Computer Graphics (CS 4731)
Linear Algebra for Graphics (Vector Spaces and Barycentric Coordinates)

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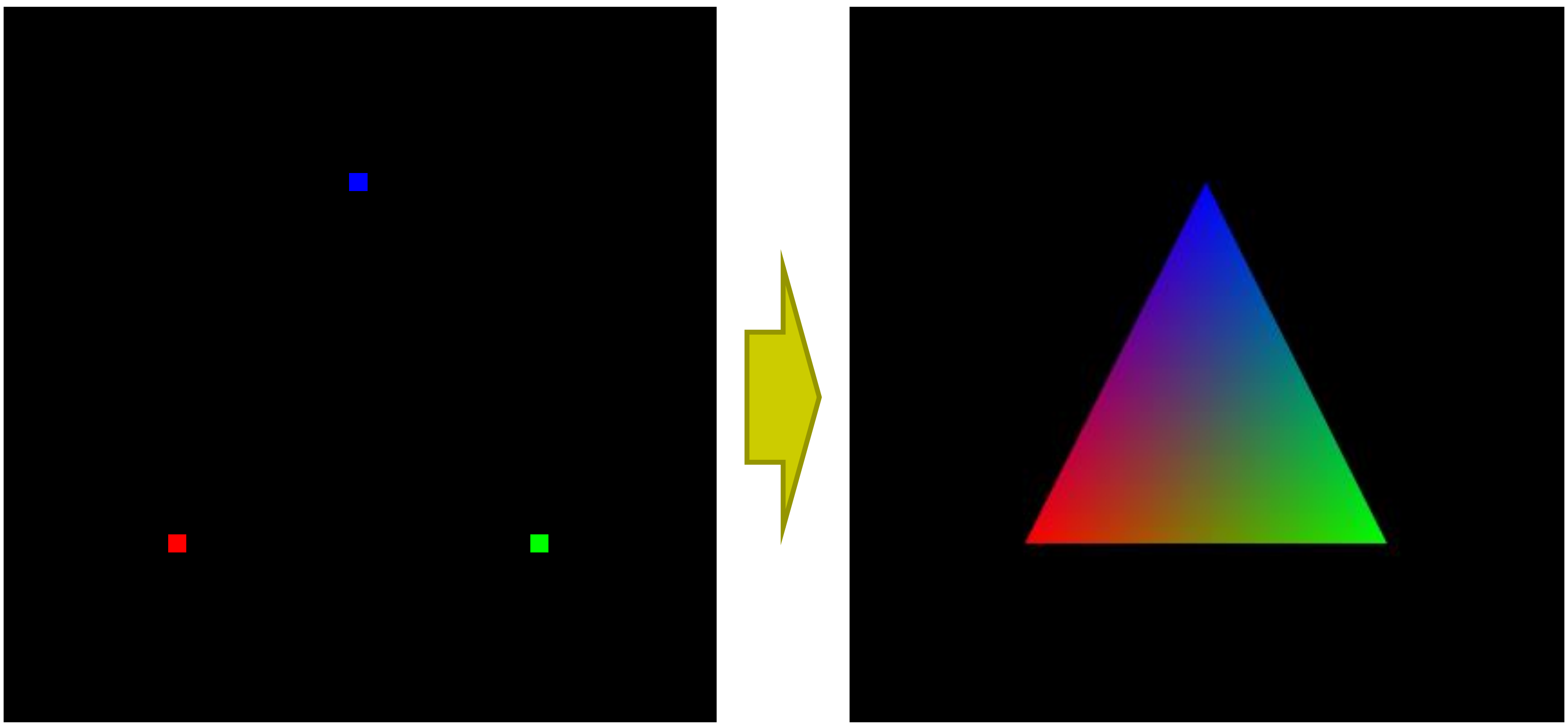
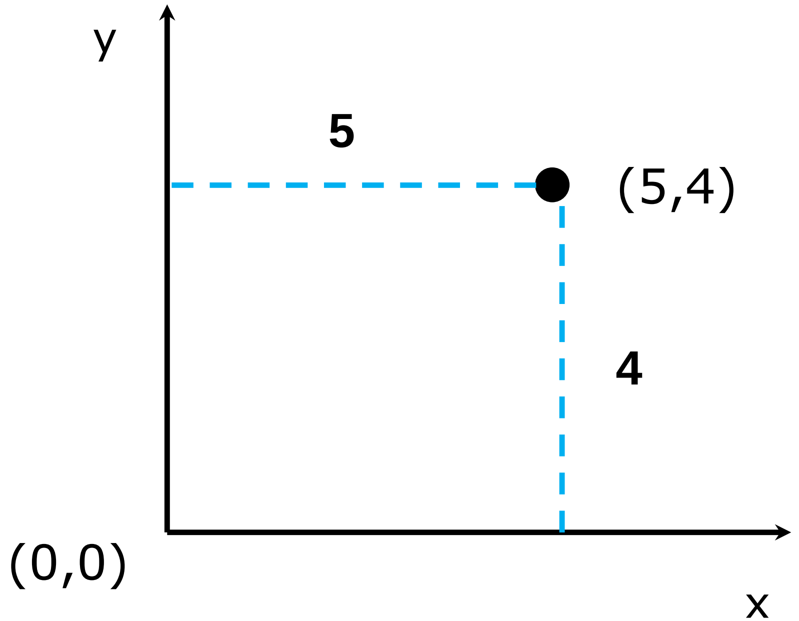
Slides Courtesy of Emmanuel Agu (WPI) and Matt Stoker (University of Utah)

Recall…

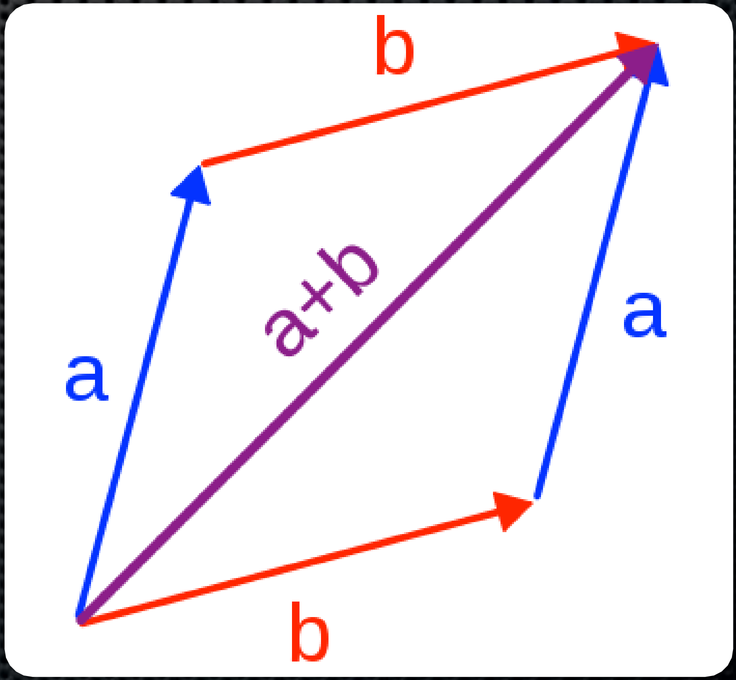


Points, Scalars and Vectors

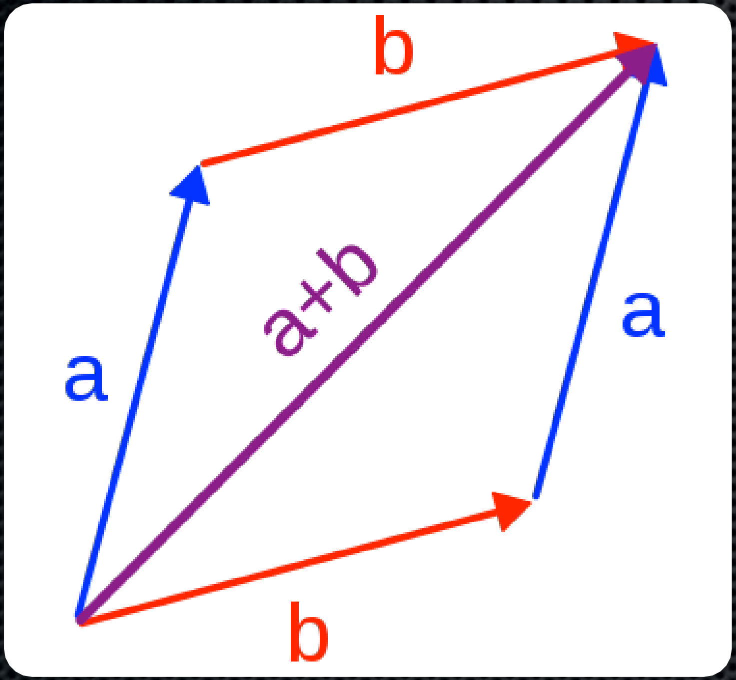


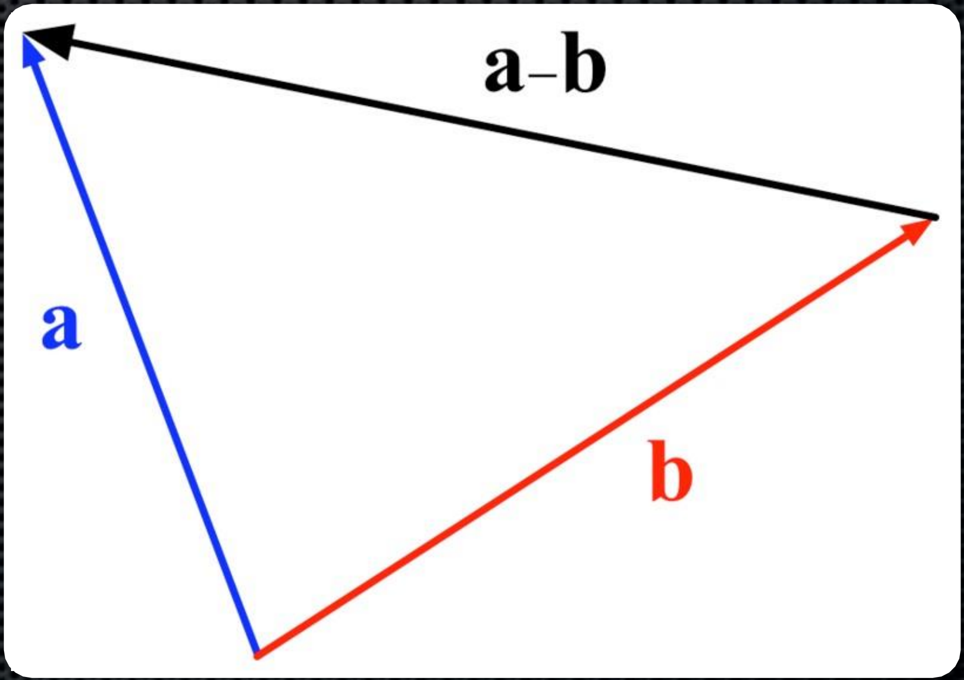
Vectors

- Vectors
  - Addition & Subtraction
  - Scalar Multiplication
  - Magnitude & Normalization
  - Dot & Cross Product

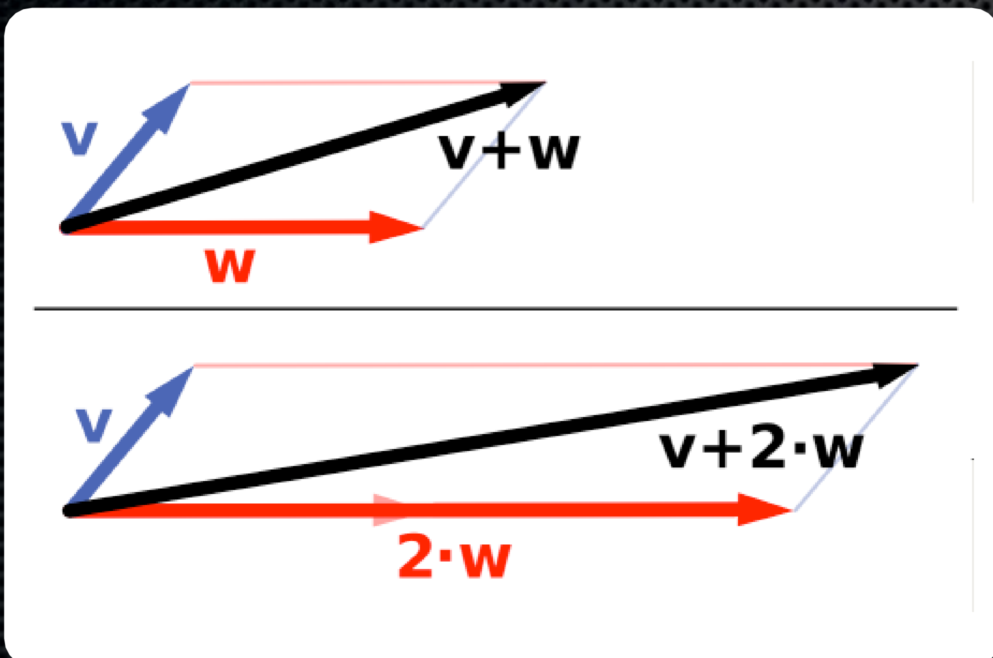


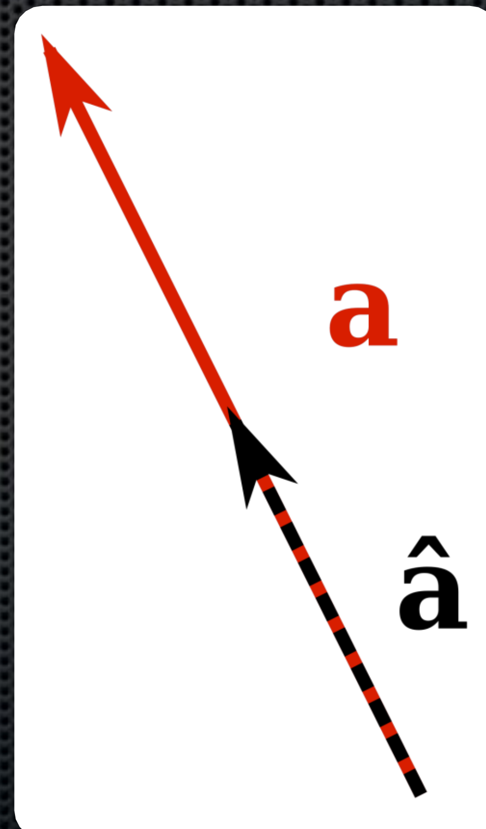
Vector Addition & Subtraction





Vector-Scalar Multiplication





$$\hat{a} = \frac{1}{|a|} a$$

Magnitude of a Vector

- Magnitude of **a**

$$|\mathbf{a}| = \sqrt{{a_1}^2 + {a_2}^2 \ldots\ldots\ldots + {a_n}^2}$$

- Normalizing a vector (unit vector)

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{vector}{magnitude}$$

- Note magnitude of normalized vector = 1. i.e

$$\sqrt{{a_1}^2 + {a_2}^2 \ldots\ldots\ldots + {a_n}^2} = 1$$

Magnitude of a Vector

- Example: if a = (2, 5, 6)

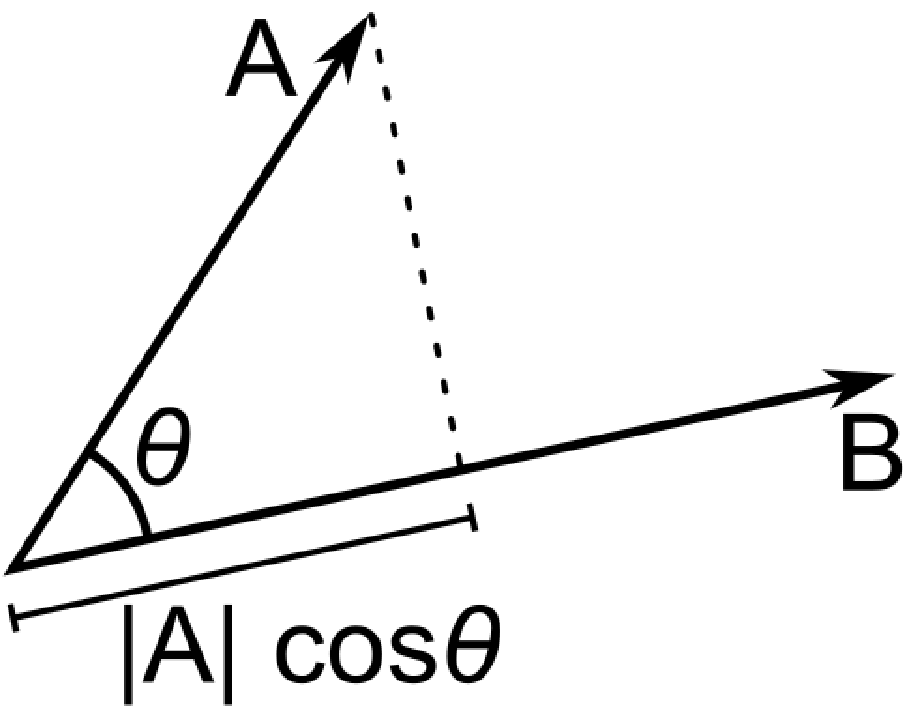
- Magnitude of **a**

$$|\mathbf{a}| = \sqrt{2^2 + 5^2 + 6^2 =} \sqrt{65}$$

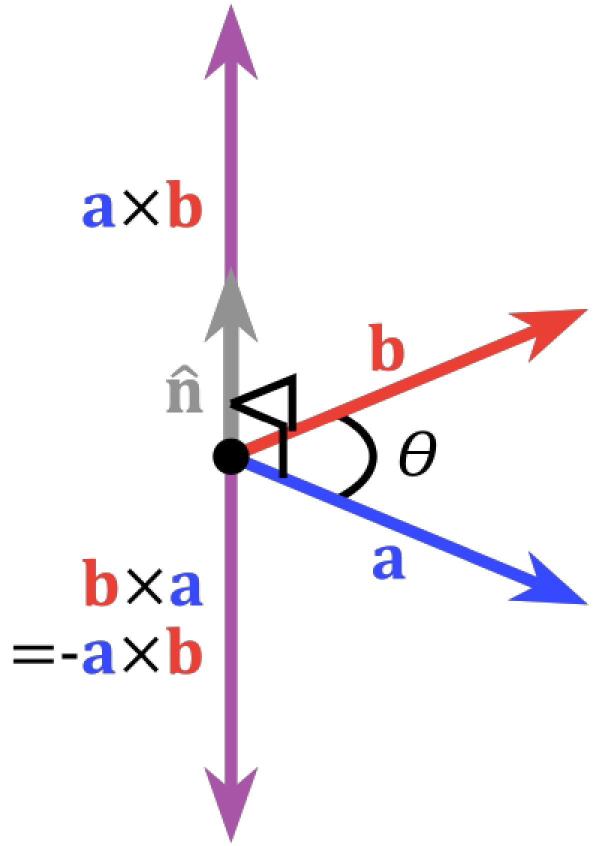
- Normalizing **a**

$$\hat{\mathbf{a}} = \left(\frac{2}{\sqrt{65}}, \frac{5}{\sqrt{65}}, \frac{6}{\sqrt{65}} \right)$$

Vector Dot & Cross Product

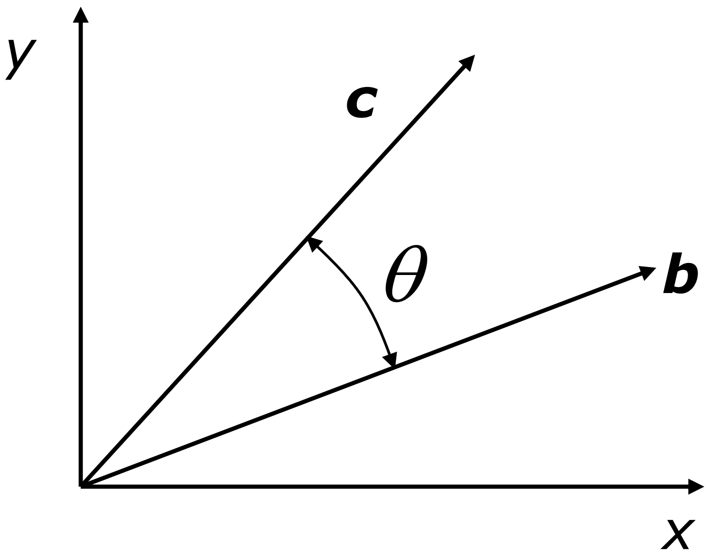


$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$$

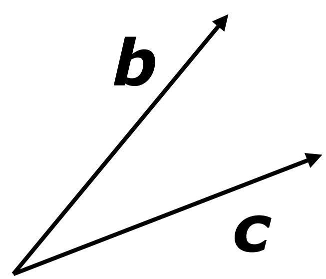


$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin(\theta)\,\mathbf{n}$$

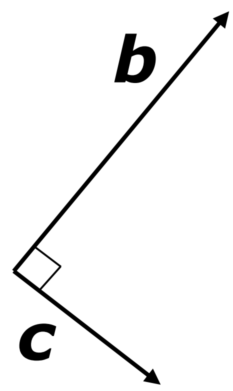
Angle Between Two Vectors



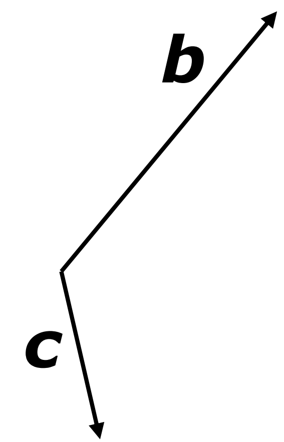
$$\mathbf{b} \cdot \mathbf{c} = |\mathbf{b}||\mathbf{c}| \cos \theta$$



