Computer Graphics (CS 4731)
Gimbal Lock and Quaternions

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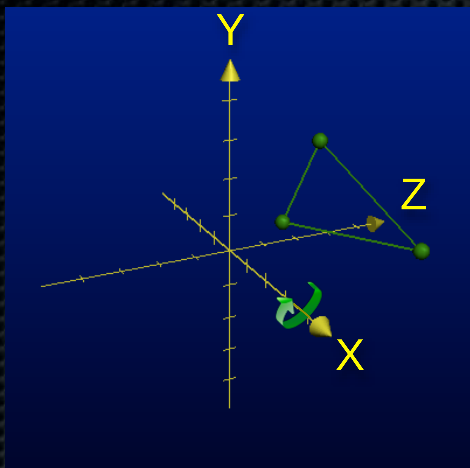
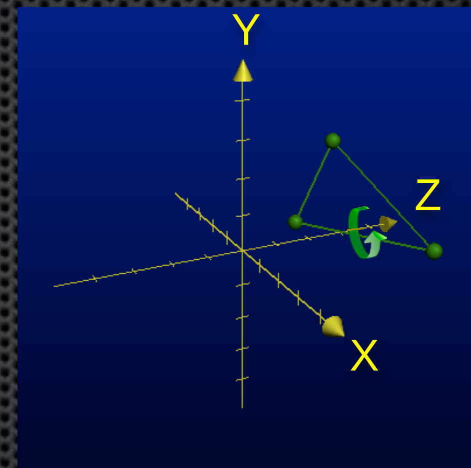
Rotation in 3D

Rotation About X-Axis

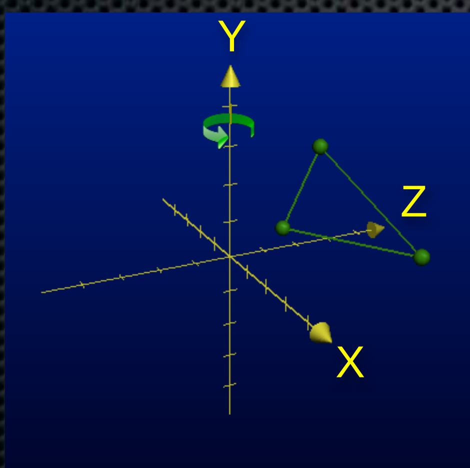


$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation About Z-Axis



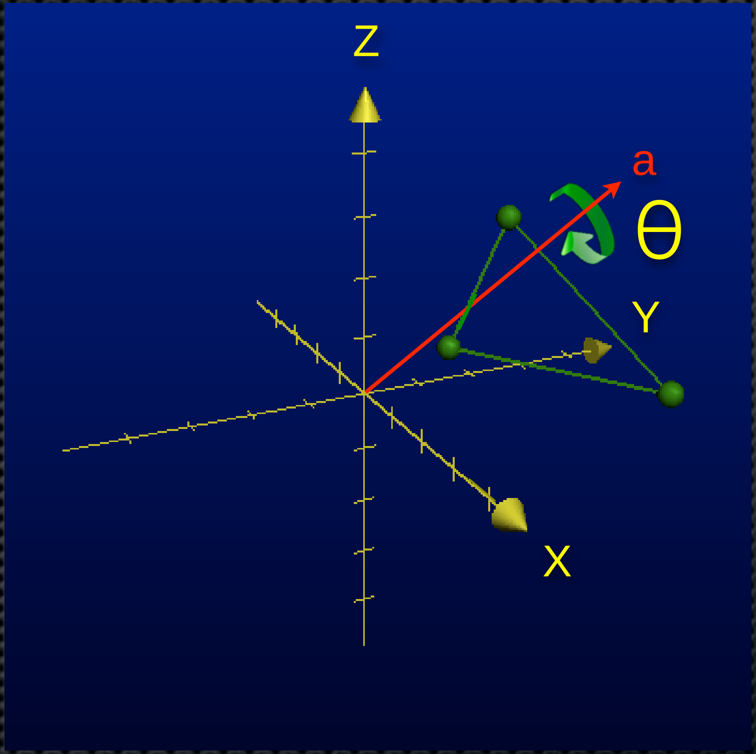
Rotation About Y-Axis



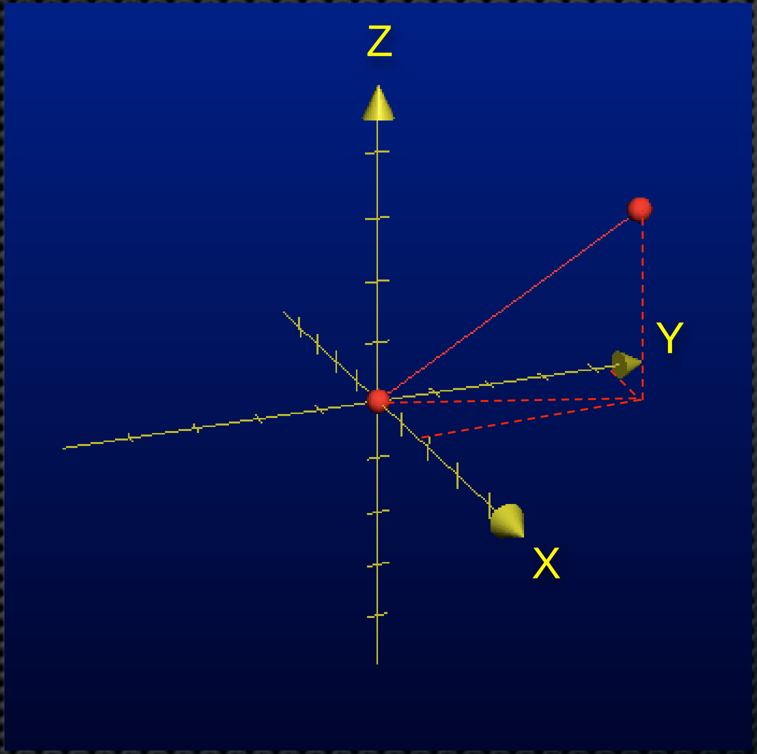
$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

