Lecture 10: Stability and frequency properties (mostly recap), error control and software

- Stability (mostly recap), frequency properties (14.6)
- Error control: Automatic adjustment of step-size (14.7)
  - Solver vs Integrator
- Interpolation and events
- Briefly:
  - Multistep methods
    - ”Tend to be more efficient than single-step methods for systems with smooth solutions and high accuracy requirements”
    - Often used in advanced modeling software (e.g. Dymola)
  - Differential-algebraic systems
  - Software

Implicit Runge-Kutta (IRK) methods

- IVP: $\dot{y} = f(y,t), \quad y(0) = y_0$
- IRK:

$$
\begin{aligned}
k_1 &= f(y_n + h(a_{1,1}k_1 + a_{1,2}k_2 + \ldots + a_{1,\sigma}k_\sigma), t_n + c_1 h) \\
k_2 &= f(y_n + h(a_{2,1}k_1 + a_{2,2}k_2 + \ldots + a_{2,\sigma}k_\sigma), t_n + c_2 h) \\
k_3 &= f(y_n + h(a_{3,1}k_1 + a_{3,2}k_2 + \ldots + a_{3,\sigma}k_\sigma), t_n + c_3 h) \\
&\vdots \\
k_\sigma &= f(y_n + h(a_{\sigma,1}k_1 + a_{\sigma,2}k_2 + \ldots + a_{\sigma,\sigma}k_\sigma), t_n + c_\sigma h) \\
y_{n+1} &= y_n + h(b_1 k_1 + b_2 k_2 + \ldots + b_\sigma k_\sigma)
\end{aligned}
$$

- Butcher array:

$\mathbf{c}$	$\mathbf{A}$
	$\mathbf{b}^\mathsf{T}$

c_1	a_{11}	a_{12}	$\cdots$		$a_{1,\sigma}$
c_2	a_{21}	a_{22}	$\cdots$		$a_{2,\sigma}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$		
c_σ	$a_{\sigma,1}$	$a_{\sigma,2}$	$\cdots$	$a_{\sigma,\sigma-1}$	$a_{\sigma,\sigma}$
	b_1	b_2	$\ldots$	$b_{\sigma-1}$	b_σ

Recap: Test system, stability function

- One step method (typically: Runge-Kutta):

$$y_{n+1} = y_n + h\phi(y_n, t_n)$$

- Apply it to scalar test system:

$$\dot{y} = \lambda y$$

- We get:

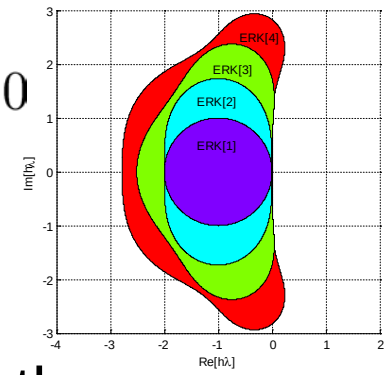
$$y_{n+1} = R(h\lambda)y_n$$

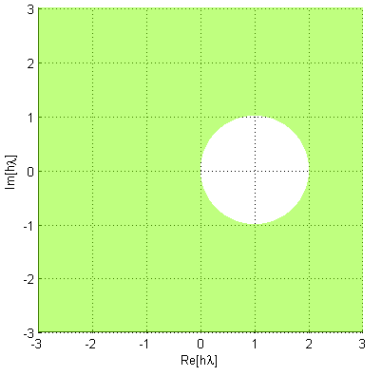
  where $R(h\lambda)$ is stability function

