International Finance

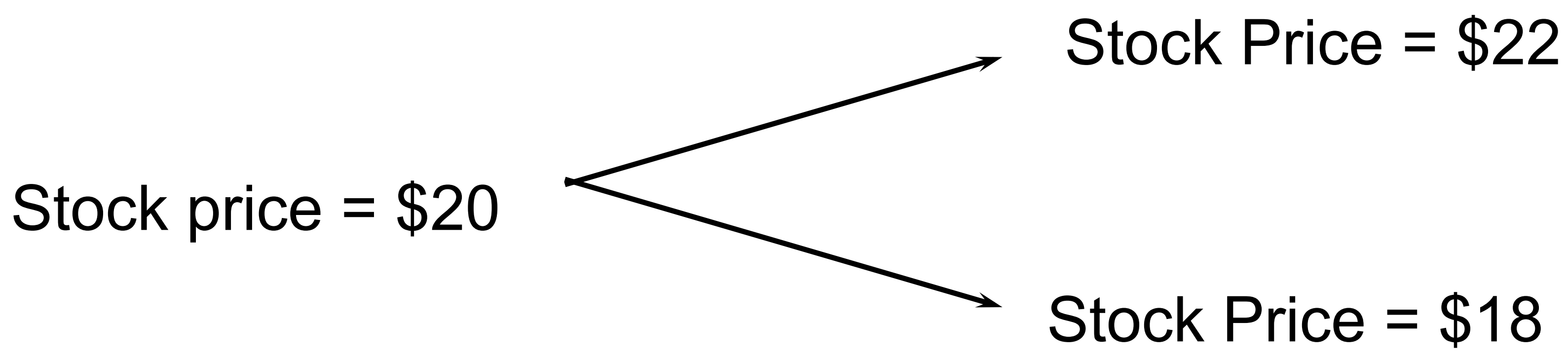
José Vicente Romero Ch.
Universidad del Rosario

Introduction to Binomial Trees

Chapter 12

A Simple Binomial Model

- A stock price is currently $20
- In three months it will be either $22 or $18



A Call Option

A 3-month call option on the stock has a strike price of 21.

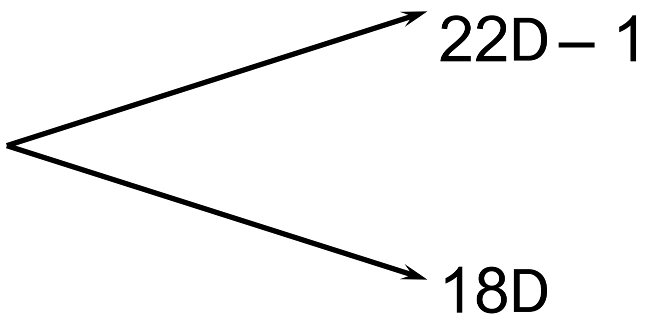
Stock price = $20
Option Price=?

Stock Price = $22
Option Price = $1

Stock Price = $18
Option Price = $0

Setting Up a Riskless Portfolio

- Consider the Portfolio: long D shares
  short 1 call option



- Portfolio is riskless when 22D – 1 = 18D or D = 0.25

Valuing the Portfolio
(Risk-Free Rate is 12%)

- The riskless portfolio is:

  long 0.25 shares
  short 1 call option

- The value of the portfolio in 3 months is

$$22 \times 0.25 - 1 = 4.50$$

- The value of the portfolio today is

$$4.5e^{-0.12 \times 0.25} = 4.3670$$

Valuing the Option

- The portfolio that is

  long 0.25 shares
  short 1 option

  is worth 4.367
- The value of the shares is

  5.000 (= 0.25 x 20)
- The value of the option is therefore

  0.633 (= 5.000 – 4.367)

Generalization

A derivative lasts for time T and is dependent on a stock

$$
\begin{tikzcd}
 & \begin{matrix} Su \\ f_u \end{matrix} \\
\begin{matrix} S \\ f \end{matrix} \arrow[ur] \arrow[dr] & \\
 & \begin{matrix} Sd \\ f_d \end{matrix}
\end{tikzcd}
$$

Generalization
(continued)

- Consider the portfolio that is long D shares and short 1 derivative

$$Su\mathrm{D} - f_u$$

$$Sd\mathrm{D} - f_d$$

- The portfolio is riskless when $Su\mathrm{D} - f_u = Sd\,\mathrm{D} - f_d$ or

$$\Delta = \frac{f_u - f_d}{Su - Sd}$$

Generalization
(continued)

- Value of the portfolio at time T is

$$Su\,\mathrm{D} - f_u$$

- Value of the portfolio today is

$$(Su\,\mathrm{D} - f_u)e^{-rT}$$

- Another expression for the portfolio value today (cost of setting up the portfolio) is

$$S\,\mathrm{D} - f$$

- Hence

$$f = S\,\mathrm{D} - (Su\,\mathrm{D} - f_u)e^{-rT}$$

Generalization
(continued)

