NTNU – Trondheim
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Exercise 11
TTK4130 Modeling and Simulation

Problem 1 (Robotic manipulator)

We wish to model a robotic manipulator with the configuration shown in Figure 1.

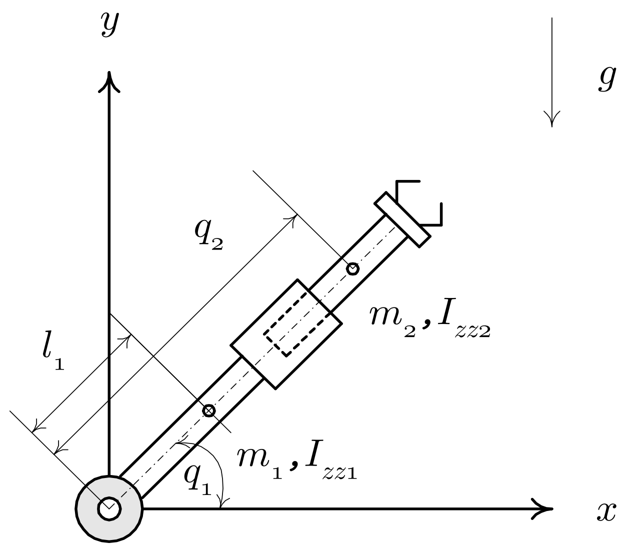

Figure 1: Manipulator

The manipulator has two degrees of freedom (that is, two generalized coordinates). We will use Lagrange's equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i, \qquad i = 1, 2$$

to set up the equations of motion for the manipulator, where

$$\mathcal{L} = T - U = \text{kinetic energy} - \text{potential energy} \tag{1}$$

and q_1 and q_2 are the generalized coordinates (see Figure 1). The axis x and y can be assumed fixed, that is, axes in an inertial system.

We will disregard mass and inertia of the motors in this problem. The moment of inertia of the first arm is denoted I_{zz1}, while the moment of inertia of the second arm is I_{zz2} (each referenced to the center of mass of the respective arm). The dots on the figure marks the centers of mass for each arm. The arrow marked g illustrates the direction of gravity.

(a) Find the total kinetic energy, T, for the manipulator, and show that it can be written on the form $T = \frac{1}{2}\dot{\mathbf{q}}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ where $\mathbf{q} = \begin{pmatrix} q_1 & q_2 \end{pmatrix}^{\mathsf{T}}$ and

$$\mathbf{M}(\mathbf{q}) = \begin{pmatrix} m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 q_2^2 & 0 \\ 0 & m_2 \end{pmatrix}.$$

Solution: The expression for kinetic energy for a rigid body (each arm) can be written

$$\frac{1}{2} m \vec{v}_c \cdot \vec{v}_c + \frac{1}{2} \vec{w}_{ib} \cdot \vec{M}_{b/c} \cdot \vec{w}_{ib}.$$

We are only interested in motion in the plane, so we disregard (assume zero) velocity in the z-direction, and angular velocity not about the z-axis.

The velocity of center of mass (superscript i denotes the base (inertial) system) and angular velocity (about z-axis) for each body becomes:

- Arm 1:

$$\mathbf{r}_{c1}^i = \begin{pmatrix} l_1 \cos q_1 \\ l_1 \sin q_1 \end{pmatrix}, \qquad \mathbf{v}_{c1}^i = \begin{pmatrix} -l_1 \sin q_1 \dot{q}_1 \\ l_1 \cos q_1 \dot{q}_1 \end{pmatrix}, \qquad w_{z1} = \dot{q}_1$$

$$\vec{v}_{c1} \cdot \vec{v}_{c1} = \left(\mathbf{v}_{c1}^i\right)^\mathsf{T} \mathbf{v}_{c1}^i = l_1^2 \sin^2 q_1 \dot{q}_1^2 + l_1^2 \cos^2 q_1 \dot{q}_1^2 = l_1^2 \dot{q}_1^2$$