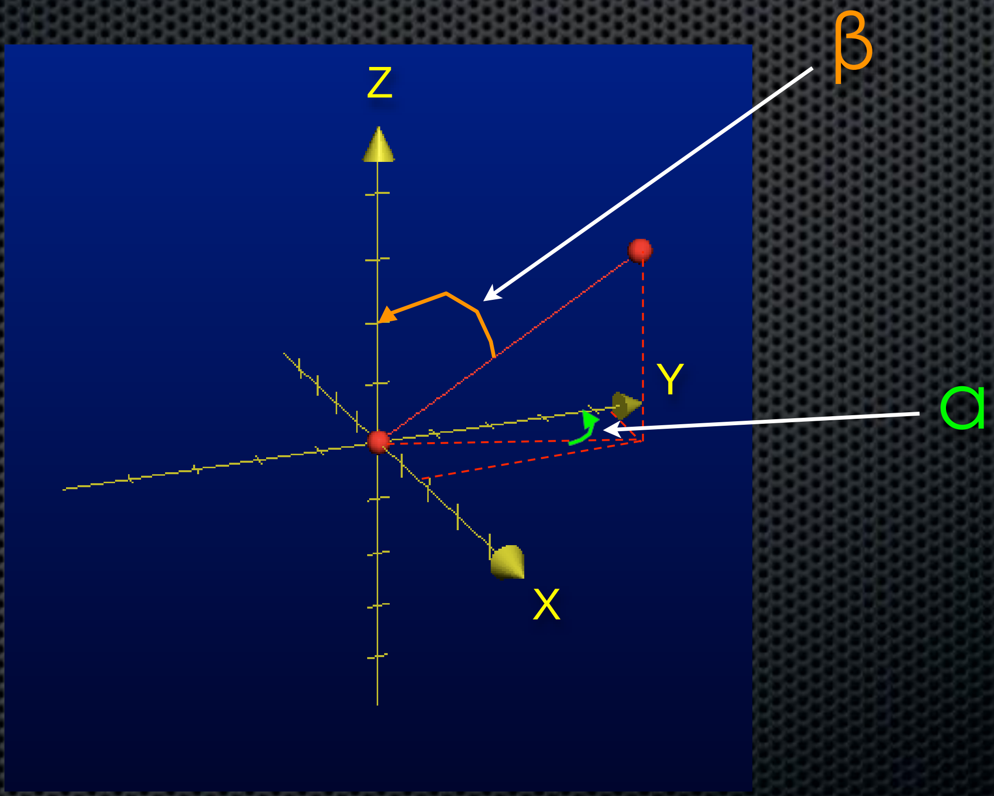
Arbitrary Axis Rotation in 3D



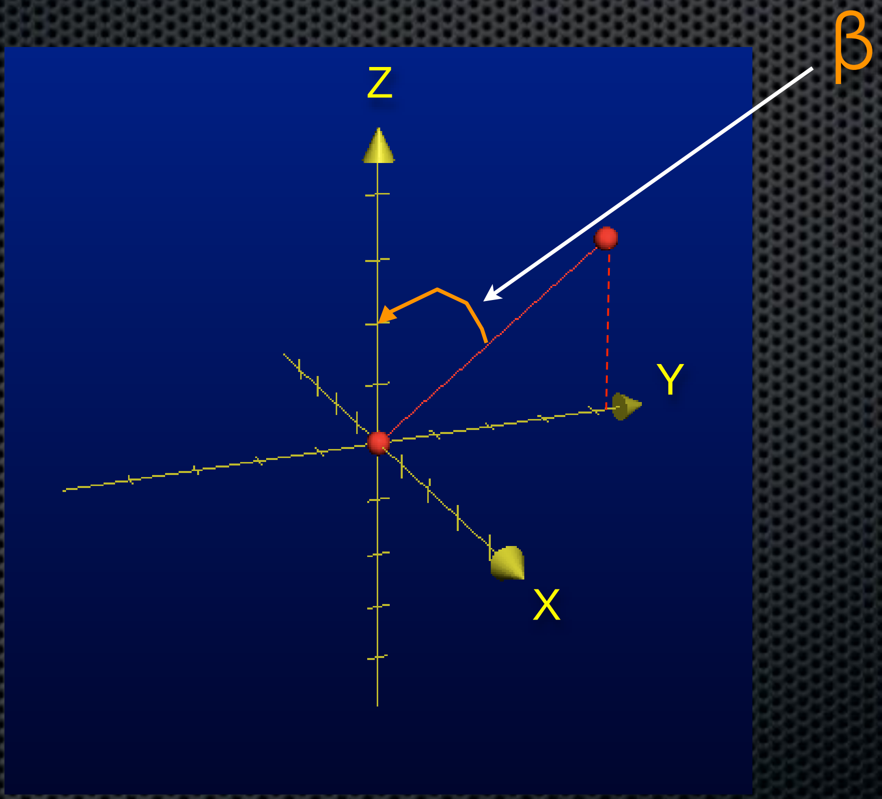
Arbitrary Axis Rotation in 3D



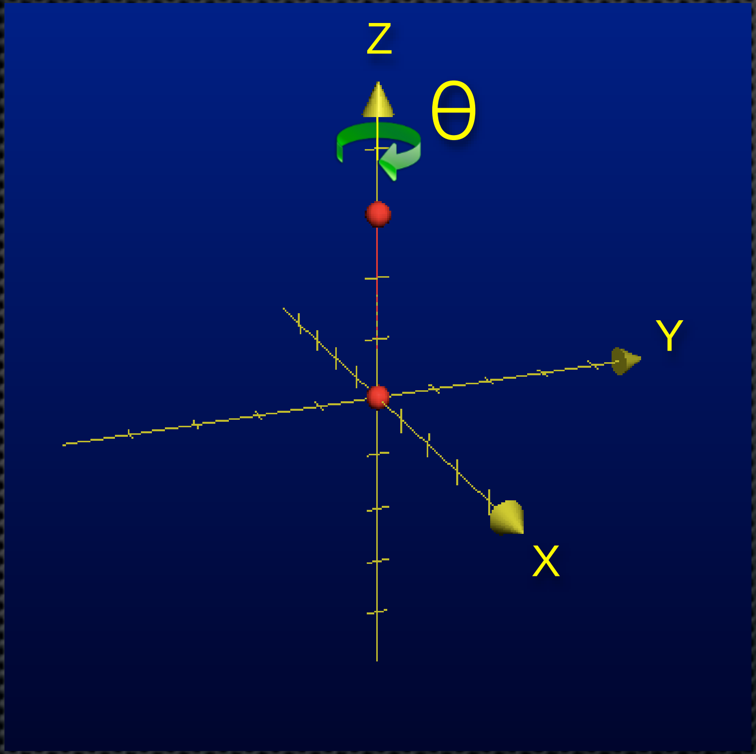
Arbitrary Axis Rotation in 3D



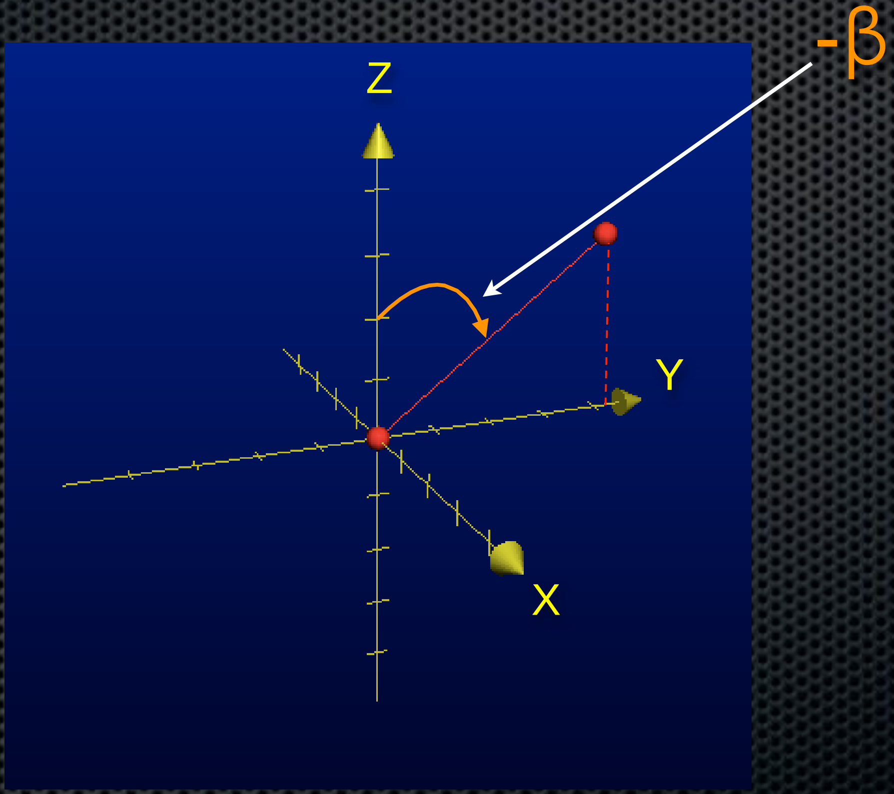
Arbitrary Axis Rotation in 3D



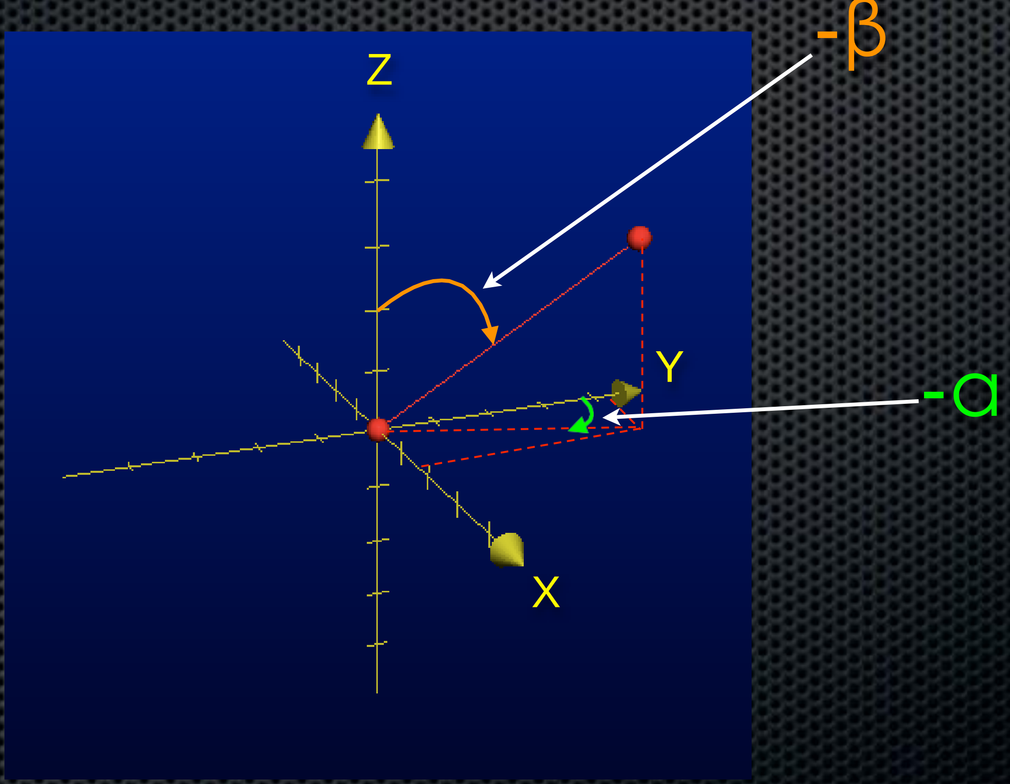
Arbitrary Axis Rotation in 3D



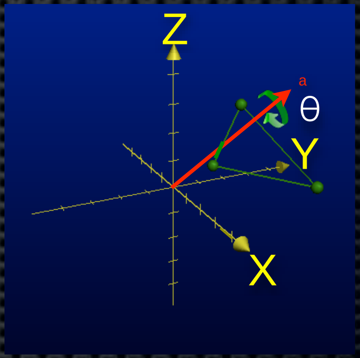
Arbitrary Axis Rotation in 3D



Arbitrary Axis Rotation in 3D



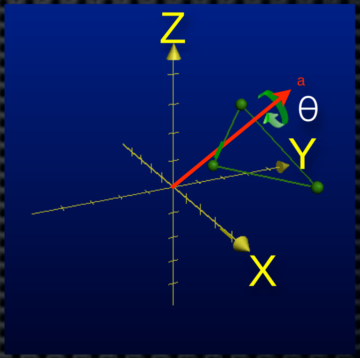
Arbitrary Axis Rotation in 3D



$$M = R_z(-\alpha) \times R_x(-\beta) \times R_z(\theta) \times R_x(\beta) \times R_z(\alpha)$$