- The method is stable (for test system!) if

$$|R(h\lambda)| \leq 1$$

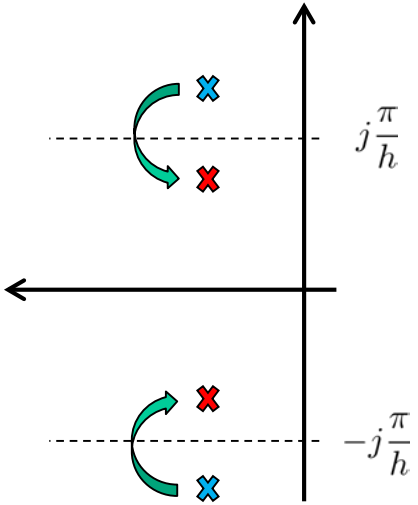
Linear stability: A- and L-stability





- A-stability: $|R(h\lambda)| \leq 1$ for all $\mathrm{Re}\,\lambda \leq 0$
  - Relevant (mostly) for **stiff** systems
  - No explicit methods are A-stable
  - Many implicit methods are A-stable
- Aliasing: Frequencies larger than the «Nyquist frequency» $\frac{\pi}{h}$ are mapped to within the Nyquist frequency
- A method is L-stable if it is A-stable, and

$$|R(hj\omega)| \to 0 \text{ when } \omega \to \infty$$



  - Relevant for (stiff) systems with **oscillatory modes**
  - Dampens out fast frequencies
  - We often want L-stability in implicit methods, but not always:
    - We typically want to suppress dynamics that are faster than step length («stiff decay»)
    - However, we may want to not dampen oscillatory modes
    - We may want to **not** dissipate energy (numerically) in simulations

Padé approximations

- The local solution to test system:

$$y_L(t_n; t_{n+1}) = e^{h\lambda} y_n$$

- Stability function:

$$y_{n+1} = R(h\lambda) y_n$$

- A method is «good» if

$$R(s) \approx e^s$$

- Explicit Runge-Kutta methods with $\sigma = p \cdot$ 4: Taylor expansion!

$$R(s) = 1 + s + \ldots + \frac{s^p}{p!}$$

- Implicit Runge-Kutta methods:

$$R(s) = \frac{1 + \beta_1 s + \ldots + \beta_k s^k}{1 + \gamma_1 s + \ldots + \gamma_m s^m}$$

