Runge-Kutta methods and their properties

Info

The Butcher array has form

$$
\begin{array}{c|c}
\mathbf{c} & \mathbf{A} \\
\hline
 & \mathbf{b}^\top
\end{array}
$$

where for explicit methods, $\mathbf{A}$ is lower triangular, with zeros on the diagonal, and for implicit methods, it is full.

The stability function for explicit methods can be calculated as

$$R_E(h\lambda) = \det\left[\mathbf{I} - \lambda h\left(\mathbf{A} - \mathbf{1}\mathbf{b}^\top\right)\right]$$

For implicit methods, the stability function is

$$R_I(h\lambda) = \left[1 + \lambda h \mathbf{b}^\top \left(\mathbf{I} - h\lambda\mathbf{A}\right)^{-1}\mathbf{1}\right]$$

or, equivalently

$$R_I(h\lambda) = \frac{\det\left[\mathbf{I} - \lambda h\left(\mathbf{A} - \mathbf{1}\mathbf{b}^\top\right)\right]}{\det\left(\mathbf{I} - \lambda h\mathbf{A}\right)}$$

A-Stability

A method is A-stable if $|R(\lambda h)| \leq 1$ for all $\mathfrak{R}(\lambda) \leq 0$.

L-stability

A method is L-stable if it is A-stable and, in addition, if $|R(j\omega h)| \to 0$ when $\omega \to \infty$ for all systems $\dot{y} = \lambda y$ where $\lambda = j\omega$.

Forward Euler

Synonyms

ERK1?

Butcher Array

$$
\begin{array}{c|c}
0 & \\
\hline
 & 1
\end{array}
$$

Stability Properties

Stability Function:

$$R(h\lambda) = 1 + h\lambda$$

Stability ensured whenever

$$|R(h\lambda)| = |1 + h\lambda| \leq 1$$

This is an explicit method, thus it is **neither A- nor L-stable**.

Improved Euler

Synonyms

ERK2.

Butcher Array

$$
\begin{array}{c|cc}
0 & & \\
1 & 1 & \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

Stability Properties

Stability Function:

$$R(h\lambda) = 1 + h\lambda + \frac{1}{2}h^2\lambda^2$$

This is an explicit method, thus it is **neither A- nor L-stable**.

Modified Euler

Synonyms

Explicit midpoint rule. ERK2.

Butcher Array

$$\begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array}$$

Stability Properties

Stability Function:

$$R(h\lambda) = 1 + h\lambda + \frac{1}{2}h^2\lambda^2$$

This has the same stability function as Improved Euler.

This is an explicit method, thus it is **neither A- nor L-stable**.

Heun's

Synonyms

ERK3.

Butcher Array

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{3} & \frac{1}{3} & & \\ \frac{2}{3} & 0 & \frac{2}{3} & \\ \hline & \frac{1}{4} & 0 & \frac{3}{4} \end{array}$$

Stability Properties

Stability Function:

$$R(h\lambda) = \frac{1}{6}h^3\lambda^3 + \frac{1}{2}h^2\lambda^2 + h\lambda + 1$$

This is an explicit method, thus it is **neither A- nor L-stable**.

RK4

Synonyms

Unknown.

