FLOYD-WARSHALL + BELLMAN-FORD ALGORITHMS

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Agenda

1 Floyd-Warshall algorithm

2 Bellman-Ford algorithm

3 Bibliography

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Floyd-Warshall algorithm

Let G be a weighted graph, finds the shortest path between all pairs of nodes in G

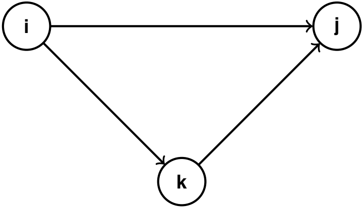
- All-pairs shortest paths

Floyd algorithm: can be used on weighted graphs with negative weights (but no negative cycles)

- Based on the Warshall algorithm for computing transitive closures

Design strategy: dynamic programming

Floyd-Warshall algorithm



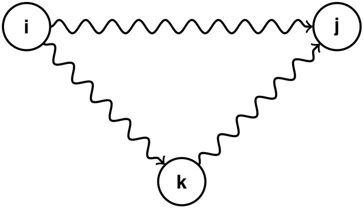
Before first iteration ($k = 0$)

- D[i][j]: direct distance between i and j

After first iteration ($k = 0$)

- D[i][j]: best option between $i \rightarrow j$ and $i \rightarrow 0 \rightarrow j$

Floyd-Warshall algorithm



Before second iteration ($k = 1$)

- D[i][j]: best option between $i \to j$ and $i \to 0 \to j$

After second iteration ($k = 1$)

- D[i][j]: best option between $i \rightsquigarrow j$ and $i \rightsquigarrow 1 \rightsquigarrow j$
  - $i \rightsquigarrow j$ the best of $i \to j$, $i \to 0 \to j$
  - $i \rightsquigarrow 1$ the best of $i \to 1$, $i \to 0 \to 1$
  - $1 \rightsquigarrow j$ the best of $1 \to j$, $1 \to 0 \to j$
- Considering the best among $i \to j, i \to 0 \to j, i \to 1 \to j,$ $i \to 0 \to 1 \to j, i \to 1 \to 0 \to j$

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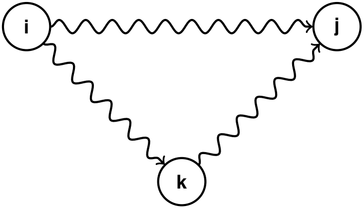
Floyd-Warshall algorithm



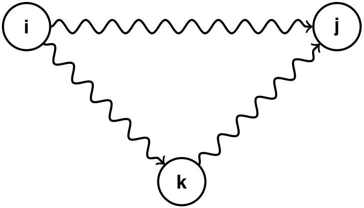
Before third iteration ($k = 2$)

- D[i][j]: best option between $i \rightsquigarrow j$ and $i \rightsquigarrow 1 \rightsquigarrow j$

After third iteration ($k = 2$)

- D[i][j]: best option between $i \rightsquigarrow j$ and $i \rightsquigarrow 2 \rightsquigarrow j$
  - $i \rightsquigarrow j$ the best of $i \to j$, $i \to 0 \to j$, $i \to 1 \to j$, $i \to 0 \to 1 \to j$, $i \to 1 \to 0 \to j$
  - $i \rightsquigarrow 2$ the best of $i \to 2$, $i \to 0 \to 2$, $i \to 1 \to 2$, $i \to 0 \to 1 \to 2$, $i \to 1 \to 0 \to 2$
  - $2 \rightsquigarrow j$ the best of ...

Floyd-Warshall algorithm



The algorithm stops when $k = n$

After the last iteration ($k = n - 1$)

- D[i][j]: best option between $i \rightsquigarrow j$ and $i \rightsquigarrow n - 1 \rightsquigarrow j$, which means the best among all possibilities

Floyd-Warshall algorithm

Algorithm: void Floyd(Graph G, int[][] D)

```
for i ← 0 to n(G) − 1 do
    for j ← 0 to n(G) − 1 do
        if i = j then D[i][j] ← 0;
        else if weight(G, i, j) ≠ 0 then D[i][j] ← weight(G, i, j);
        else D[i][j] ← ∞;
for k ← 0 to n(G) − 1 do
    for i ← 0 to n(G) − 1 do
        for j ← 0 to n(G) − 1 do
            if D[i][k] ≠ ∞ ∧ D[k][j] ≠ ∞ ∧ D[i][j] > D[i][k] + D[k][j] then
                D[i][j] ← D[i][k] + D[k][j];
```

