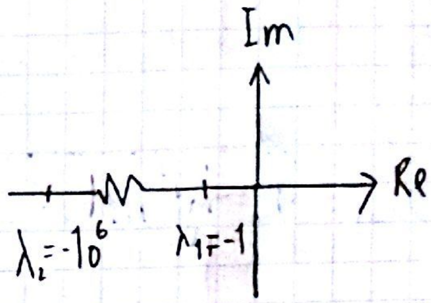
14.5 Implisitte Runge-Kutta metoder 05.02.16

Motivasjon: Stive systemer

Eks.

$$\dot{y}_1 = -y_1$$

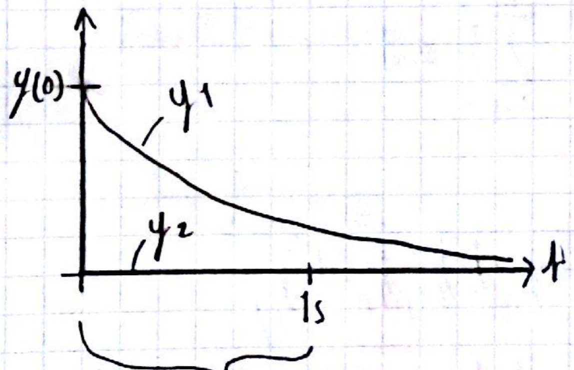
$$\dot{y}_2 = -10^6 y_2$$



Stabilitetskrav: $h|\lambda| \leq 2$

λ_1: $h \leq 2$

λ_2: $h \leq 2 \cdot 10^{-6}$



$>10^6$ tidsskritt m/Euler

Def: stive systemer

- Systemer med stor spredning i egenverdier

alt:

- Systemer som ikke kan simuleres effektivt med eksplisitt metoder

Implisitte Runge-Kutta metoder

$$k_1 = f(y_n + h(a_{11}k_1 + \dots + a_{1\sigma}k_\sigma), t_n + c_1 h)$$

$$\vdots$$

$$k_\sigma = f(y_n + h(a_{\sigma 1}k_1 + \dots a_{\sigma\sigma}k_\sigma), t_n + c_\sigma h)$$

$$y_{n+1} = y_n + h(b_1 k_1 + \dots + b_\sigma k_\sigma)$$

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1\sigma} \\ \vdots & \vdots & \ddots & \vdots \\ c_\sigma & a_{\sigma 1} & \cdots & a_{\sigma\sigma} \\ \hline & b_1 & \cdots & b_\sigma \end{array}$$

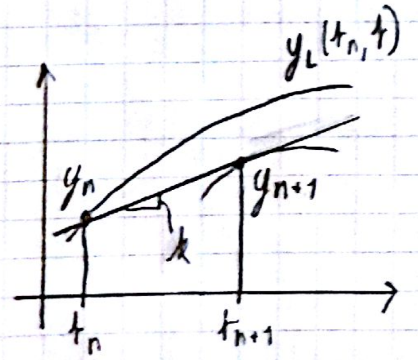
krav:

$$0 \leq c_i \leq 1$$

$$\sum_{j=1}^{\sigma} a_{ij} = c_i$$

$$\sum_{i=1}^{\sigma} b_i = 1$$

Implisitt Euler



$$\underbrace{\begin{aligned} k_1 &= f(y_n + hk_1, t_n + h) \\ y_{n+1} &= y_n + hk_1 \end{aligned}}$$

$$y_{n+1} = y_n + hf(y_{n+1}, t_{n+1})$$

Orden:

$$k_1 = f(y_n, t_n) + h\frac{df}{dt}(y_n, t_n) + O(h^2)$$

$$y_{n+1} = y_n + hf(y_n, t_n) + h^2\frac{df}{dt} + O(h^3)$$

Feil: $O(h^2) \Rightarrow$ orden 1

Stabilitet:

$$\dot{y} = \lambda y$$

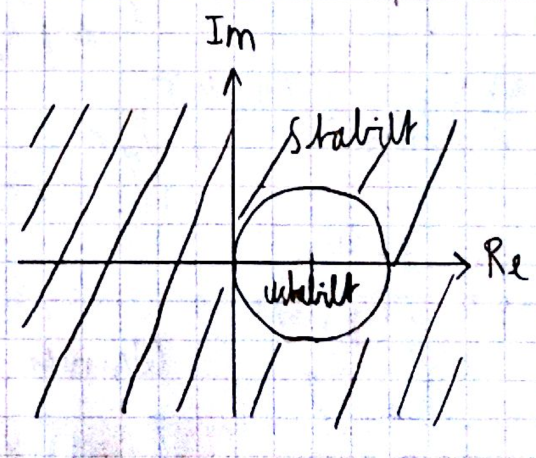
$$k_1 = \lambda(y_n + hk_1) = \lambda y_{n+1}$$

