Lecture 20: Rigid body dynamics, summing up

- Brief recap: Newton-Euler equations of motion
- Brief recap: Lagrange’s equation of motion

- Pendulum example using both Newton-Euler and Lagrange

- Old exam(s) (using Lagrange)

Lagrange vs Newton-Euler

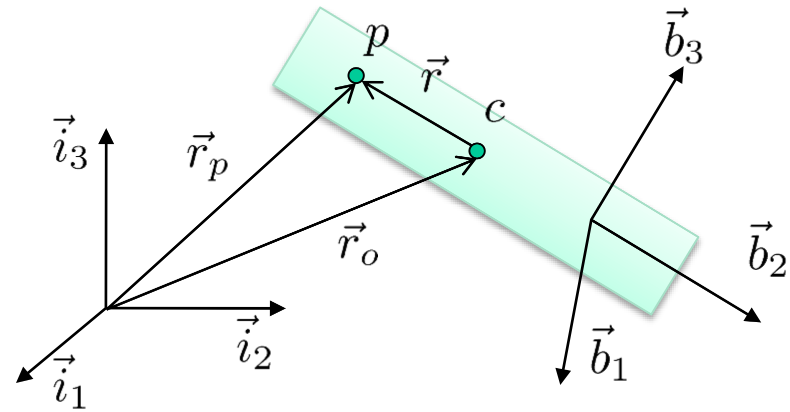
Newton-Euler

- Vectors
- Forces and moments
- Does not eliminate forces of constraints:
  - Obtains solutions for all forces and kinematic variables
  - ”Inefficient” (large DAE models)
- More general
  - Large systems can be handled (but for some configurations tricks are needed)
  - Used in advanced modeling software

Lagrange

- Algebraic
- Energy
- Eliminates forces of constraints
  - Solutions only for generalized coordinates (and forces)
  - ”Efficient” (smaller ODE models)
- Less general
  - Need independent generalized coordinates
  - Difficult to automate for large/complex problems

Newton-Euler EoM for rigid bodies



- Velocities and accelerations (Ch. 6.12)

$$\vec{v}_c := \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r}_c, \quad \vec{v}_p := \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r}_p$$

$$\vec{a}_c := \frac{{}^i\mathrm{d}^2}{\mathrm{d}t^2}\vec{r}_c, \quad \vec{a}_p := \frac{{}^i\mathrm{d}^2}{\mathrm{d}t^2}\vec{r}_p$$

$$\begin{aligned}\vec{v}_p &= \vec{v}_c + \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r} \\ &= \vec{v}_c + \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{r} + \vec{\omega}_{ib} \times \vec{r} \\ &= \vec{v}_c + \vec{\omega}_{ib} \times \vec{r}, \quad \vec{r} \text{ fixed.}\end{aligned}$$

$$\boxed{\frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{u} = \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{u} + \vec{\omega}_{ib} \times \vec{u}}$$

$$\boxed{\vec{a}_p = \vec{a}_c + \vec{\alpha}_{ib} \times \vec{r} + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r}), \quad \vec{r} \text{ fixed.}}$$

- Newton-Euler equations of motion (Ch. 7.3)

$$\boxed{\begin{aligned}\vec{F}_{bc} &= m\vec{a}_c \\ \vec{T}_{bc} &= \vec{M}_{b/c} \cdot \vec{\alpha}_{ib} + \vec{\omega}_{ib} \times \left(\vec{M}_{b/c} \cdot \vec{\omega}_{ib}\right)\end{aligned}}$$

Newton-Euler equations of motion

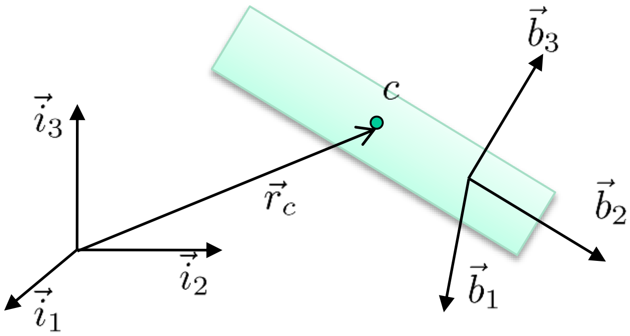
- Newton's law (for particle *k*)

$$m_k \vec{a}_k = \vec{F}^{(r)}$$

- Newton-Euler EoM for rigid bodies:
  - Integrate Newton's law over body, define center of mass
  - Define torque/moment and angular momentum to handle forces that give rotation about center of mass
  - Define inertia dyadic/matrix

$$\vec{F}_{bc} = m\vec{a}_c$$

$$\vec{T}_{bc} = \vec{M}_{b/c} \cdot \vec{\alpha}_{ib} + \vec{\omega}_{ib} \times \left(\vec{M}_{b/c} \cdot \vec{\omega}_{ib} \right)$$