- Substituting for D we obtain

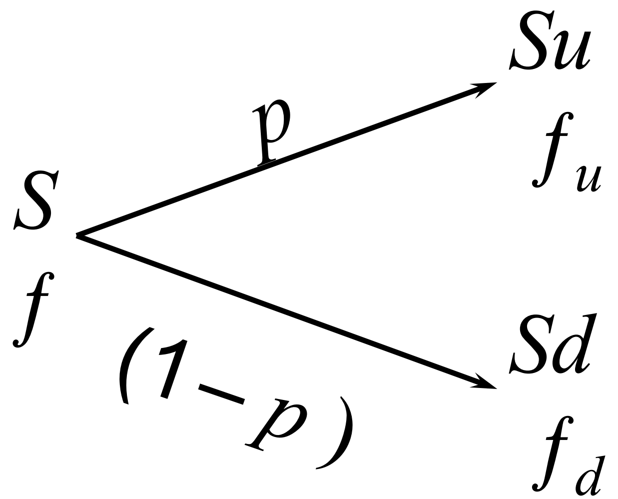
$$f = [\, p f_u + (1 - p) f_d \,] e^{-rT}$$

where

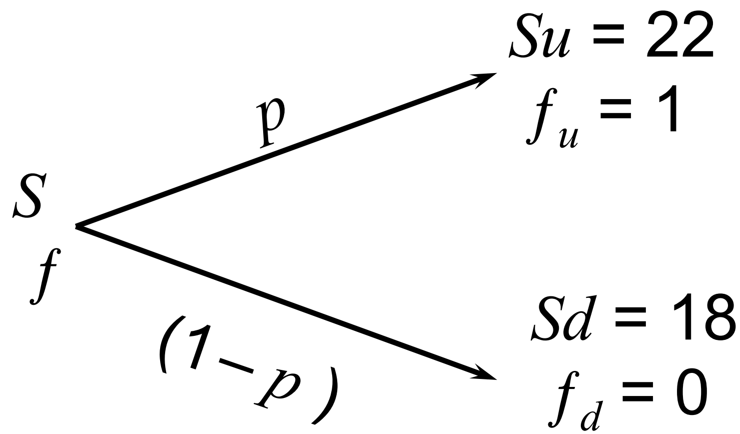
$$p = \frac{e^{rT} - d}{u - d}$$

Risk-Neutral Valuation

- $f = [\, p f_u + (1 - p) f_d \,] e^{-rT}$
- The variables p and $(1 - p)$ can be interpreted as the risk-neutral probabilities of up and down movements
- The value of a derivative is its expected payoff in a risk-neutral world discounted at the risk-free rate



Original Example Revisited



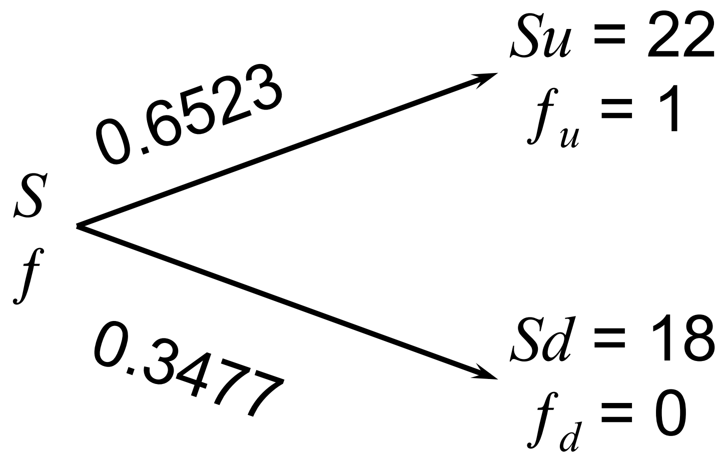
- Since p is a risk-neutral probability

$20e^{0.12 \times 0.25} = 22p + 18(1-p)$; $p = 0.6523$

- Alternatively, we can use the formula

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12\times0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