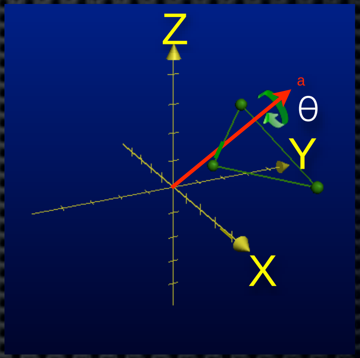
Arbitrary Axis Rotation in 3D

$$M = \begin{bmatrix} \cos{-\alpha} & -\sin{-\alpha} & 0 & 0 \\ \sin{-\alpha} & \cos{-\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos{-\beta} & -\sin{-\beta} & 0 \\ 0 & \sin{-\beta} & \cos{-\beta} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta & 0 \\ 0 & \sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Arbitrary Axis Rotation in 3D

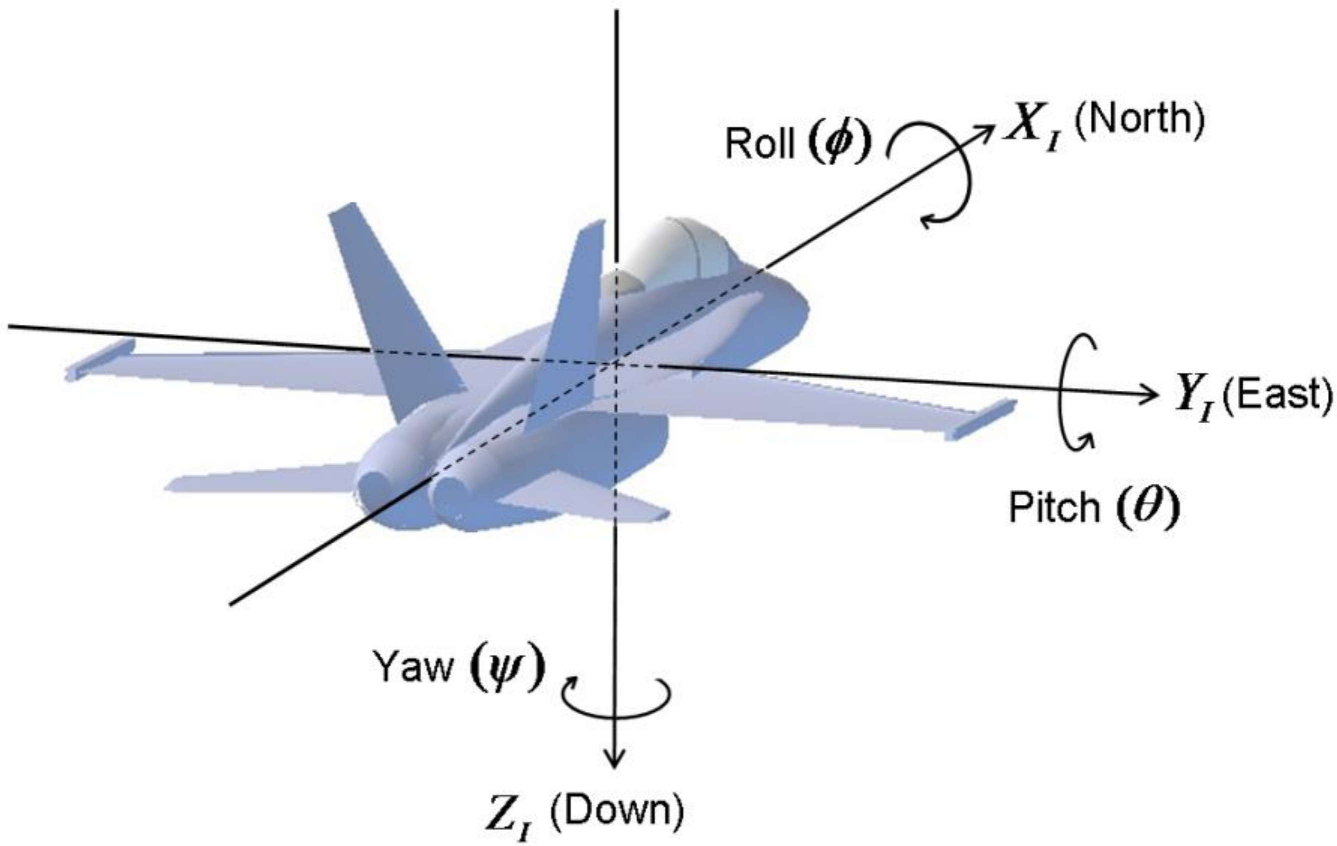


$$M = \begin{bmatrix} m & m & m & 0 \\ m & m & m & 0 \\ m & m & m & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