- Best approximation (for given *k* and *m*): Padé-approximation

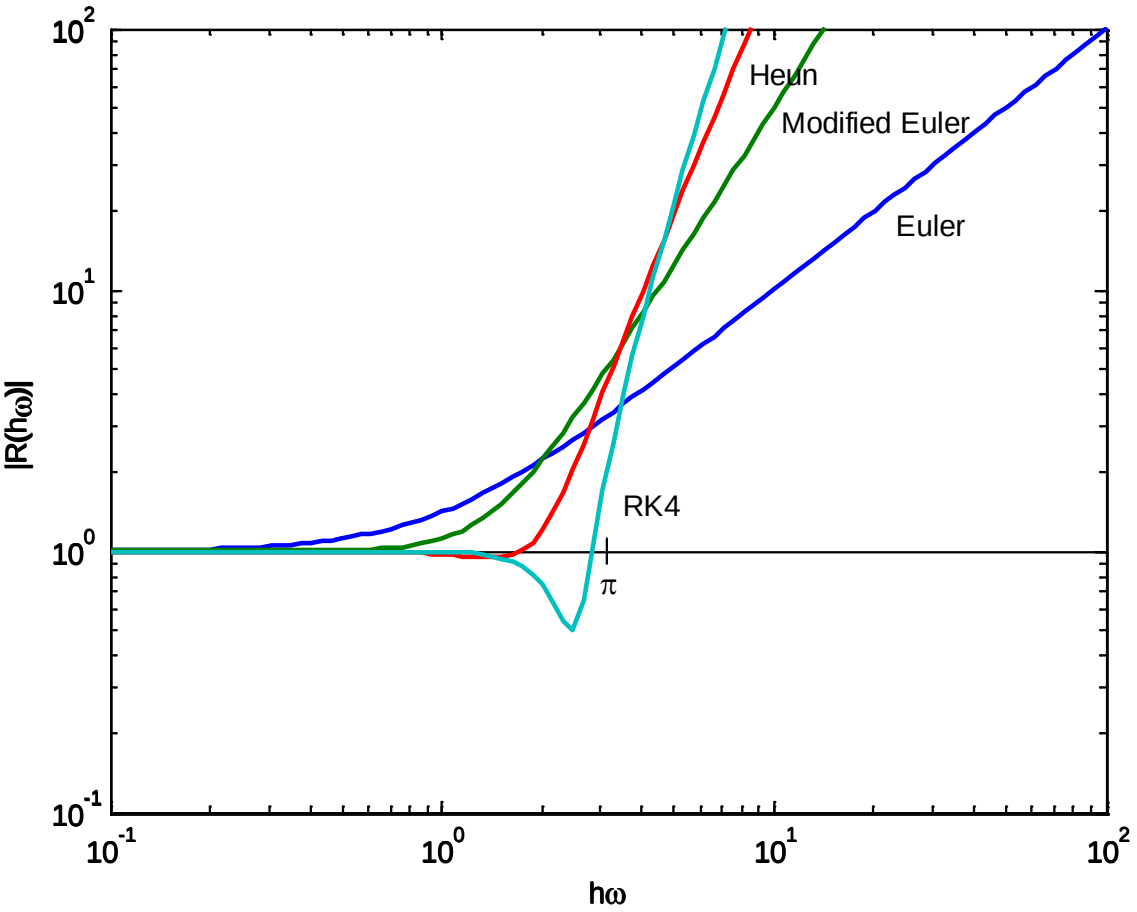
Padé approximations to e^s

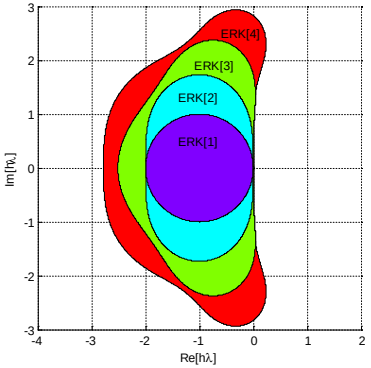
m \ k	0	1	2	3
0	$\frac{1}{1}$	$\frac{1+s}{1}$	$\frac{1+s+\frac{1}{2}s^2}{1}$	$\frac{1+s+\frac{1}{2}s^2+\frac{1}{6}s^3}{1}$
1	$\frac{1}{1-s}$	$\frac{1+\frac{1}{2}s}{1-\frac{1}{2}s}$	$\frac{1+\frac{2}{3}s+\frac{1}{6}s^2}{1-\frac{1}{3}s}$	$\frac{1+\frac{3}{4}s+\frac{1}{4}s^2+\frac{1}{24}s^3}{1-\frac{1}{4}s}$
2	$\frac{1}{1-s+\frac{1}{2}s^2}$	$\frac{1+\frac{1}{3}s}{1-\frac{2}{3}s+\frac{1}{6}s^2}$	$\frac{1+\frac{1}{2}s+\frac{1}{12}s^2}{1-\frac{1}{2}s+\frac{1}{12}s^2}$	$\frac{1+\frac{3}{5}s+\frac{3}{20}s^2+\frac{1}{60}s^3}{1-\frac{2}{5}s+\frac{1}{20}s^2}$
3	$\frac{1}{1-s+\frac{1}{2}s^2-\frac{1}{6}s^3}$	$\frac{1+\frac{1}{4}s}{1-\frac{3}{4}s+\frac{1}{4}s^2-\frac{1}{24}s^3}$	$\frac{1+\frac{2}{5}s+\frac{1}{20}s^2}{1-\frac{3}{5}s+\frac{3}{20}s^2-\frac{1}{60}s^3}$	$\frac{1+\frac{1}{2}s+\frac{1}{10}s^2+\frac{1}{120}s^3}{1-\frac{1}{2}s+\frac{1}{10}s^2-\frac{1}{120}s^3}$

L-stable L-stable A-stable

- $m = 0$: Explicit Runge-Kutta methods with $p = \sigma$
- $m = k$: Gauss, Lobatto IIIA/IIIB (incl. implicit mid-point, trapezoidal)
- $m = k+1$: Radau-methods (incl. implicit Euler)
- $m = k+2$: Lobatto IIIC