(Here: Referenced to center of mass)

- Implemented in e.g. Dymola (Modelica.Multibody library)

Lagrange equations of motion

Generalized coordinates

- Find n generalized coordinates that parametrize "degrees of freedom" (allowed motion).
  - That is, all positions are function of generalized coordinates

$$\vec{r}_k = \vec{r}_k(\mathbf{q}) \qquad \mathbf{q} = \begin{pmatrix} q_1 & q_2 & \dots & q_n \end{pmatrix}^{\mathsf{T}}$$

- Differentiate to find velocity

$$\vec{v}_k(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{r}_k(\mathbf{q}) = \sum_{i=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i}\dot{q}_i$$

  - For rigid bodies: velocity of center(s) of mass, and also angular velocity $\vec{\omega}_{ib}(\mathbf{q}, \dot{\mathbf{q}})$

- Find the generalized (actuator) forces τ_i associated with q_i
  - If q_i angle, then τ_i torque
  - If q_i displacement, then τ_i force

$$\tau_i = \sum_{i=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i} \cdot \vec{F}_k$$

- On coordinate form:

$$k = 1, \dots, N \text{ particles: } \mathbf{r}_k^i(\mathbf{q}), \quad \mathbf{v}_k^i(\mathbf{q}, \dot{\mathbf{q}})$$

$$k = 1, \dots, N \text{ rigid bodies: } \mathbf{r}_{ck}^i(\mathbf{q}), \quad \mathbf{v}_{ck}^b(\mathbf{q}, \dot{\mathbf{q}}), \quad \boldsymbol{\omega}_{ik}^b(\mathbf{q}, \dot{\mathbf{q}}), \quad \mathbf{M}_{k/c}^b$$

Lagrange equations of motion

Kinetic and potential energy

- Find kinetic energy:
  - N particles: $T = \sum_{k=1}^{N} \frac{1}{2} m_k \vec{v}_k \cdot \vec{v}_k$
  - Each rigid body (p. 273):

$$T = \int_b \frac{1}{2}\vec{v}_p \cdot \vec{v}_p \mathrm{d}m = \frac{1}{2} m \vec{v}_c \cdot \vec{v}_c + \frac{1}{2}\vec{\omega}_{ib} \cdot \vec{M}_{b/c} \cdot \vec{\omega}_{ib}$$

- On coordinate form:

$$N \text{ particles: } T = \sum T_k, \quad T_k(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2} m_k (\mathbf{v}_k^i)^\mathsf{T} \mathbf{v}_k^i = \frac{1}{2} m_k (\mathbf{v}_k^b)^\mathsf{T} \mathbf{v}_k^b$$

$$N \text{ rigid bodies: } T = \sum T_k, \quad T_k(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2} m_k (\mathbf{v}_{ck}^b)^\mathsf{T} \mathbf{v}_{ck}^b + \frac{1}{2} (\boldsymbol{\omega}_{ik}^b)^\mathsf{T} \mathbf{M}_{k/c}^b \boldsymbol{\omega}_{ik}^b$$

- Find (total) potential energy $U = U(\mathbf{q}) = \sum U_k(\mathbf{q})$
  - Gravity: $U_k(\mathbf{q}) = m_k g h(\mathbf{q})$
  - Spring: $U_k(\mathbf{q}) = \frac{1}{2} k x^2(\mathbf{q})$
  - ...