b.c > 0



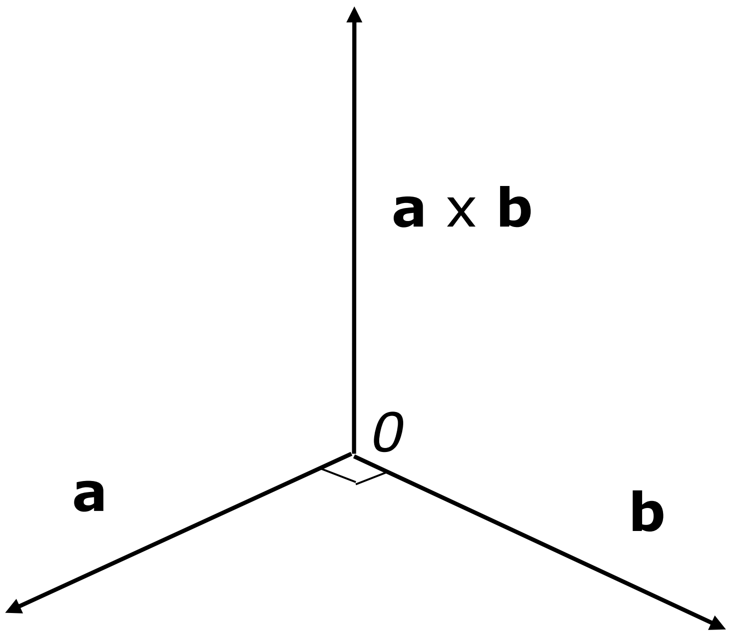
b.c = 0



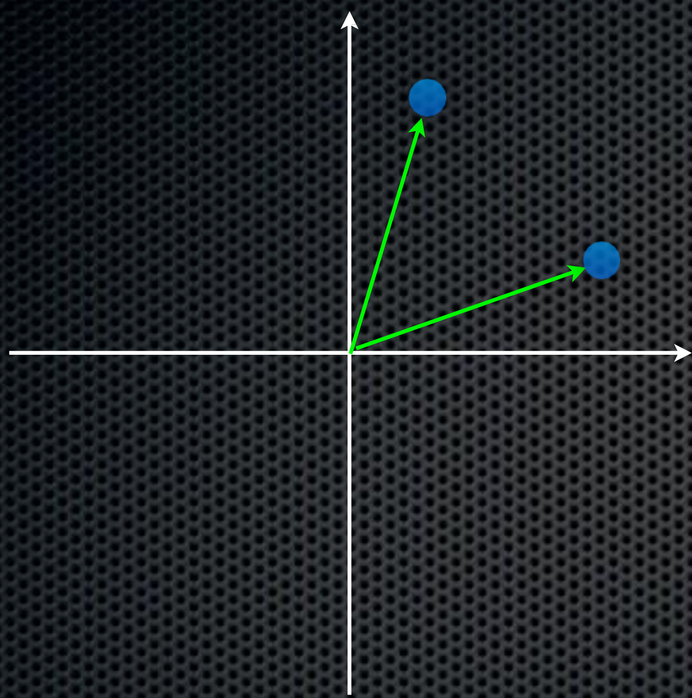
b.c < 0

Cross Product

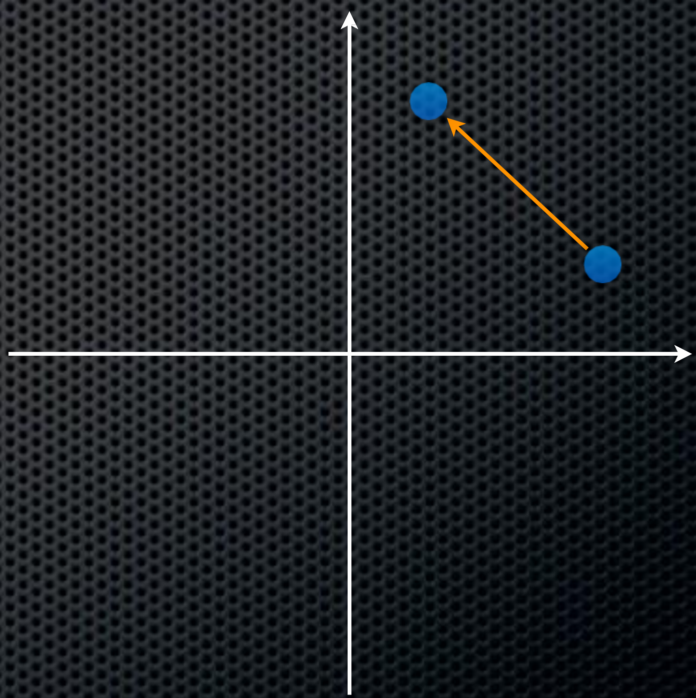
a x **b** is perpendicular to both **a** and **b**



Point and Free Vectors

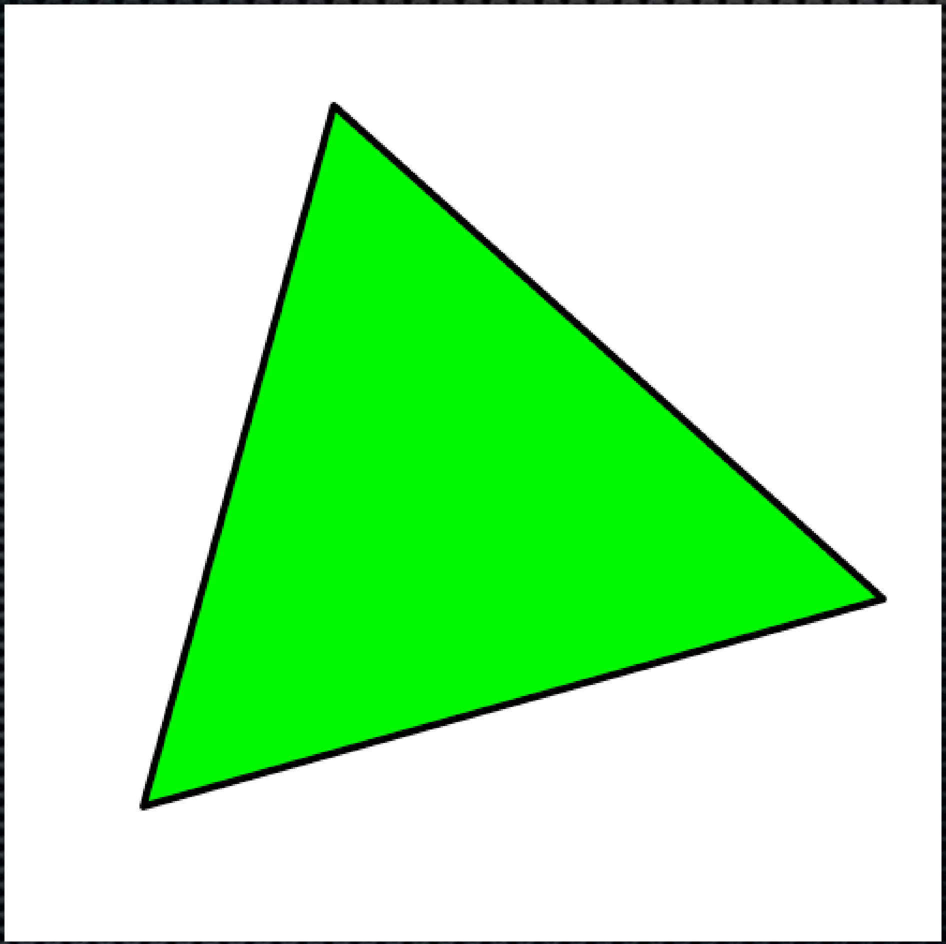


Point

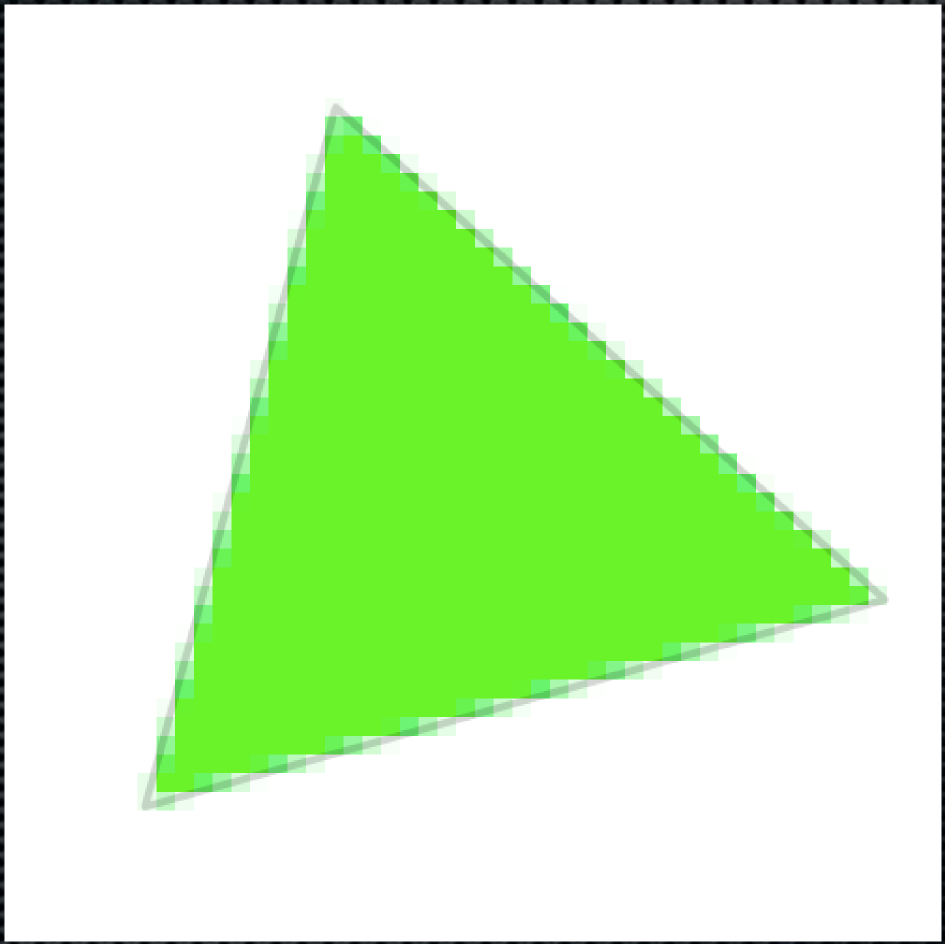


Free

Rasterization



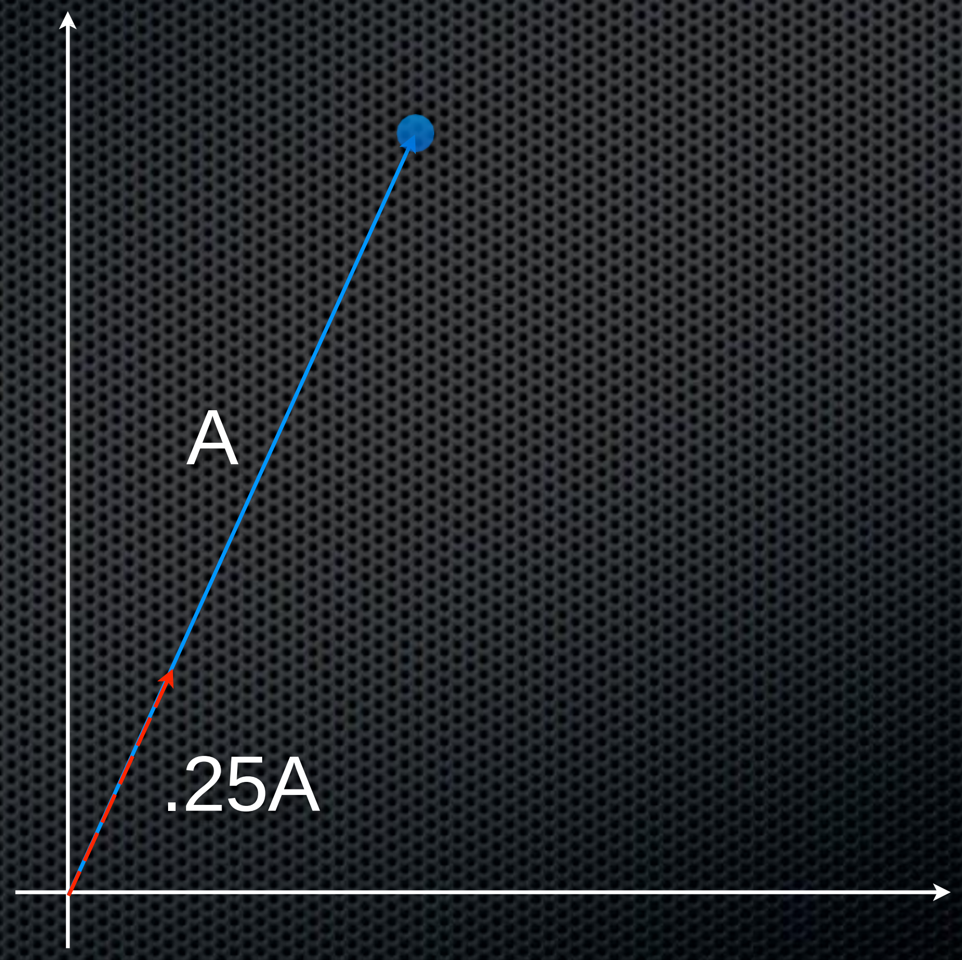
Rasterization



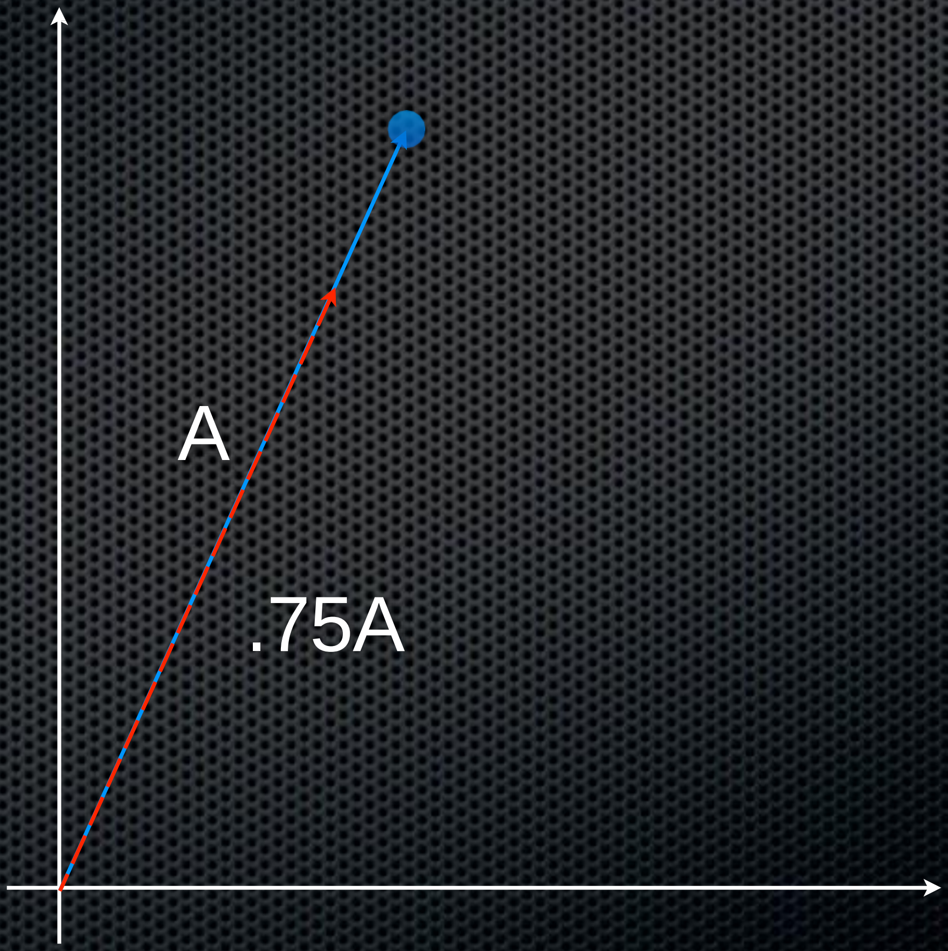
Think-Pair-Share

Given three points, how do we decide which pixels are inside a triangle and which aren’t?

