DYNAMIC PROGRAMMING

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Agenda

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Dynamic programming

Technique for solving problems with overlapping subproblems

Fibonacci numbers

- $F(n) = F(n-1) + F(n-2)$, for $n > 1$
- $F(0) = 0$ e $F(1) = 1$

Assuming that $n \geq 0$:

Algorithm: int Fib(n)

if $n \leq 1$ **then return** n;
else return $Fib(n-1) + Fib(n-2)$;

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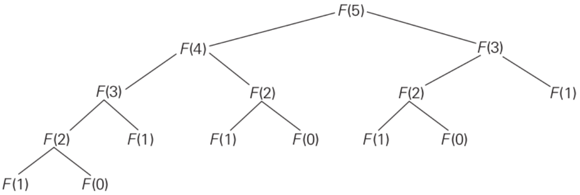
Dynamic programming[1]



Better approach (assuming that $n \geq 0$):

Algorithm: int Fib2(n)

$F[0], F[1] \leftarrow 0, 1;$
for $i \leftarrow 2$ **to** n **do**
  $F[i] \leftarrow F[i-1] + F[i-2];$
return $F[n];$

[1] Source: A. Levitin. Introduction to the Design and Analysis of Algorithms. 2011.

Dynamic programming

This technique can be seen as a variation of space-for-time trade-off

In some situations, it is even possible to avoid the extra space

- Fibonacci: store only the last two elements

Approaches for implementing dynamic programming

- Bottom-up: solves all subproblems
- Top-down: solves all required subproblems

Example: coin-row problem

There is a row of n coins whose values are some positive integers $c_1, c_2, ..., c_n$ not necessarily distinct. Goal: pick up the maximum amount of money, but not picking up two adjacent coins

Recurrence relation:

- $F(n) = max\{c_n + F(n-2), F(n-1)\}$, for $n > 1$
- $F(0) = 0$ e $F(1) = c_1$

Algorithm: int CoinRow(C[1..n])

```
F[0], F[1] ← 0, C[1];
for i ← 2 to n do
    F[i] ← max(C[i] + F[i − 2], F[i − 1]);
return F[n];
```

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Example: coin-row problem[2]

Considering: 5, 1, 2, 10, 6, and 2

$F[0] = 0$, $F[1] = c_1 = 5$

index	0	1	2	3	4	5	6
C		5	1	2	10	6	2
F	0	5					

$F[2] = \max\{1 + 0, 5\} = 5$

index	0	1	2	3	4	5	6
C		5	1	2	10	6	2
F	0	5	5				

$F[3] = \max\{2 + 5, 5\} = 7$

index	0	1	2	3	4	5	6
C		5	1	2	10	6	2
F	0	5	5	7			

$F[4] = \max\{10 + 5, 7\} = 15$

index	0	1	2	3	4	5	6
C		5	1	2	10	6	2
F	0	5	5	7	15		

$F[5] = \max\{6 + 7, 15\} = 15$

index	0	1	2	3	4	5	6
C		5	1	2	10	6	2
F	0	5	5	7	15	15	

$F[6] = \max\{2 + 15, 15\} = 17$

index	0	1	2	3	4	5	6
C		5	1	2	10	6	2
F	0	5	5	7	15	15	**17**

[2] Source: A. Levitin. Introduction to the Design and Analysis of Algorithms. 2011.

Example: change-making problem

Give change for amount n using the minimum number of coins $d_1 < d_2 < \dots < d_m$ where $d_1 = 1$ (assuming unlimited coins)

Recurrence relation:

- $F(n) = min_{j:n \geq d_j}\{F(n - d_j)\} + 1$, para $n > 0$
- $F(0) = 0$

Algorithm: int ChangeMaking(D[1..m], n)

```
F[0] ← 0;
for i ← 1 to n do
    temp, j ← ∞, 1;
    while j ≤ m ∧ i ≥ D[j] do
        temp ← min(F[i − D[j]], temp);
        j ← j + 1;
    F[i] ← temp + 1;
return F[n];
```

Example: change-making problem[3]

Considering: $n = 6$ and the following denominations $1, 3$ e 4

$F[0] = 0$

n	0	1	2	3	4	5	6
F	0						

$F[1] = \min\{F[1-1]\} + 1 = 1$

n	0	1	2	3	4	5	6
F	0	1					

$F[2] = \min\{F[2-1]\} + 1 = 2$

n	0	1	2	3	4	5	6
F	0	1	2				

$F[3] = \min\{F[3-1], F[3-3]\} + 1 = 1$

n	0	1	2	3	4	5	6
F	0	1	2	1			

$F[4] = \min\{F[4-1], F[4-3], F[4-4]\} + 1 = 1$

n	0	1	2	3	4	5	6
F	0	1	2	1	1		

$F[5] = \min\{F[5-1], F[5-3], F[5-4]\} + 1 = 2$

n	0	1	2	3	4	5	6
F	0	1	2	1	1	2	

$F[6] = \min\{F[6-1], F[6-3], F[6-4]\} + 1 = 2$

n	0	1	2	3	4	5	6
F	0	1	2	1	1	2	**2**

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[3] Source: A. Levitin. Introduction to the Design and Analysis of Algorithms. 2011.