Valuing the Option Using Risk-Neutral Valuation

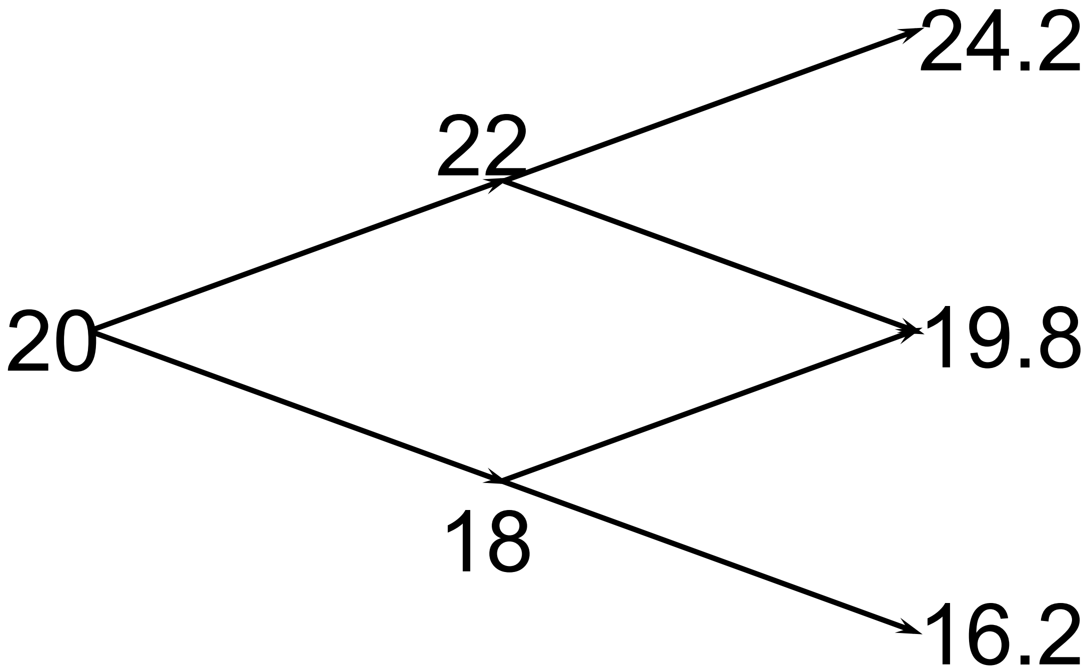


The value of the option is

$e^{-0.12x0.25}$ [0.6523x1 + 0.3477x0]

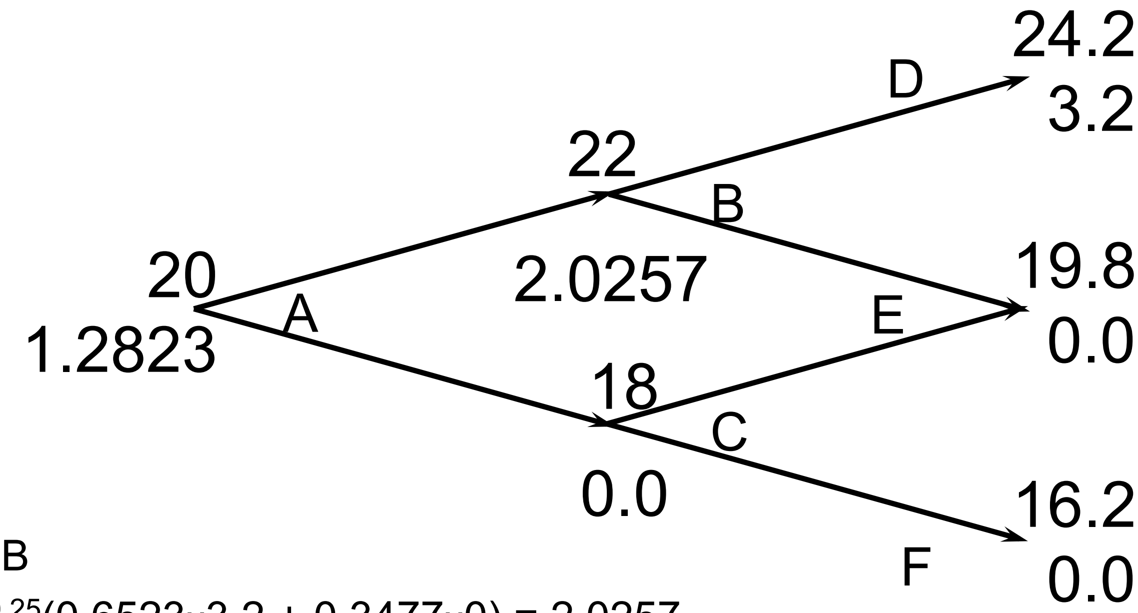
= 0.633

A Two-Step Example



- Each time step is 3 months
- K=21, r =12%

Valuing a Call Option



- Value at node B

  $= e^{-0.12 \times 0.25}(0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257$