Time efficiency in $\Theta(|V|^3)$

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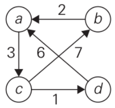
Floyd-Warshall algorithm[1]



$$D^{(0)} = \begin{array}{c|cccc|} & a & b & c & d \\ \hline a & 0 & \infty & 3 & \infty \\ b & 2 & 0 & \infty & \infty \\ c & \infty & 7 & 0 & 1 \\ d & 6 & \infty & \infty & 0 \\ \hline \end{array}$$

$$D^{(1)} = \begin{array}{c|cccc|} & a & b & c & d \\ \hline a & 0 & \infty & 3 & \infty \\ b & 2 & 0 & \mathbf{5} & \infty \\ c & \infty & 7 & 0 & 1 \\ d & 6 & \infty & \mathbf{9} & 0 \\ \hline \end{array}$$

$$D^{(2)} = \begin{array}{c|cccc|} & a & b & c & d \\ \hline a & 0 & \infty & 3 & \infty \\ b & 2 & 0 & 5 & \infty \\ c & \mathbf{9} & 7 & 0 & 1 \\ d & 6 & \infty & 9 & 0 \\ \hline \end{array}$$

$$D^{(3)} = \begin{array}{c|cccc|} & a & b & c & d \\ \hline a & 0 & \mathbf{10} & 3 & \mathbf{4} \\ b & 2 & 0 & 5 & \mathbf{6} \\ c & 9 & 7 & 0 & 1 \\ d & 6 & \mathbf{16} & 9 & 0 \\ \hline \end{array}$$

$$D^{(4)} = \begin{array}{c|cccc|} & a & b & c & d \\ \hline a & 0 & 10 & 3 & 4 \\ b & 2 & 0 & 5 & 6 \\ c & \mathbf{7} & 7 & 0 & 1 \\ d & 6 & 16 & 9 & 0 \\ \hline \end{array}$$

[1] Source: A. Levitin. Introduction to the Design and Analysis of Algorithms. 2011.

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Bellman-Ford algorithm

Let G be a weighted graph and $v \in V$ (*source*), finds the shortest path from v to all other nodes in V

- Single-source shortest paths

Bellman-Ford algorithm: can be used on weighted graphs with negative cycles

Intuition:

- There can be maximum $|V| - 1$ edges in any simple path
- Processes all edges $|V| - 1$ times
- Iteration $k = 0$: all shortest paths which are at most 1 edge long
- Iteration $k = |V| - 2$: all shortest paths
- Iteration $k = |V| - 1$: if something better is found, there is a negative cycle

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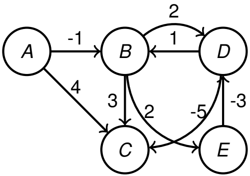
Bellman-Ford algorithm

Algorithm: void BellmanFord(Graph G, int s, int[] D)

for $i \leftarrow 0$ **to** $n(G) - 1$ **do** $D[i] \leftarrow \infty$;
$D[s] \leftarrow 0$;
for $k \leftarrow 0$ **to** $n(G) - 2$ **do**
    for $i \leftarrow 0$ **to** $n(G) - 1$ **do**
        $j \leftarrow \mathit{first}(G, i)$;
        while $j < n(G)$ **do**
            if $D[j] > D[i] + \mathit{weight}(G, i, j)$ **then**
                $D[j] \leftarrow D[i] + \mathit{weight}(G, i, j)$;
            $j \leftarrow \mathit{next}(G, i, j)$;
for $i \leftarrow 0$ **to** $n(G) - 1$ **do**
    $j \leftarrow \mathit{first}(G, i)$;
    while $j < n(G)$ **do**
        if $D[j] > D[i] + \mathit{weight}(G, i, j)$ **then**
            negative cycle detected
        $j \leftarrow \mathit{next}(G, i, j)$;

Let $s = A$



	A	B	C	D	E
Distance	0	∞	∞	∞	∞

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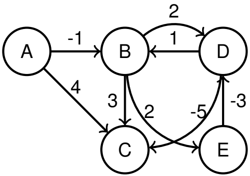
Bellman-Ford algorithm

Algorithm: void BellmanFord(Graph G, int s, int[] D)

for $i \leftarrow 0$ **to** $n(G) - 1$ **do** $D[i] \leftarrow \infty$;
$D[s] \leftarrow 0$;
for $k \leftarrow 0$ **to** $n(G) - 2$ **do**
    for $i \leftarrow 0$ **to** $n(G) - 1$ **do**
        $j \leftarrow first(G, i)$;
        while $j < n(G)$ **do**
            if $D[j] > D[i] + weight(G, i, j)$ **then**
                $D[j] \leftarrow D[i] + weight(G, i, j)$;
            $j \leftarrow next(G, i, j)$;
for $i \leftarrow 0$ **to** $n(G) - 1$ **do**
    $j \leftarrow first(G, i)$;
    while $j < n(G)$ **do**
        if $D[j] > D[i] + weight(G, i, j)$ **then**
            negative cycle detected
        $j \leftarrow next(G, i, j)$;

