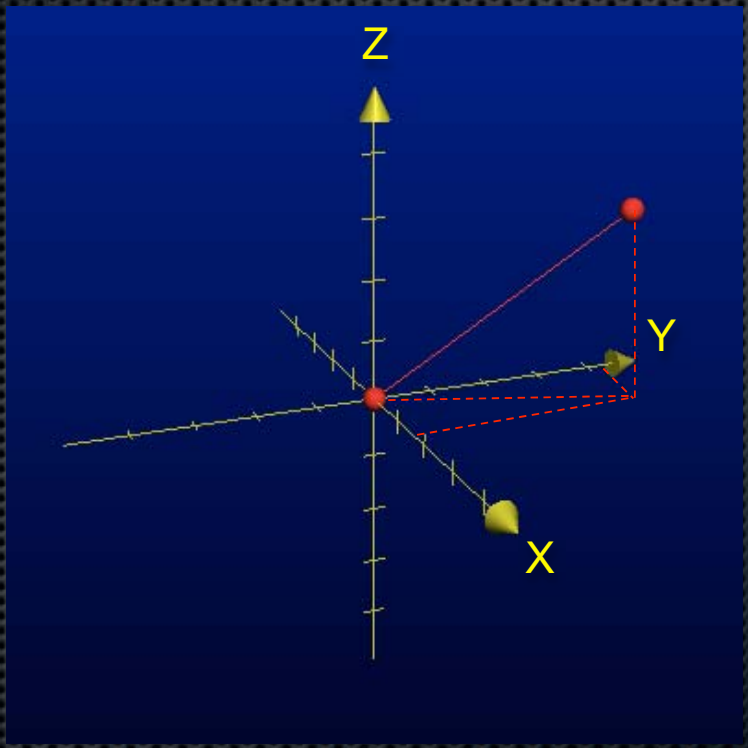
$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