Example: knapsack (0-1)[4]

Given n items of weight w_i and value v_i, where $i = 1 \cdots n$, and a capacity W, find the most valuable subset of items that fits into the knapsack's capacity

Recurrence relation:

- $F(i, j) = max\{F(i-1, j), v_i + F(i-1, j-w_i)\}$, if $j - w_i \geq 0$
- $F(i, j) = F(i-1, j)$, if $j - w_i < 0$
- $F(0, j) = 0$, for $j \geq 0$, $F(i, 0) = 0$, for $i \geq 0$

		0	$j-w_i$	j	W
	0	0	0	0	0
	$i-1$	0	$F(i-1, j-w_i)$	$F(i-1, j)$	
w_i, v_i	i	0		$F(i, j)$	
	n	0			goal

[4] Source: A. Levitin. Introduction to the Design and Analysis of Algorithms. 2011.

Example: knapsack (0-1) – bottom-up[5]

		capacity j					
	i	0	1	2	3	4	5
	0	0	0	0	0	0	0
$w_1 = 2,\ v_1 = 12$	1	0	0	12	12	12	12
$w_2 = 1,\ v_2 = 10$	2	0	10	12	22	22	22
$w_3 = 3,\ v_3 = 20$	3	0	10	12	22	30	32
$w_4 = 2,\ v_4 = 15$	4	0	10	15	25	30	**37**

Algorithm: int Knapsack(n,W,w[1..n],v[1..n],F[0..n,0..W])

for $i \leftarrow 0$ **to** n **do**
  for $j \leftarrow 0$ **to** W **do**
    if $i = 0 \lor j = 0$ **then** $F[i][j] \leftarrow 0$;
    else if $w[i] \leq j$ **then**
      $F[i][j] \leftarrow max(F[i-1][j], v[i] + F[i-1][j-w[i]])$;
    else $F[i][j] \leftarrow F[i-1][j]$;

return $F[n][W]$;

[5] Source: A. Levitin. Introduction to the Design and Analysis of Algorithms. 2011.

Example: knapsack (0-1) – top-down | 0,j/i,0=0, i,j=-1[6]

		capacity j					
	i	0	1	2	3	4	5
	0	0	0	0	0	0	0
$w_1 = 2,\ v_1 = 12$	1	0	0	12	12	12	12
$w_2 = 1,\ v_2 = 10$	2	0	—	12	22	—	22
$w_3 = 3,\ v_3 = 20$	3	0	—	—	22	—	32
$w_4 = 2,\ v_4 = 15$	4	0	—	—	—	—	**37**

Algorithm: int MFKnapsack(i,j,w[1..n],v[1..n],F[0..n,0..W])

if $F[i,j] < 0$ **then**
  if $j < w[i]$ **then** $value \leftarrow MFKnapsack(i-1, j, w, v, F)$;
  else
    $value \leftarrow max(MFKnapsack(i-1, j, w, v, F),$
      $v[i] + MFKnapsack(i-1, j-w[i], w, v, F));$
  $F[i,j] \leftarrow value;$
return $F[i,j];$

[6] Source: A. Levitin. Introduction to the Design and Analysis of Algorithms. 2011.

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Approximation algorithms

Sometimes it is even harder to find a solution

- NP-Complete $\subset$ NP-hard

Alternative: “solve” the problem using approximation algorithms

- The solution does not need to be optimal, but only acceptable

Examples: greed approximation for TSP

- Nearest-neighbour algorithm
- MST-based algorithm

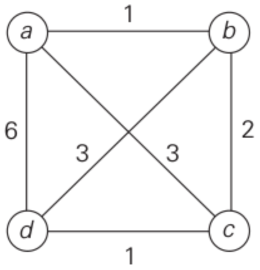
TSP: nearest-neighbour algorithm[7]

Initial node: a

- $s_a = a - b - c - d - a$,
  cost = 10

Optimal solution

- $s^* = a - b - d - c - a$,
  cost = 8
- Accuracy ration:
  $r(s_a) = \frac{10}{8} = 1,25$



Changing the weight of (a, d) to w:

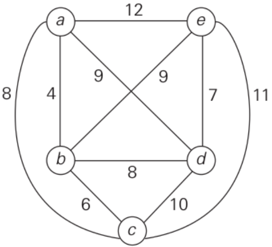
- $r(s_a) = \frac{4+w}{8}$

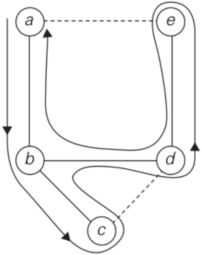
[7] Source: A. Levitin. Introduction to the Design and Analysis of Algorithms. 2011.

TSP: MST-based algorithm[8]

In general terms:

1. Construct an MST
2. Perform a walk around the MST
   (recording the order of nodes visited)
3. Eliminate all repeated occurrences except the initial node
   (making shortcuts in a walk)





[8] Source: A. Levitin. Introduction to the Design and Analysis of Algorithms. 2011.

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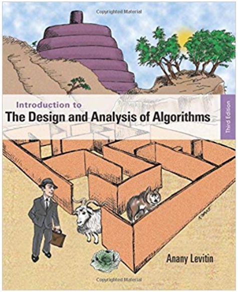

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