- Arm 2:

$$\mathbf{r}_{c2}^i = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}, \qquad \mathbf{v}_{c2}^i = \begin{pmatrix} \dot{q}_2 \cos q_1 - q_2 \sin q_1 \dot{q}_1 \\ \dot{q}_2 \sin q_1 + q_2 \cos q_1 \dot{q}_1 \end{pmatrix}, \qquad w_{z2} = \dot{q}_1$$

$$\begin{aligned} \vec{v}_{c2} \cdot \vec{v}_{c2} &= \left(\mathbf{v}_{c2}^i\right)^\mathsf{T} \mathbf{v}_{c2}^i \\ &= \dot{q}_2^2 \cos^2 q_1 - \dot{q}_2 \cos q_1 q_2 \sin q_1 \dot{q}_1 + q_2^2 \sin^2 q_1 \dot{q}_1^2 \\ &\quad + \dot{q}_2^2 \sin^2 q_1 + \dot{q}_2 \sin q_1 q_2 \cos q_1 \dot{q}_1 + q_2^2 \cos^2 q_1 \dot{q}_1^2 \\ &= \dot{q}_2^2 + q_2^2 \dot{q}_1^2 \end{aligned}$$

The kinetic energy for each body becomes

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 l_1^2 \dot{q}_1^2 + \frac{1}{2} I_{zz1} \dot{q}_1^2, \\ T_2 &= \frac{1}{2} m_2 \left(q_2^2 \dot{q}_1^2 + \dot{q}_2^2\right) + \frac{1}{2} I_{zz2} \dot{q}_1^2, \end{aligned}$$

and the total kinetic energy for the system is

$$T = T_1 + T_2.$$

Remark: For a simple system like the first body, one can "see" directly that the velocity is $|\vec{v}_{c1}| = l_1 \dot{q}_1$. This may also be possible for the second body, but as setups become more complicated, it is easy to make mistakes when using the "see"-method.

(b) Find the potential energy, U, for the manipulator.

Solution: The potential energy for each arm is

$$\begin{aligned} U_1 &= m_1 g l_1 \sin q_1, \\ U_2 &= m_2 g q_2 \sin q_1. \end{aligned}$$

The total potential energy is

$$U = U_1 + U_2.$$

(c) Derive the equations of motion for the manipulator by use of Lagrange's equation.

Solution: The Lagrangian of the manipulator is

$$\begin{aligned}
\mathcal{L} &= T - U \\
&= T_1 + T_2 - U_1 - U_2 \\
&= \frac{1}{2} m_1 l_1^2 \dot{q}_1^2 + \frac{1}{2} I_{zz1} \dot{q}_1^2 + \frac{1}{2} m_2 \left(q_2^2 \dot{q}_1^2 + \dot{q}_2^2 \right) + \frac{1}{2} I_{zz2} \dot{q}_1^2 - (m_1 l_1 + m_2 q_2) g \sin q_1
\end{aligned}$$

We use Lagrange's equation,

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i, \qquad i = 1, 2$$

where

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{q}_1} &= m_1 l_1^2 \dot{q}_1 + I_{zz1} \dot{q}_1 + m_2 q_2^2 \dot{q}_1 + I_{zz2} \dot{q}_1 \\
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_1} &= m_1 l_1^2 \ddot{q}_1 + I_{zz1} \ddot{q}_1 + m_2 q_2^2 \ddot{q}_1 + I_{zz2} \ddot{q}_1 + 2 m_2 q_2 \dot{q}_2 \dot{q}_1 \\
\frac{\partial \mathcal{L}}{\partial q_1} &= - (m_1 l_1 + m_2 q_2)\, g \cos q_1 \\
\frac{\partial \mathcal{L}}{\partial \dot{q}_2} &= m_2 \dot{q}_2 \\
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_2} &= m_2 \ddot{q}_2 \\
\frac{\partial \mathcal{L}}{\partial q_2} &= m_2 q_2 \dot{q}_1^2 - m_2 g \sin q_1
\end{aligned}$$

which gives these equations of motion:

$$\begin{aligned}
\left(m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 q_2^2 \right) \ddot{q}_1 + 2 m_2 q_2 \dot{q}_2 \dot{q}_1 + (m_1 l_1 + m_2 q_2)\, g \cos q_1 &= \tau_1 \\
m_2 \ddot{q}_2 - m_2 q_2 \dot{q}_1^2 + m_2 g \sin q_1 &= \tau_2
\end{aligned}$$

Here, τ_1 is the generalized force corresponding to q_1, that is, a motor torque giving rotation, and τ_2 is the generalized force corresponding to q_2, a motor force giving translational motion of arm 2.