When implementing remember to check for degeneracies when $a_y = 0$ and $a_z = 0$ (gimbal lock)

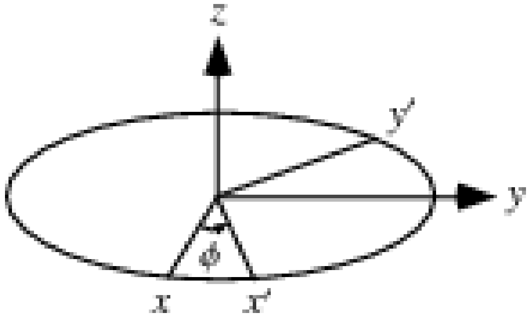
Alternative Approach: Quaternions

- Resolves the gimbal lock problem
- Non-intuitive
- Later, time-permitting

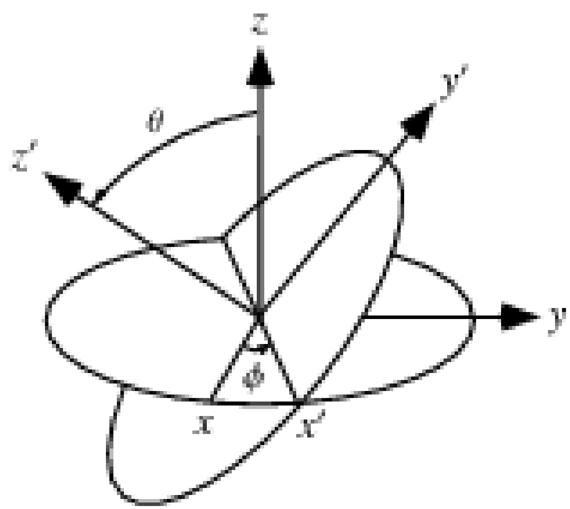


Roll (ϕ)
X_I (North)
Y_I (East)
Pitch (θ)
Yaw (ψ)
Z_I (Down)

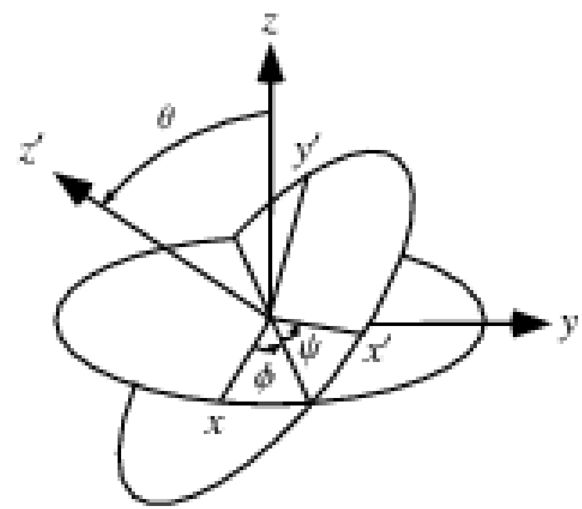
D



C



B



$$\mathsf{D} \equiv \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathsf{C} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad \mathsf{B} \equiv \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

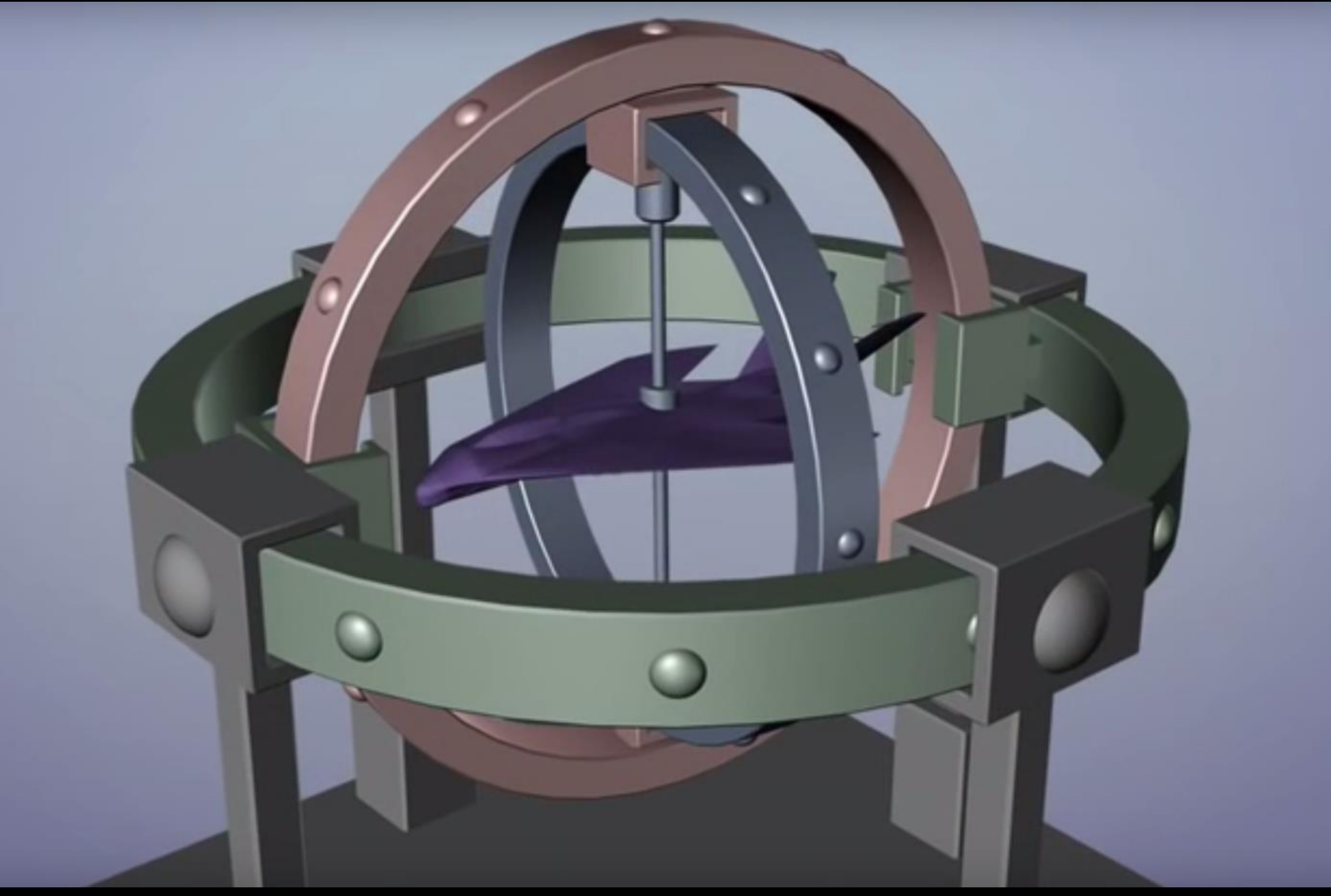
$$D \equiv \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad B \equiv \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A = BCD