$$y_{n+1} = y_n + h\lambda y_{n+1}$$

$$(1 - h\lambda) y_{n+1} = y_n$$

$$y_{n+1} = \frac{1}{1 - h\lambda} y_n$$

$$R(h\lambda) = \frac{1}{1 - h\lambda}$$



$$|R(h\lambda)| \leq 1$$

$$\Downarrow$$

$$|1 - h\lambda| \geq 1$$

Trapes-metoden:

$$k_1 = f(y_n, t_n)$$

$$k_2 = f\left(y_n + \frac{h}{2}(k_1 + k_2),\ t_n + h\right)$$

Butcher tablå

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2) = y_n + \frac{h}{2}\left[f(y_n, t_n) + f(y_{n+1}, t_{n+1})\right]$$

Orden:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(y_n, t_n) + f(y_n, t_n) + h\frac{df}{dt} + O(h^3)\right]$$

$$= y_n + h f(y_n, t_n) + \frac{h^2}{2}\frac{df}{dt} + O(h^3)$$

Orden: 2

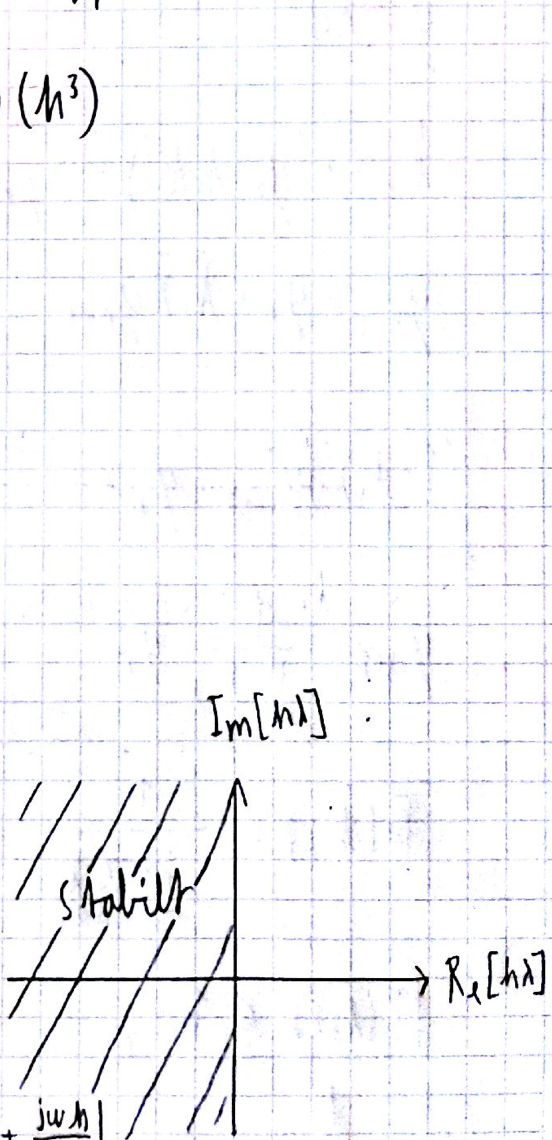
Stabilitet:

$$\dot{y} = \lambda y$$

$$y_{n+1} = y_n + \frac{h\lambda}{2}\left[y_n + y_{n+1}\right]$$

$$\left(1 - \frac{h\lambda}{2}\right) y_{n+1} = \left(1 + \frac{h\lambda}{2}\right) y_n$$

$$y_{n+1} = \underbrace{\frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}}_{R(h\lambda)} y_n$$



Observer

$$|R(j\omega h)| = \left|\frac{1 + \frac{j\omega h}{2}}{1 - \frac{j\omega h}{2}}\right| = 1$$

Implisitt midtpunktmetode

Butcher tablå

$$\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}$$

$$k_1 = f(y_n + \frac{h}{2}k_1, t_n + \frac{h}{2})$$

$$y_{n+1} = y_n + hk_1$$

Orden:

$$y_{n+1} = y_n + h\left[f(y_n, t_n) + \frac{h}{2}\frac{df}{dt} + O(h^2)\right]$$

$$= y_n + hf + \frac{h^2}{2}\frac{df}{dt} + O(h^3)$$

Orden 2

Same stabilitet som trapes metode

Stabilitetsfunksjon for IRK

Som for ERK

$$R(h\lambda) = 1 + h\lambda b^T(I - h\lambda A)^{-1}\mathbb{1}$$

evt.

$$R(h\lambda) = \frac{\det(I - h\lambda(A - b^T\mathbb{1}))}{\det(I - h\lambda A)}$$

dvs.

$$R(h\lambda) = \frac{\text{polynom i } h\lambda, \text{ grad} \leq 0}{\text{polynom i } h\lambda, \text{ grad} \leq 0}$$