Bayesian Observer: Threshold and Slope Proofs

1 Setup

We consider a Bayesian observer estimating two stimuli s_1 and s_2 drawn from the same prior

$$s \sim \mathcal{N}(\mu_0, \sigma_0^2), \qquad \tau_0 = 1/\sigma_0^2.$$

Each stimulus produces a noisy sensory measurement

$$y_i \mid s_i \sim \mathcal{N}(s_i, \sigma_{s_i}^2), \qquad \tau_{s_i} = 1/\sigma_{s_i}^2.$$

Posterior for each stimulus follows Gaussian–Gaussian conjugacy:

$$\mu_{i,\text{post}}(y_i) = \frac{\tau_0 \mu_0 + \tau_{s_i} y_i}{\tau_0 + \tau_{s_i}}, \tag{1}$$

$$\sigma_{i,\text{post}}^2 = \frac{1}{\tau_0 + \tau_{s_i}}. \tag{2}$$

The observer samples

$$\hat{s}_i \sim \mathcal{N}(\mu_{i,\text{post}}, \sigma_{i,\text{post}}^2)$$

and chooses stimulus 1 whenever

$$D := \hat{s}_1 - \hat{s}_2 > 0.$$

2 Distribution of the Decision Variable

Since $\hat{s}_1$ and $\hat{s}_2$ are independent Gaussians and linear combinations preserve normality,

$$D \sim \mathcal{N}(\mu_D, \sigma_D^2),$$

with

$$\mu_D = E[\hat{s}_1] - E[\hat{s}_2] \tag{3}$$

$$= \frac{\tau_0 \mu_0 + \tau_{s_1} s_1}{\tau_0 + \tau_{s_1}} - \frac{\tau_0 \mu_0 + \tau_{s_2} s_2}{\tau_0 + \tau_{s_2}}, \tag{4}$$

and decision variance

$$\sigma_D^2 = \sigma_{1,\text{post}}^2 + \sigma_{2,\text{post}}^2 = \frac{1}{\tau_0 + \tau_{s_1}} + \frac{1}{\tau_0 + \tau_{s_2}}.$$

3 Psychometric Function

The psychometric function gives the choice probability We define the decision variable

$$D \sim \mathcal{N}(\mu_D(s_1), \sigma_D^2).$$

The psychometric function is the choice probability

$$\Psi(s_1) = P(D > 0) \tag{5}$$

$$= P\left(\frac{D - \mu_D(s_1)}{\sigma_D} > \frac{0 - \mu_D(s_1)}{\sigma_D}\right) \tag{6}$$

$$= P\left(Z > -\frac{\mu_D(s_1)}{\sigma_D}\right), \tag{7}$$

where $Z = \dfrac{D - \mu_D(s_1)}{\sigma_D} \sim \mathcal{N}(0, 1)$.

By definition of the standard normal CDF Φ,

$$\Phi(a) = P(Z \le a), \quad Z \sim \mathcal{N}(0, 1),$$

we have

$$\Psi(s_1) = P\left(Z > -\frac{\mu_D(s_1)}{\sigma_D}\right) \tag{8}$$

$$= 1 - P\left(Z \le -\frac{\mu_D(s_1)}{\sigma_D}\right) \tag{9}$$

$$= 1 - \Phi\left(-\frac{\mu_D(s_1)}{\sigma_D}\right). \tag{10}$$

Using the symmetry of the standard normal density,

$$\phi(z) = \phi(-z) \quad \Longrightarrow \quad \Phi(-x) = 1 - \Phi(x),$$

we obtain

$$\Psi(s_1) = 1 - \Phi\left(-\frac{\mu_D(s_1)}{\sigma_D}\right) \tag{11}$$

$$= \Phi\left(\frac{\mu_D(s_1)}{\sigma_D}\right). \tag{12}$$

Hence,

$$\boxed{\Psi(s_1) = \Phi\left(\frac{\mu_D(s_1)}{\sigma_D}\right)}$$

is a probit (Gaussian CDF) psychometric function.

4 Threshold Equals Prior Mean (Special Case)

The threshold s_1^* satisfies $\Psi(s_1^*) = 0.5$, which is equivalent to The psychometric function is

$$\Psi(s_1) = P(D > 0) = \Phi\left(\frac{\mu_D(s_1)}{\sigma_D}\right),$$

where Φ is the CDF of the standard normal distribution and $D \sim \mathcal{N}(\mu_D(s_1), \sigma_D^2)$.