(d) In this problem you should show that the equations of motion in (c) can be written

$$\mathbf{M}(\mathbf{q})\, \ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\, \dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}. \tag{2}$$

Explain how several choices are possible for $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$. Show that when you use the Christoffel symbols (cf. eq. (8.57)–(8.58) in the book), then

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{pmatrix} m_2 q_2 \dot{q}_2 & m_2 q_2 \dot{q}_1 \\ -m_2 q_2 \dot{q}_1 & 0 \end{pmatrix}.$$

What is the vector $\mathbf{g}(\mathbf{q})$?

Solution: We first find the "mass matrix" $\mathbf{M}(\mathbf{q})$ and $\mathbf{g}(\mathbf{q})$,

$$\mathbf{M}(\mathbf{q}) = \begin{pmatrix} m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 q_2^2 & 0 \\ 0 & m_2 \end{pmatrix}$$
$$\mathbf{g}(\mathbf{q}) = \begin{pmatrix} (m_1 l_1 + m_2 q_2)\, g \cos q_1 \\ m_2 g \sin q_1 \end{pmatrix}$$

We see that we have term containing multiplications of derivatives of q_i, that is, $\dot{q}_1 \dot{q}_2$. This term can be placed two places in $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$, depending on if we extract $\dot{q}_1$ or $\dot{q}_2$.
We find the Christoffel symbols c_{ijk} by using the equation

$$c_{ijk} = \frac{1}{2}\left(\frac{\partial m_{kj}}{q_i} + \frac{\partial m_{ik}}{q_j} - \frac{\partial m_{ij}}{q_k}\right)$$

which gives

$$c_{111} = \frac{1}{2}\left(\frac{\partial m_{11}}{\partial q_1} + \frac{\partial m_{11}}{\partial q_1} - \frac{\partial m_{11}}{\partial q_1}\right) = 0$$
$$c_{112} = \frac{1}{2}\left(\frac{\partial m_{21}}{\partial q_1} + \frac{\partial m_{12}}{\partial q_1} - \frac{\partial m_{11}}{\partial q_2}\right) = -m_2 q_2$$
$$c_{122} = \frac{1}{2}\left(\frac{\partial m_{22}}{\partial q_1} + \frac{\partial m_{12}}{\partial q_2} - \frac{\partial m_{12}}{\partial q_2}\right) = 0$$
$$c_{211} = \frac{1}{2}\left(\frac{\partial m_{11}}{\partial q_2} + \frac{\partial m_{21}}{\partial q_1} - \frac{\partial m_{21}}{\partial q_1}\right) = m_2 q_2$$
$$c_{222} = \frac{1}{2}\left(\frac{\partial m_{12}}{\partial q_2} + \frac{\partial m_{21}}{\partial q_2} - \frac{\partial m_{22}}{\partial q_1}\right) = 0$$
$$c_{222} = \frac{1}{2}\left(\frac{\partial m_{22}}{\partial q_2} + \frac{\partial m_{22}}{\partial q_2} - \frac{\partial m_{22}}{\partial q_2}\right) = 0$$

Note that, for a fixed k, we have where $c_{ijk} = c_{jik}$. Consequently, $c_{211} = c_{121}$ and $c_{212} = c_{122}$.
We can now find the elements in the $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ matrix by

$$c_{ij} = \sum_{k=1}^{n=2} c_{ijk} \dot{q}_k$$

which gives

$$c_{11} = c_{111}\dot{q}_1 + c_{211}\dot{q}_2 = m_2 q_2 \dot{q}_2$$
$$c_{12} = c_{121}\dot{q}_1 + c_{212}\dot{q}_2 = m_2 q_2 \dot{q}_1$$
$$c_{21} = c_{112}\dot{q}_1 + c_{212}\dot{q}_2 = -m_2 q_2 \dot{q}_1$$
$$c_{22} = c_{122}\dot{q}_1 + c_{222}\dot{q}_2 = 0$$