$$A \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

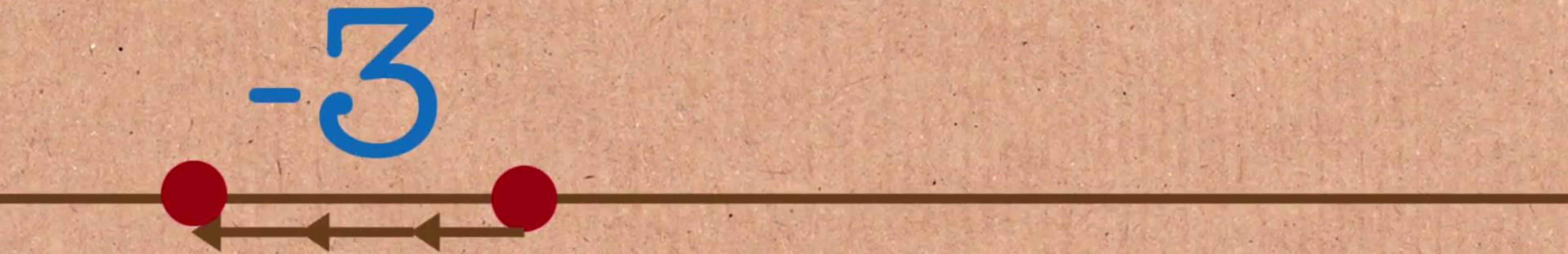
where

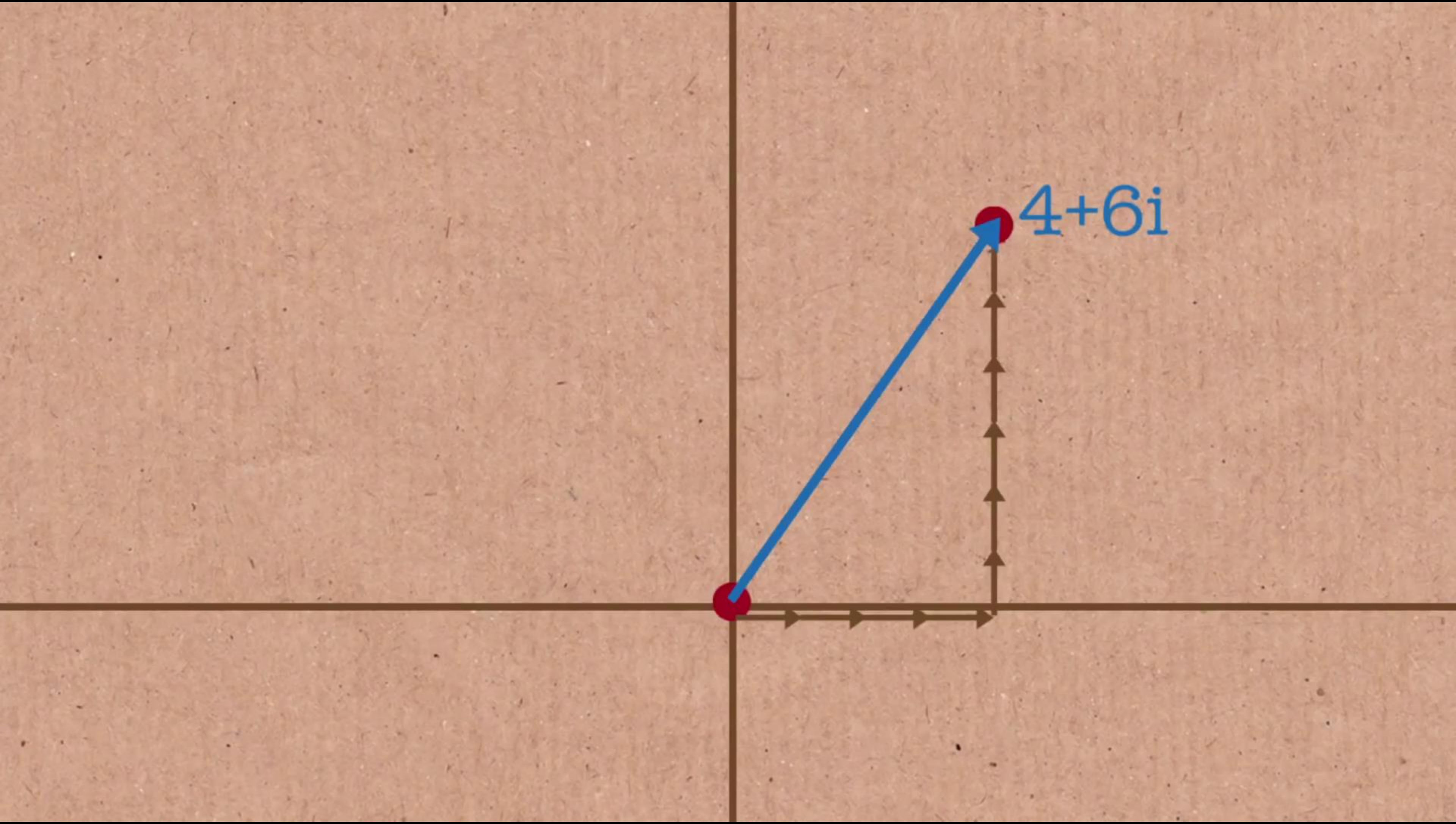
$$\begin{aligned}
a_{11} &= -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi \\
a_{12} &= \sin\psi\cos\phi + \cos\theta\sin\phi\cos\psi \\
a_{13} &= -\cos\psi\sin\theta \\
a_{21} &= -\cos\psi\sin\phi - \cos\theta\cos\phi\sin\psi \\
a_{22} &= \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi \\
a_{23} &= \sin\psi\sin\theta \\
a_{31} &= \sin\theta\cos\phi \\
a_{32} &= \sin\theta\sin\phi \\
a_{33} &= \cos\theta.
\end{aligned}$$