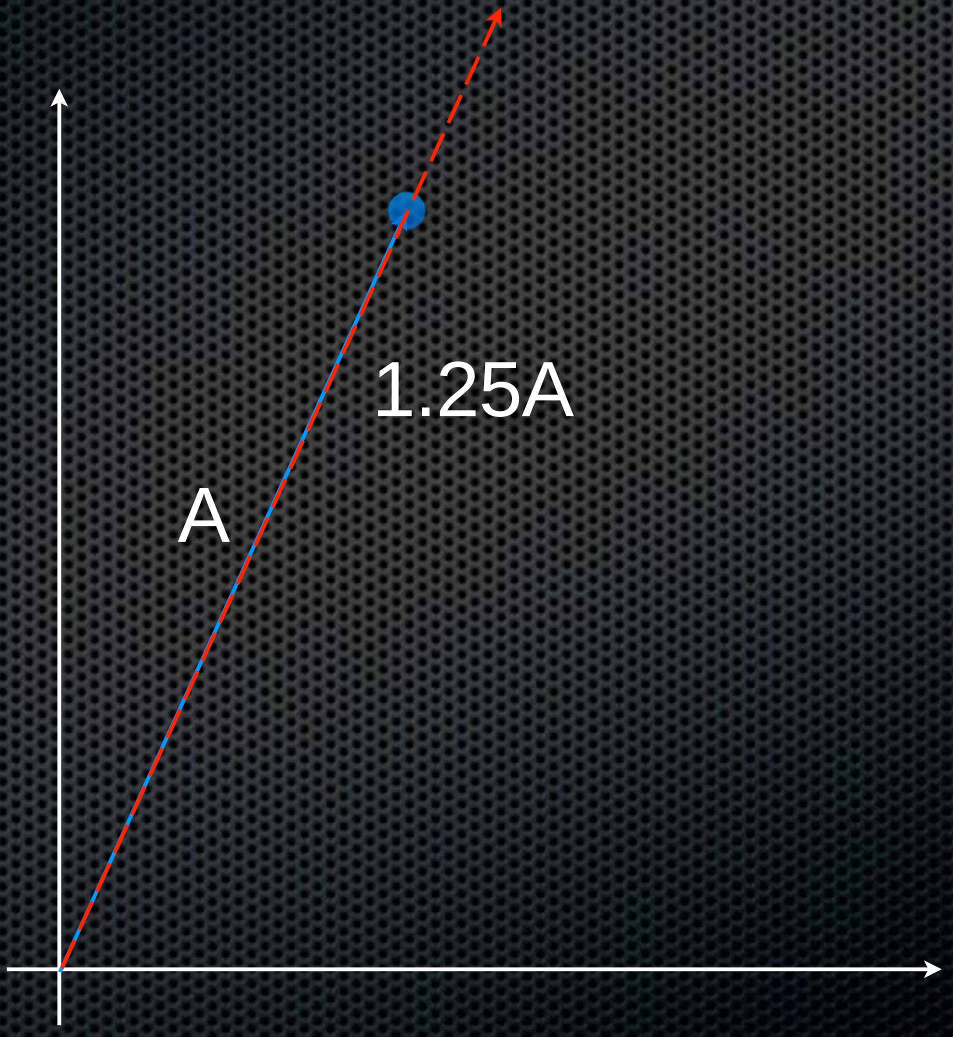
An Interesting Idea



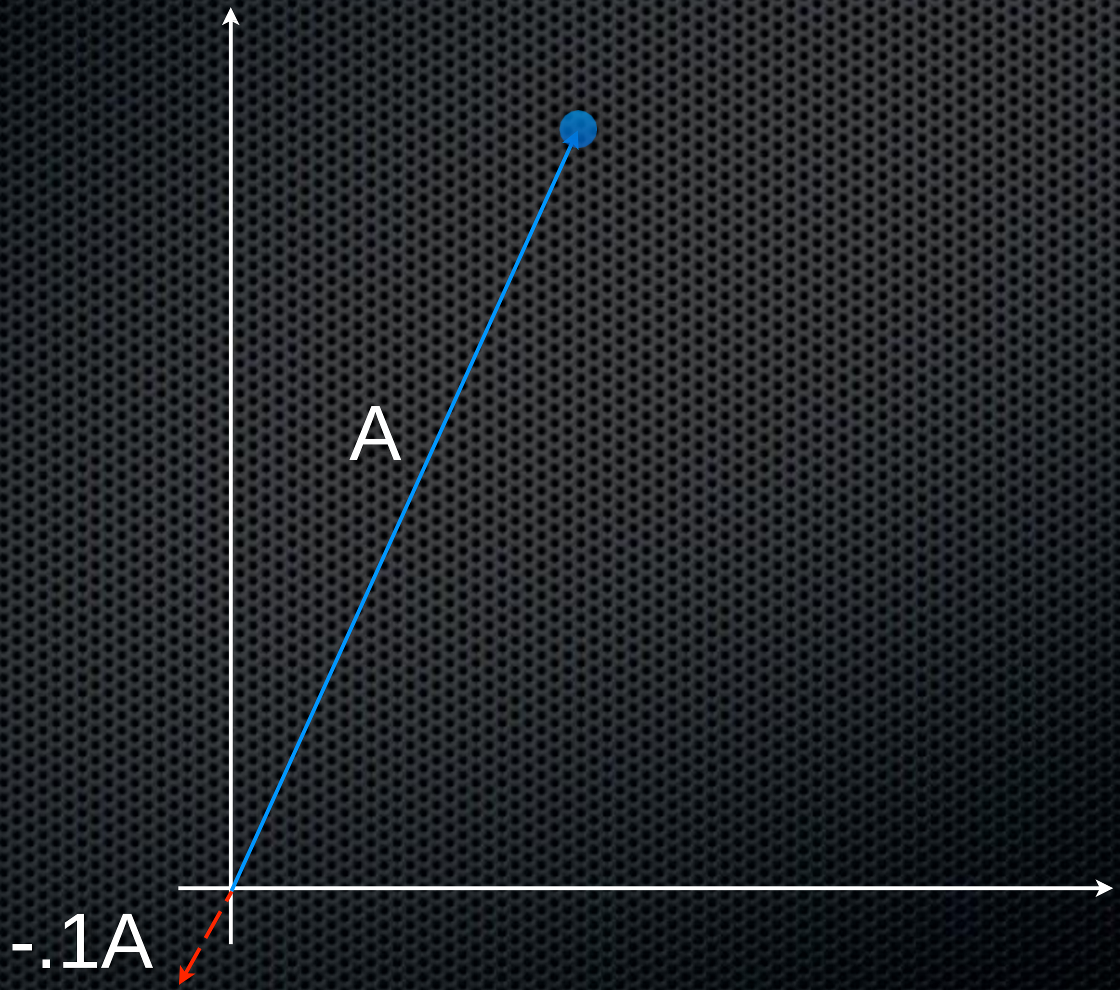
An Interesting Idea



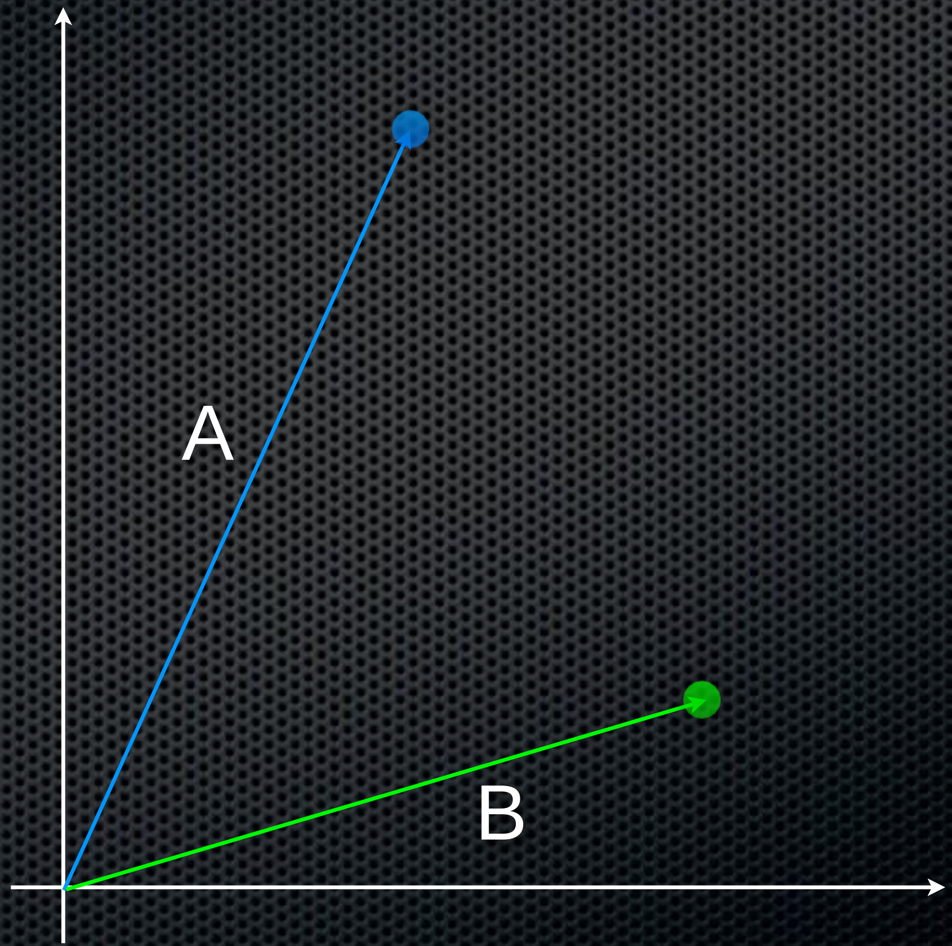
An Interesting Idea



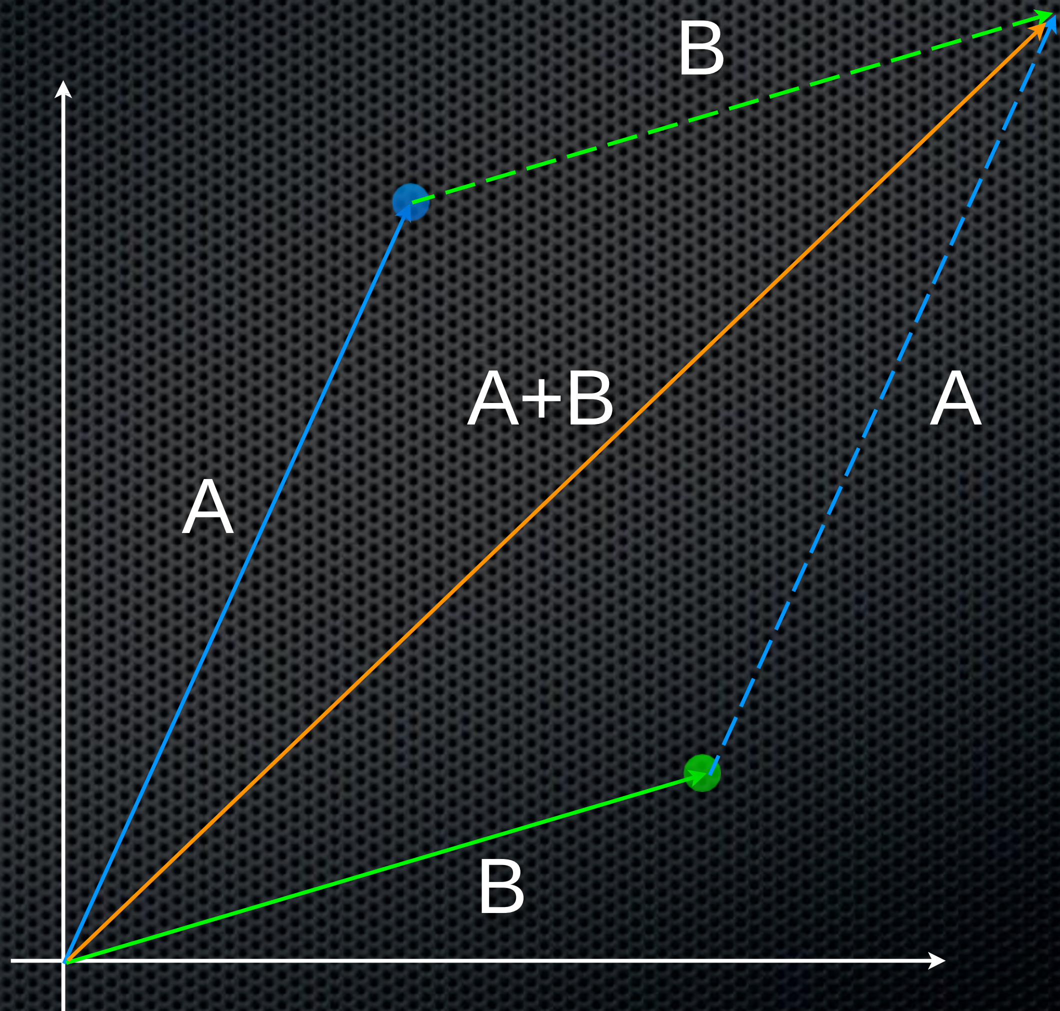
An Interesting Idea



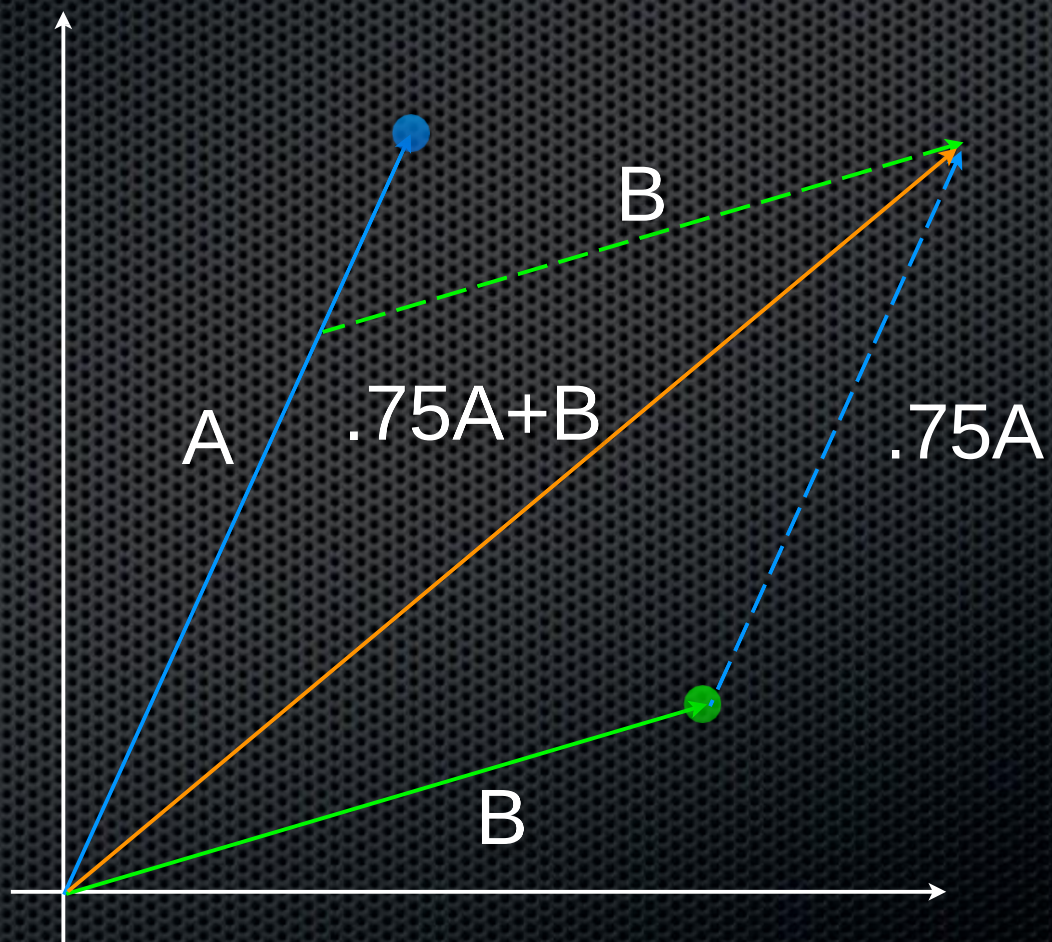
An Interesting Idea



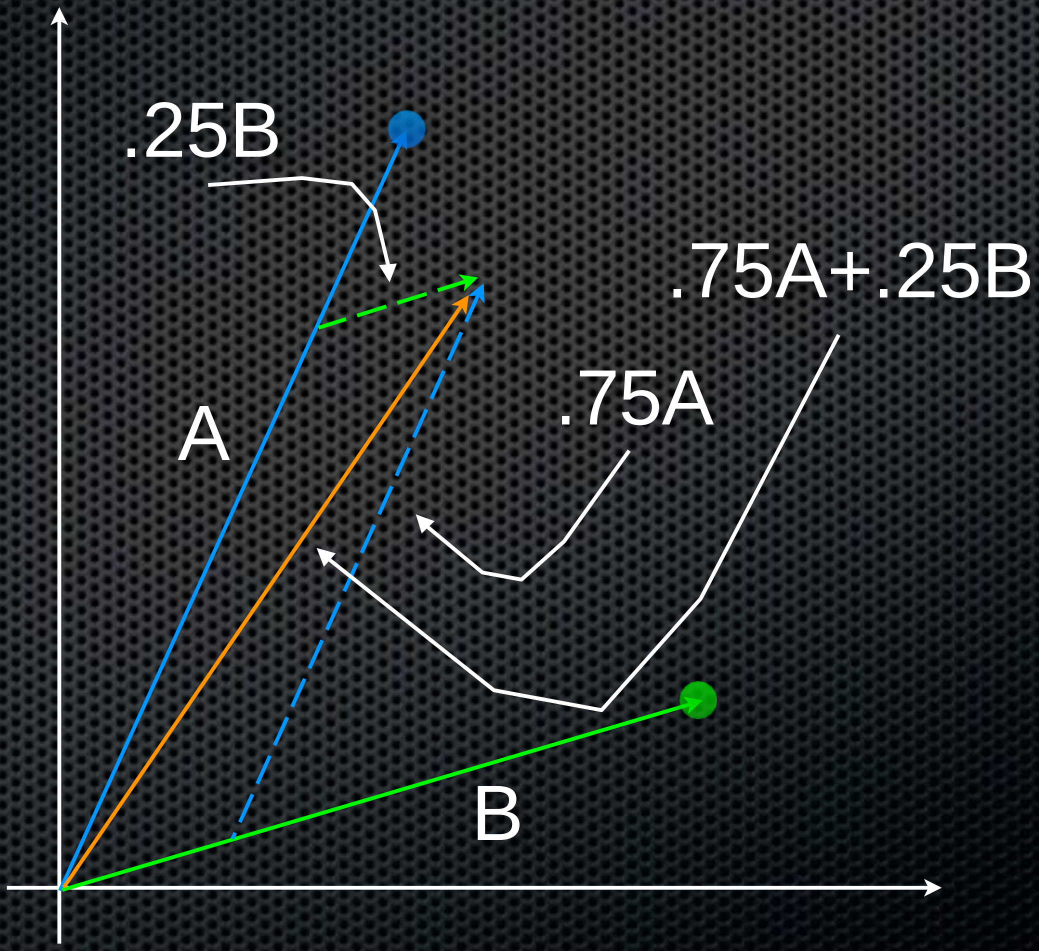
An Interesting Idea



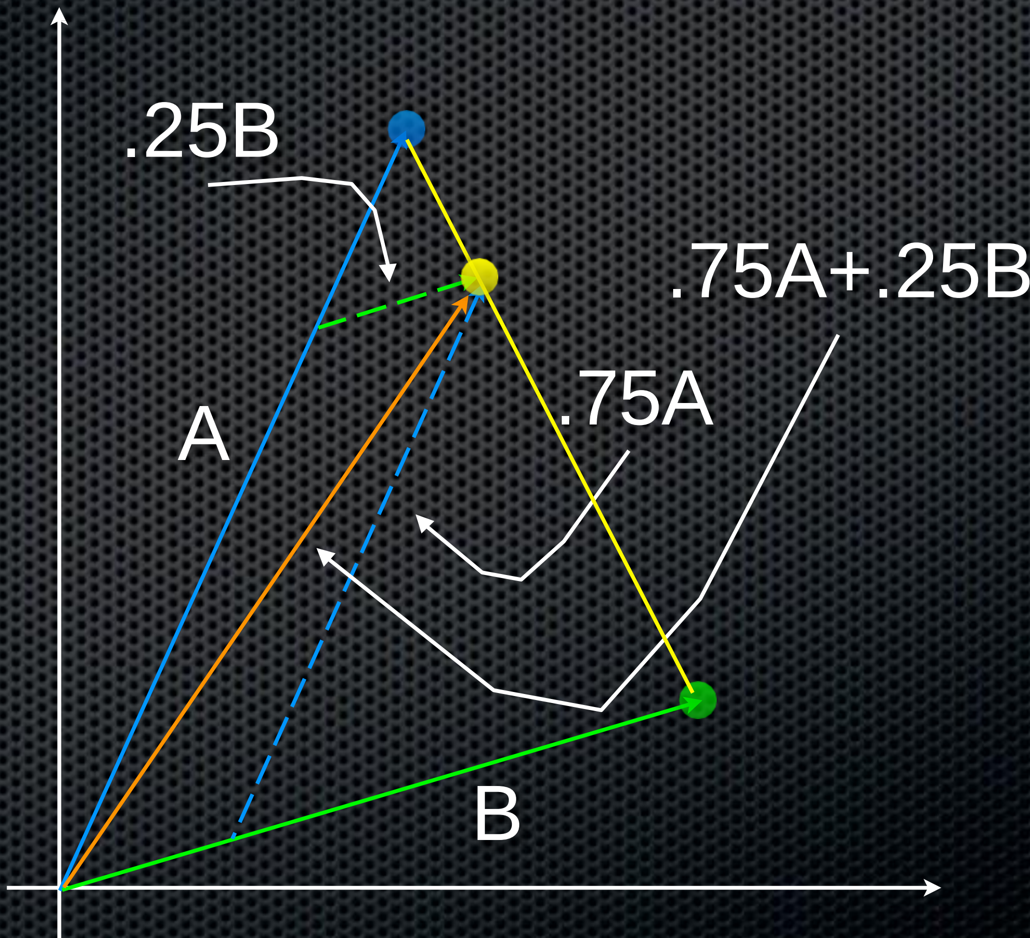
Linear Combination



Linear Combination

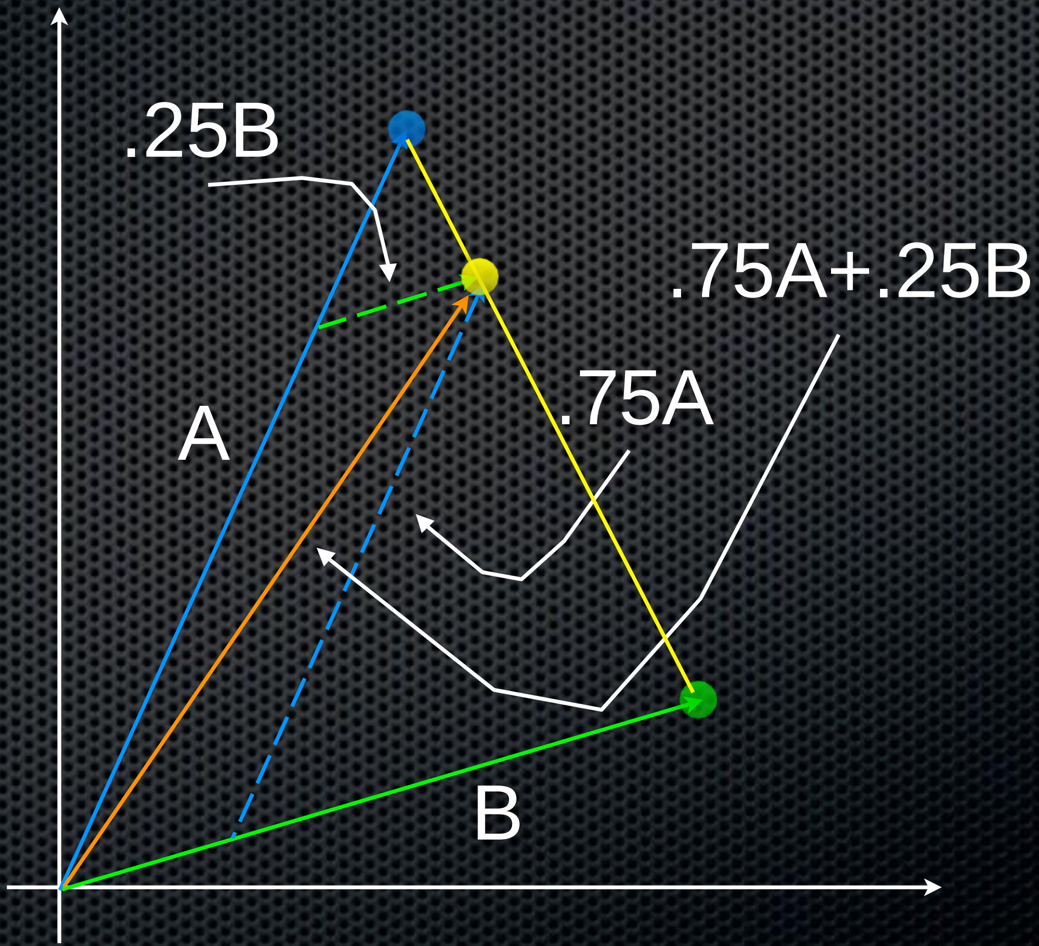


Linear Combination



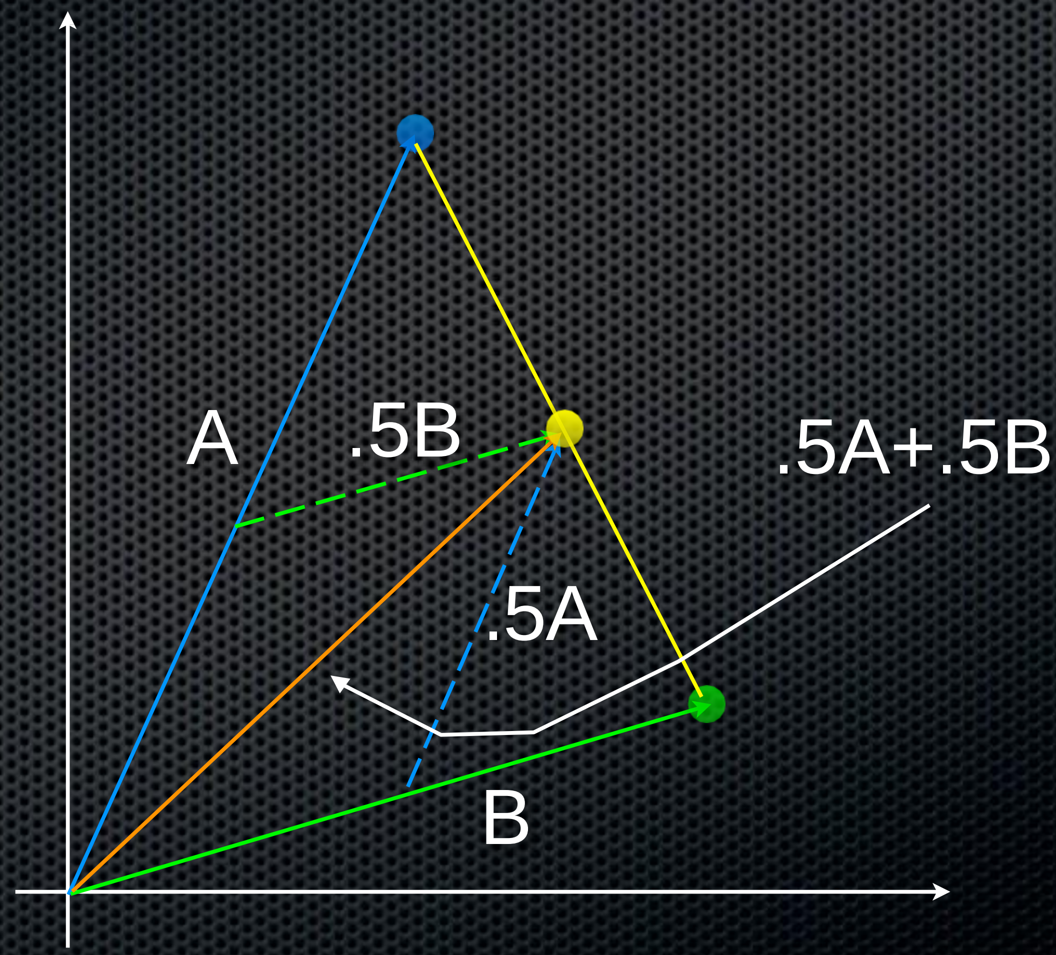
Linear Combination or Blend

.75 + .25 = 1



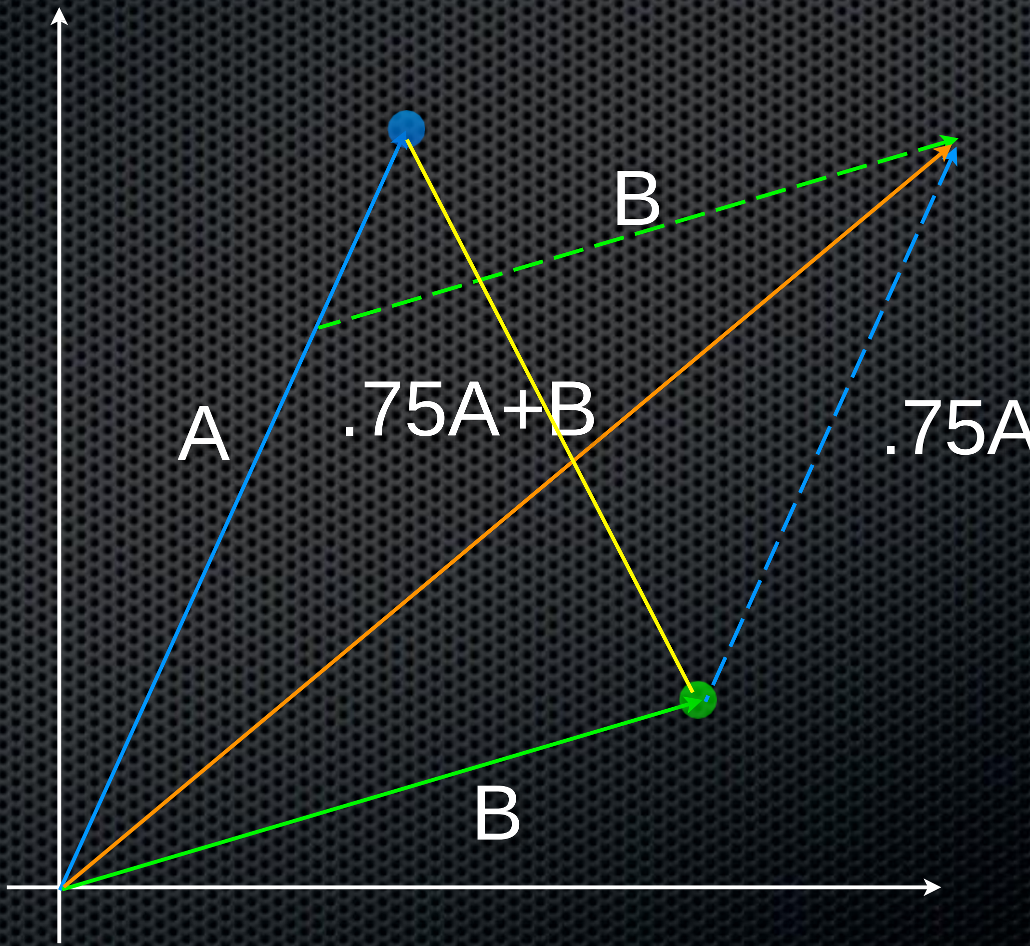
Linear Combination or Blend

.5 + .5 = 1

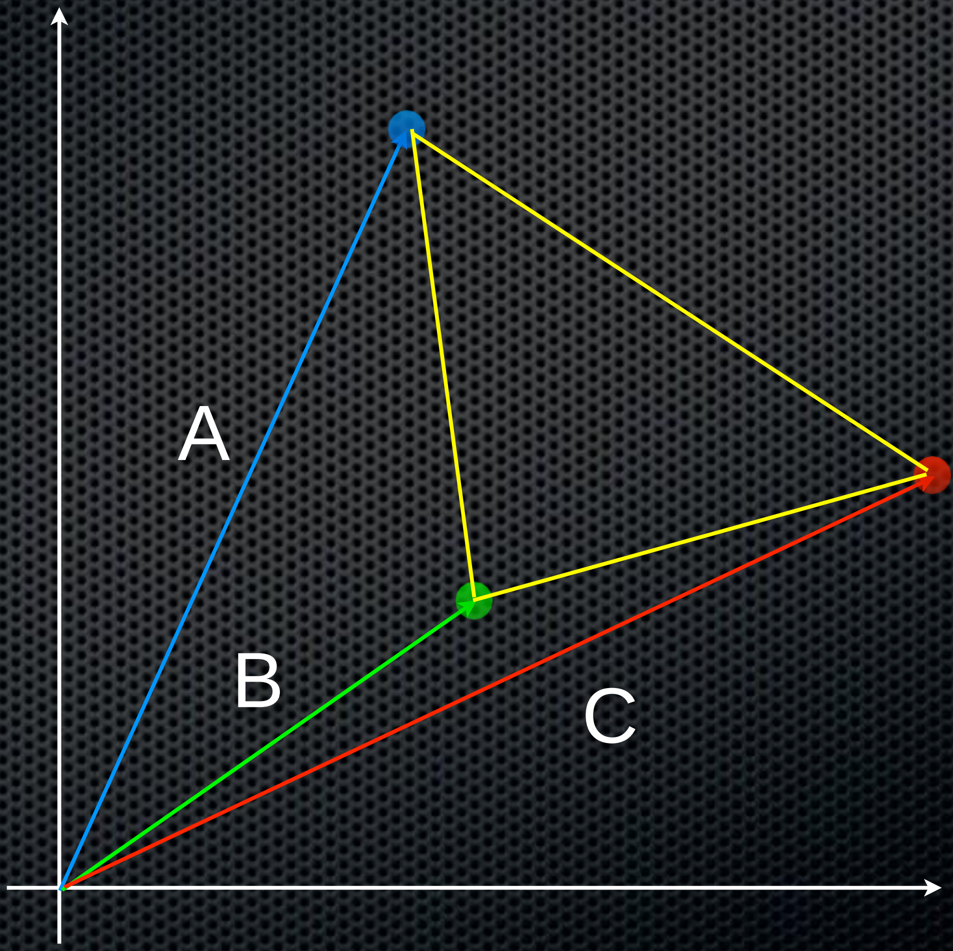


Linear Combination