The point of subjective equality (PSE), denoted s_1^*, is defined as the stimulus value for which the observer chooses either option with equal probability:

$$\Psi(s_1^*) = 0.5.$$

Since $\Phi(0) = 0.5$ and Φ is strictly increasing, we have

$$\Psi(s_1^*) = 0.5 \quad \Longleftrightarrow \quad \frac{\mu_D(s_1^*)}{\sigma_D} = 0.$$

Because $\sigma_D > 0$, this is equivalent to

$$\mu_D(s_1^*) = 0.$$

Thus, the psychometric midpoint occurs precisely when the mean of the decision variable D crosses zero. Solving this equation yields

$$s_1^* = \frac{\tau_{s_2} s_2 (\tau_0 + \tau_{s_1}) + \tau_0 \mu_0 (\tau_{s_1} - \tau_{s_2})}{\tau_{s_1} (\tau_0 + \tau_{s_2})}.$$

Pure-Prior Reference $(\tau_{s_2} \to 0)$

When s_2 is maximally noisy $(\sigma_{s_2}^2 \to \infty)$, $\tau_{s_2} \to 0$ and the formula reduces to

$$s_1^* = \mu_0.$$

Thus the threshold directly equals the prior mean. But with the current constraints on the deviations of the sigma we will choose $\tau_{s_2} \to 0.0625$ and $\tau_{s_1} \to 100000$ or even bigger.

5   Slope of the Psychometric Function

Differentiate the psychometric function:

$$\Psi'(s_1) = \Phi'\left(\frac{\mu_D}{\sigma_D}\right) \cdot \frac{\mu_D'}{\sigma_D} \tag{13}$$

$$= \phi\left(\frac{\mu_D}{\sigma_D}\right) \cdot \frac{\mu_D'}{\sigma_D}, \tag{14}$$

where ϕ is the standard normal PDF.

At threshold, $\mu_D(s_1^*) = 0$, so $\phi(0) = 1/\sqrt{2\pi}$ and therefore:

$$\boxed{\Psi'(s_1^*) = \frac{1}{\sqrt{2\pi}} \frac{\mu_D'(s_1^*)}{\sigma_D}.}$$

For stimulus increments linear in s_1, $\mu_D'(s_1^*) = 1$, giving

$$\boxed{\Psi'(s_1^*) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_D}.}$$

Using the expression for σ_D^2,

$$\boxed{\Psi'(s_1^*) \propto \frac{1}{\sqrt{\frac{1}{\tau_0 + \tau_{s_1}} + \frac{1}{\tau_0 + \tau_{s_2}}}}.}$$

Thus the psychometric slope is proportional to the square root of the posterior precision.

6   Relationship to the Gumbel Psychometric

The Gumbel psychometric function is

$$\Psi(x) = 1 - \exp\{-\exp[b(x - x_0)]\}.$$

Its slope at threshold x_{50} (where $\Psi(x_{50}) = 1/2$) satisfies

$$\exp[b(x_{50} - x_0)] = \ln 2.$$

Differentiating,

$$\Psi'(x) = e^{-e^{bt}} e^{bt} b, \qquad t = x - x_0, \tag{15}$$

$$\Psi'(x_{50}) = \frac{\ln 2}{2} b. \tag{16}$$

Since $\Psi'(x_{50}) \propto 1/\sigma_D$, we obtain

$$b_k = \frac{A}{\sqrt{\frac{1}{\tau_0 + \tau_{s_1,k}} + \frac{1}{\tau_0 + \tau_{s_2,k}}}},$$

with $A > 0$ an arbitrary scale constant. This expression allows recovery of the prior precision τ_0 from slopes across noise conditions.

7 Conclusion

We have shown that the slope of the psychometric function at threshold is inversely proportional to the standard deviation of the decision variable. Because the Gumbel steepness parameter b is proportional to this slope, each noise condition k obeys

$$b_k = \frac{A}{\sqrt{\frac{1}{\tau_0 + \tau_{s_1,k}} + \frac{1}{\tau_0 + \tau_{s_2,k}}}},$$

where $A > 0$ is a scale factor and $\tau_0 = 1/\sigma_0^2$ is the prior precision. Given empirical Gumbel slopes b_k and known sensory precisions $\tau_{s_1,k}, \tau_{s_2,k}$, one fits the parameters (A, τ_0) via nonlinear least squares. The prior variance is then recovered as

$$\hat{\sigma}_0^2 = \frac{1}{\hat{\tau}_0}.$$

In the symmetric case $\tau_{s_1,k} = \tau_{s_2,k} = \tau_k$, the mapping reduces to

$$b_k^2 = c_1 \tau_k + c_0,$$

with $c_1 = A^2/2$ and $c_0 = (A^2/2)\tau_0$. Thus,

$$\tau_0 = \frac{c_0}{c_1},$$

allowing recovery of the prior variance via simple linear regression.