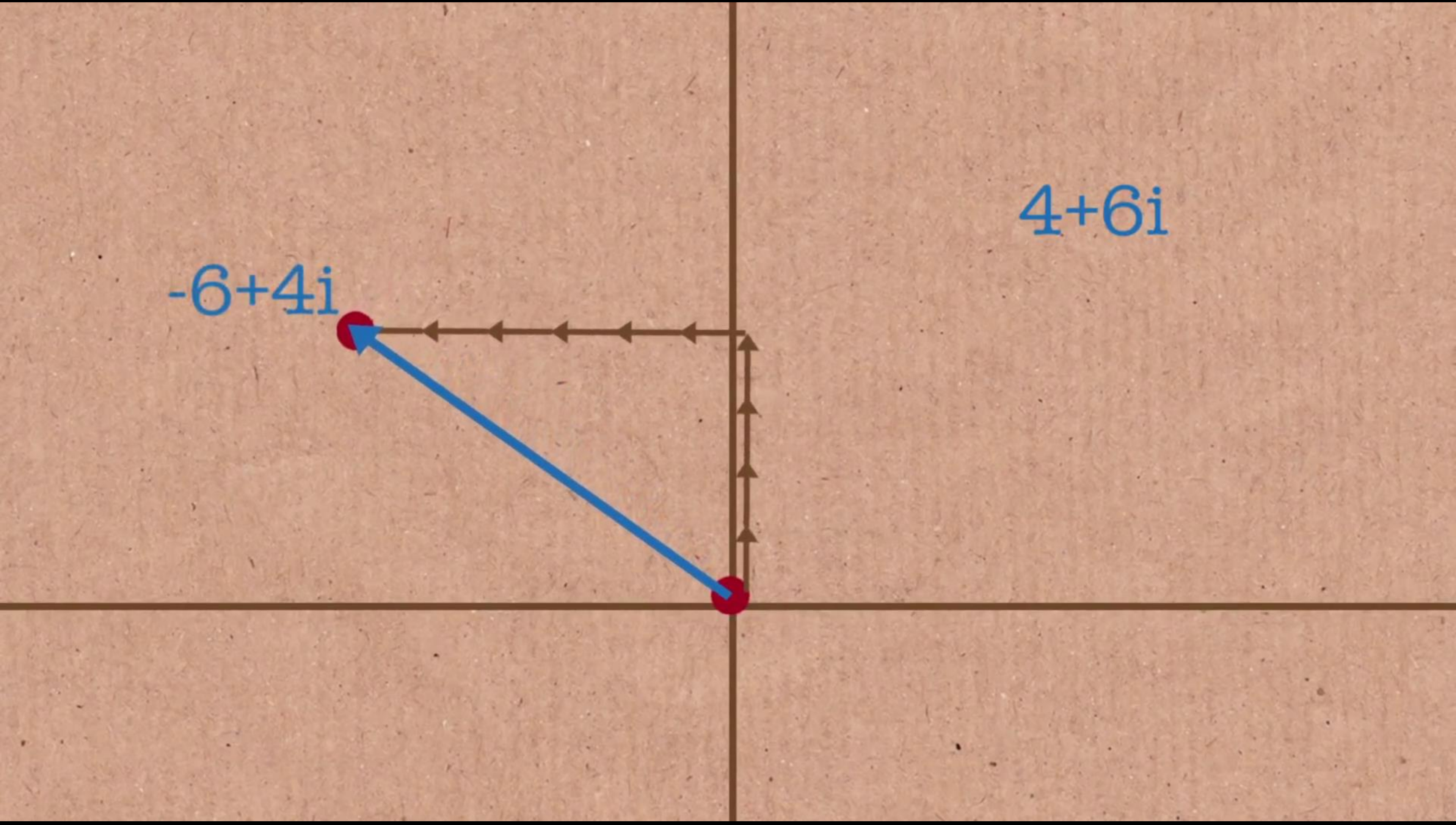


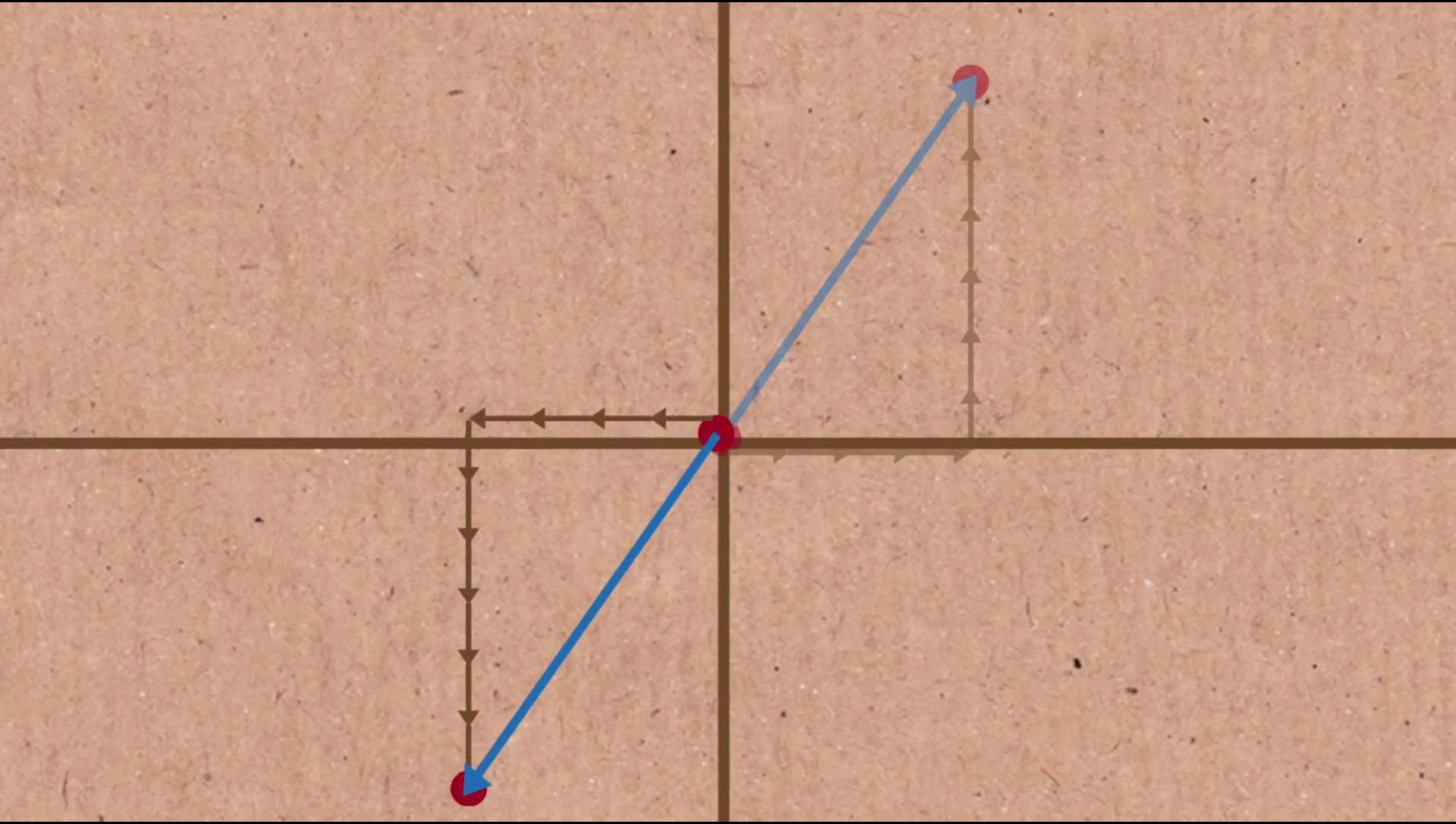

Gimbal Lock


+5

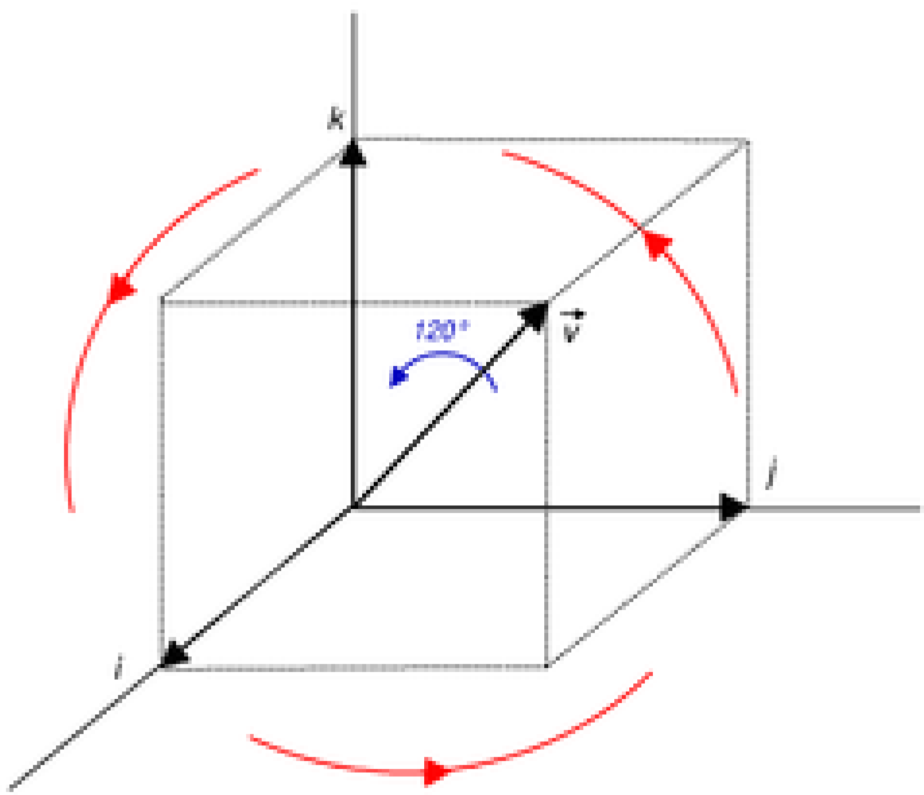

-3


4+6i

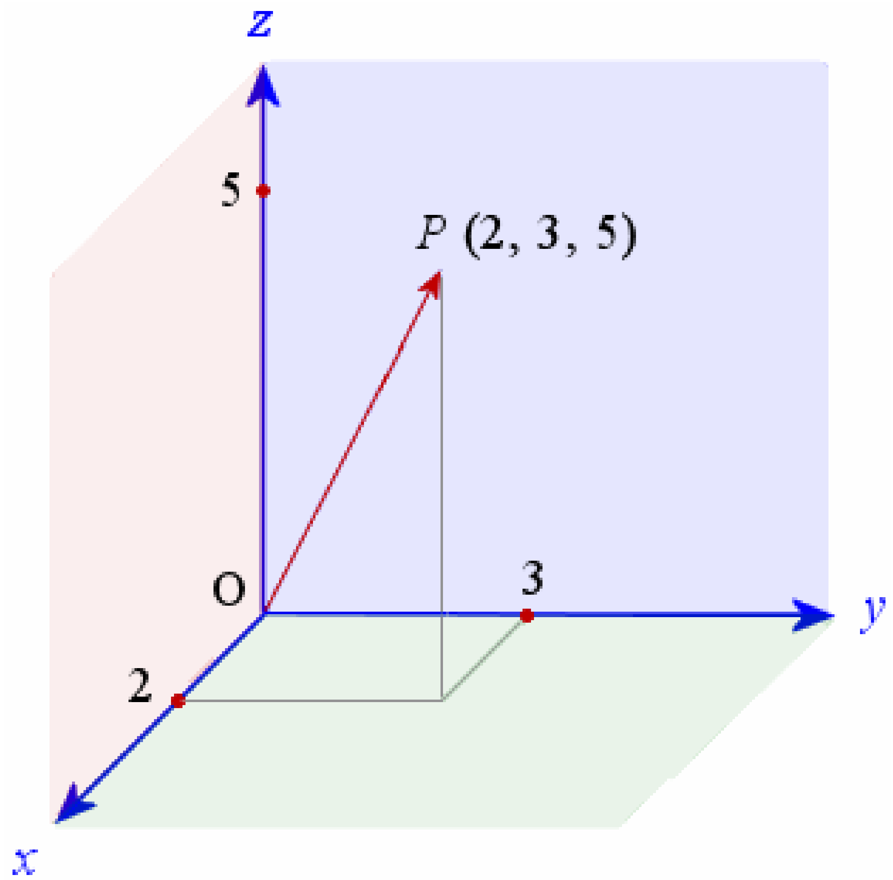

4+6i
-6+4i

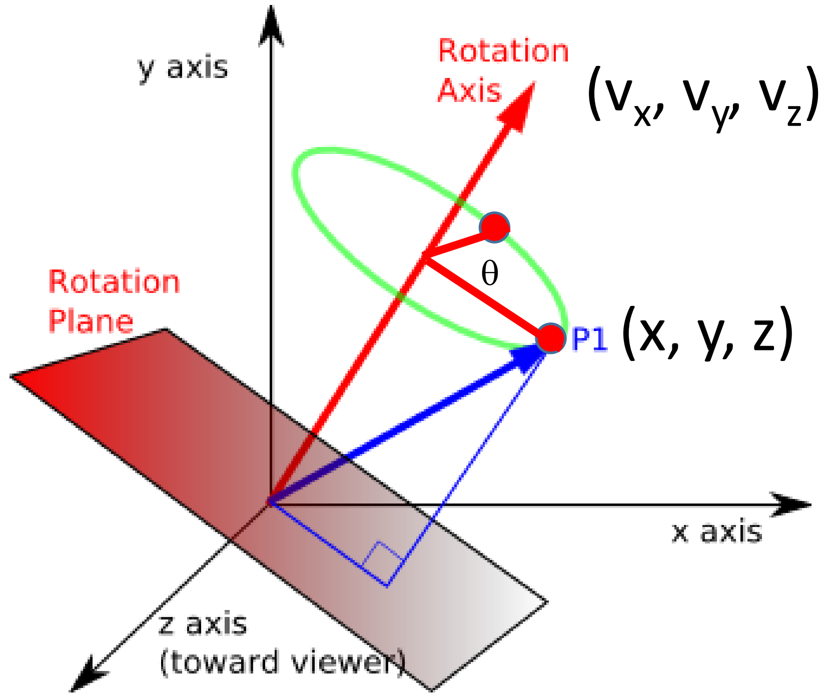


$$6 + 5i + 11j + 2k$$


k
j
i
120°
$\vec{v}$

$$i^2 = j^2 = k^2 = ijk = -1$$


z
5
P (2, 3, 5)
O
3
y
2
x

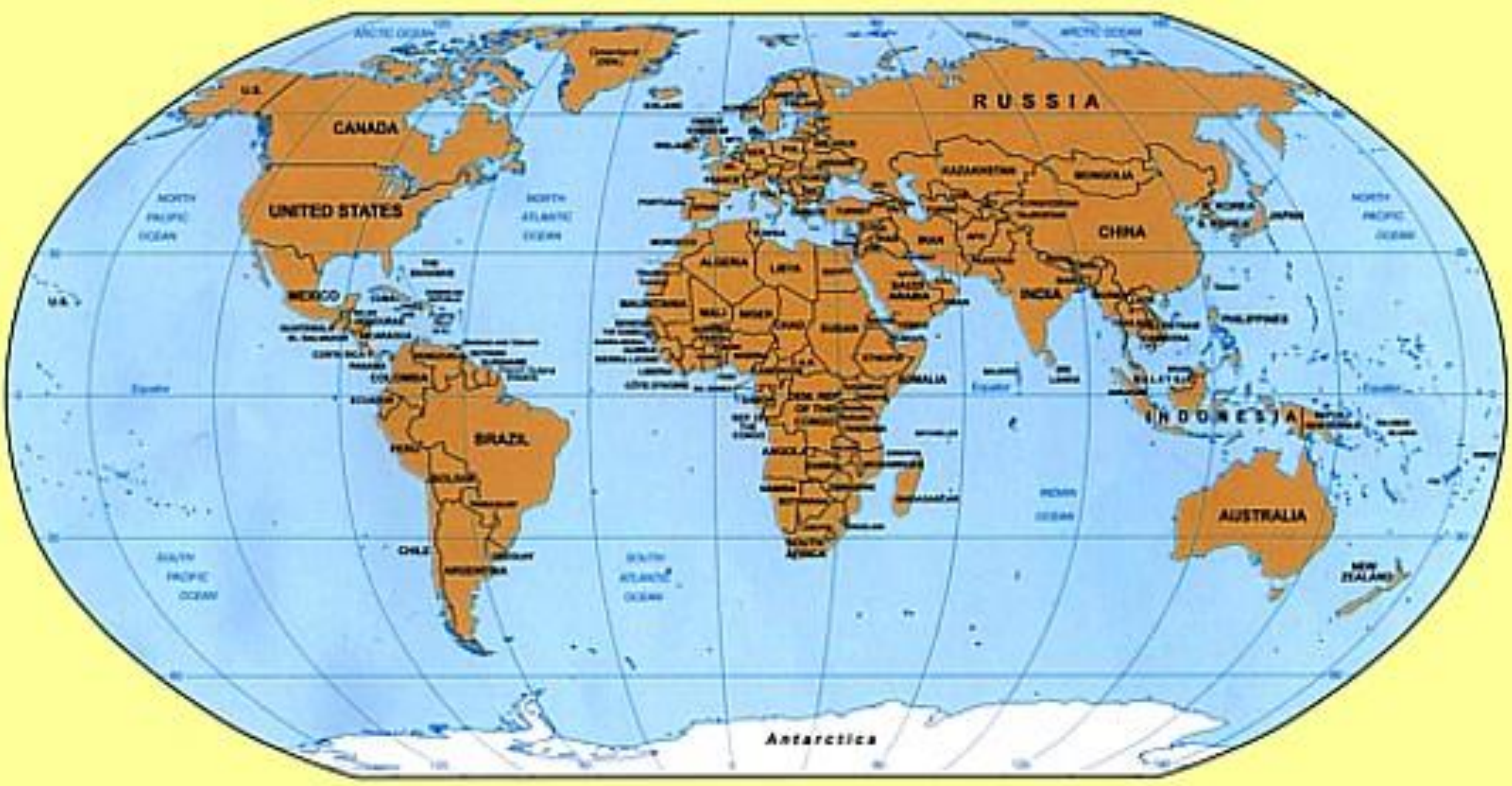


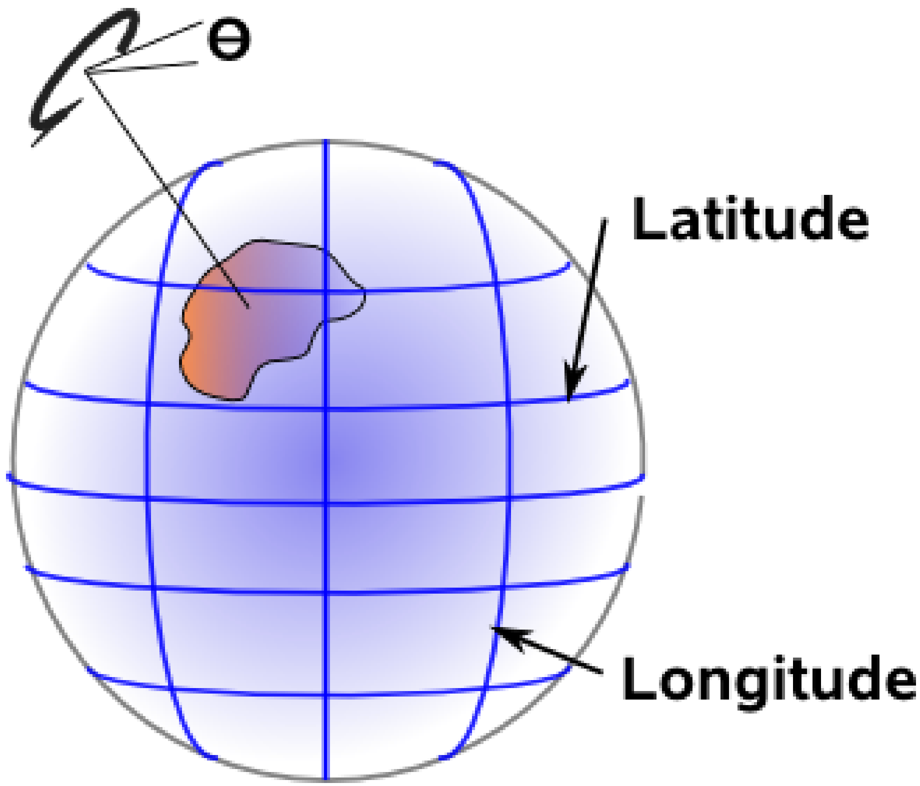
$$q = \cos(\theta/2) + i\,(v_x * \sin(\theta/2)) + j\,(v_y * \sin(\theta/2)) + k\,(v_z * \sin(\theta/2))$$

$$p1 = xi + yj + zk$$

$$q^{-1} = \cos(\theta/2) - i\,(v_x * \sin(\theta/2)) - j\,(v_y * \sin(\theta/2)) - k\,(v_z * \sin(\theta/2))$$

rotation = $q * p1 * q^{-1}$


RUSSIA
CANADA
UNITED STATES
MEXICO
KAZAKHSTAN
MONGOLIA
CHINA
JAPAN
INDIA
PHILIPPINES
INDONESIA
