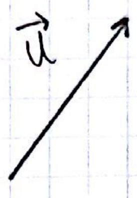
Vektorer

03.03.16

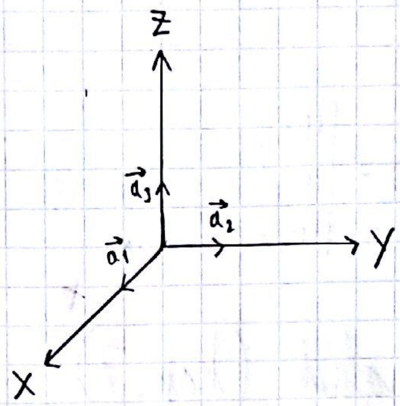
Vektor: <u>størrelse</u> & <u>retning</u>



Notasjon:

Koordinatfri: $\vec{u}$

Koordinatvektor: $\underline{u}$



$$\vec{u} = u_1\vec{a}_1 + u_2\vec{a}_2 + u_3\vec{a}_3$$

$$\underline{u}^a = \begin{bmatrix} u_1^a \\ u_2^a \\ u_3^a \end{bmatrix}$$

<u>Eks 78</u>

<u>Fact:</u> $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{b})\vec{c}$

<u>Dvs.:</u> $\underline{a}^\times \underline{b}^\times \underline{c} = \underline{b}(\underline{a}^T \underline{c}) - (\underline{a}^T \underline{b})\underline{c} = (\underline{b}\,\underline{a}^T - \underline{a}^T \underline{b} I)\underline{c}$

$$\underline{a}^\times \underline{b}^\times = \underline{b}\,\underline{a}^T - \underline{a}^T \underline{b} I$$

$$\underline{a}^\times \underline{a}^\times = \underline{a}\,\underline{a}^T - \underline{a}^T \underline{a} I$$

Dyade

Eks. Treghetsdyaden

Spinn: $\vec{h} = \sum_{i=1}^{3} h_i \vec{a}_i$, $\underline{h} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$

Vinkelhastighet: $\vec{\omega} = \sum_{i=1}^{3} \omega_i \vec{a}_i$, $\underline{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$

Sammenheng spinn og vinkelhastighet (kap. 7)

$$\boxed{\underline{h} = M \underline{\omega}}, \quad h_i = \sum_{j=1}^{3} m_j \omega_j$$

M: treghetsmoment $\quad M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & \vdots & \vdots \\ \vdots & & \end{bmatrix}$

Definerer dyaden $\vec{M}$

$$\vec{M} = \sum_{i=1}^{3} \sum_{j=1}^{3} m_{ij} \overbrace{\vec{a}_i \vec{a}_j}^{\text{Ikke multiplisert, men et vektorpar}}$$

$$\vec{M} \cdot \vec{\omega} = \sum_i \sum_j m_{ij} \vec{a}_i \vec{a}_j \cdot \sum_k \omega_k \vec{a}_k$$

$$= \sum_i \sum_j \sum_k m_{ij} \vec{a}_i \omega_k \underbrace{\vec{a}_j \cdot \vec{a}_k}_{= 1 \text{ når } k=j, \ 0 \text{ ellers}}$$

$\Rightarrow$

$\Rightarrow$

$$\vec{M} \cdot \vec{\omega} = \sum_i \sum_j m_{ij} \omega_j \vec{a}_i = \vec{h}$$

Dvs. $\boxed{\vec{h} = \vec{M}\vec{\omega}}$

Generell dyade: $\vec{D} = \sum_i \sum_j d_{ij} \vec{a}_i \vec{a}_j$

$d_{ij} = \vec{a}_i \cdot \vec{D} \cdot a_j$ matriserepr. $D = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$

Premultiplikasjon:

$$\vec{w} = \vec{u} \cdot \vec{D} = \sum_k u_k \vec{a}_k \cdot \sum_i \sum_j d_{ij} \vec{a}_i \vec{a}_j$$

$$= \underbrace{\sum_i \sum_j u_i d_{ij}}_{w_j} \vec{a}_j$$

$\vec{w} = \vec{u}\,\vec{D}$ $\quad \vec{z} = \vec{D} \cdot \vec{u}$

$\Updownarrow$

$\underline{w} = \underline{u}^T D$ $\quad \underline{z} = D\underline{u}$

Eks

$$\vec{\vec{I}} = \vec{a}_1\vec{a}_1 + \vec{a}_2\vec{a}_2 + \vec{a}_3\vec{a}_3$$

$$\vec{\vec{I}}\cdot\vec{v} = \left(\text{———} \;''\; \text{———}\right)\left(v_1\vec{a}_1 + v_2\vec{a}_2 + v_3\vec{a}_3\right)$$