Butcher Array

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6}
\end{array}
$$

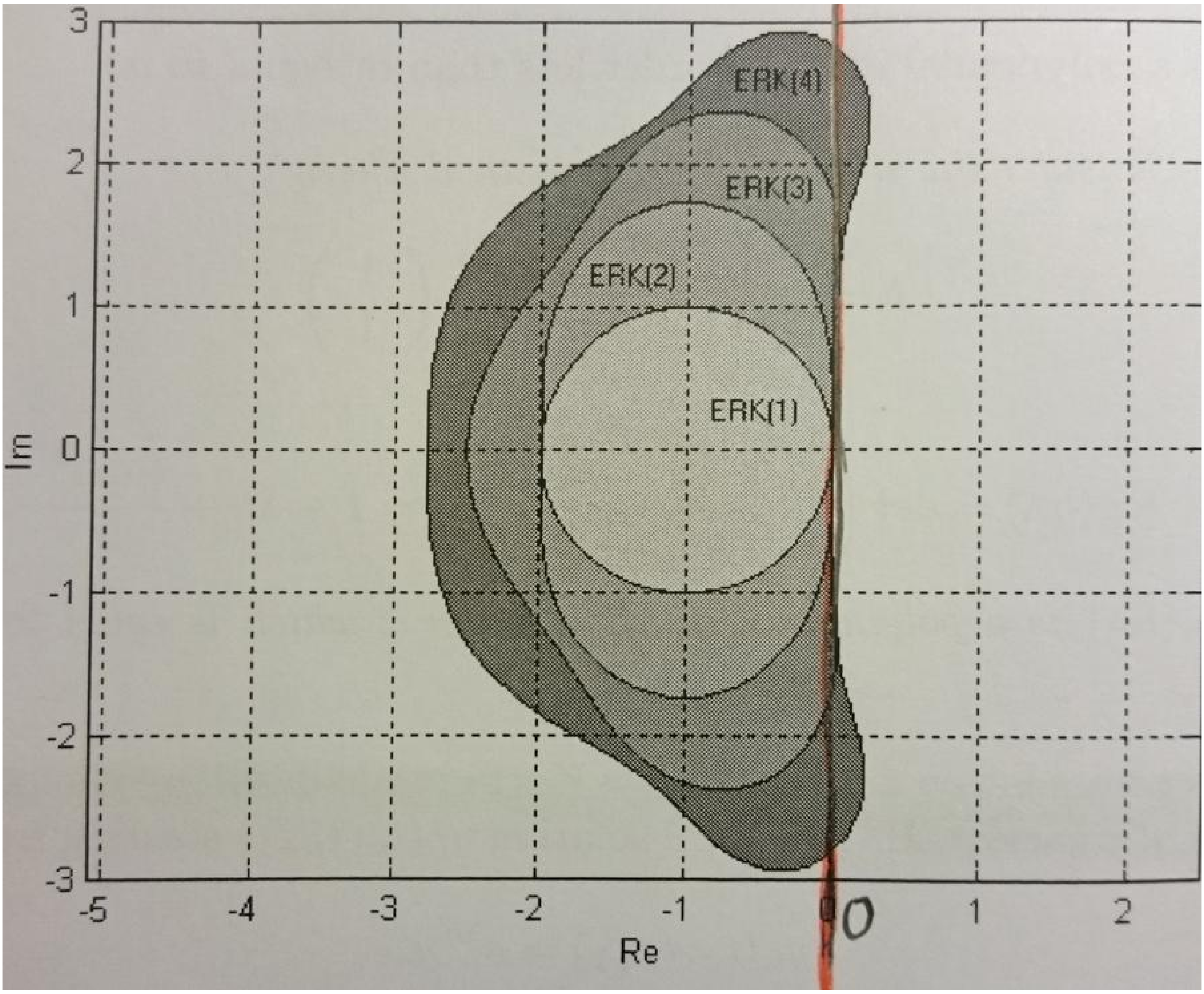
Stability Properties

Stability Function:

$$R(h\lambda) = \frac{h^4\lambda^4}{24} + \frac{h^3\lambda^3}{6} + \frac{h^2\lambda^2}{2} + h\lambda + 1$$

This is an explicit method, thus it is **neither A- nor L-stable**.

Regions of Stability for Explicit Methods



Implicit Euler

Synonyms

Unknown.

Butcher Array

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

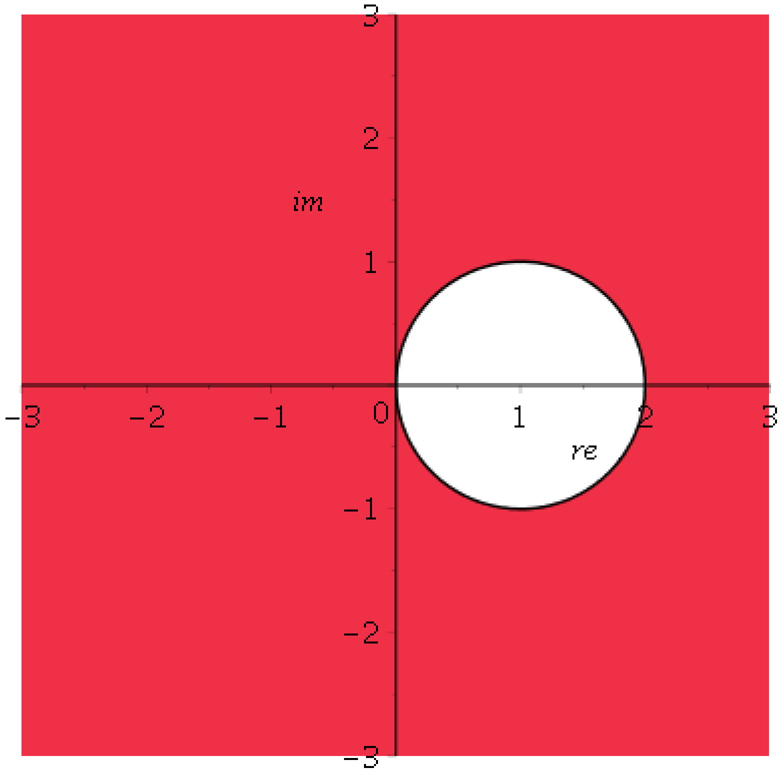
Stability Properties

Stability Function:

$$R(h\lambda) = \frac{1}{1 - h\lambda}$$

This method is **A-stable** and **L-stable**.