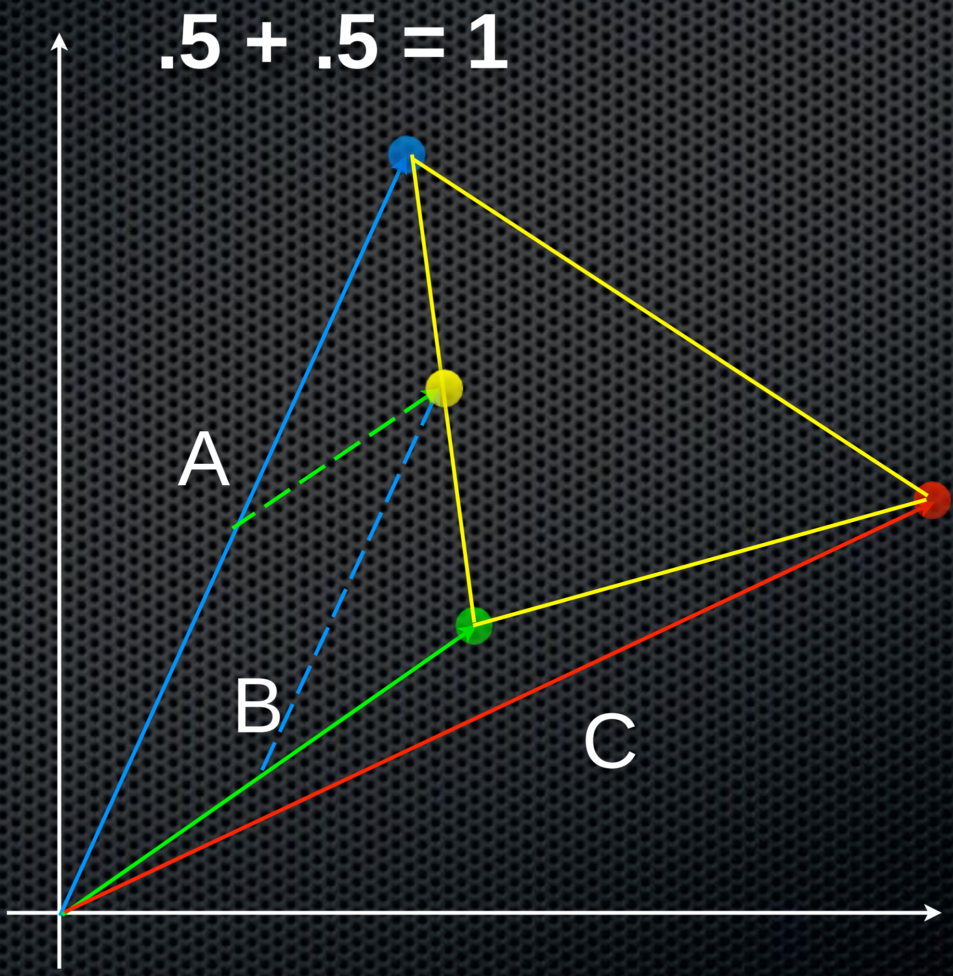
.75 + 1 ≠ 1



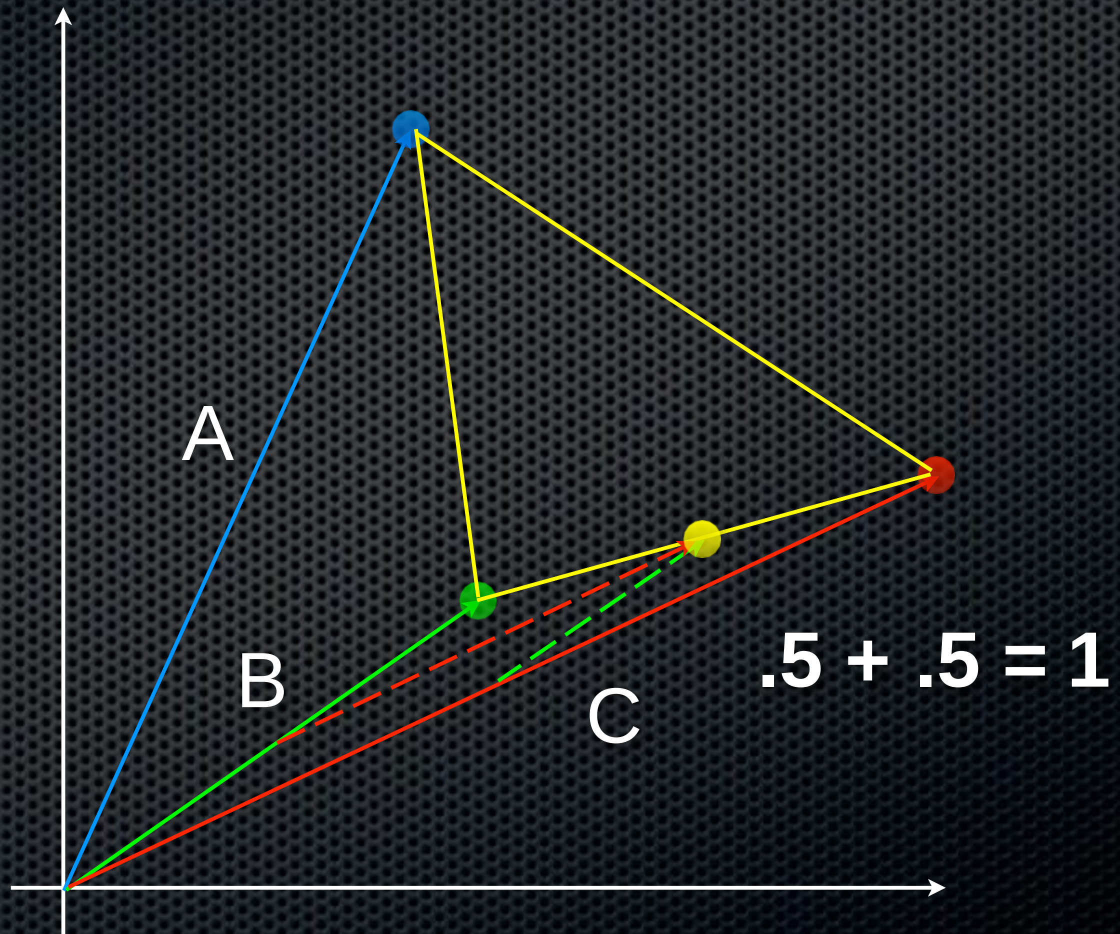
Triangle



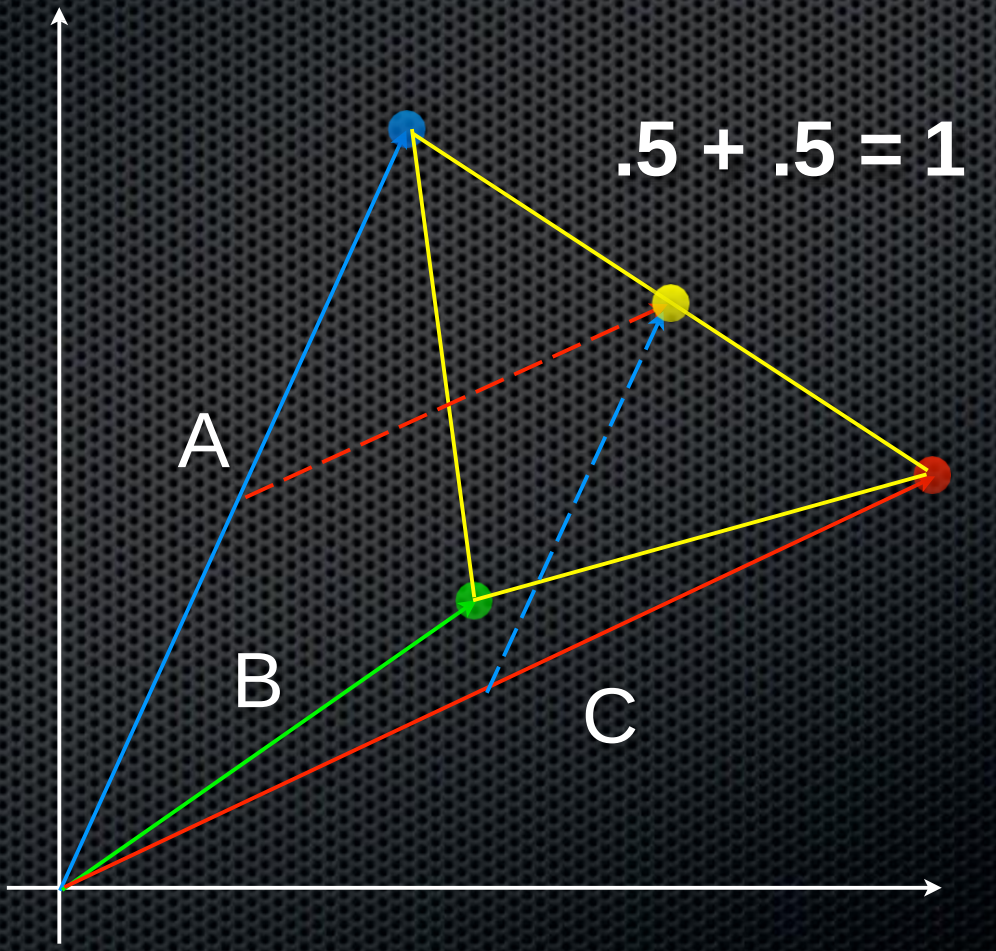
Triangle



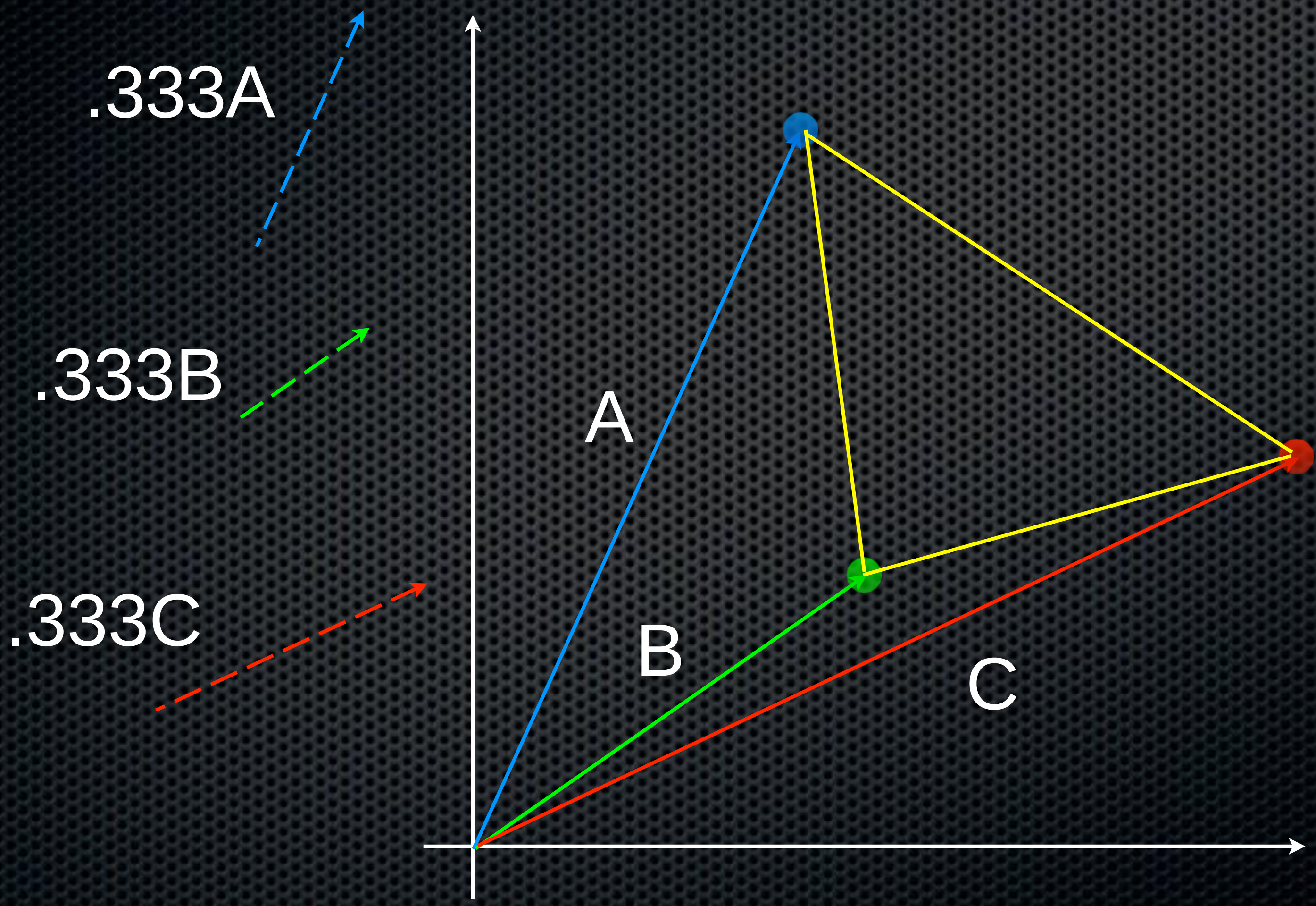
Triangle



Triangle

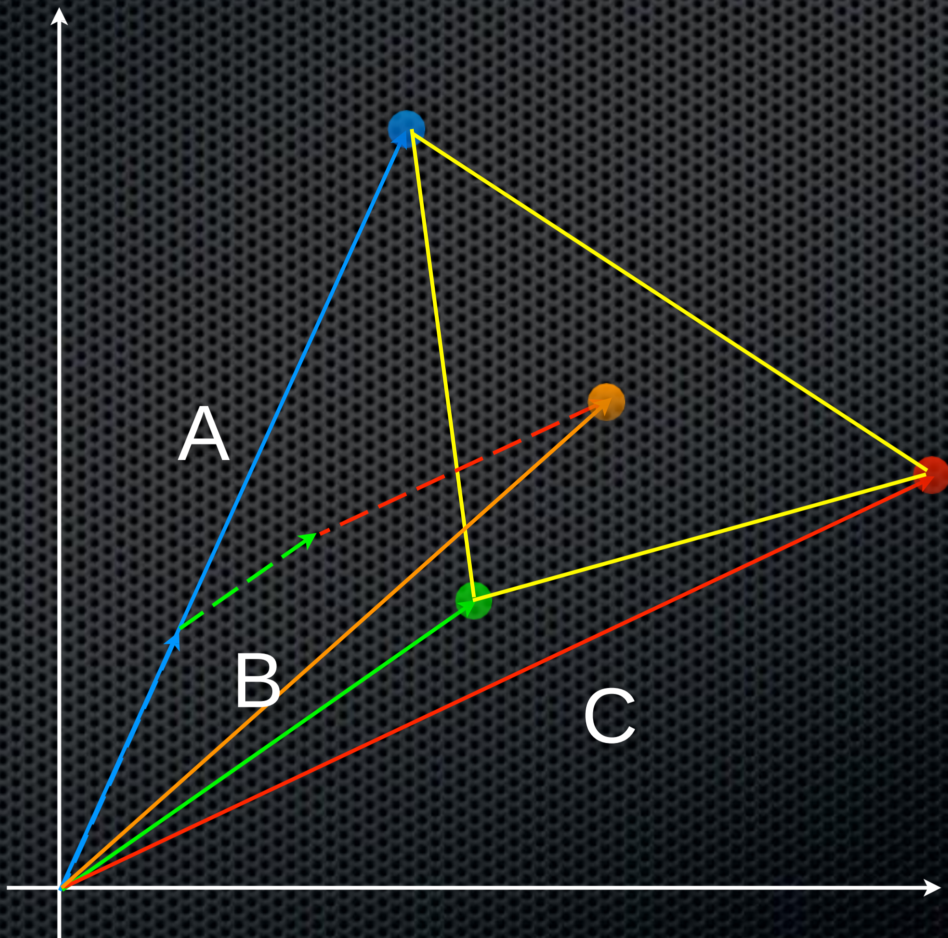


Triangle

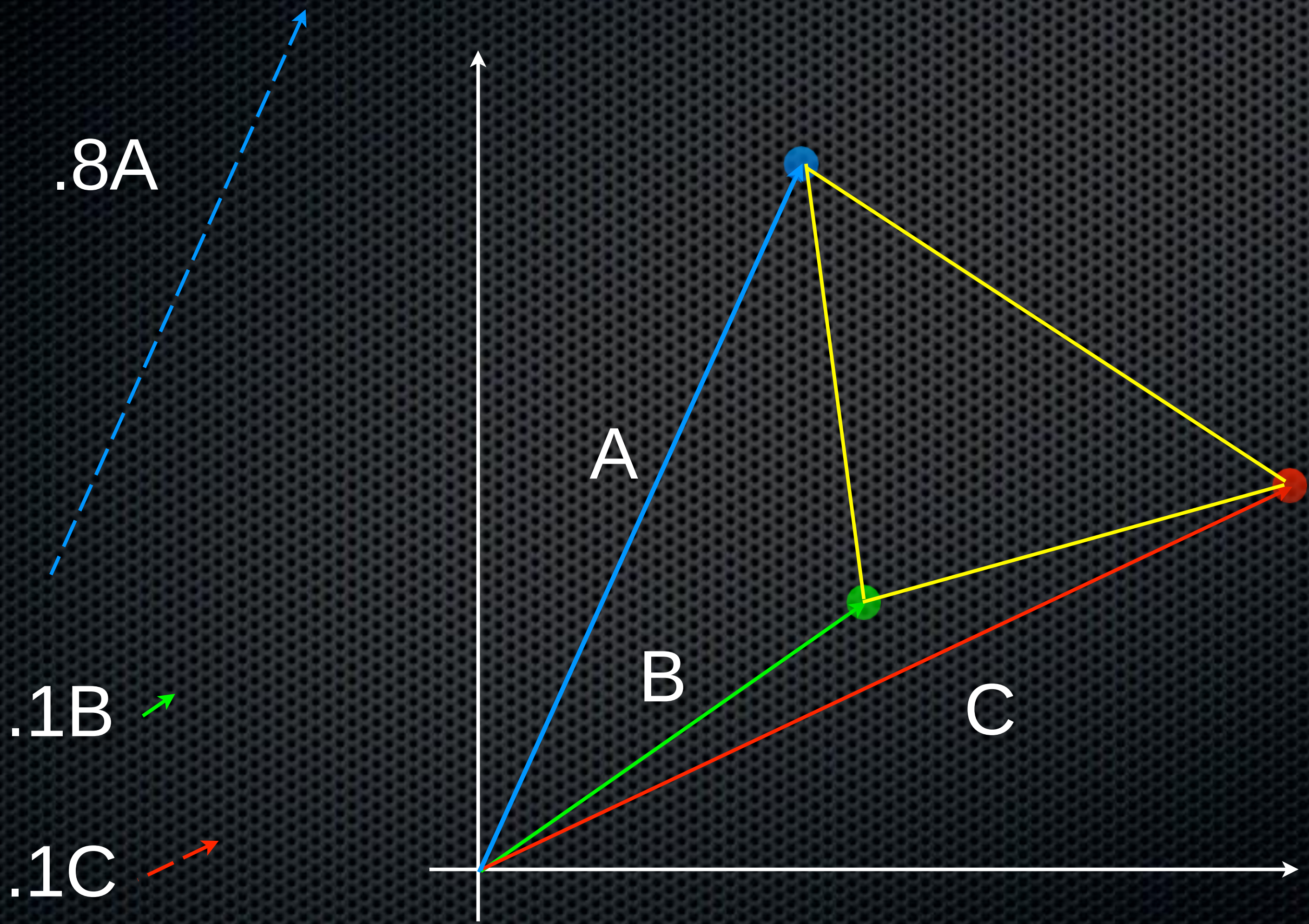


Triangle

$.333 + .333 + .333 = 1$

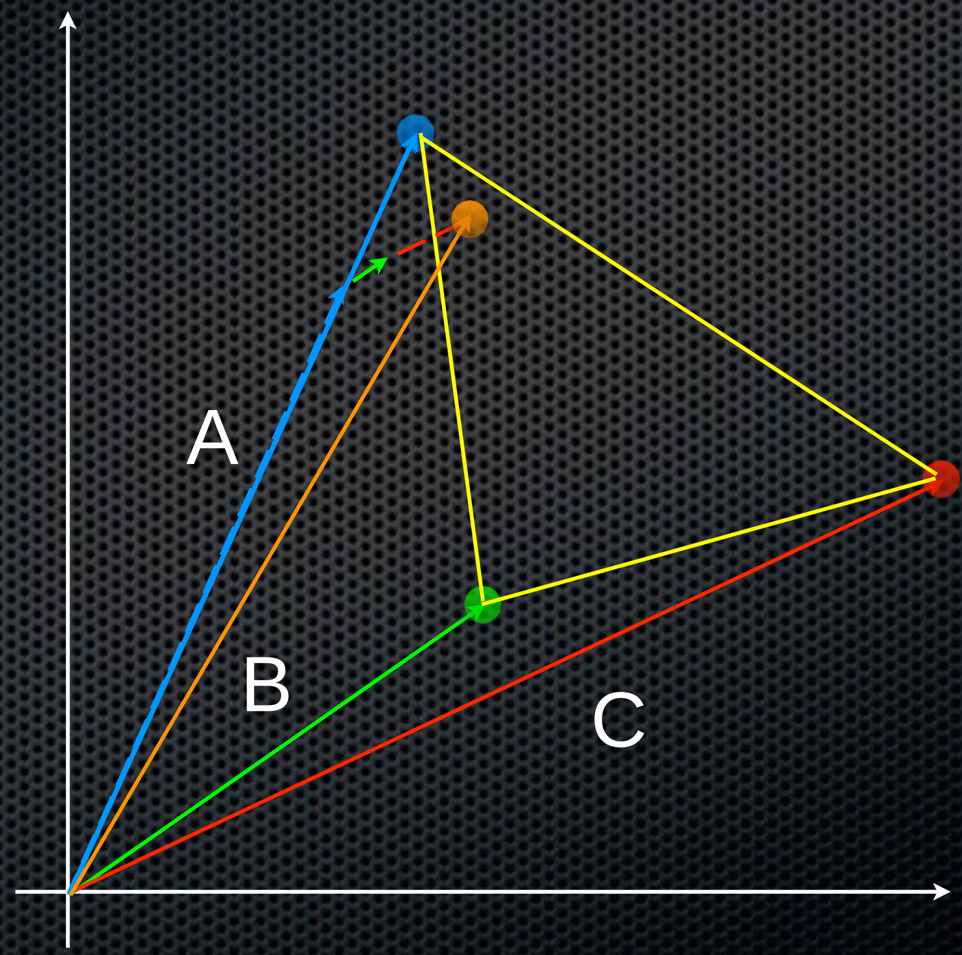


Triangle



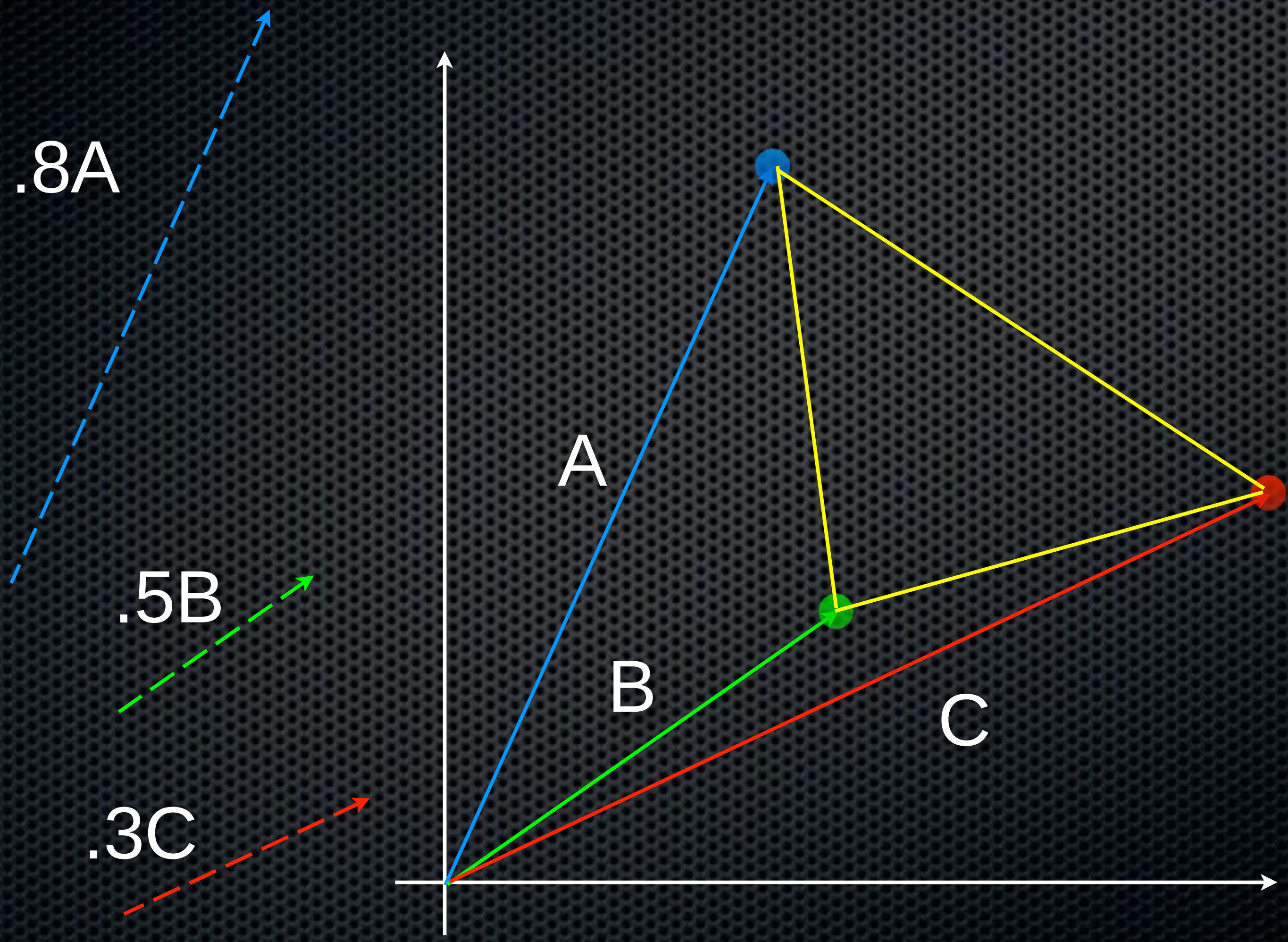
Triangle

$.8 + .1 + .1 = 1$



Triangle

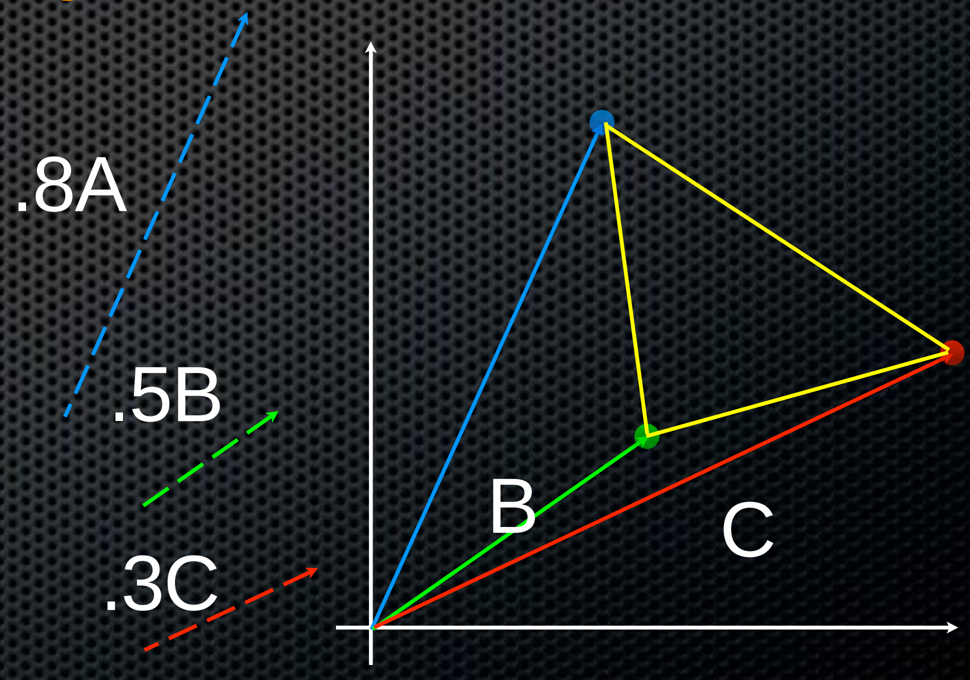
$.8 + .5 - .3 = 1$



Think-Pair-Share

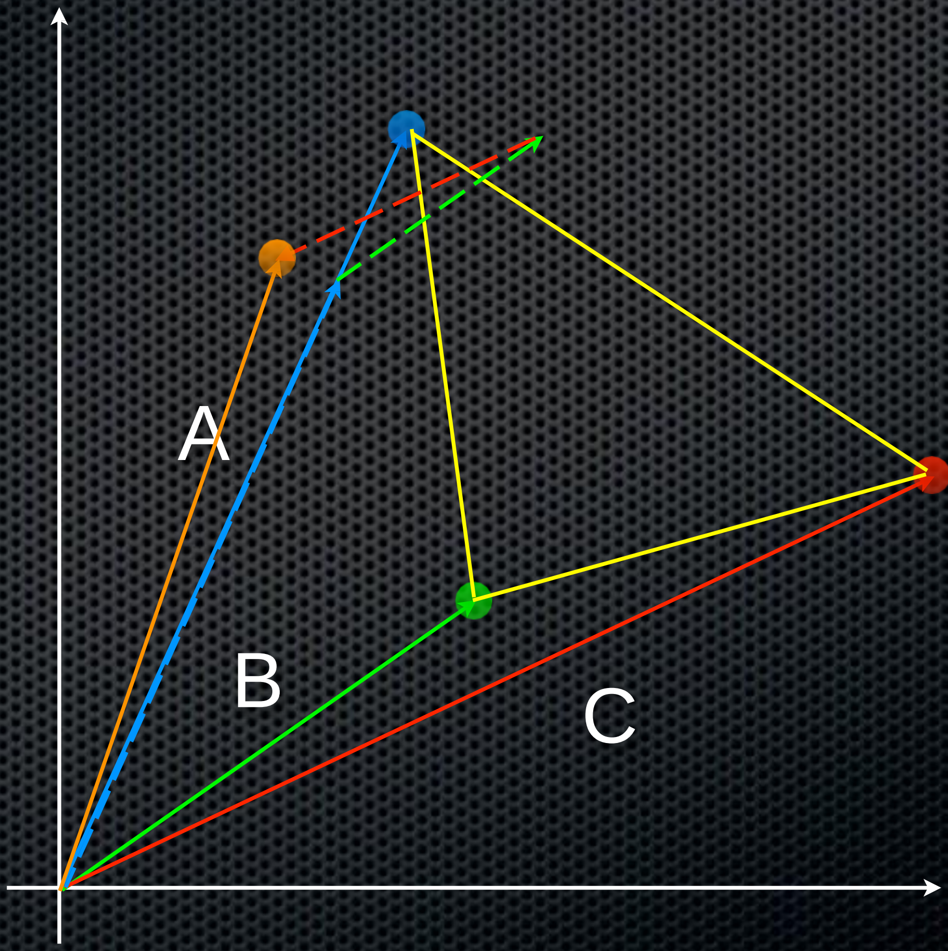
If we added these vertices together in this way, would they yield a point inside a triangle? Why or why not?

$.8 + .5 - .3 = 1$



Triangle

.8 + .5 - .3 = 1