We then get

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{pmatrix} m_2 q_2 \dot{q}_2 & m_2 q_2 \dot{q}_1 \\ -m_2 q_2 \dot{q}_1 & 0 \end{pmatrix}$$

(e) What matrix properties do the matrices $\mathbf{M}(\mathbf{q})$ and $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ possess?

Solution: The mass matrix $\mathbf{M}(\mathbf{q})$ is

$$\begin{aligned} &\text{symmetric :} \quad \mathbf{M} = \mathbf{M}^\top \\ &\text{positive definite :} \quad \mathbf{x}^\top \mathbf{M}(\mathbf{q})\mathbf{x} > 0 \qquad \forall \mathbf{x} \neq 0 \end{aligned}$$

The matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ has no specific property, but as we will see next, we can choose it so $\dot{\mathbf{M}}(\mathbf{q}) - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric.

(f) Show (using the matrices developed in this problem) that the matrix $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric when $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ has been defined by use of the Christoffel symbols.

Solution:

$$\dot{\mathbf{M}}(\mathbf{q}) = \begin{pmatrix} 2m_2 q_2 \dot{q}_2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{pmatrix} 2m_2 q_2 \dot{q}_2 & 2m_2 q_2 \dot{q}_1 \\ -2m_2 q_2 \dot{q}_1 & 0 \end{pmatrix}$$

which gives

$$\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{pmatrix} 2m_2 q_2 \dot{q}_2 & 2m_2 q_2 \dot{q}_1 \\ -2m_2 q_2 \dot{q}_1 & 0 \end{pmatrix} - \begin{pmatrix} 2m_2 q_2 \dot{q}_2 & 2m_2 q_2 \dot{q}_1 \\ -2m_2 q_2 \dot{q}_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2m_2 q_2 \dot{q}_1 \\ 2m_2 q_2 \dot{q}_1 & 0 \end{pmatrix}$$

We see that $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew symmetric.

(g) Show that the derivative of the energy function $E(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) + U(\mathbf{q})$ is

$$\dot{E}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^\top \boldsymbol{\tau}.$$

Hint: Use $T = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ and (2), do not insert the detailed model. Use that $\frac{\partial U}{\partial \mathbf{q}} = \mathbf{g}^\top(\mathbf{q})$. What can we say about passivity of the manipulator?

Solution:

$$\begin{aligned} \dot{E}(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \frac{\partial U}{\partial \mathbf{q}}\dot{\mathbf{q}} \\ &= \dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^\top \dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} + \frac{\partial U}{\partial \mathbf{q}}\dot{\mathbf{q}} \end{aligned}$$

Inserting (2) and $\frac{\partial U}{\partial \mathbf{q}} = \mathbf{g}^\top(\mathbf{q})$ gives

$$\begin{aligned} \dot{E}(\mathbf{q}, \dot{\mathbf{q}}) &= \dot{\mathbf{q}}^\top \left(\boldsymbol{\tau} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} - \mathbf{g}(\mathbf{q})\right) + \frac{1}{2}\dot{\mathbf{q}}^\top \dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} + \mathbf{g}^\top(\mathbf{q})\dot{\mathbf{q}} \\ &= \dot{\mathbf{q}}^\top \boldsymbol{\tau} + \frac{1}{2}\dot{\mathbf{q}}^\top \left(\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\right)\dot{\mathbf{q}} \\ &= \dot{\mathbf{q}}^\top \boldsymbol{\tau} \end{aligned}$$

where we used that $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric.
This almost implies that the manipulator is passive with applied generalized forces $\boldsymbol{\tau}$ (torque on arm 1 and force on arm 2) as input and generalized velocities $\dot{\mathbf{q}}$ as output (angular velocity of arm 1 and velocity of arm 2).
The only glitch is that the energy function that proves passivity should be positive, while the potential energy seems to be unbounded below. However, if we know that $U > U_{\text{min}}$ for some constant U_{min}, we can use $V = E - U_{\text{min}} = T + U - U_{\text{min}}$ as energy function, which fulfill

$$\dot{V}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^\top \boldsymbol{\tau}.$$

The potential energy is bounded in this problem if $0 < q_2 < q_{2,\text{max}}$.

