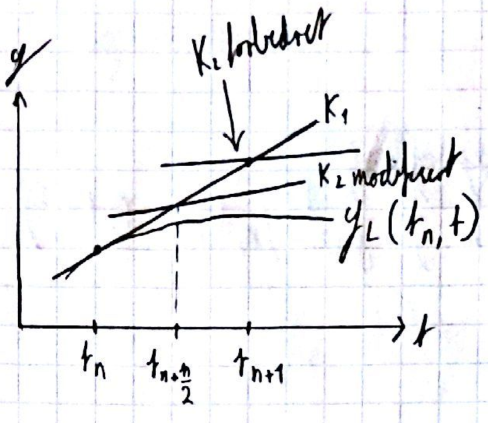
14.3 Eulermetoder

29.01.16



Euler

$$K = f(y_n, t_n)$$

$$y_{n+1} = y_n + hK$$

Modifisert euler:

$$K_1 = f(y_n, t_n)$$

$$K_2 = f\left(y_n + \frac{h}{2}K_1,\ t_n + \frac{h}{2}\right)$$

$$y_{n+1} = y_n + hK_2$$

Forbedret euler:

$$K_1 = f(y_n, t_n)$$

$$K_2 = f(y_n + hK_1,\ t_n + h)$$

$$y_{n+1} = y_n + \frac{h}{2}(K_1 + K_2)$$

Euler $y_{n+1} = y_n + hf(y_n, t_n)$

$$l_{n+1} = O(h^2), \text{ orden } 1$$

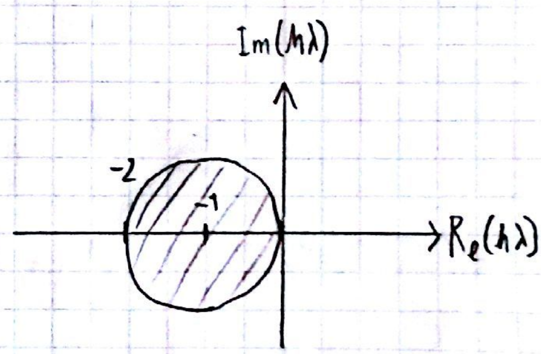
Stabilitet av Euler

$$y_{n+1} = y_n + h\lambda y_n = \underbrace{(1 + h\lambda)}_{R(h\lambda)} y_n$$

Stabil: $|1 + h\lambda| \leq 1$

λ reell: $-1 \leq 1 + h\lambda \leq 1$

$$-2 \leq h\lambda \leq 0$$



Orden av forbedret Euler

Taylor rekke utv. av K_2:

$$K_2 = f(y_n, t_n) + h \frac{d}{dt} f(y_n, t_n) + \frac{h^2}{2} \frac{d^2}{dt^2} f(y_n, t_n) + O(h^3)$$

Sett inn:

$$y_{n+1} = y_n + \frac{h}{2}(K_1 + K_2)$$

$$= y_n + \frac{h}{2} f(y_n, t_n) + \frac{h}{2} f(y_n, t_n) + \frac{h^2}{2} \frac{d}{dt} f(y_n, t_n) + \frac{h^3}{4} \frac{d^2}{dt^2} f(y_n, t_n) + O(h^4)$$

$$= \underbrace{y_n}_{\text{OK}} + \underbrace{h f(y_n, t_n) +}_{\text{OK}} \underbrace{\frac{h^3}{4} \frac{d^2}{dt^2} f(y_n, t_n) + O(h^4)}_{\text{IKKE OK}}$$

Dvs: $e_{n+1} = O(h^3)$ metode orden 2

Stabilitet av forbedret Euler

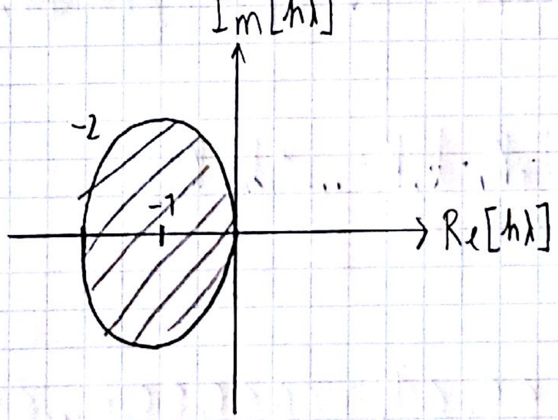
Test systemet: $\dot{y} = \lambda y$

$$K_1 = \lambda y_n$$

$$K_2 = \lambda (y_n + h \lambda y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}(K_1 + K_2) = \underbrace{\left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right)}_{R(h\lambda)} y_n$$