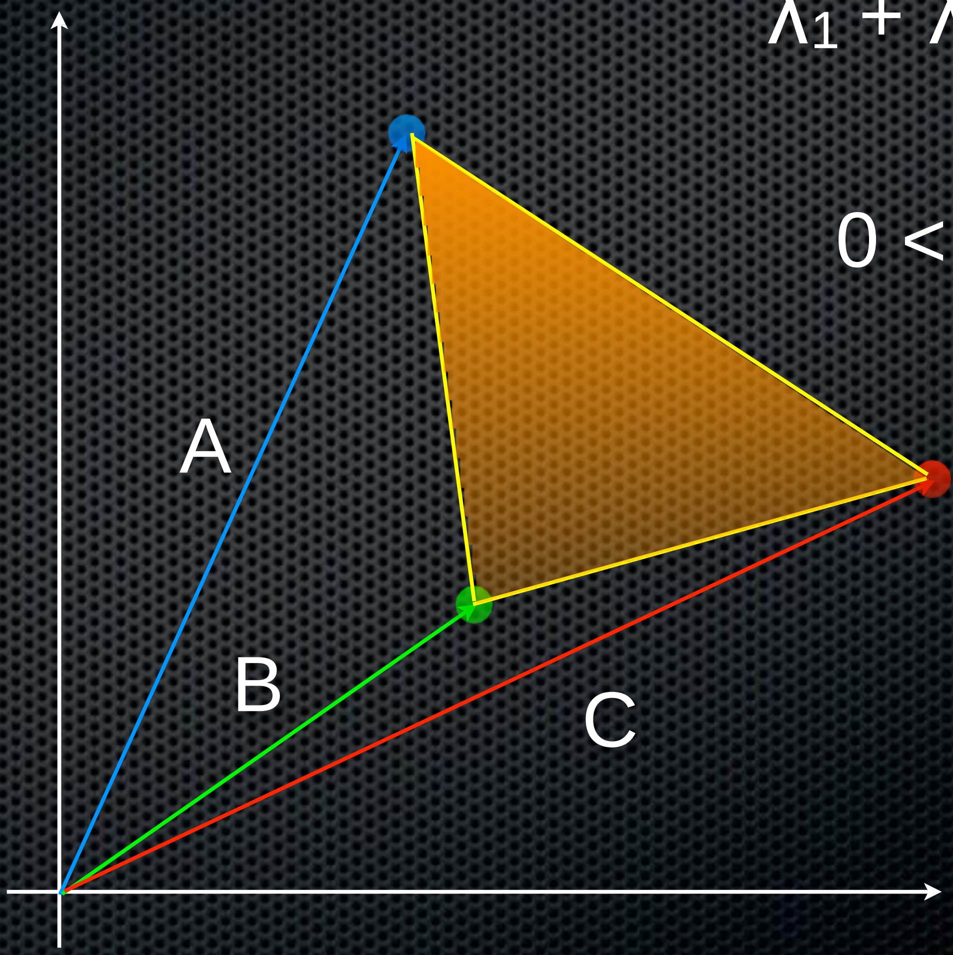


Barycentric Coordinates

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

and

$$0 < \lambda_x < 1$$



Affine Combination

- Given a vector

$$\mathbf{a} = (a_{1,} a_2, a_3, \ldots, a_n)$$

$$a_1 + a_2 + \ldots\ldots\ldots a_n = 1$$

- Affine combination: Sum of all components = 1

- Convex affine = affine + no negative component i.e

$$a_1, a_2, \ldots\ldots\ldots a_n = non - negative$$

Barycentric Coordinates

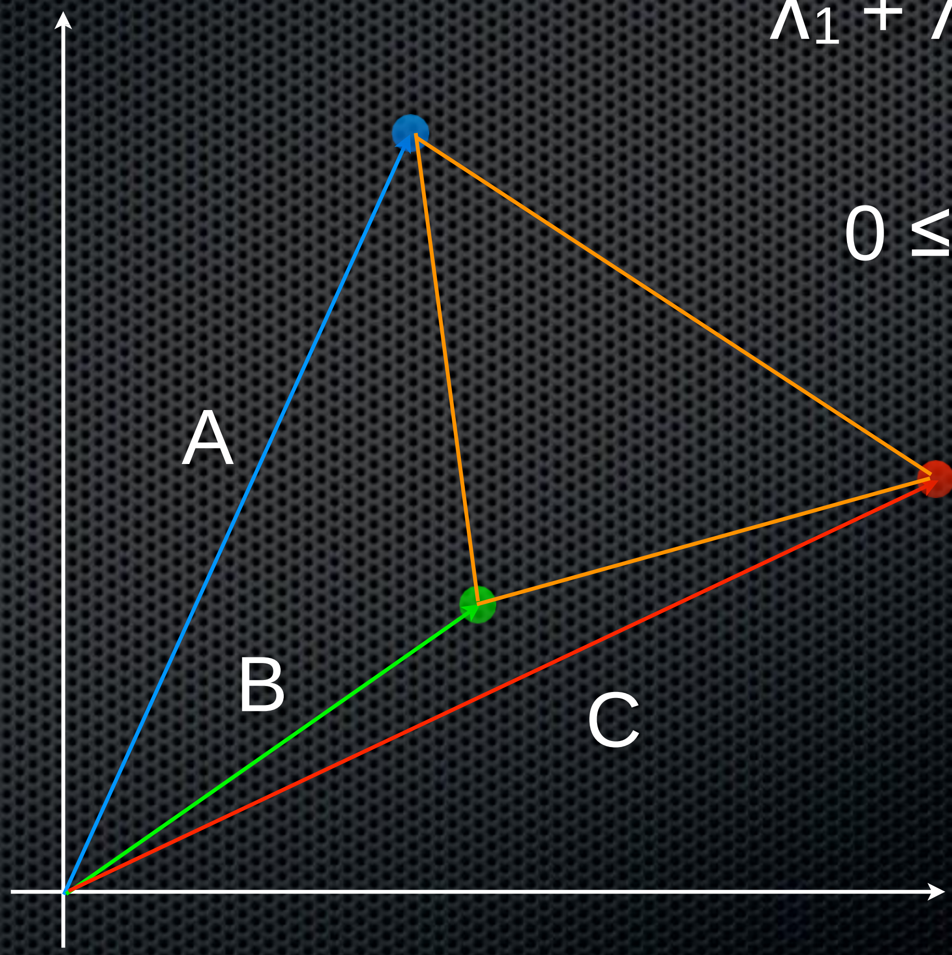
$\lambda_1 + \lambda_2 + \lambda_3 = 1$

and

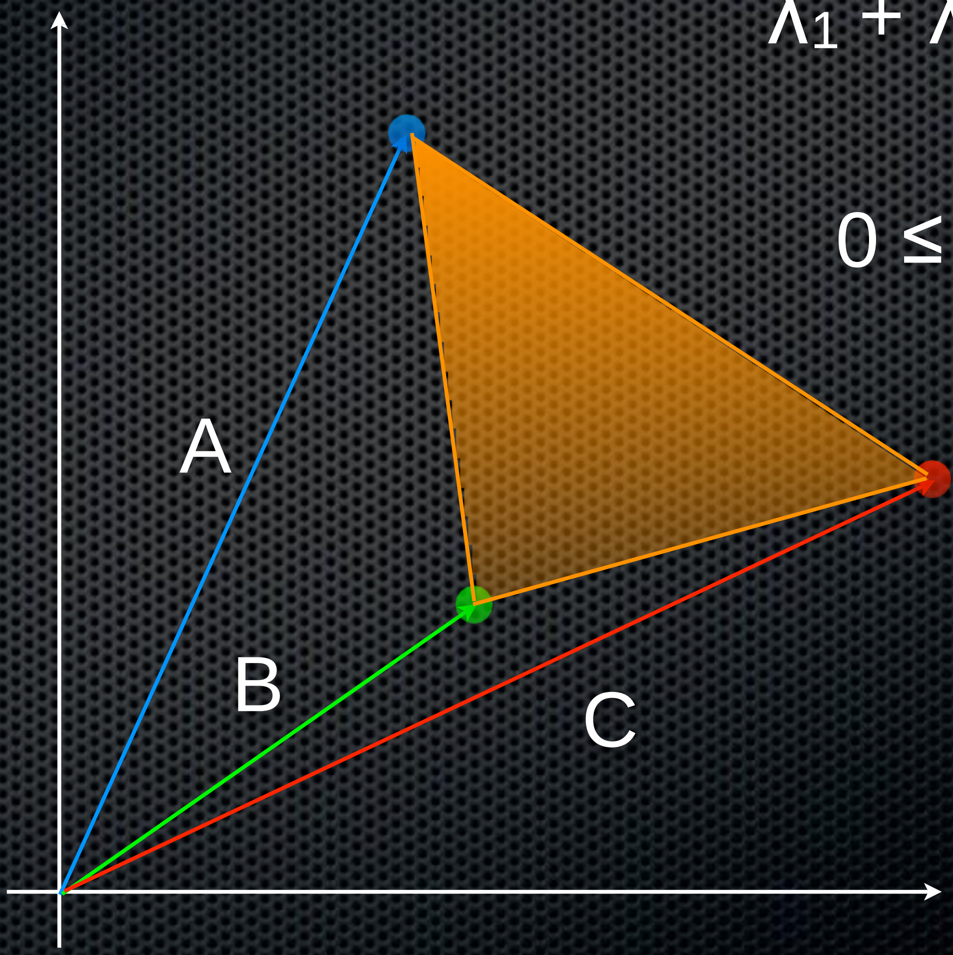
$0 \leq \lambda_x \leq 1$

and

1 or 2 $\lambda_x = 0$



Barycentric Coordinates



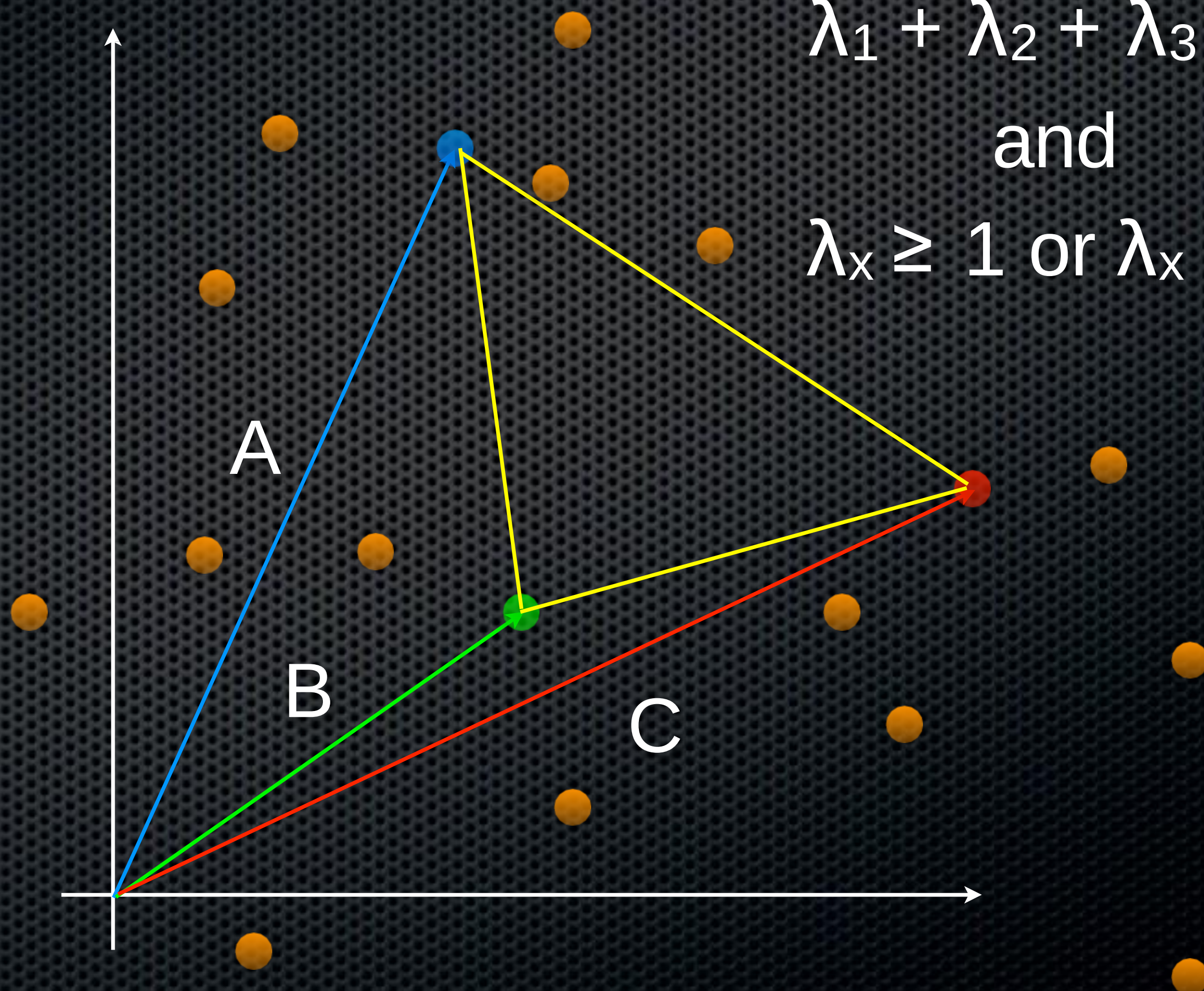
$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