ALGERIA
LIBYA
SUDAN
SOMALIA
ANGOLA
BRAZIL
ARGENTINA
CHILE
AUSTRALIA
NEW ZEALAND
NORTH PACIFIC OCEAN
NORTH ATLANTIC OCEAN
SOUTH ATLANTIC OCEAN
SOUTH PACIFIC OCEAN
INDIAN OCEAN
Antarctica



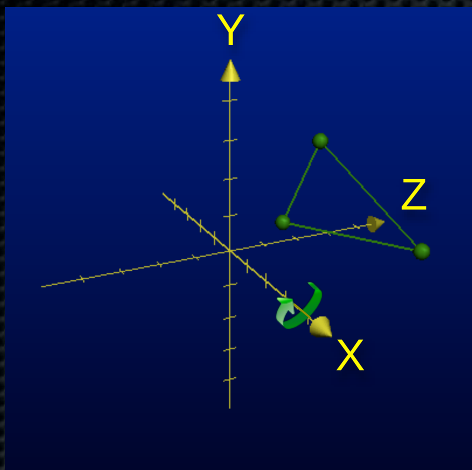
θ
Latitude
Longitude

$$
\mathbf{R} = \begin{bmatrix}
1-2\sin^2\frac{\theta}{2}(v_y^2+v_z^2) & 2\sin^2\frac{\theta}{2}v_xv_y - 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}v_z & 2\sin^2\frac{\theta}{2}v_xv_z + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}v_y & 0 \\
2\sin^2\frac{\theta}{2}v_xv_y + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}v_z & 1-2\sin^2\frac{\theta}{2}(v_x^2+v_z^2) & 2\sin^2\frac{\theta}{2}v_yv_z - 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}v_x & 0 \\