- Value at node A

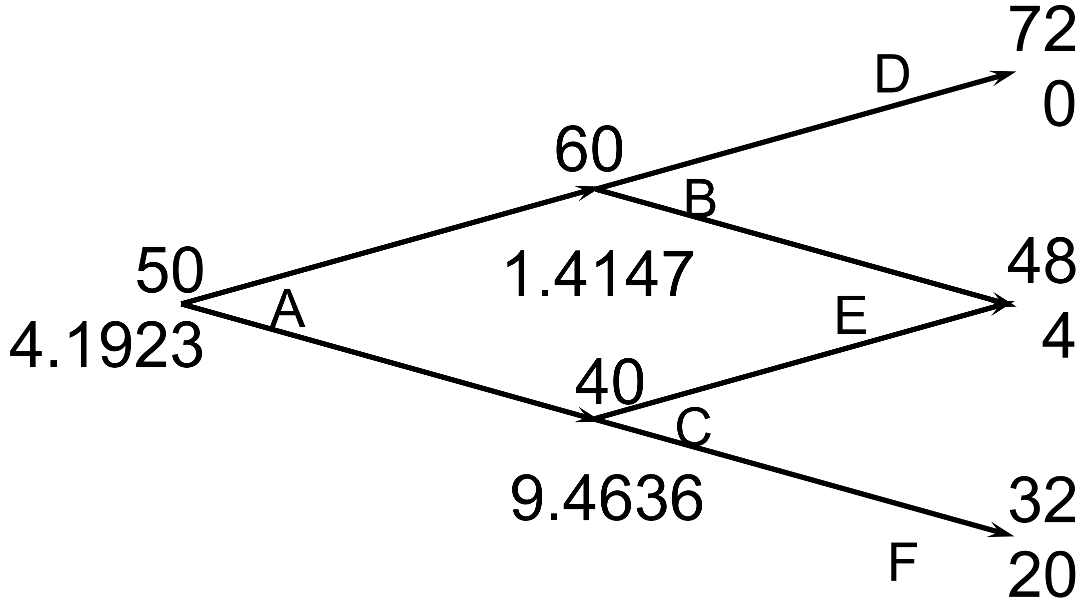
  $= e^{-0.12 \times 0.25}(0.6523 \times 2.0257 + 0.3477 \times 0)$

  $= 1.2823$

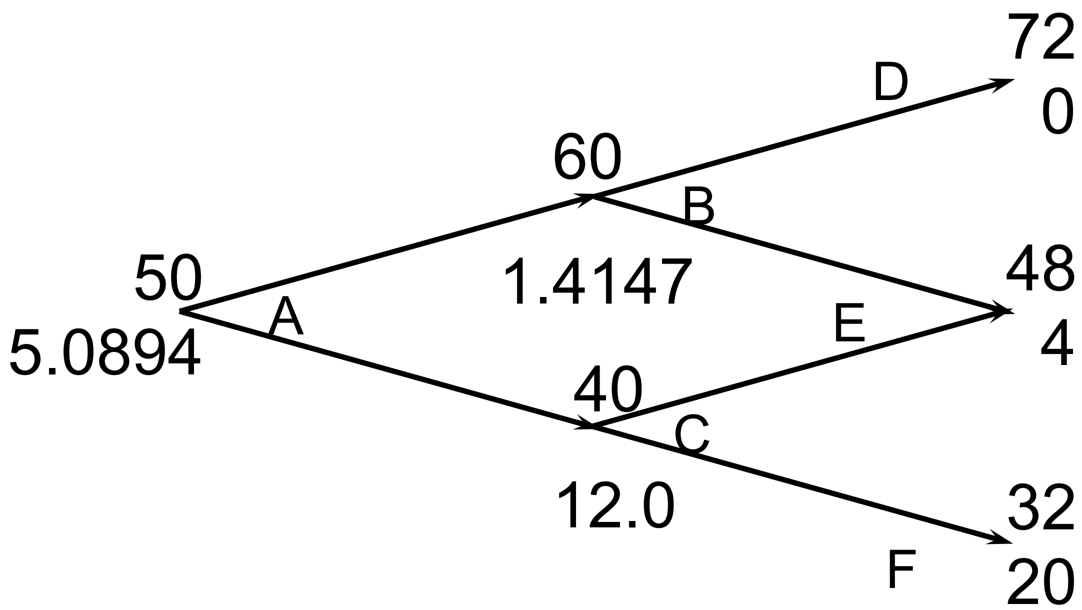
A Put Option Example; K=52

K = 52, Δt = 1yr

r = 5%



What Happens When an Option is American (Figure 12.8, page 278)



Delta

- Delta (D) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock
- The value of D varies from node to node

Choosing *u* and *d*

One way of matching the volatility is to set

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

where s is the volatility and Dt is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein

The Probability of an Up Move

$$p = \frac{a-d}{u-d}$$

$a = e^{r\Delta t}$ for a nondividend paying stock

$a = e^{(r-q)\Delta t}$ for a stock index where q is the dividend yield on the index

$a = e^{(r-r_f)\Delta t}$ for a currency where r_f is the foreign risk-free rate

$a = 1$ for a futures contract