Let $s = A$



	A	B	C	D	E
Distance	0	-1	4	∞	∞

Considering $k = 0$

- After $i = 0$ (node A)

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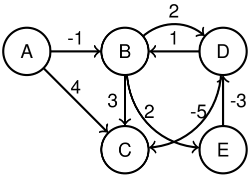
Bellman-Ford algorithm

Algorithm: void BellmanFord(Graph G, int s, int[] D)

```
for i ← 0 to n(G) − 1 do  D[i] ← ∞ ;
D[s] ← 0;
for k ← 0 to n(G) − 2 do
    for i ← 0 to n(G) − 1 do
        j ← first(G, i);
        while j < n(G) do
            if D[j] > D[i] + weight(G, i, j) then
                D[j] ← D[i] + weight(G, i, j);
            j ← next(G, i, j);
for i ← 0 to n(G) − 1 do
    j ← first(G, i);
    while j < n(G) do
        if D[j] > D[i] + weight(G, i, j) then
            negative cycle detected
        j ← next(G, i, j);
```

Let $s = A$



	A	B	C	D	E
Distance	0	-1	2	1	1

Considering $k = 0$

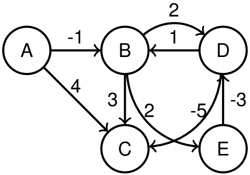
- After $i = 1$ (node B)

Bellman-Ford algorithm

Algorithm: void BellmanFord(Graph G, int s, int[] D)

```
for i ← 0 to n(G) − 1 do  D[i] ← ∞ ;
D[s] ← 0;
for k ← 0 to n(G) − 2 do
    for i ← 0 to n(G) − 1 do
        j ← first(G, i);
        while j < n(G) do
            if D[j] > D[i] + weight(G, i, j) then
                D[j] ← D[i] + weight(G, i, j);
            j ← next(G, i, j);
for i ← 0 to n(G) − 1 do
    j ← first(G, i);
    while j < n(G) do
        if D[j] > D[i] + weight(G, i, j) then
            negative cycle detected
        j ← next(G, i, j);
```

Let $s = A$



	A	B	C	D	E
Distance	0	-1	2	1	1

Considering $k = 0$

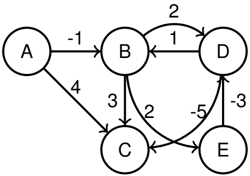
- After $i = 2$ (node C)

Bellman-Ford algorithm

Algorithm: void BellmanFord(Graph G, int s, int[] D)

for $i \leftarrow 0$ **to** $n(G) - 1$ **do** $D[i] \leftarrow \infty$;
$D[s] \leftarrow 0$;
for $k \leftarrow 0$ **to** $n(G) - 2$ **do**
  for $i \leftarrow 0$ **to** $n(G) - 1$ **do**
    $j \leftarrow first(G, i)$;
    while $j < n(G)$ **do**
      if $D[j] > D[i] + weight(G, i, j)$ **then**
        $D[j] \leftarrow D[i] + weight(G, i, j)$;
      $j \leftarrow next(G, i, j)$;
for $i \leftarrow 0$ **to** $n(G) - 1$ **do**
  $j \leftarrow first(G, i)$;
  while $j < n(G)$ **do**
    if $D[j] > D[i] + weight(G, i, j)$ **then**
      negative cycle detected
    $j \leftarrow next(G, i, j)$;

Let $s = A$



	A	B	C	D	E
Distance	0	-1	-4	1	1

Considering $k = 0$

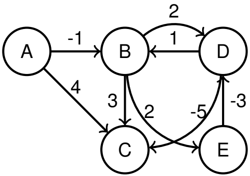
- After $i = 3$ (node D)

Bellman-Ford algorithm

Algorithm: void BellmanFord(Graph G, int s, int[] D)

```
for i ← 0 to n(G) − 1 do  D[i] ← ∞ ;
D[s] ← 0;
for k ← 0 to n(G) − 2 do
    for i ← 0 to n(G) − 1 do
        j ← first(G, i);
        while j < n(G) do
            if D[j] > D[i] + weight(G, i, j) then
                D[j] ← D[i] + weight(G, i, j);
            j ← next(G, i, j);
for i ← 0 to n(G) − 1 do
    j ← first(G, i);
    while j < n(G) do
        if D[j] > D[i] + weight(G, i, j) then
            negative cycle detected
        j ← next(G, i, j);
```