2\sin^2\frac{\theta}{2}v_xv_z - 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}v_y & 2\sin^2\frac{\theta}{2}v_yv_z + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}v_x & 1-2\sin^2\frac{\theta}{2}(v_x^2+v_y^2) & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\begin{aligned}
\mathbf{p} &= <1,0,0> \\
\mathbf{a} &= <0,1,0> \\
\mathbf{p}' &= \mathbf{R}\mathbf{p} \\
\mathbf{R} &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \\
\mathbf{p}' &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= <0,0,-1>
\end{aligned}
$$

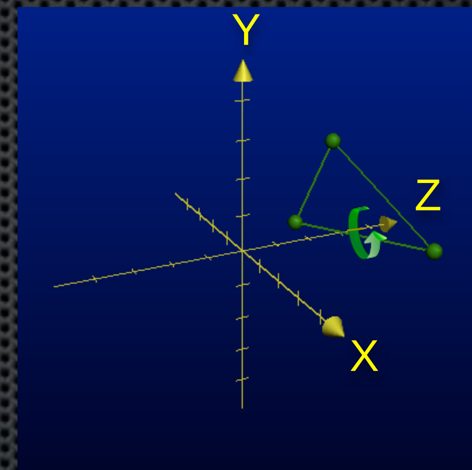
Rotation in 3D

Rotation About X-Axis



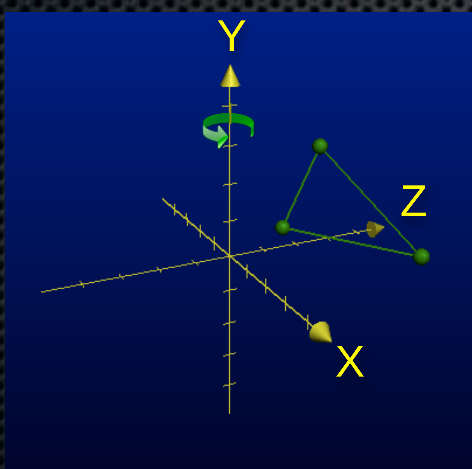
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation About Z-Axis



$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation About Y-Axis



$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$