Problem 2 (Tank with liquid)

A tank with area A is filled with an incompressible liquid with (constant) density ρ and level h. The liquid volume is then $V = Ah$ and the mass of the liquid in the tank is $m = V\rho$. Liquid enters the tank through a pipe with mass flow $w_i = \rho A_i v_i$, where A_i is the pipe cross section, and v_i is the velocity (constant over the cross section). Liquid leaves the tank through a second pipe with mass flow $w_u = \rho A_u v_u$ where A_u is the cross section of the pipe and v_u is the velocity.

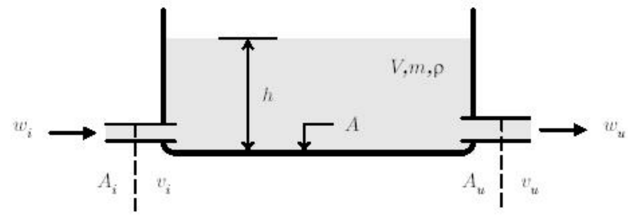

Figure 2: Tank with liquid

Use a mass balance for the tank to set up a differential equation for the level h.

Solution: The principle of mass conservation is

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint_V \rho dV = 0,$$

that is, the mass is constant in a material volume. Using eq. (10.90) in the book, the liquid mass balance for a fixed volume V_f (the total volume of the tank) becomes (eq. (11.8) in the book):

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_f}\rho \mathrm{d}V}_{\substack{\text{rate of change}\\ \text{of liquid mass}\\ \text{in } V_f}} = - \underbrace{\iint_{\partial V_f}\rho \mathbf{v}^{\mathsf{T}}\mathbf{n}\mathrm{d}A}_{\substack{\text{net increase of mass}\\ \text{by flow in and out}\\ \text{of } V_f}}$$

(Alternatively, one could assume the liquid volume as "control volume" and use eq. (11.10).)

We have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_f}\rho \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\rho V = \rho A\dot{h},$$

(that is, the total mass of liquid in the tank), and

$$-\iint_{\partial V_f}\rho \mathbf{v}^{\mathsf{T}}\mathbf{n}\mathrm{d}A = \rho v_i A_i - \rho v_u A_u$$

(flow in and out of the tank).

Inserted into the mass balance above, this gives

$$\begin{aligned} A\rho\dot{h} &= \rho v_i A_i - \rho v_u A_u \\ \dot{h} &= \frac{A_i}{A}v_i - \frac{A_u}{A}v_u. \end{aligned}$$

Problem 3 (Stirred tank (Exam 2015))

In this problem, we consider a stirred tank that cools an inlet stream, see Figure 3. The tank is cooled by a "jacket" that contains a fluid of (presumably) lower temperature than the tank. The inlet stream to the tank has density ρ, temperature T_1, and massflowrate w_1. The outflow from the tank is

$$w_2 = Cu\sqrt{h},$$

where C is a constant and u is the valve opening. The liquid level is h. You can assume that the outflow is controlled such that the level does not exceed the height of the jacket.

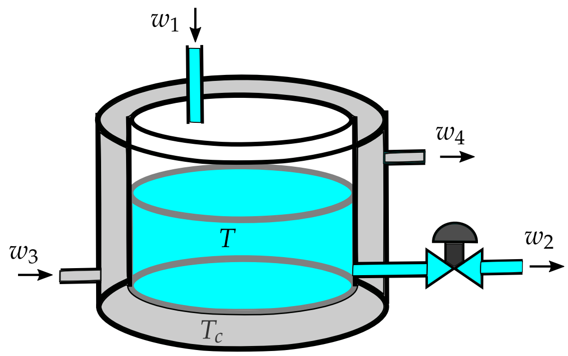

Figure 3: Tank with cooling jacket.