and

$$0 \leq \lambda_x \leq 1$$

Barycentric Coordinates

Outside



$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

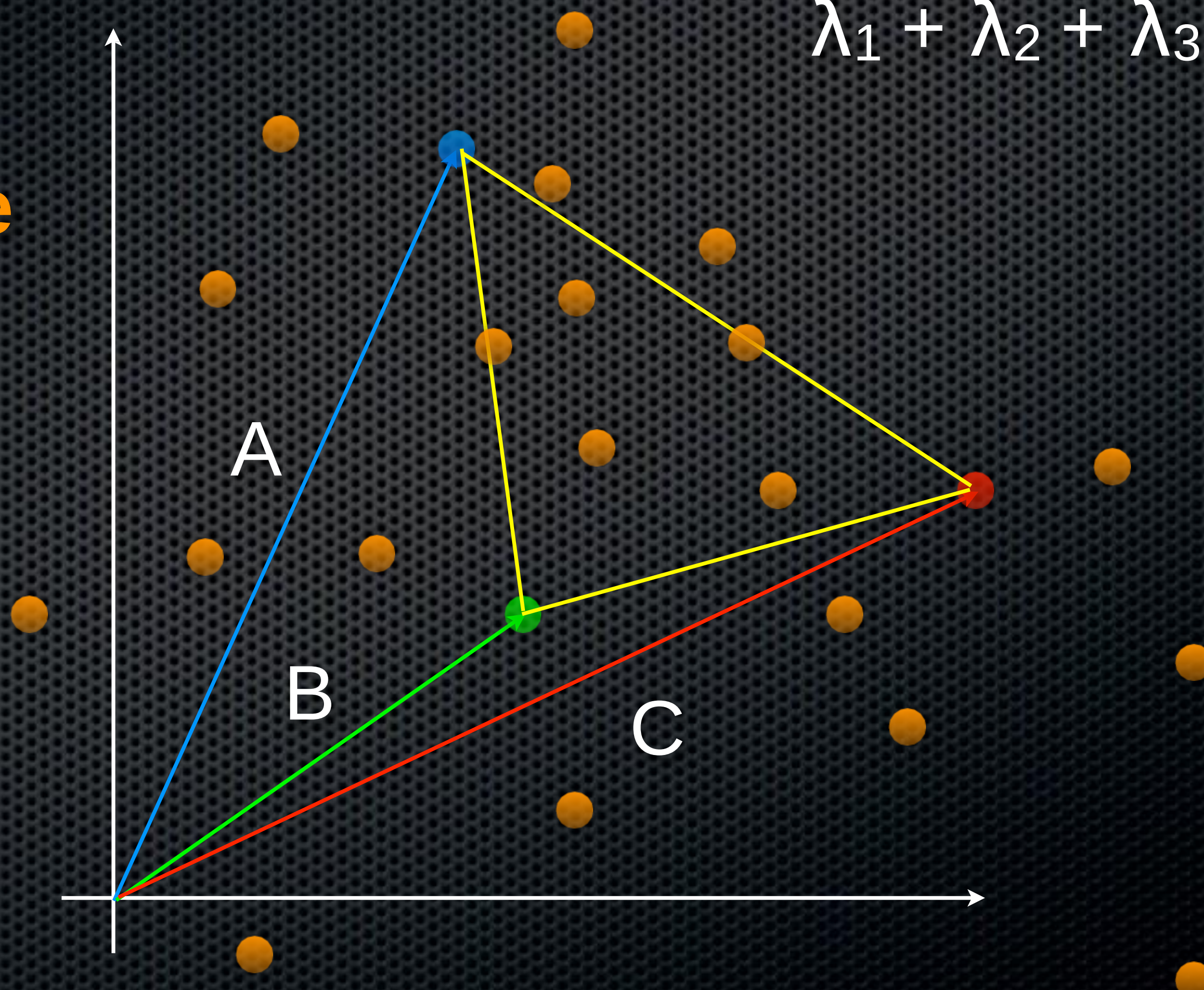
and

$$\lambda_x \geq 1 \text{ or } \lambda_x \leq 0$$

Not Barycentric Coordinates

$$\lambda_1 + \lambda_2 + \lambda_3 \neq 1$$

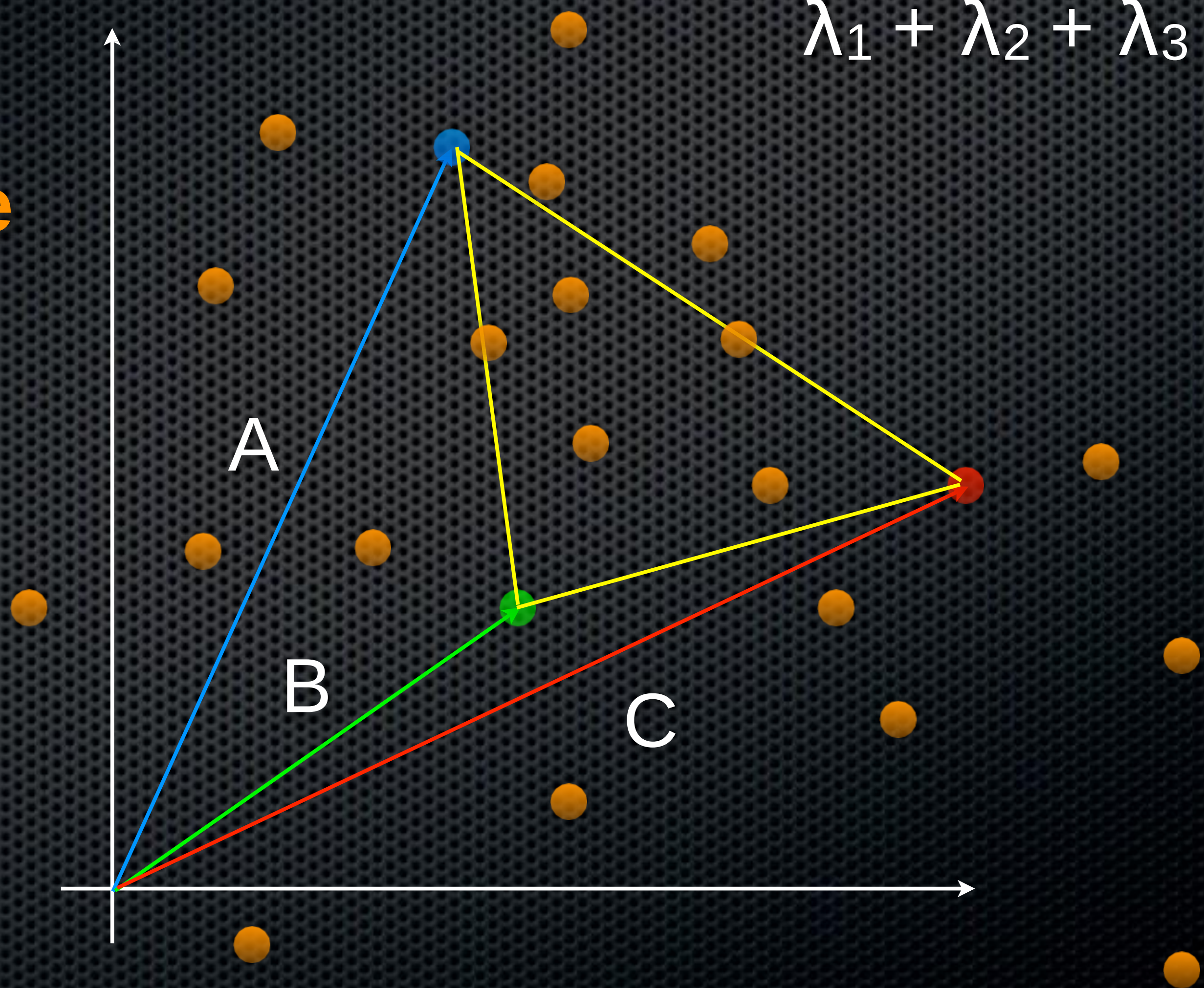
Anywhere



Linear Combination

$\lambda_1 + \lambda_2 + \lambda_3 \neq 1$

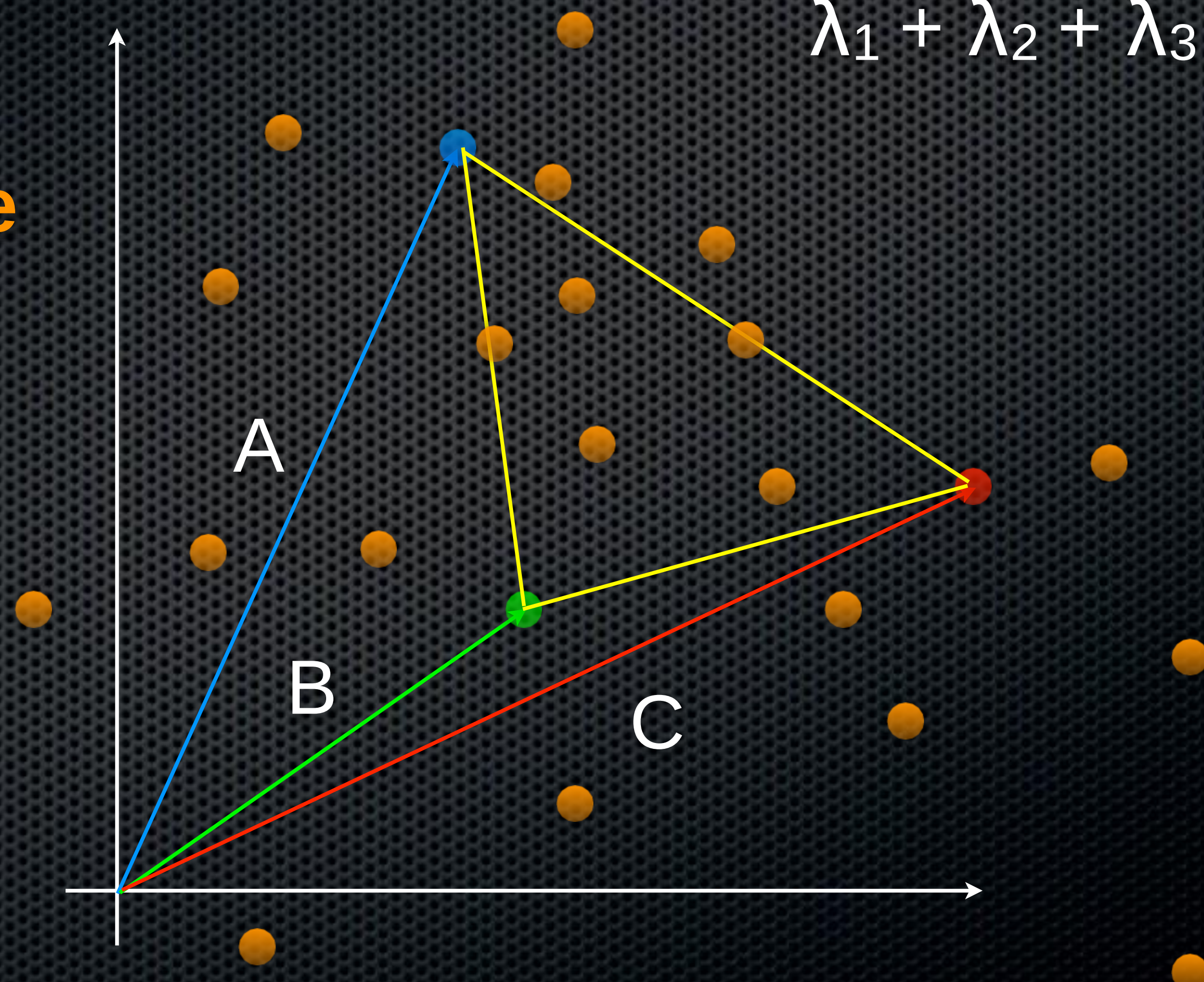
Anywhere



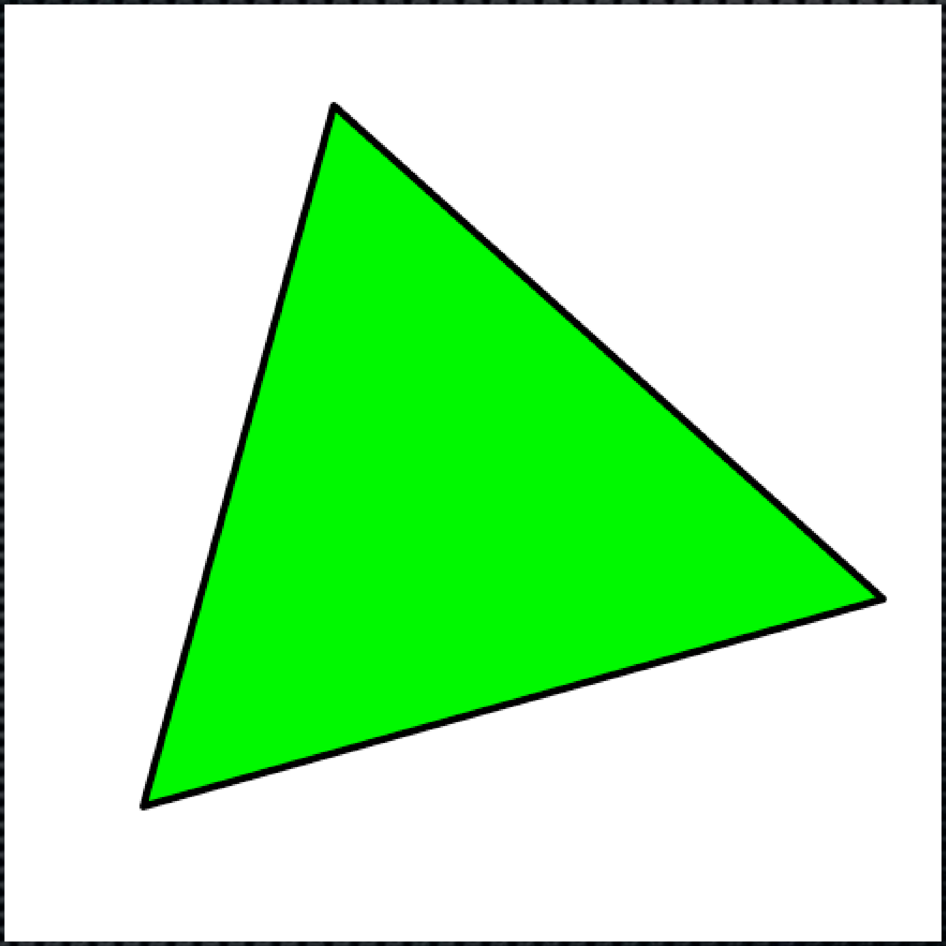
Linear Combination but **Not** a Blend

$\lambda_1 + \lambda_2 + \lambda_3 \neq 1$

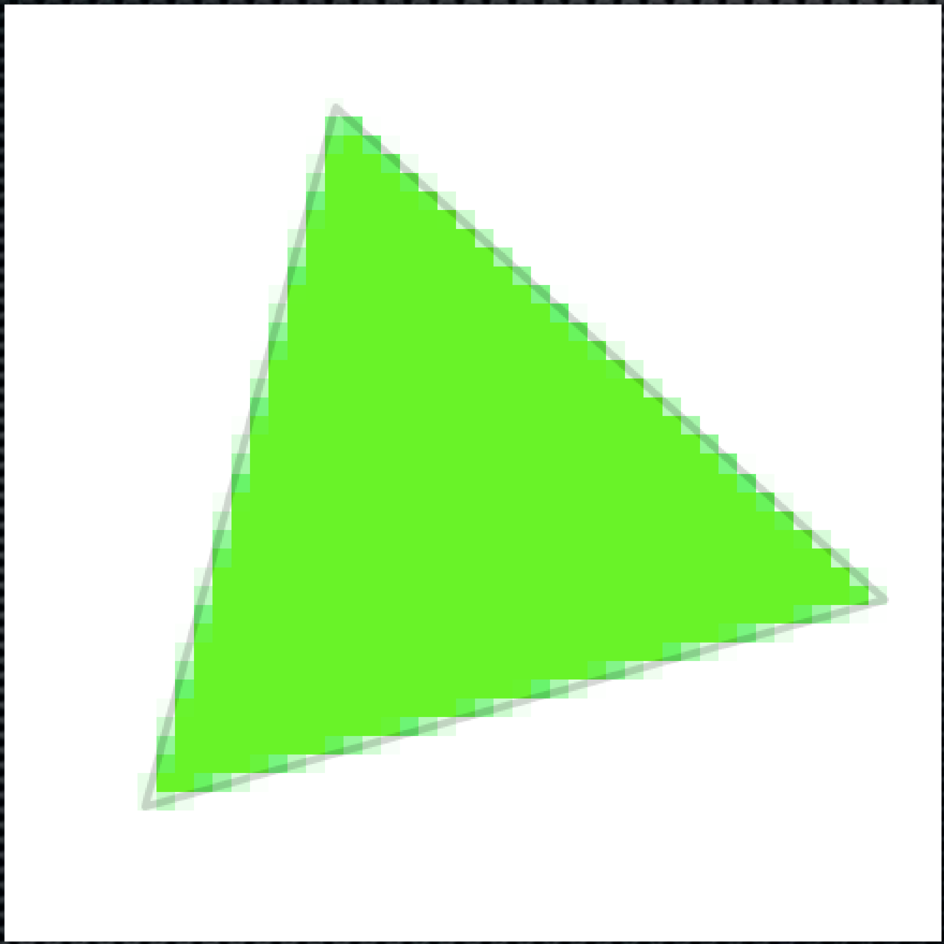
Anywhere



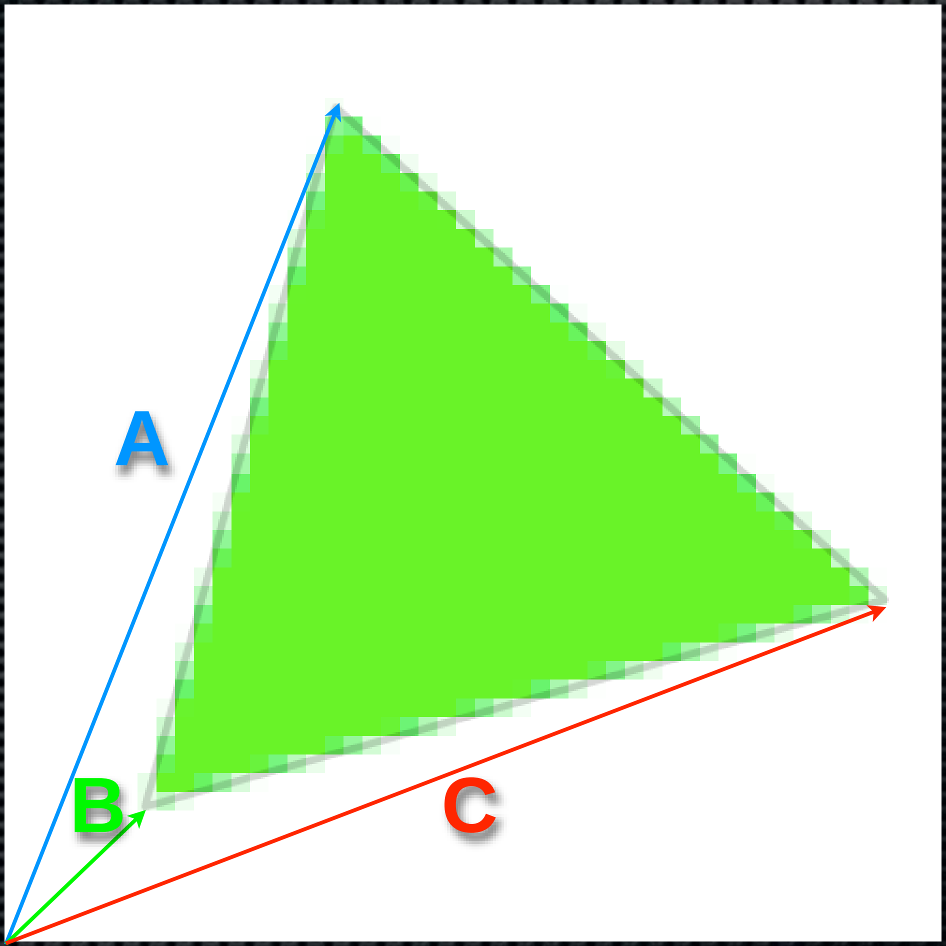
Rasterization



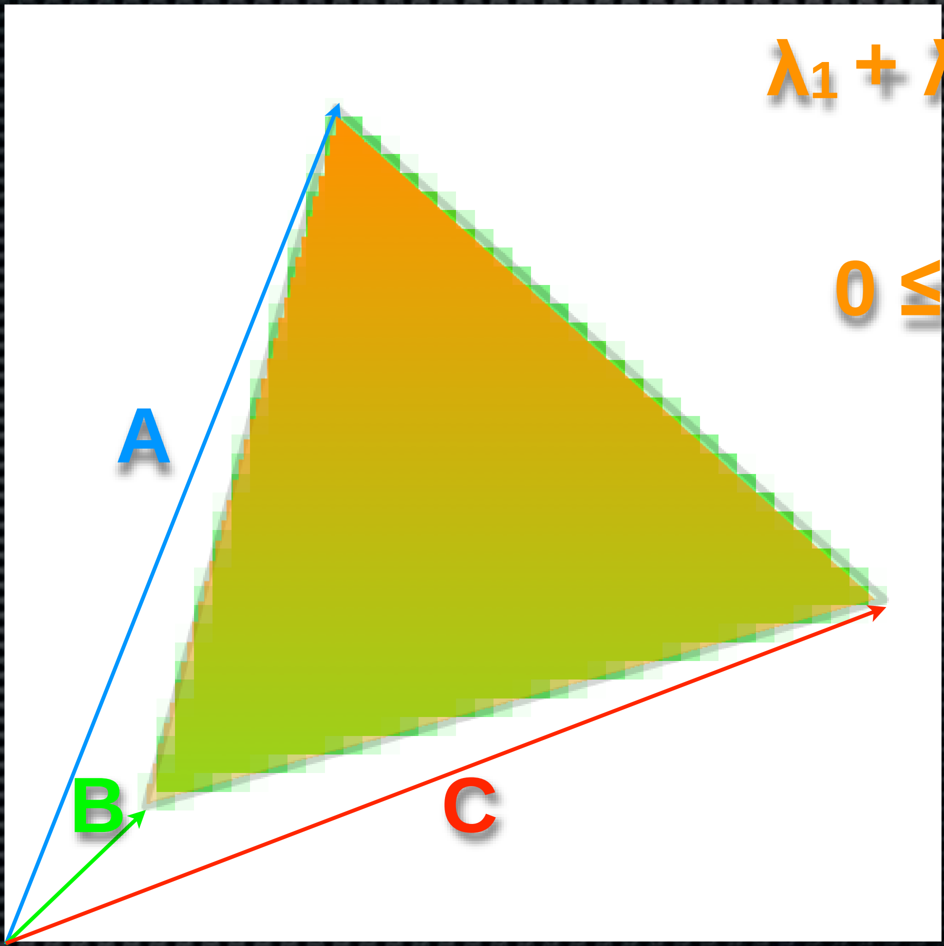
Rasterization



Rasterization



Barycentric Coordinates

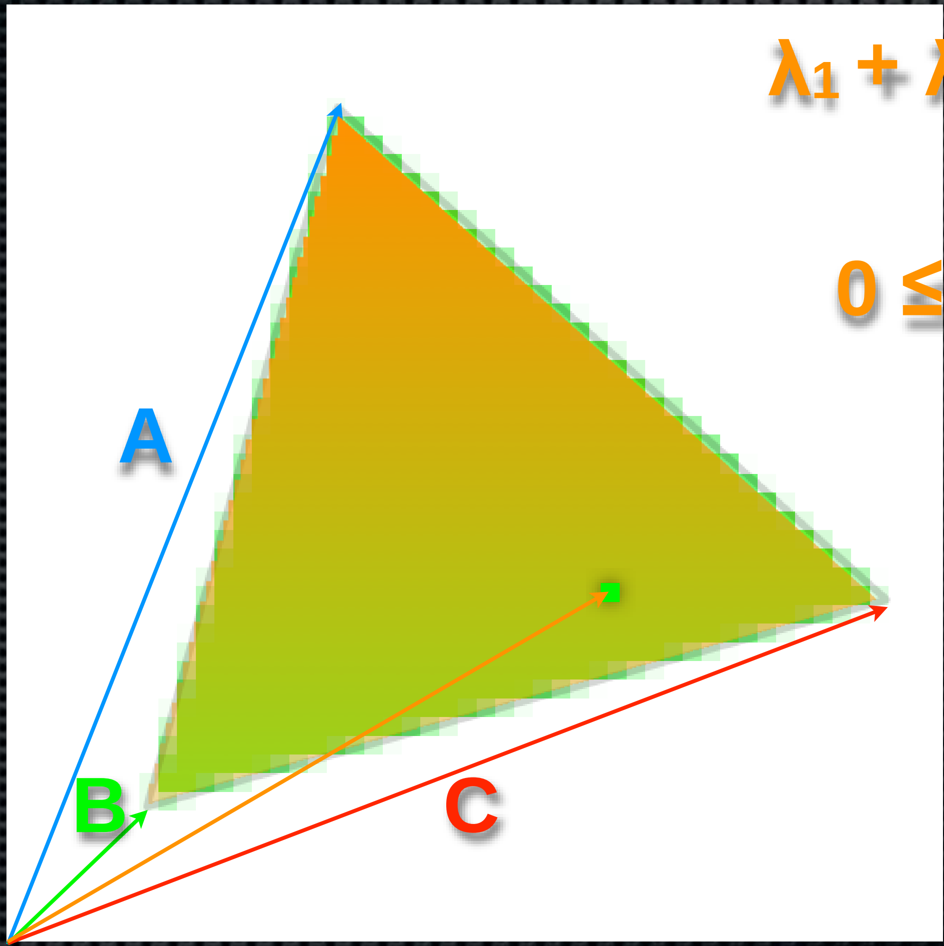


$\lambda_1 + \lambda_2 + \lambda_3 = 1$

and

$0 \leq \lambda_x \leq 1$

Barycentric Coordinates



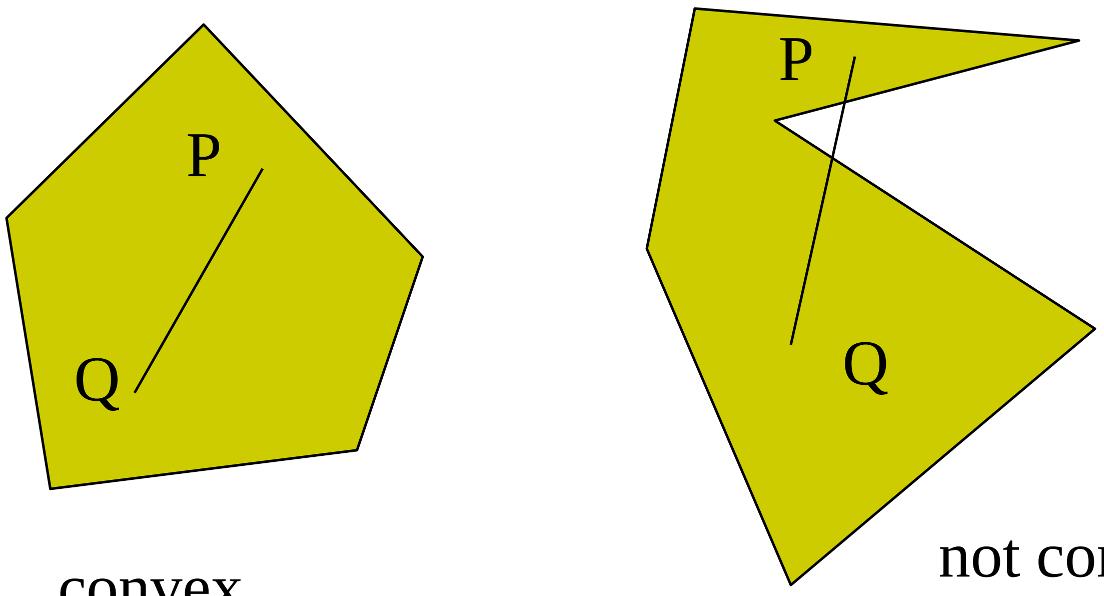
$\lambda_1 + \lambda_2 + \lambda_3 = 1$

and

$0 \leq \lambda_x \leq 1$

Convexity

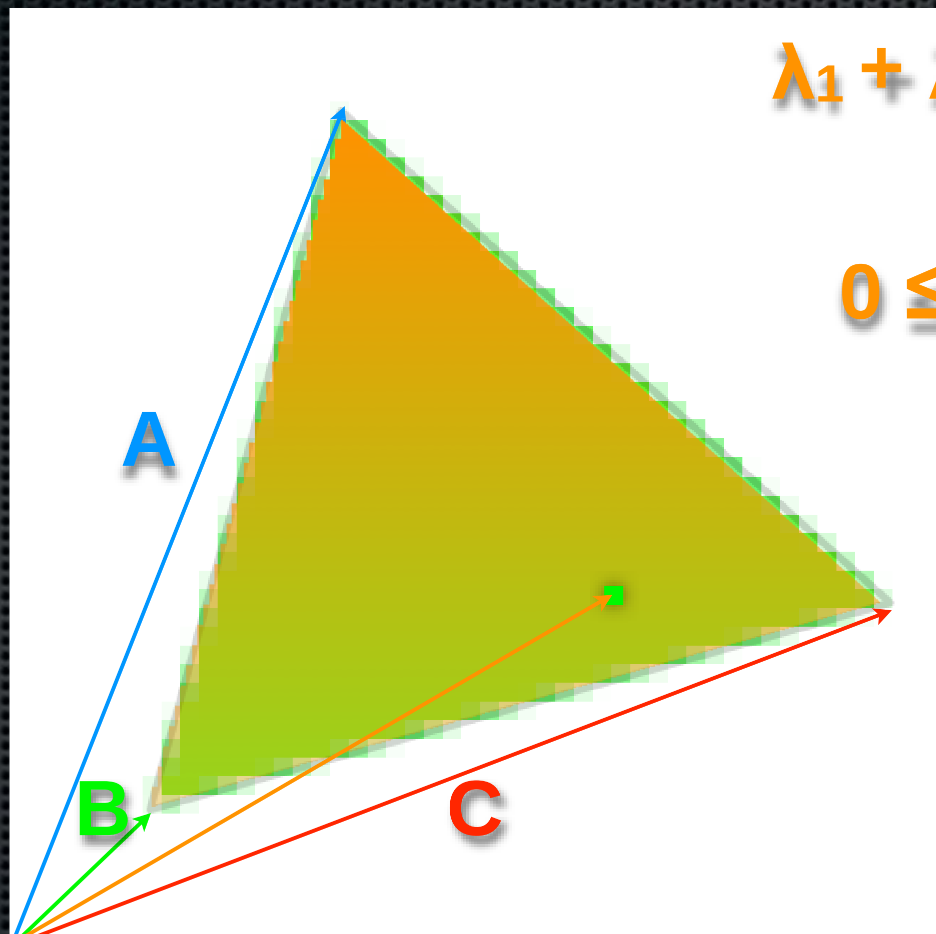
- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object



convex

not convex

Barycentric Coordinates



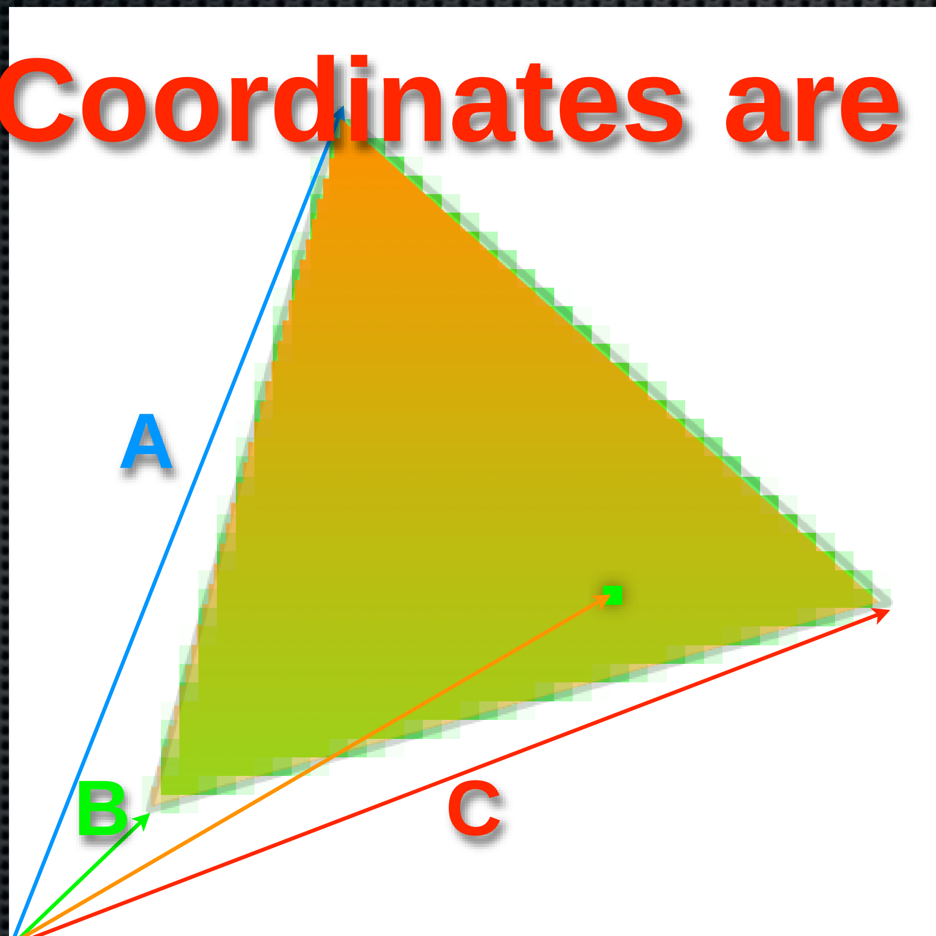
$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

and

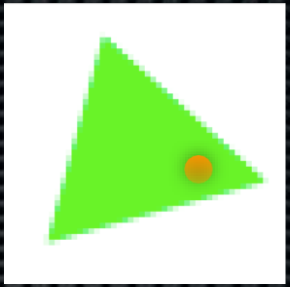
$$0 \leq \lambda_x \leq 1$$

Barycentric Coordinates

Pixel Coordinates are in X,Y