$$= v_1\vec{a}_1 + v_2\vec{a}_2 + v_3\vec{a}_3 = \vec{v}$$

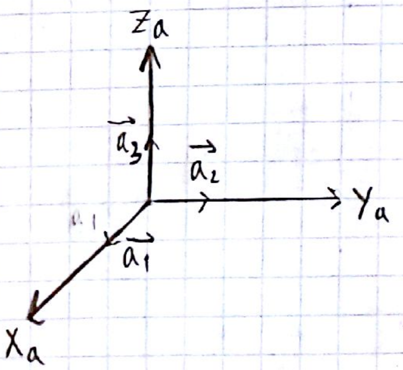
$$\vec{\vec{I}}\cdot\vec{v} = \vec{v} \iff \underline{I}\,\underline{v} = \underline{v}$$

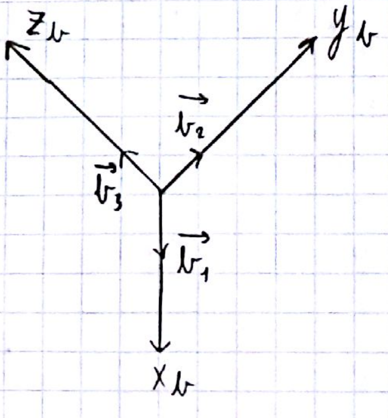
$$\vec{v}\cdot\vec{\vec{I}} = \vec{v} \iff \underline{v}^T\underline{I} = \underline{v}^T$$

Eks.

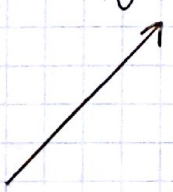
$$\vec{v}\times\vec{u} = \underbrace{\vec{\vec{u}}^{\times}}_{\substack{\text{Dyaderepr.}\\ \text{av matrise } \underline{u}^{\times}}}\cdot\,\vec{v}$$

Rotasjonsmatrisen





$\vec{v}$



$$\vec{v} = \sum_i v_i^a\,\vec{a}_i$$

$$= \sum_i v_i^b\,\vec{b}_i$$

$$v_i^a = \vec{v}\cdot\vec{a}_i$$

$$v_i^b = \vec{v}\cdot\vec{b}_i$$

$$\underline{v}^a = \begin{bmatrix} v_1^a \\ v_2^a \\ v_3^a \end{bmatrix} \qquad \underline{v}^b = \begin{bmatrix} v_1^b \\ v_2^b \\ v_3^b \end{bmatrix}$$

$\Longrightarrow$

$\Rightarrow$

$$v_i^a = \vec{v}\,\vec{a}_i = (v_1^b \vec{b}_1 + v_2^b \vec{b}_2 + v_3^b \vec{b}_3)\,\vec{a}_i$$

$$= (\vec{b}_1 \vec{a}_i)\,v_1^b + (\vec{b}_2 \cdot \vec{a}_i)\,v_2^b + (\vec{b}_3 \cdot \vec{a}_i)\,v_3^b$$

$$\begin{bmatrix} v_1^a \\ v_2^a \\ v_3^a \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \vec{b}_1 & \vec{a}_1 \cdot \vec{b}_2 & \vec{a}_1 \vec{b}_3 \\ \vec{a}_2 \vec{b}_1 & \vec{a}_2 \cdot \vec{b}_2 & \vec{a}_2 \vec{b}_3 \\ \vec{a}_3 \vec{b}_1 & \vec{a}_3 \vec{b}_2 & \vec{a}_3 \vec{b}_3 \end{bmatrix} \begin{bmatrix} v_1^b \\ v_2^b \\ v_3^b \end{bmatrix}$$

$$\underline{v}^a = R_b^a \quad \underline{v}^b$$

$$\underline{v}^a = R_b^a\,\underline{v}^b$$

R_b^a: "koordinattransformasjon fra b til a"