The inlet and outlet massflowrates for the jacket is matched such that the jacket is always filled with fluid ($w_3 = w_4$). The cooling fluid has density ρ_c, and the inlet stream to the jacket has temperature T_3. Since the tank is stirred, we assume homogenous conditions, that is, the temperature T is the same everywhere in the tank. Similarly, we assume that the temperature T_c is the same everywhere in the jacket.

The cross-sectional area of the tank is A. The volume of the jacket is V_c.

The heat transfer from the tank to the jacket is

$$Q = Gh(T - T_c),$$

where h is the height of the liquid in the tank, and G a (constant) heat transfer coefficient. We assume that the jacket (and tank) is well insulated from the surroundings, meaning there are no other heat losses.

We assume both fluids incompressible, meaning that specific internal energy and enthalpy both can be assumed equal and proportional to temperature, with constant of proportionality being c_p and c_{pc} for the two fluids, respectively.

(a) Set up differential equations for the temperatures T in the tank and T_c in the jacket, and the level h in the tank.

Solution: We must first set up mass balances. For the tank, the mass balance

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho A h) = w_1 - w_2$$

gives

$$\boxed{\dot{h} = \frac{1}{\rho A}\left(w_1 - Cu\sqrt{h}\right)}$$

For later use, the mass balance for the jacket:

$$\frac{\mathrm{d}}{\mathrm{d}t}m = w_3 - w_4 = 0.$$

(*The volumn could be balanced instead of the mass. However, in that case it has to be explicitly mentioned that the density is constant and, therefore, it is possible to balance volumn.*)

Then, the energy balance for the tank gives

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho c_p T A h\right) &= w_1 c_p T_1 - w_2 c_p T - Gh(T - T_c) \\
\rho c_p A h \frac{\mathrm{d}}{\mathrm{d}t} T + \rho c_p A T \frac{\mathrm{d}}{\mathrm{d}t} h &= w_1 c_p T_1 - w_2 c_p T - Gh(T - T_c) \\
\rho c_p A h \frac{\mathrm{d}}{\mathrm{d}t} T + \rho c_p A T \frac{1}{\rho A}\left(w_1 - w_2\right) &= w_1 c_p T_1 - w_2 c_p T - Gh(T - T_c) \\
\rho c_p A h \frac{\mathrm{d}}{\mathrm{d}t} T &= w_1 c_p (T_1 - T) - Gh(T - T_c)
\end{aligned}
$$

$$
\boxed{\frac{\mathrm{d}}{\mathrm{d}t} T = \frac{w_1}{\rho A h}(T_1 - T) - \frac{G}{\rho c_p A}(T - T_c)}
$$

For the jacket:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_c c_{p,c} T_c V_c\right) &= w_3 c_{p,c} T_3 - w_4 c_{p,c} T_c + Gh(T - T_c) \\
\rho_c c_{p,c} V_c \frac{\mathrm{d}}{\mathrm{d}t} T_c &= w_3 c_{p,c} T_3 - w_4 c_{p,c} T_c + Gh(T - T_c)
\end{aligned}
$$

Since $w_3 = w_4$:

$$
\boxed{\frac{\mathrm{d}}{\mathrm{d}t} T_c = \frac{w_3}{\rho_c V_c}(T_3 - T_c) + \frac{Gh}{\rho_c c_{p,c} V_c}(T - T_c)}
$$

Problem 4 (Compressor, momentum balance, Bernoulli's equation)

A compressor takes in air with pressure p_0 and velocity $v_0 = 0$ from the surroundings. The air flows through a duct into the compressor. For control, it would be beneficial to have a measurement of the mass flow into the compressor. However, this measurement is not available.