Let $s = A$



	A	B	C	D	E
Distance	0	-1	-4	-2	1

Considering $k = 0$

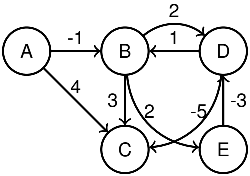
- After $i = 4$ (node E)

Bellman-Ford algorithm

Algorithm: void BellmanFord(Graph G, int s, int[] D)

```
for i ← 0 to n(G) − 1 do  D[i] ← ∞ ;
D[s] ← 0;
for k ← 0 to n(G) − 2 do
    for i ← 0 to n(G) − 1 do
        j ← first(G, i);
        while j < n(G) do
            if D[j] > D[i] + weight(G, i, j) then
                D[j] ← D[i] + weight(G, i, j);
            j ← next(G, i, j);
for i ← 0 to n(G) − 1 do
    j ← first(G, i);
    while j < n(G) do
        if D[j] > D[i] + weight(G, i, j) then
            negative cycle detected
        j ← next(G, i, j);
```

Let $s = A$



	A	B	C	D	E
Distance	0	-1	-7	-2	1

Considering $k = 1$

- After $i = 0..4$

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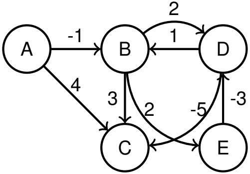
Bellman-Ford algorithm

Algorithm: void BellmanFord(Graph G, int s, int[] D)

```
for i ← 0 to n(G) − 1 do  D[i] ← ∞ ;
D[s] ← 0;
for k ← 0 to n(G) − 2 do
    for i ← 0 to n(G) − 1 do
        j ← first(G, i);
        while j < n(G) do
            if D[j] > D[i] + weight(G, i, j) then
                D[j] ← D[i] + weight(G, i, j);
            j ← next(G, i, j);
for i ← 0 to n(G) − 1 do
    j ← first(G, i);
    while j < n(G) do
        if D[j] > D[i] + weight(G, i, j) then
            negative cycle detected
        j ← next(G, i, j);
```

Let $s = A$



	A	B	C	D	E
Distance	0	-1	-7	-2	1

Considering $k = 2..3$

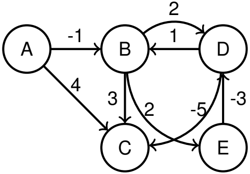
- Nothing changes

Bellman-Ford algorithm

Algorithm: void BellmanFord(Graph G, int s, int[] D)

for $i \leftarrow 0$ **to** $n(G) - 1$ **do** $D[i] \leftarrow \infty$;
$D[s] \leftarrow 0$;
for $k \leftarrow 0$ **to** $n(G) - 2$ **do**
  for $i \leftarrow 0$ **to** $n(G) - 1$ **do**
    $j \leftarrow first(G, i)$;
    while $j < n(G)$ **do**
      if $D[j] > D[i] + weight(G, i, j)$ **then**
        $D[j] \leftarrow D[i] + weight(G, i, j)$;
      $j \leftarrow next(G, i, j)$;
for $i \leftarrow 0$ **to** $n(G) - 1$ **do**
  $j \leftarrow first(G, i)$;
  while $j < n(G)$ **do**
    if $D[j] > D[i] + weight(G, i, j)$ **then**
      negative cycle detected
    $j \leftarrow next(G, i, j)$;

Let $s = A$



	A	B	C	D	E
Distance	0	-1	-7	-2	1

One more loop

- Nothing changes, implies no neg. cycle

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Bellman-Ford algorithm

Another example of dynamic programming

Time efficiency in $\Theta(\mid V \mid\mid E \mid) = \Theta(\mid V \mid^3)$, since $\mid E \mid \in O(\mid V \mid^2)$

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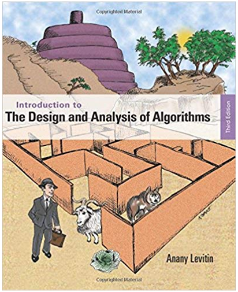

Chapter 9 (pp. 308–311)
Anany Levitin.
Introduction to the Design and Analysis of Algorithms.
3rd edition. Pearson. 2011.

Chapter 16 (pp. 513–515)
Clifford Shaffer.
Data Structures and Algorithm Analysis. Dover, 2013.

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