Stabil for $\left|1 + h\lambda + \frac{(h\lambda)^2}{2}\right| \leq 1$



14.4 Eksplisitte Runge-Kutta (ERK)-metoder

$$\dot{y} = f(y,t) \quad , \quad y(t_0) = y_0$$

ERK-metoder med σ trinn

$$K_1 = f(y_n, t_n)$$

$$K_2 = f(y_n + h a_{21} K_1, t_n + c_2 h)$$

$$K_3 = f(y_n + h a_{31} K_1 + h a_{32} K_2, t_n + c_3 h)$$

$$\vdots$$

$$K_\sigma = f(y_n + h(a_{\sigma 1} K_1 + a_{\sigma 2} K_2 + a_{\sigma,\sigma-1} K_{\sigma-1}), t_n + c_\sigma h)$$

$$y_{n+1} = y_n + h(b_1 K_1 + b_2 K_2 + \ldots + b_\sigma K_\sigma)$$

hvor

$$c_i : \; 0 \leq c_i \leq 1, \quad 0 \leq c_1 \leq c_2 \leq \ldots \leq c_\sigma$$

$$a_{ij} : \; \sum_{j=1}^{i-1} a_{ij} = c_i \leq 1$$

$$b_i : \; \sum_{i=1}^{\sigma} b_i = 1$$

Butcher-tablå

$$
\begin{array}{c|cccccc}
0 & 0 & 0 & 0 & \cdots & & \\
c_2 & a_{21} & 0 & 0 & \cdots & & \\
c_3 & a_{31} & a_{32} & 0 & \cdots & & \\
\vdots & \vdots & \vdots & \vdots & & & \\
c_\sigma & a_{\sigma 1} & a_{\sigma 2} & \cdots & & a_{\sigma,\sigma-1} & 0 \\
\hline
 & b_1 & b_2 & \cdots & & b_{\sigma-1} & b_\sigma
\end{array}
=
\begin{array}{c|c}
c & A \\
\hline
 & b^T
\end{array}
$$

Euler $K_1 = f(y_n, t_n)$

$$y_{n+1} = y_n + hK_1$$

$$
\begin{array}{c|c}
0 & 0 \\
\hline
 & 1
\end{array}
$$

Forbedret Euler

$K_1 = f(y_n, t_n)$

$K_2 = f(y_n + hK_1, t_n + h)$

$y_{n+1} = y_n + \frac{h}{2}(K_1 + K_2)$

$$
\begin{array}{c|cc}
0 & 0 & 0 \\
1 & 1 & 0 \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

Heun (ERK3)

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & 0 \\
\frac{1}{3} & 0 & \frac{1}{3} & 0 \\
\hline
 & \frac{1}{4} & 0 & \frac{3}{4}
\end{array}
$$

ERK4

$$
\begin{array}{c|cccc}
0 & & & & \\
1/2 & 1/2 & & & \\
1/2 & 0 & 1/2 & & \\
1 & 0 & 0 & 1 & \\
\hline
 & 1/6 & 1/3 & 1/3 & 1/6
\end{array}
$$

Stabilitetsfunksjon for ERK-metoder

Testsystem $\dot{y} = \lambda y$

$$(*)\begin{cases} K_1 = \lambda y_n \\ K_2 = \lambda(y_n + h a_{21} K_1) \\ \vdots \\ K_\sigma = \lambda(y_n + h(a_{\sigma 1} K_1 + a_{\sigma 2} K_2 + \dots + a_{\sigma,\sigma-1} K_{\sigma-1})) \end{cases}$$

$$(**)\ y_{n+1} = y_n + h(b_1 K_1 + \dots + b_\sigma K_\sigma)$$

Definerer $\mathbb{K} = \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_\sigma \end{pmatrix}$, $\mathbb{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

$$(*)\quad \mathbb{K} = \lambda(\mathbb{1} y_n + hA\mathbb{K})$$

$$(**)\quad y_{n+1} = y_n + hb^T\mathbb{K}$$

Løs $(*)$ mhp $\mathbb{K}$

$$\mathbb{K} = (I - h\lambda A)^{-1} \lambda \mathbb{1} y_n$$

Sett inn i (**)

$$y_{n+1} = (1 + h\lambda b^T (I - h\lambda A)^{-1} \mathbb{1}) y_n$$

$$\Rightarrow R_E(h\lambda) = 1 + h\lambda b^T (I - h\lambda A)^{-1} \mathbb{1}$$

<u>Alt.</u>

$$y_{n+1} = \underbrace{\frac{\det(I - h\lambda(A - \mathbb{1} b^T))}{\det(I - h\lambda A)}}_{R_E(h\lambda)} y_n$$

For ERK: $\det(I - h\lambda A) = \det \begin{bmatrix} 1 & 0 & 0 & \cdots \\ & 1 & 0 & \cdots \\ & & 1 & \ddots \\ \cdots & & & \ddots \end{bmatrix} = 1$

$$\Rightarrow R_E(h\lambda) = \det(I - h\lambda(A - \mathbb{1} b^T))$$

Observerer:

1) $|R_E(h\lambda)| \to \infty$ når $h\lambda \to \infty$

2) $R_E(h\lambda)$ er polynom i $h\lambda$ av orden $\leq \sigma$