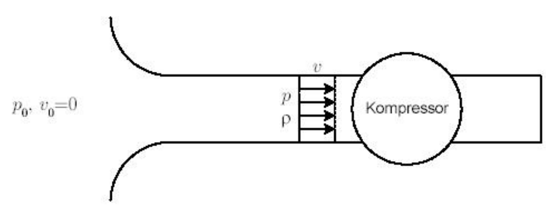

Figure 4: Compressor

Instead, there is a pressure measurement in the duct, giving a measurement p. How can the mass flow w and velocity v be found from this measurement? Assume that the density ρ in the duct is constant and known, there is no friction, and that the velocity is uniform over the cross-section where the pressure transmitter is located.

Hint: Use (the stationary) Bernoulli's equation.

Solution: Bernoulli's equation for frictionless, incompressible flow along a streamline relates pressure, velocity and elevation in two points:

$$\frac{p_1 - p_0}{\rho} + \frac{1}{2}\left(v_1^2 - v_0^2\right) + (z_1 - z_0)g = 0.$$

Choosing point 1 to be the location of the pressure transmitter ($p_1 = p$, $v_1 = v$ and $z_1 = 0$) and point 0 to be the duct inlet ($p_0 = p_0$, $v_0 = 0$ and $z_0 = 0$), we get

$$p = p_0 - \frac{1}{2}v^2\rho$$

that can be solved to

$$v = \sqrt{\frac{2\left(p_0 - p\right)}{\rho}}$$

(note that p must be smaller than p_0). This gives mass flow

$$w = A\rho v = A\sqrt{2\rho\left(p_0 - p\right)}$$

Problem 5 (Mixing, reactions (Exam 2010))

An incompressible liquid of substance C enters a perfectly mixed tank (a continuous stirred tank reactor, CSTR) with mass flow w_C and temperature T_C. In the tank, the substance reacts (e.g. due to the presence of a catalyst) to form the substance D with a rate JV, where J is the reaction rate per unit volume, and $V = Ah$ is the volume of the tank. The tank then consists of a mixture of C and D, which leaves the tank with mass flow w and temperature T. The mass of substance C in the tank is denoted m_C, and the mass of substance D is denoted m_D.

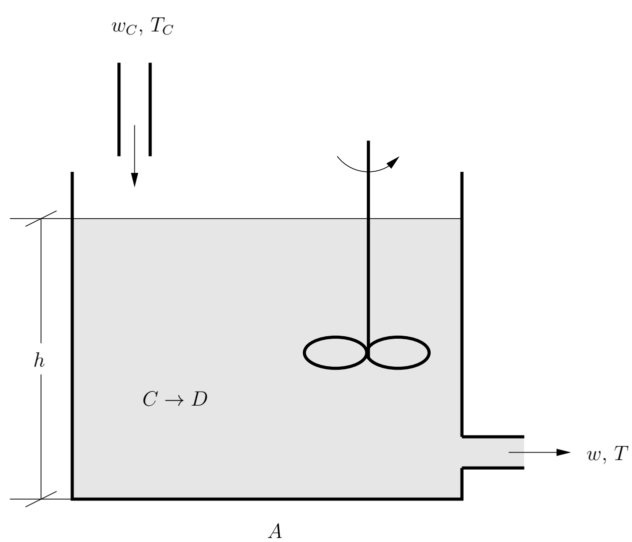

Figure 5: Tank reactor

(a) Set up a differential equation for the level of the tank. (Hint: Use the ordinary overall mass balance. Assume that the average density ρ is constant.)

Solution: The mass balance equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_f}\rho\mathrm{d}V = -\iint\limits_{\partial V_f}\rho\vec{v}\cdot\vec{n}\mathrm{d}A$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho A h) = w_C - w$$
$$\frac{\mathrm{d}}{\mathrm{d}t}h = \frac{w_C - w}{\rho A}$$

(b) In a material volume V_m, the following holds:

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint\limits_{V_m}\rho_C\mathrm{d}V = -\iiint\limits_{V_m}J\mathrm{d}V.$$

Use this together with the appropriate form of the transport theorem to explain that the mass balance for substance C on integral form in a fixed control volume V_f is

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_f}\rho_C\mathrm{d}V = -\iiint\limits_{V_f}J\mathrm{d}V - \iint\limits_{\partial V_f}\rho_C\vec{v}\cdot\vec{n}\mathrm{d}A.$$

(In this particular case, the natural control volume, the volume of liquid in the tank, is not fixed, but this can be ignored since $\rho_C\vec{v}_c\cdot\vec{n} = 0$ – expansion of the volume does not accumulate more of substance C.)