This is the stable region:



Trapezoidal Rule

Synonyms

Lobatto IIIA order 2.

Butcher Array

$$
\begin{array}{c|cc}
0 & 0 & 0 \\
1 & \frac{1}{2} & \frac{1}{2} \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

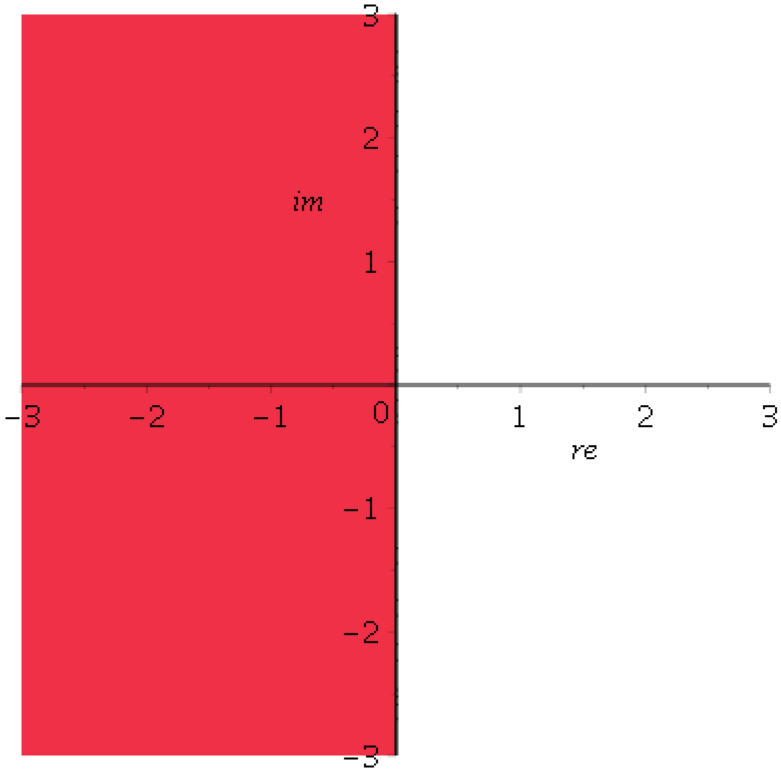
Stability Properties

Stability Function:

$$
R(h\lambda) = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}
$$

This method is **A-stable** (for sure), but **not L-stable** (I think, based on the stability region).

This is the stable region:



Implicit Midpoint Rule

Synonyms

Gauss order 2.

Butcher Array

$$\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}$$

Stability Properties

Stability Function:

$$R(h\lambda) = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}$$

This method has the same stability function, thus the same region of stability, as the Trapezoidal Rule. However, it turns out to have better stability properties for nonlinear systems.

Theta Method

Synonyms

Unknown.

Butcher Array

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \theta & 1-\theta \\ \hline & \theta & 1-\theta \end{array}$$

This is equal to Euler's method if $\theta = 1$, and the trapezoidal rule if $\theta = 0$.

Stability Properties

Stability Function:

$$R(h\lambda) = \frac{1 + h\lambda\theta}{1 - h\lambda(1-\theta)}$$

Hammer and Hollingsworth 4

Synonyms

Gauss order 4.