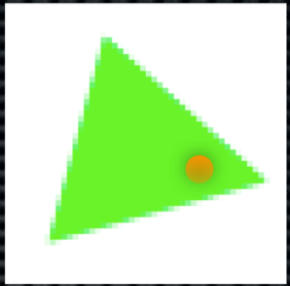
Barycentric Coordinates



$$x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$$
$$y = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

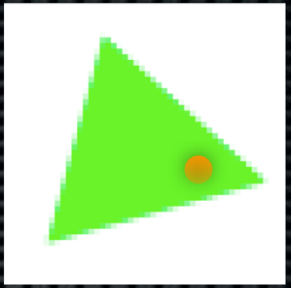
Barycentric Coordinates



$$x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$$
$$y = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3$$

$$\lambda_3 = 1 - \lambda_1 - \lambda_2$$

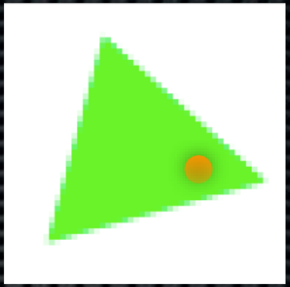
Barycentric Coordinates



$$x = \lambda_1 x_1 + \lambda_2 x_2 + (1 - \lambda_1 + \lambda_2) x_3$$
$$y = \lambda_1 y_1 + \lambda_2 y_2 + (1 - \lambda_1 + \lambda_2) y_3$$

$$\lambda_3 = 1 - \lambda_1 - \lambda_2$$

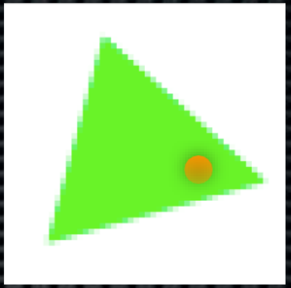
Barycentric Coordinates



Lots of rearranging...

$$\lambda_3 = 1 - \lambda_1 - \lambda_2$$

Barycentric Coordinates

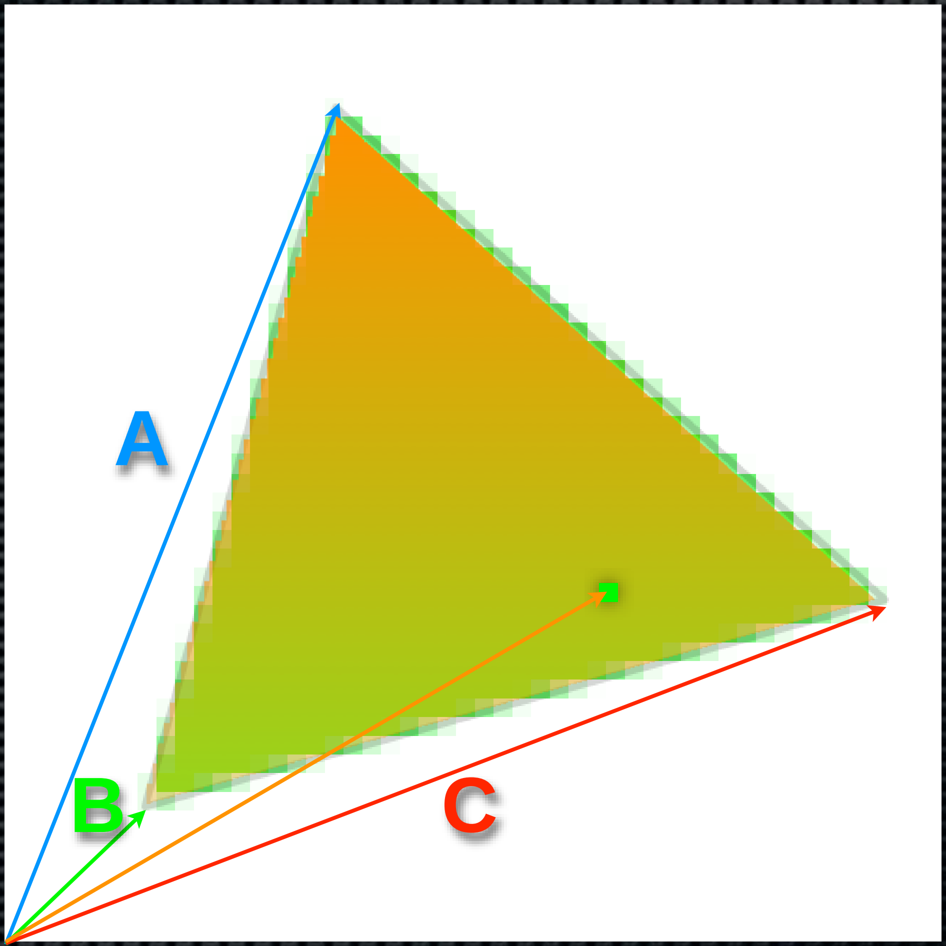


$$\lambda_1 = \frac{(y_2 - y_3)(x - x_3) + (x_3 - x_2)(y - y_3)}{(y_2 - y_3)(x_1 - x_3) + (x_3 - x_2)(y_1 - y_3)}$$

$$\lambda_2 = \frac{(y_3 - y_1)(x - x_3) + (x_1 - x_3)(y - y_3)}{(y_2 - y_3)(x_1 - x_3) + (x_3 - x_2)(y_1 - y_3)}$$

$$\lambda_3 = 1 - \lambda_1 - \lambda_2$$

Barycentric Coordinates

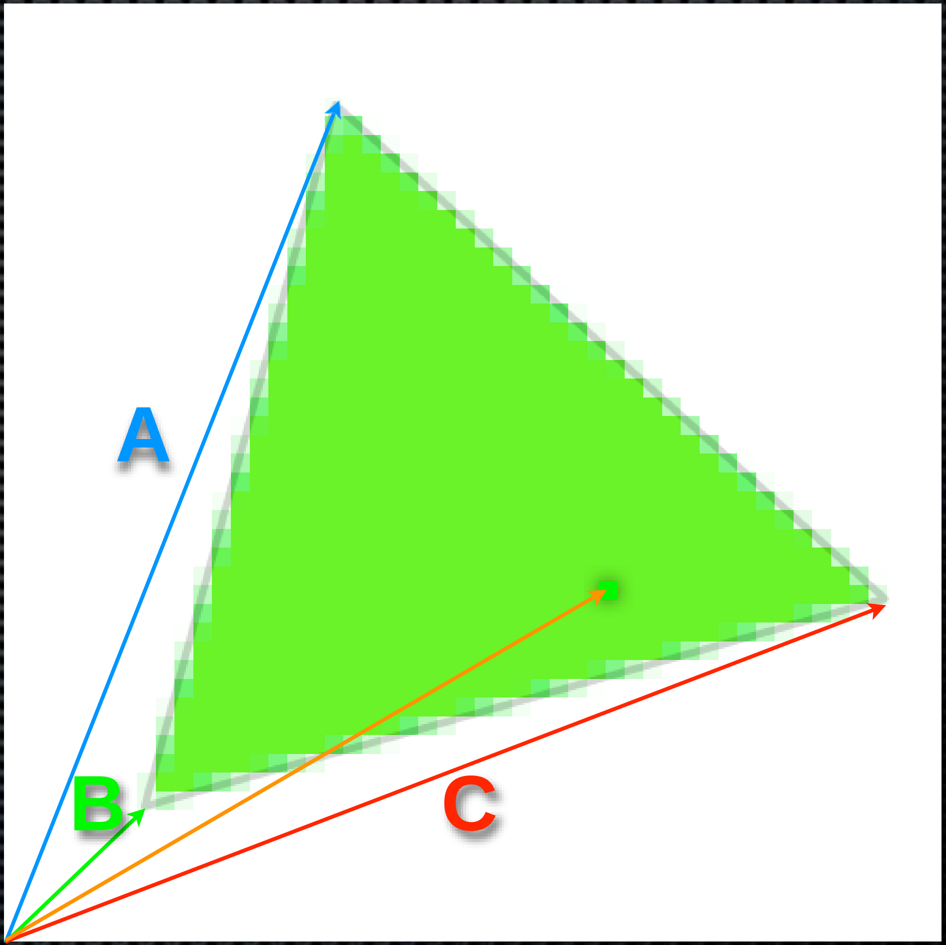


Barycentric Coordinates

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

and

$$0 \leq \lambda_x \leq 1$$



Think-Pair-Share

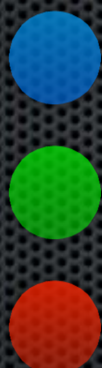
How can we use the logic of Barycentric coordinates to determine pixel color?