Solution: Take eq. (10.90) in the book, set $\phi = \rho_C$ and insert the first equation to obtain the result.

(c) Use this to write up the mass balance for the mass of substance C in the tank ($\frac{\mathrm{d}}{\mathrm{d}t}m_C = \ldots$). Assume here, and for the rest of the problem, that J is proportional to the density of substance C, $J = k\frac{m_C}{V}$, and that the outflow of substance C is proportional to the mass ratio of substance C to the total mass in the tank, and the total outflow, $w_{C,out} = \frac{m_C}{m_C+m_D}w$.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_C A h) = w_C - JV - w_{C,out}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}m_C = w_C - km_C - \frac{m_C}{m_C+m_D}w$$

(d) What is the mass balance equation on integral form for substance D (in a fixed volume)? Use this to write up the mass balance of substance D.

Solution: For substance D, we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint\limits_{V_m}\rho_D\mathrm{d}V = \iiint\limits_{V_m}J\mathrm{d}V.$$

Insertion into (10.90) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_f}\rho_D\mathrm{d}V = \iiint\limits_{V_f}J\mathrm{d}V - \iint\limits_{\partial V_f}\rho_D\vec{v}\cdot\vec{n}\mathrm{d}A.$$

Solving the integrals, give

$$\frac{\mathrm{d}}{\mathrm{d}t}m_D = JV - w_D = km_C - \frac{m_D}{m_C + m_D}w.$$

(e) Check that the solution in (c) and (d) agrees with the answer in (a).

Solution:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}m &= \frac{\mathrm{d}}{\mathrm{d}t}m_C + \frac{\mathrm{d}}{\mathrm{d}t}m_D \\ &= w_C - km_C - \frac{m_C}{m_C + m_D}w + km_C - \frac{m_D}{m_C + m_D}w \\ &= w_C - w.\end{aligned}$$

This agrees with the solution to (a).

The final question is optional:

(f) Set up a differential equation for the temperature in the tank. Assume that the heat generated by the reaction is proportional to J, with proportionality constant c. Disregard kinetic energy, potential energy and pressure work. Assume no 'heat flux' (the tank is well insulated). Assume the internal energy is $u = c_p T$.

Solution: The book does not treat energy balances with "internally generated" energy. We must therefore derive the energy balance on integral form for this (as we did for the mass balance above).

Under the assumptions made, (11.164) takes the form

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint_{V_m} \rho u \mathrm{d}V = \iiint_{V_m} cJ\mathrm{d}V$$

($e = u$, pressure work and heat flux ignored, but heat from reaction added.) Insertion into (11.169) (for a fixed volume) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_f} \rho u \mathrm{d}V = \iiint_{V_f} cJ\mathrm{d}V - \iint_{\partial V_f} \rho u \vec{v}\cdot\vec{n}\mathrm{d}A.$$

Inserting $u = c_p T$ and resolving the integrals, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho c_p T V\right) &= cJV + w_C c_p T_C - w c_p T \\ \rho c_p V \frac{\mathrm{d}}{\mathrm{d}t}T + \rho c_p T A \frac{\mathrm{d}}{\mathrm{d}t}h &= cJV + w_C c_p T_C - w c_p T\end{aligned}$$

Insertion of the result in (a), and using $JV = km_C$,

$$\begin{aligned}\rho c_p A h \frac{\mathrm{d}}{\mathrm{d}t}T + c_p T(w_C - w) &= ckm_C + w_C c_p T_C - w c_p T \\ \frac{\mathrm{d}}{\mathrm{d}t}T &= \frac{ckm_C + c_p w_C (T_C - T)}{\rho c_p A h}\end{aligned}$$

Correct result (without derivation of the energy balance) will give full score.