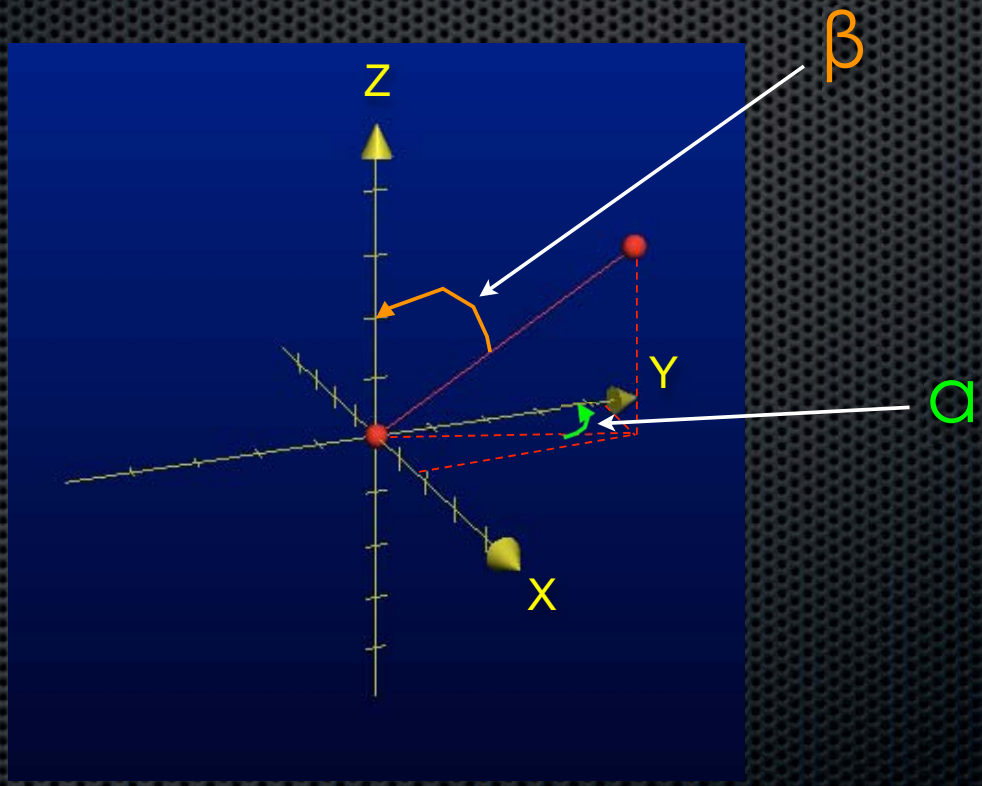
Arbitrary Axis Rotation in 3D



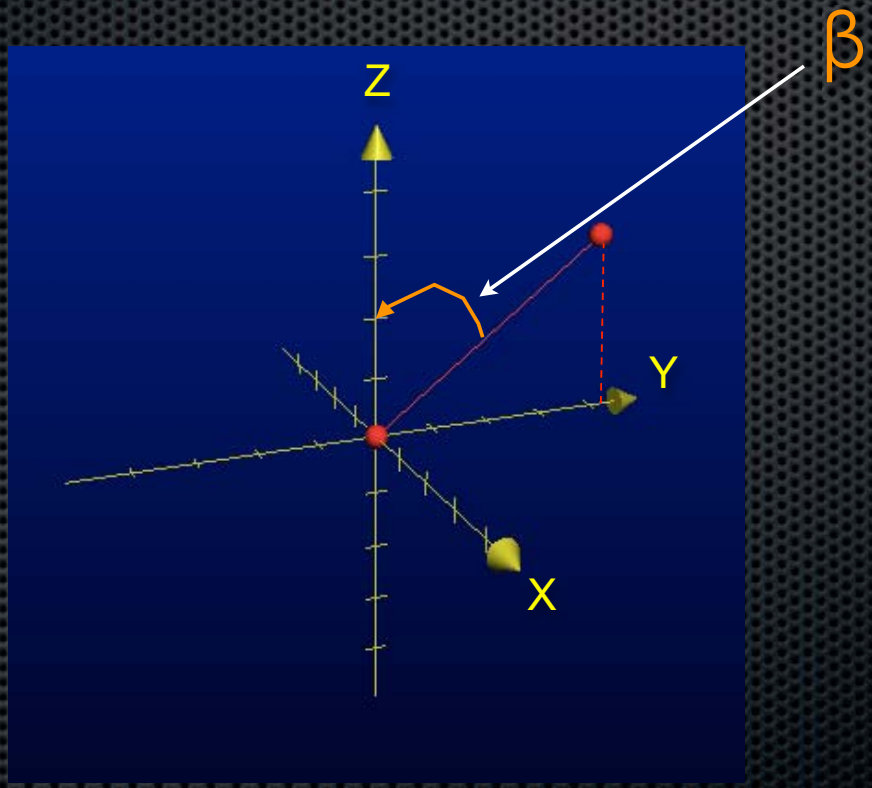
Arbitrary Axis Rotation in 3D



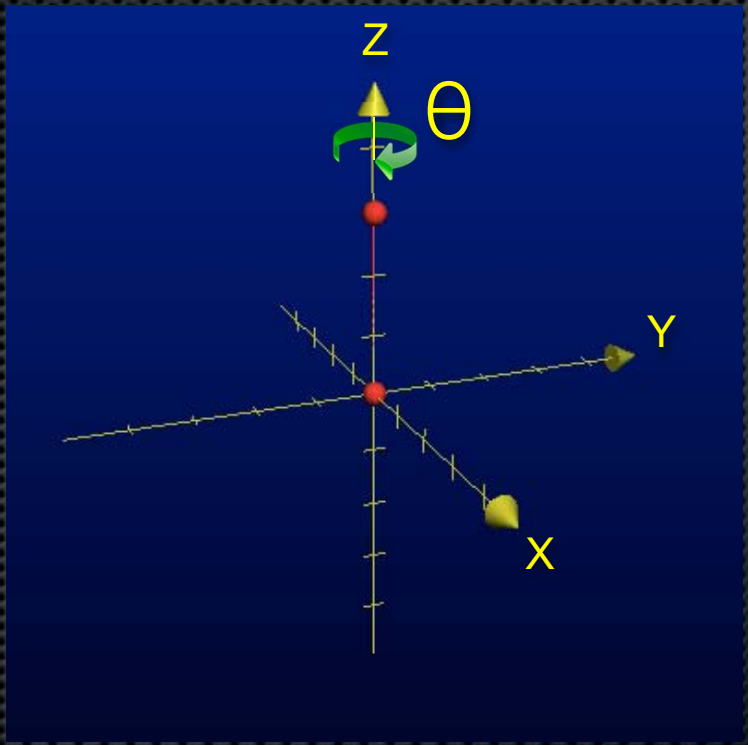
Arbitrary Axis Rotation in 3D



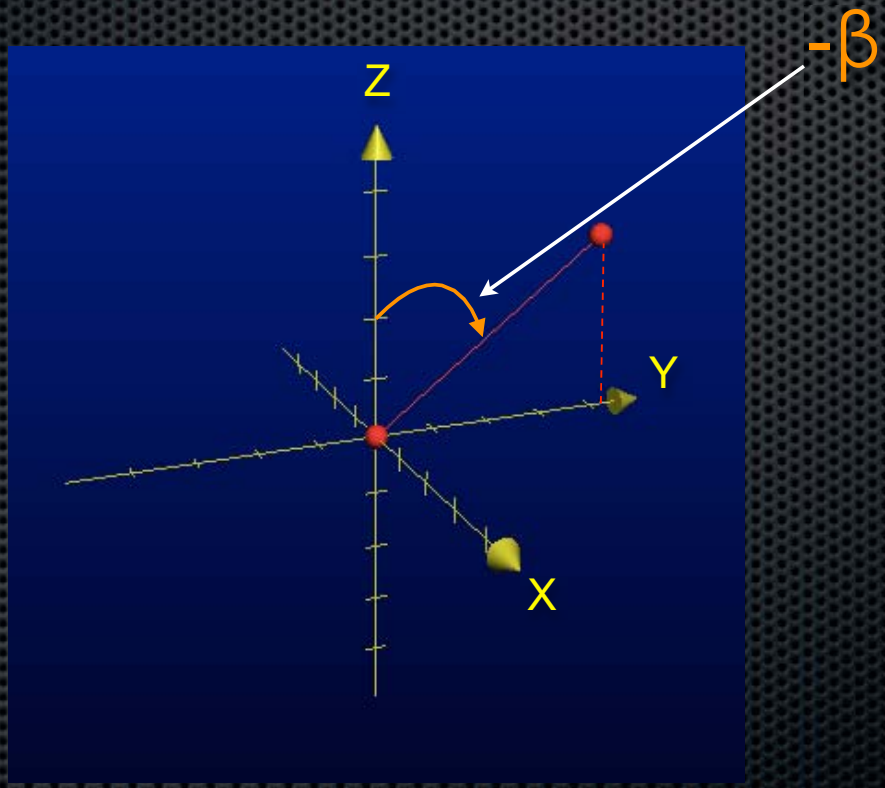
Arbitrary Axis Rotation in 3D



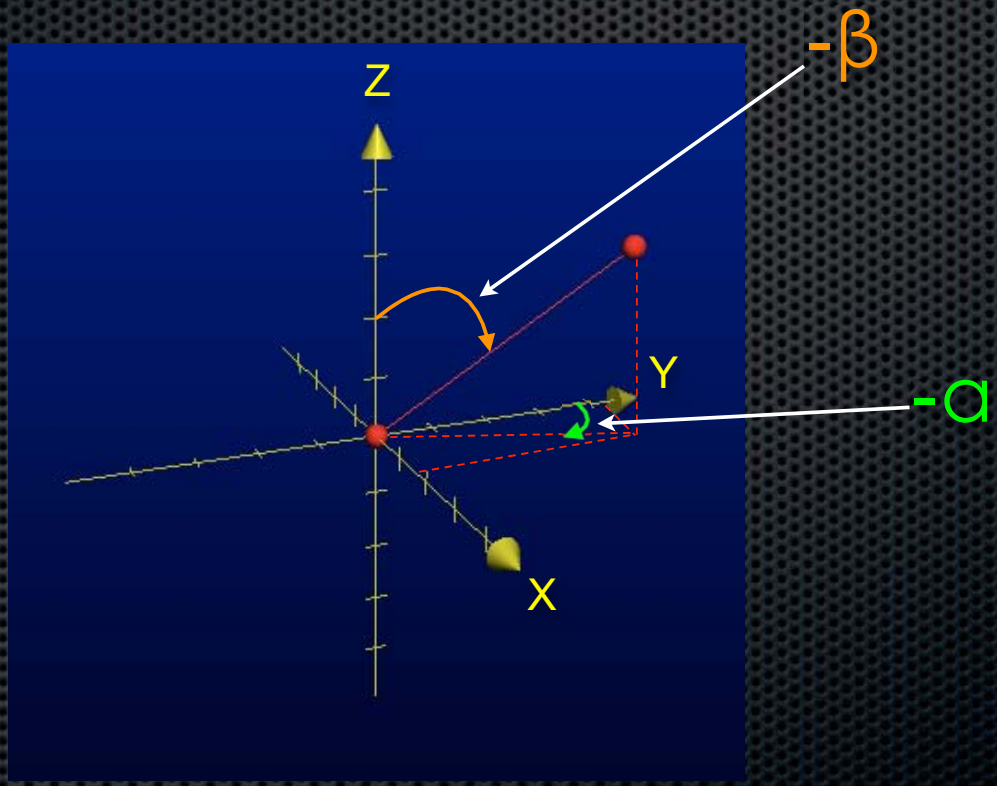
Arbitrary Axis Rotation in 3D



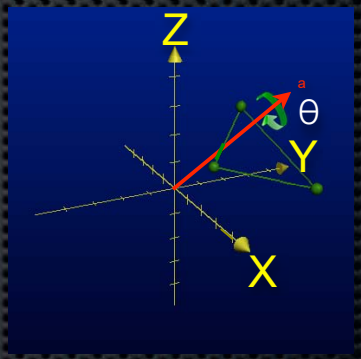
Arbitrary Axis Rotation in 3D



Arbitrary Axis Rotation in 3D