Lagrange equations of motion

- Construct Lagrangian

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q})$$

- Find 2*n* partial derivatives (scalars)

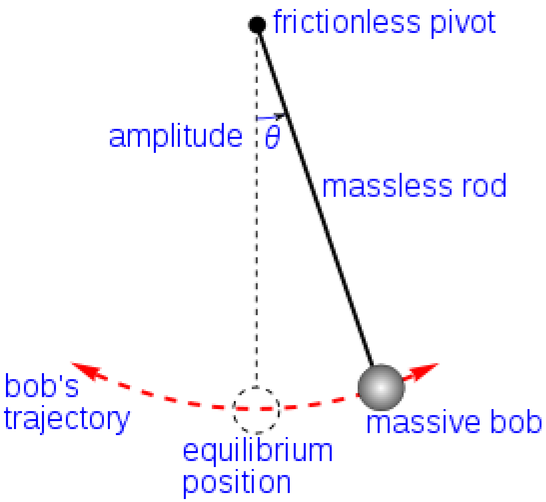
$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad \frac{\partial \mathcal{L}}{\partial q_i}$$

- Write up *n* equations of motion
  - That is, *n* 2nd order differential equations

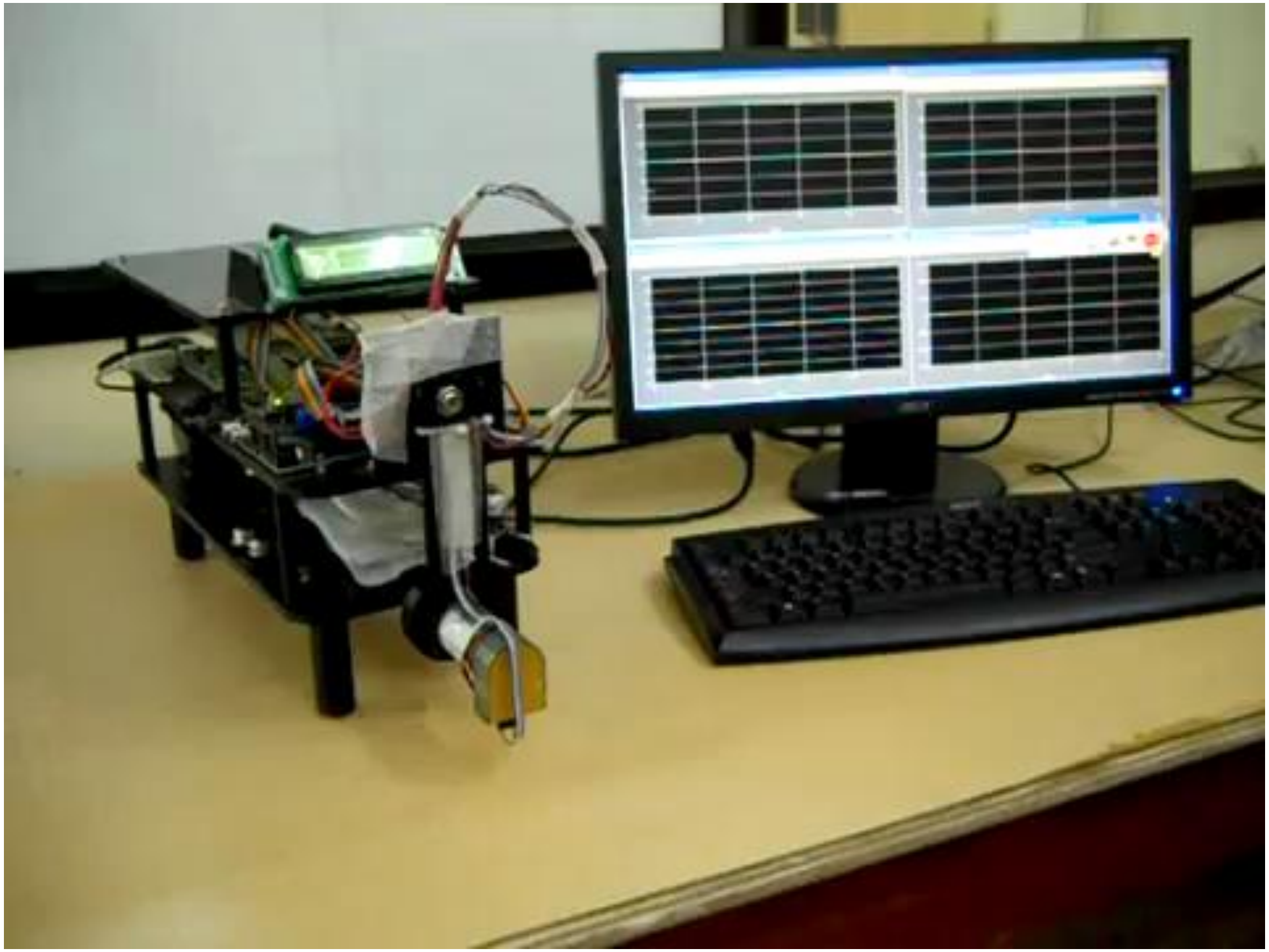
$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i}$$