R_b^a: "Rotasjonsmatrisen fra a til b"

$$\underline{v}^a = R_b^a\,\underline{v}^b, \quad R_b^a = \begin{bmatrix} \vec{a}_1 \vec{b}_1 & \vec{a}_1 \vec{b}_2 & \cdots \\ \vec{a}_2 \vec{b}_1 & \ddots & \\ \vdots & & \end{bmatrix}$$

$$\underline{v}^b = R_a^b\,\underline{v}^a, \quad R_a^b = \begin{bmatrix} \vec{b}_1 \vec{a}_1 & \vec{b}_1 \vec{a}_2 & \cdots \\ \vec{b}_2 \vec{a}_1 & \ddots & \\ \vdots & & \end{bmatrix}$$

$$\Rightarrow \boxed{R_a^b = (R_b^a)^T} \quad (*)$$

$$\underline{v}^b = R_a^b \underline{v}^a = R_a^b R_b^a \underline{v}^b$$

$$\Rightarrow \boxed{R_a^b R_b^a = I} \quad (**)$$

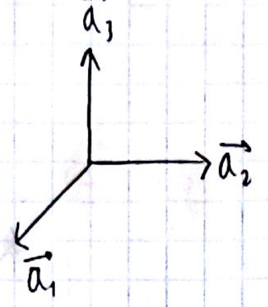
$(*) + (**)$:

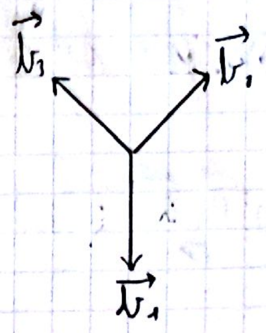
$$\boxed{R_a^b = (R_b^a)^{-1} = (R_b^a)^T}$$

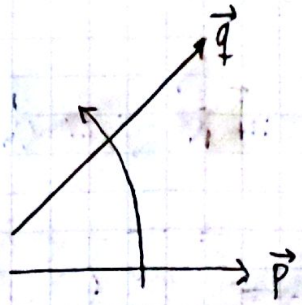
$$(R_b^a)(R_b^a)^T = I$$

R_b^a: ortogonal matrice

$$SO(3): \{R \mid R \in \mathbb{R}^{3\times 3},\ R^T R = I,\ \det(R) = 1\}$$







Definer $\vec{q}$: $\underline{q}^a = R^a_b \, \underline{p}^a$

$$\underline{q}^b = R^b_a \, \underline{q}^a = \underbrace{R^b_a \, R^a_b}_{I} \, \underline{p}^a = \underline{p}^a$$

Dvs.: R^a_b roterer $\vec{p}$ til $\vec{q}$ slik at $\underline{q}^b = \underline{p}^a$

Sammensatte rotasjoner

$$\underline{v}^b = R^b_c \, \underline{v}^c$$

$$\underline{v}^a = R^a_b \, \underline{v}^b = R^a_b \, R^b_c \, \underline{v}^c$$

$$\underline{v}^a = R^a_c \, \underline{v}^c$$

$$\boxed{R^a_c = R^a_b \, R^b_c}$$

$$\boxed{R^a_d = R^a_b \, R^b_c \, R^c_d}$$

Koordinat-transformasjon av dyade

dyade $\vec{D} = \sum_i \sum_j d_{ij}^a \vec{a}_i \vec{a}_j$

$$D^a = \begin{bmatrix} d_{11}^a & d_{21}^a & \cdots \\ d_{12}^a & \ddots & \vdots \\ \cdots & \cdots & \ddots \end{bmatrix}$$

$$\vec{D} = \sum_i \sum_j d_{ij}^b \vec{b}_i \vec{b}_j$$

$$D^b = \begin{bmatrix} d_{11}^b & d_{21}^b & \cdots \\ d_{12}^b & \ddots & \vdots \\ \cdots & \cdots & \ddots \end{bmatrix}$$

Koordinatfri: $\vec{z} = \vec{D} \cdot \vec{u}$

Koordinatversjon: $\underline{z}^a = D^a \underline{u}^a$ eller $\underline{z}^b = D^b \underline{u}^b$

$$D^a \underline{u}^a = \underline{z}^a = R_b^a \underline{z}^b = R_b^a D^b \underline{u}^b = R_b^a D^b R_a^b \underline{u}^a \Rightarrow \boxed{D^a = R_b^a D^b R_a^b}$$

Eks. Dyade $\vec{u}^\times$

$$\vec{w} = \vec{u} \times \vec{v} = (\vec{u}^\times) \vec{v}$$

$$(\underline{u}^a)^\times = R_b^a (\underline{u}^b)^\times R_a^b$$