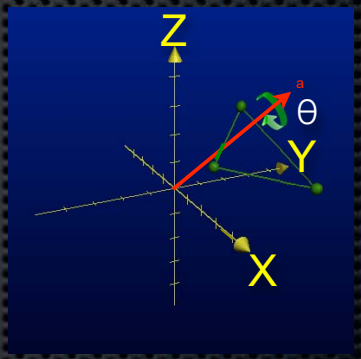
Arbitrary Axis Rotation in 3D



$$M = R_z(-\alpha) \times R_x(-\beta) \times R_z(\theta) \times R_x(\beta) \times R_z(\alpha)$$

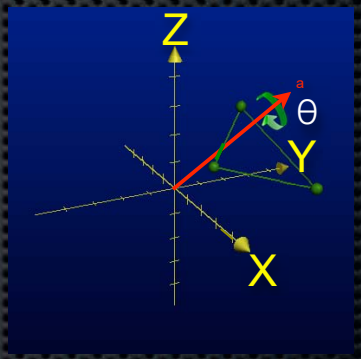
Arbitrary Axis Rotation in 3D

$$M = \begin{bmatrix} \cos-\alpha & -\sin-\alpha & 0 & 0 \\ \sin-\alpha & \cos-\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos-\beta & -\sin-\beta & 0 \\ 0 & \sin-\beta & \cos-\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta & 0 \\ 0 & \sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Arbitrary Axis Rotation in 3D



$$M = \begin{bmatrix} m & m & m & 0 \\ m & m & m & 0 \\ m & m & m & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

When implementing remember to check for degeneracies when $a_y = 0$ and $a_z = 0$ (gimbal lock)

Alternative Approach: Quaternions

- Resolves the gimbal lock problem
- Non-intuitive
- Later, time-permitting