Example: Pendulum

- Pendulum (bob) as particle:
  - Using Newton-Euler EoM, in inertial and body system
  - Using Lagrange EoM
- Pendulum as rigid body
  - Using Lagrange EoM



Gyroscopic pendulum
(Inertia wheel pendulum)



Problem 1 (26 %)

The gyroscopic pendulum consists of a physical pendulum with a rotating symmetric disc at the end, spinning about an axis parallel to the axis of rotation of the pendulum. See Figure 1. The stiff rod has mass m_1, length ℓ_1 and moment of inertia I_1. The position of the rod's center of gravity is given by ℓ_{c1} (cf. figure). The disc has mass m_2 and moment of inertia I_2. The pendulum is attached to a fixed coordinate system (axis x and y).

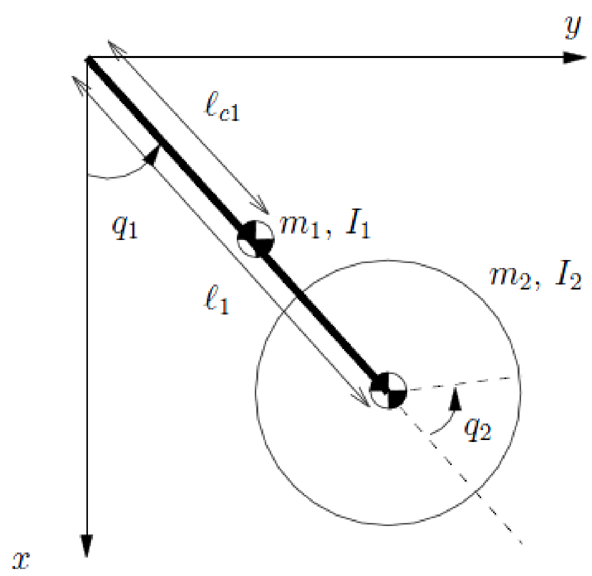


Figure 1: Gyroscopic pendulum

The rotating disc is actuated by a torque τ (which could be generated e.g. by a DC-motor). The gyroscopic pendulum is sometimes used as an experiment to illustrate nonlinear control theory.

We will develop the equations of motion for the gyroscopic pendulum.

(4 %) (a) Choose appropriate generalized coordinates for this system. The figure should give you some hints. What are the corresponding generalized forces?

(6 %) (b) What is the angular velocity of the disc (that is, of a coordinate system fixed in the disc) in the earth-fixed coordinate system?

(10 %) (c) Find the kinetic and potential energy for the system as functions of the generalized coordinates.

(6 %) (d) Derive the equations of motion for the system.

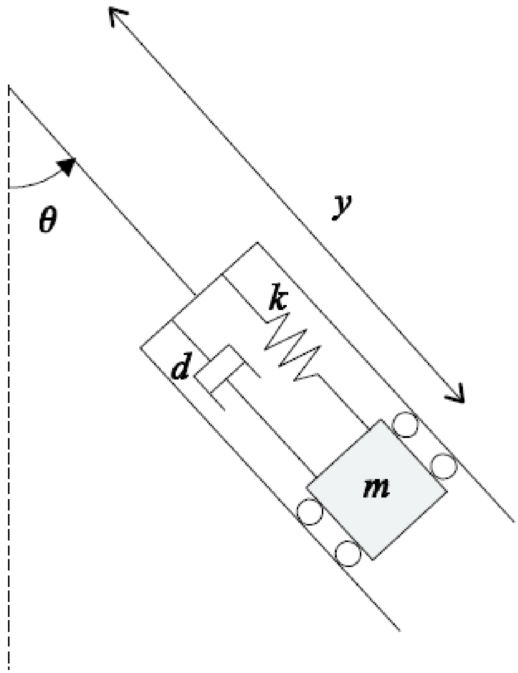


Figure 1: Kloss i rør

Oppgave 3) (15 %)

Figur (1) viser en kloss inne i et rør som svinger om et opphengspunkt. Anta at all masse bortsett fra klossen er neglisjerbar, og at klossens masse er m med massesenter gitt av y som er avtanden mellom massesenteret og opphengspunktet. Videre er fjærkonstanten k og dempekonstanten d. Fjæra er kraftløs når $y = y_0$. Det er ingen friksjon i systemet.

Velg passende generaliserte koordinater **q** og bruk Lagranges formulering for å sette opp en matematisk modell.