Barycentric Coordinates

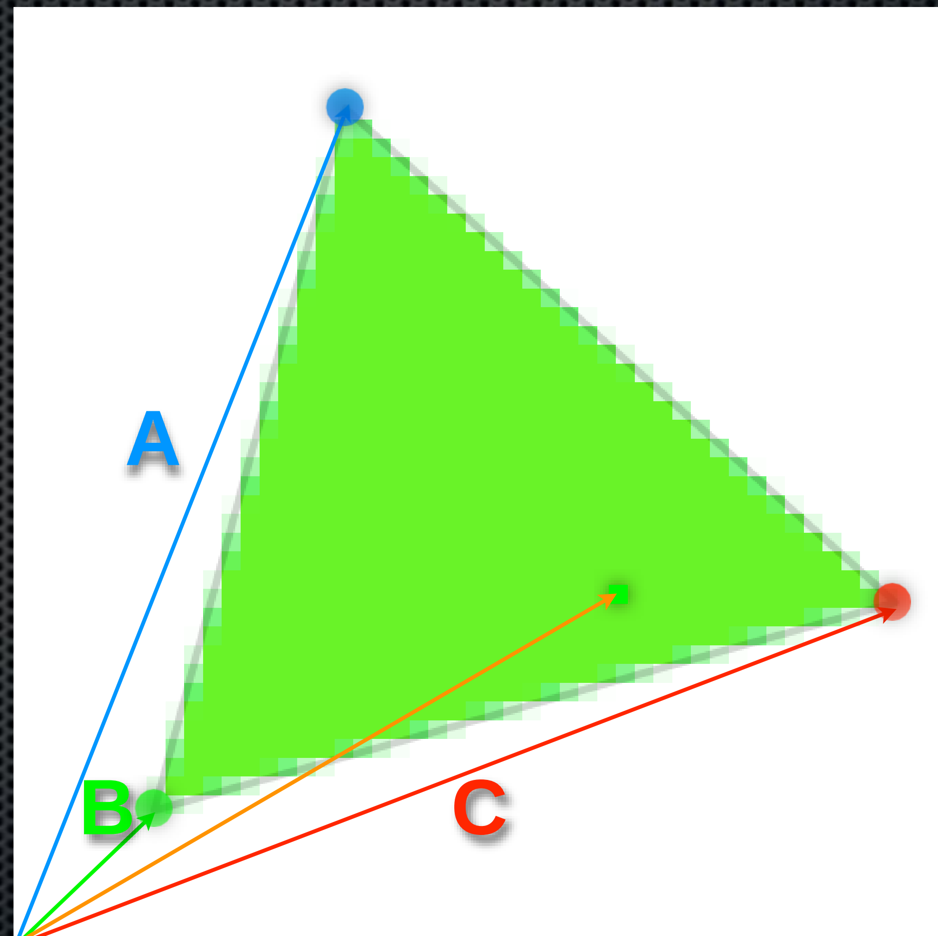
$\lambda_1 + \lambda_2 + \lambda_3 = 1$

and

$0 \leq \lambda_x \leq 1$



Associate a color with each point



Barycentric Coordinates

$\lambda_1 + \lambda_2 + \lambda_3 = 1$

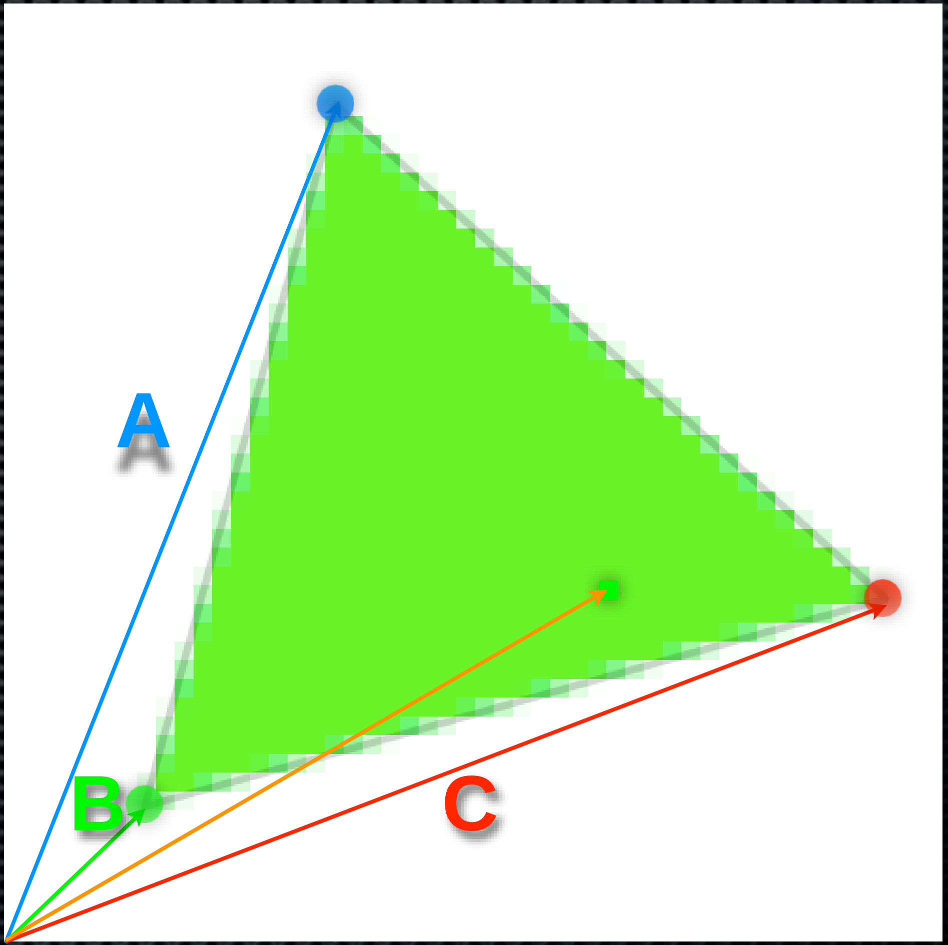
and

$0 \leq \lambda_x \leq 1$

$0 \leq \lambda_1 \leq 1$

$0 \leq \lambda_2 \leq 1$

$0 \leq \lambda_3 \leq 1$



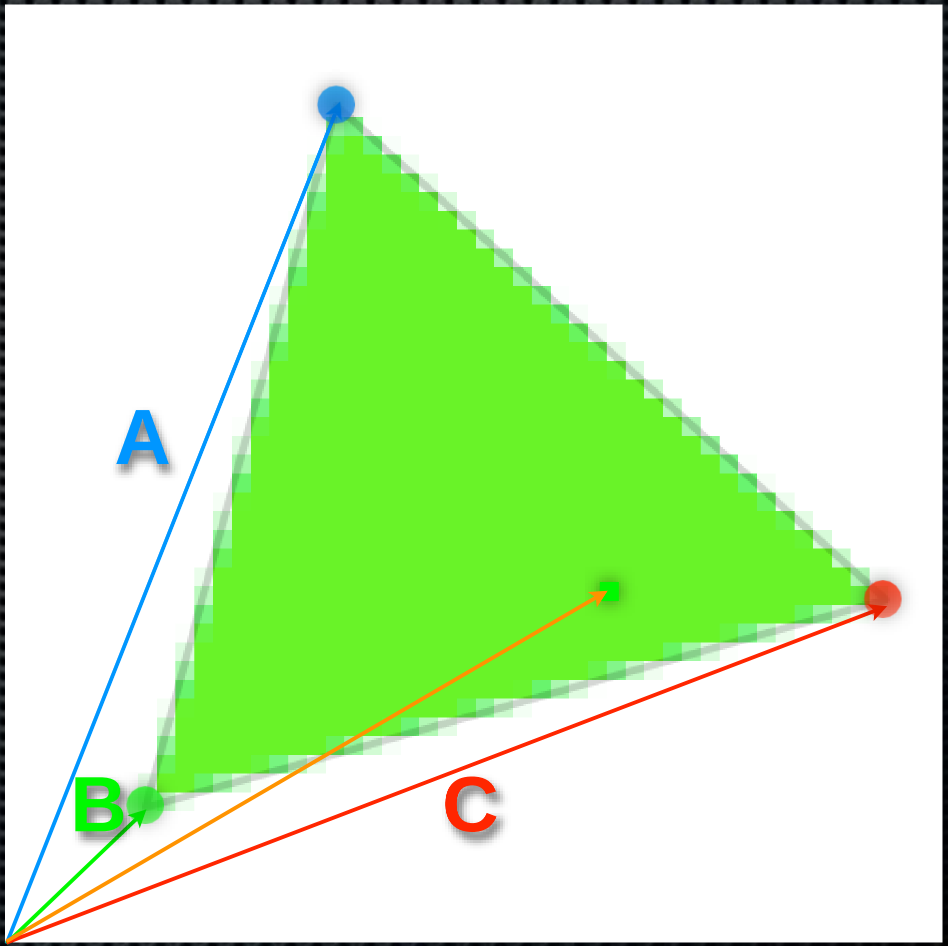
Barycentric Coordinates

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

and

$$0 \leq \lambda_x \leq 1$$

- λ_1 = **Weight 1**
- λ_2 = **Weight 2**
- λ_3 = **Weight 3**



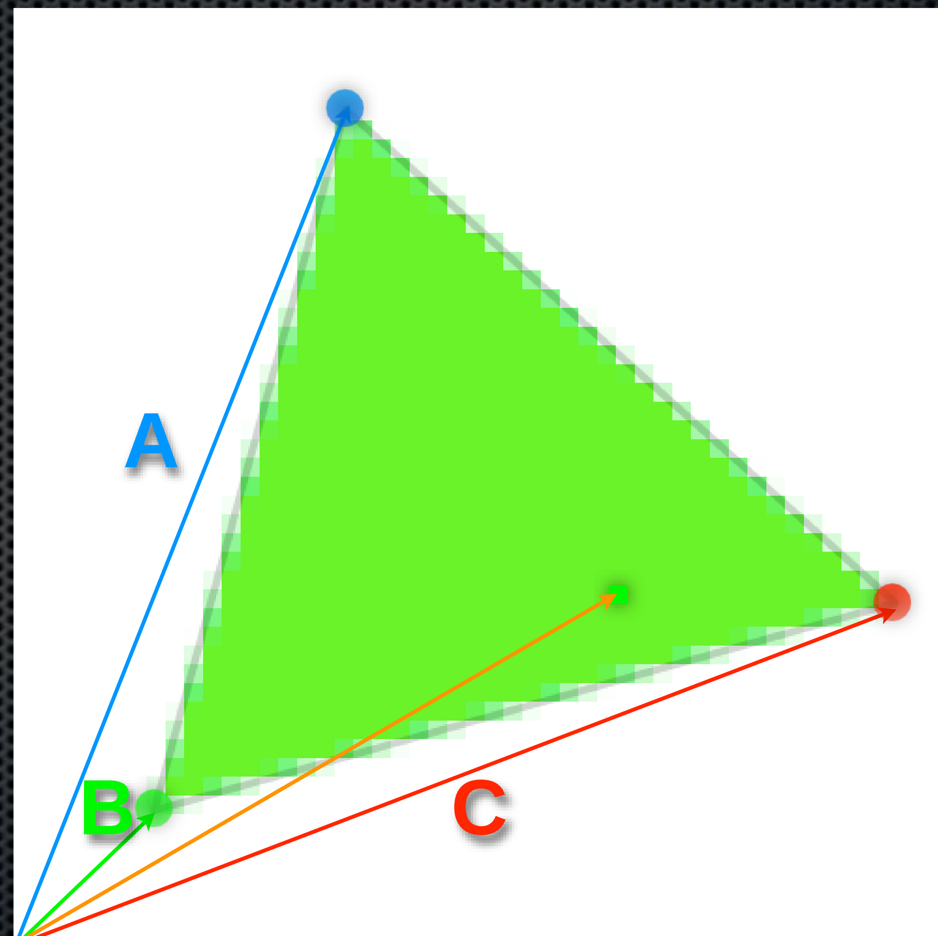
Barycentric Coordinates

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

and

$$0 \leq \lambda_x \leq 1$$

- $\lambda_1 =$
- $\lambda_2 =$
- $\lambda_3 =$



Barycentric Coordinates

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

and

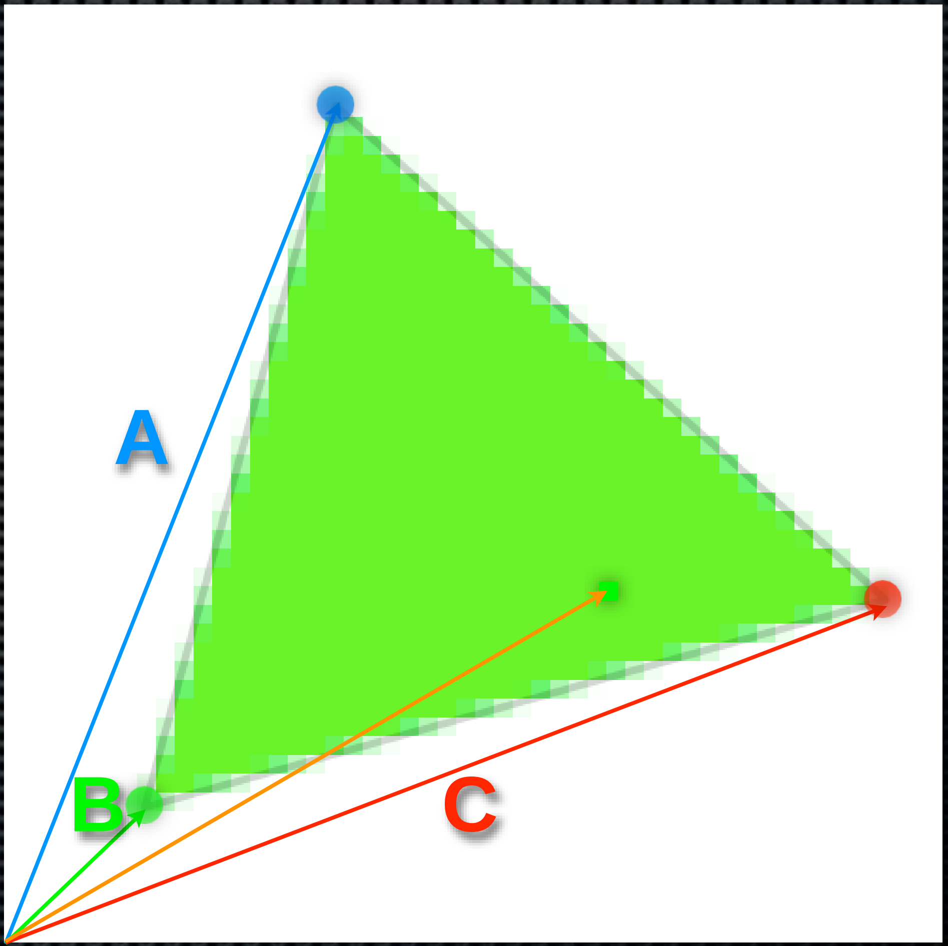
$$0 \leq \lambda_x \leq 1$$

$\cdot \; \lambda_1 +$

$\cdot \; \lambda_2 +$

$\cdot \; \lambda_3 +$

$=$



Barycentric Coordinates

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

and

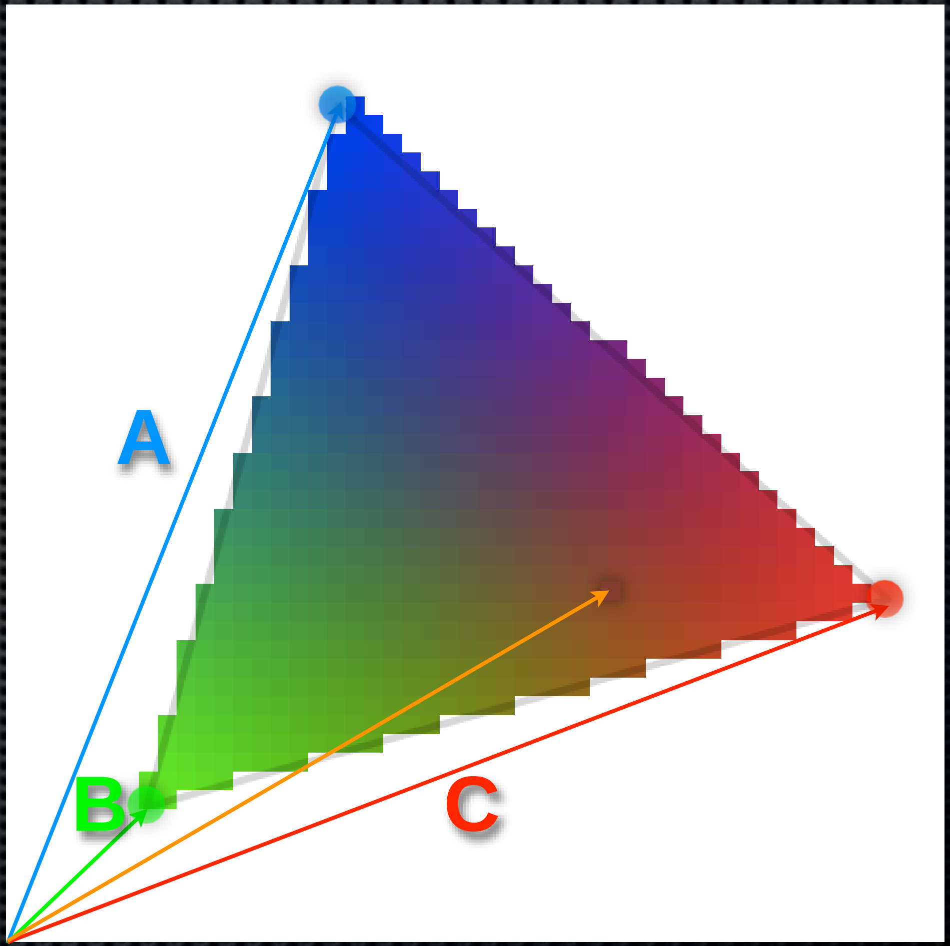
$$0 \leq \lambda_x \leq 1$$

 • λ_1 +

 • λ_2 +

 • λ_3 +

=



Colors

0xFF3D4A4C

Alpha Channel

Green Channel

Red Channel

Blue Channel

Colors

0xFF3D4A4C

Alpha Channel

Green Channel

Red Channel

Blue Channel

Add and scale are applied to **channels**

This data form doesn't scale and add well

Colors

1.00f , 0.24f , 0.29f , 0.30f

Alpha Channel

Green Channel

Red Channel

Blue Channel

Add and scale are applied to **channels**

This data form does much better

Barycentric Coordinates

Color Scale

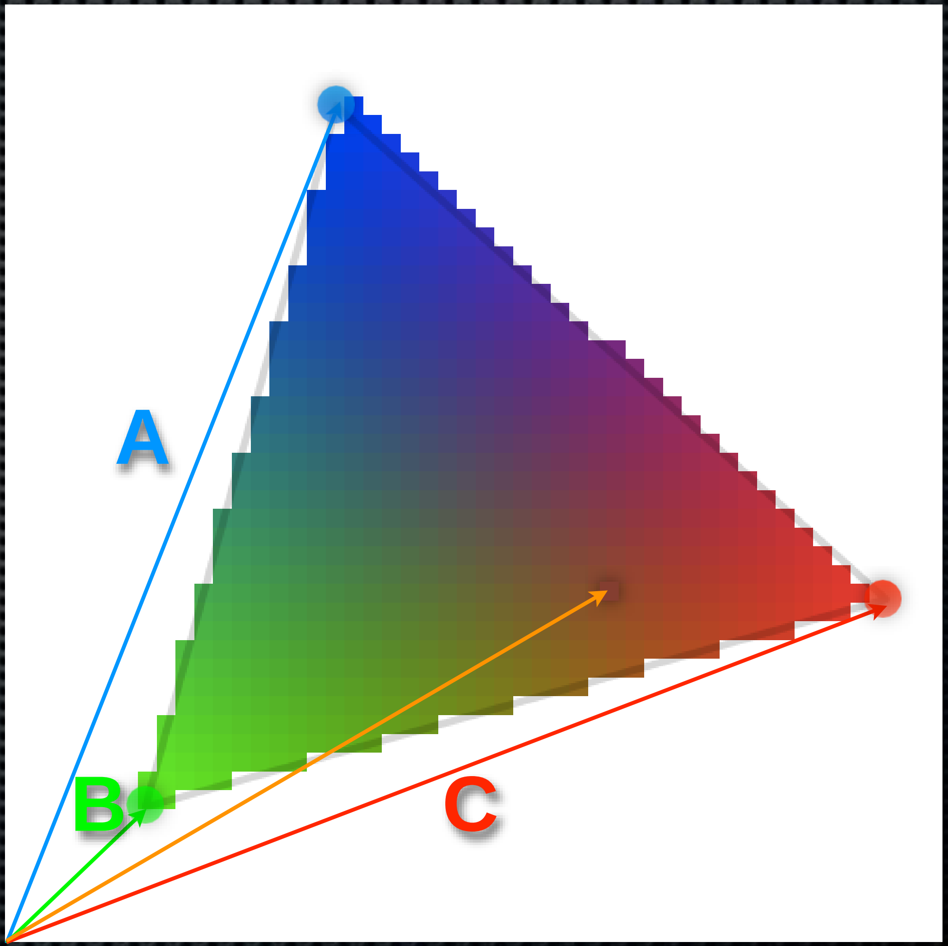
Color Add

$\cdot\ \lambda_1 +$

$\cdot\ \lambda_2 +$

$\cdot\ \lambda_3 +$

$=$

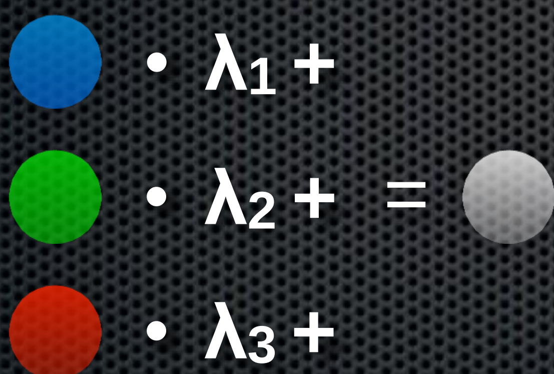


Barycentric Coordinates

$\lambda_1 + \lambda_2 + \lambda_3 = 1$

and

$0 \leq \lambda_x \leq 1$



Associate a color with each point