Butcher Array

$$\begin{array}{c|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

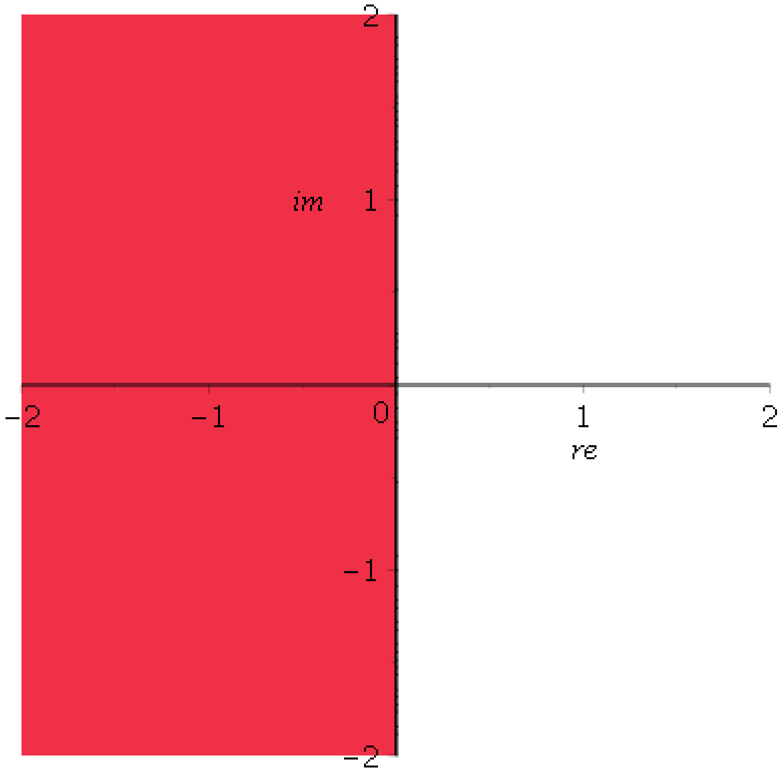
Stability Properties

Stability Function:

$$R(h\lambda) = \frac{h^2\lambda^2 + 6\,h\lambda + 12}{h^2\lambda^2 - 6\,h\lambda + 12}$$

This method is **A-stable** (fore sure), but **not L-stable** (I think, based on the stability region). Have checked that $\lim_{\Im(h\lambda)\to\infty} |R(h\lambda)| = 1$, which I think might imply that it does not dampen out high frequencies, which implies that it is not L-stable.

Region of stability:



Radau IIA, Order 3

Butcher Array

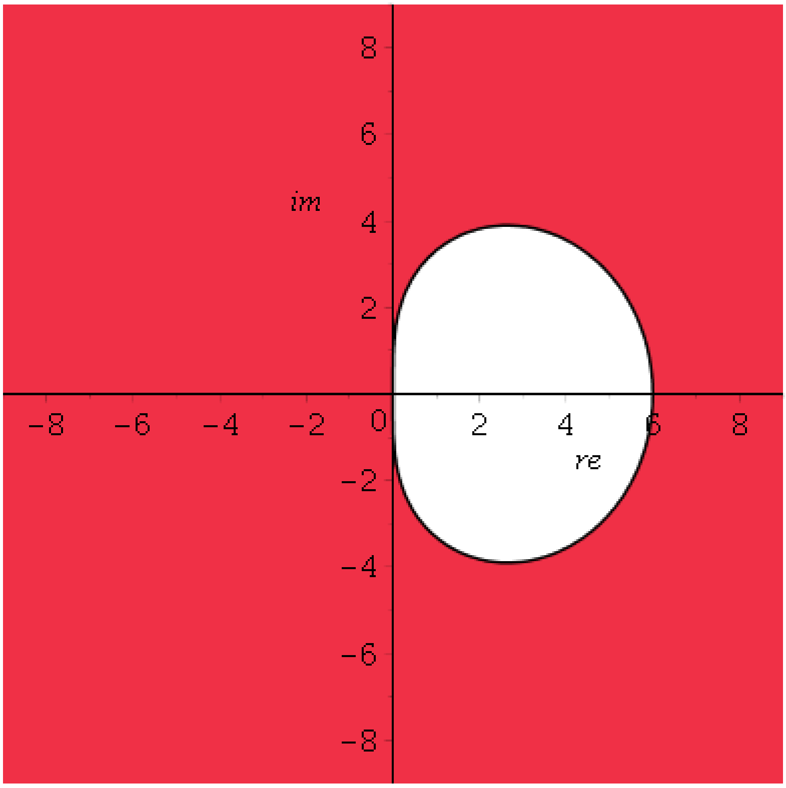
$$\begin{array}{c|cc} \frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array}$$

Stability Properties

Stability Function:

$$R(h\lambda) = 2\,\frac{h\lambda + 3}{h^2\lambda^2 - 4\,h\lambda + 6}$$

This method is **A-stable**. Don't know about L-stable. Have checked that $\lim_{\Im(h\lambda)\to\infty} R(h\lambda) = 0$, that might imply that it is L-stable.



im
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