Frequency properties, explicit methods

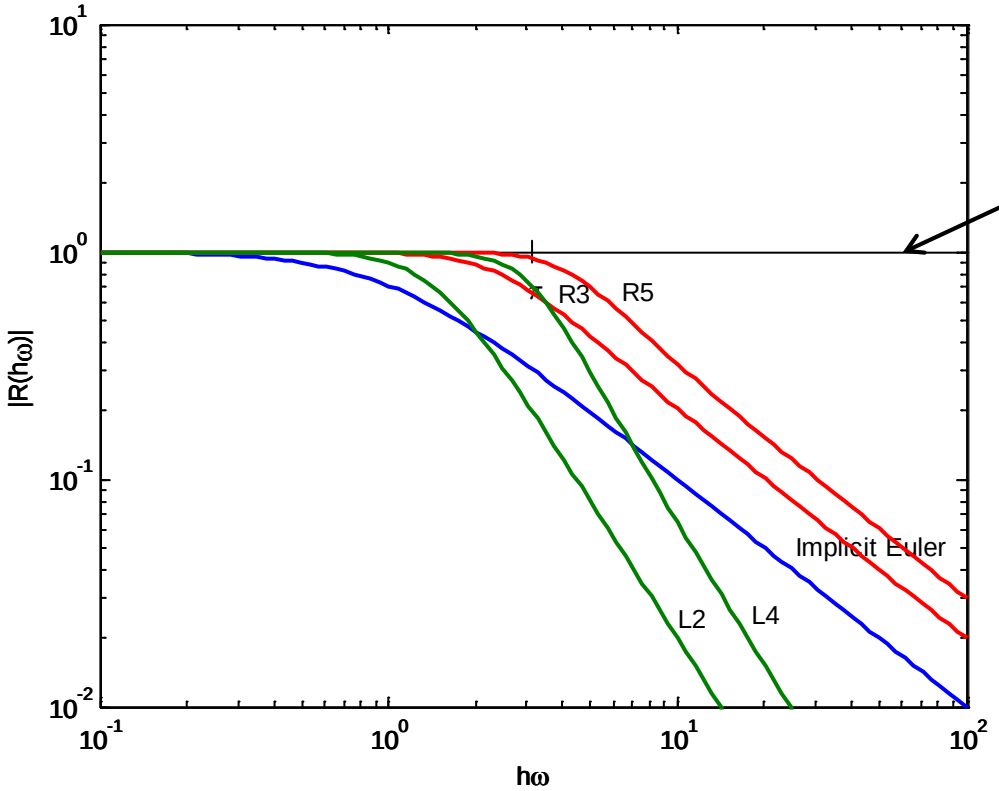




- Stability functions for explicit Runge-Kutta methods plotted as a function of $s = jh\omega$
- «Nyquist frequency» $h\omega = \pi$ indicated

Frequency properties, implicit methods

m \ k	0	1	2	3
0	$\frac{1}{1}$	$\frac{1+s}{1}$	$\frac{1+s+\frac{1}{2}s^2}{1}$	$\frac{1+s+\frac{1}{2}s^2+\frac{1}{6}s^3}{1}$
1	$\frac{1}{1-s}$	$\frac{1+\frac{1}{2}s}{1-\frac{1}{2}s}$	$\frac{1+\frac{2}{3}s+\frac{1}{6}s^2}{1-\frac{1}{3}s}$	$\frac{1+\frac{3}{4}s+\frac{1}{4}s^2+\frac{1}{24}s^3}{1-\frac{1}{4}s}$
2	$\frac{1}{1-s+\frac{1}{2}s^2}$	$\frac{1+\frac{1}{3}s}{1-\frac{2}{3}s+\frac{1}{6}s^2}$	$\frac{1+\frac{1}{2}s+\frac{1}{12}s^2}{1-\frac{1}{2}s+\frac{1}{12}s^2}$	$\frac{1+\frac{3}{5}s+\frac{3}{20}s^2+\frac{1}{60}s^3}{1-\frac{2}{5}s+\frac{1}{20}s^2}$
3	$\frac{1}{1-s+\frac{1}{2}s^2-\frac{1}{6}s^3}$	$\frac{1+\frac{1}{4}s}{1-\frac{3}{4}s+\frac{1}{4}s^2-\frac{1}{24}s^3}$	$\frac{1+\frac{2}{5}s+\frac{1}{20}s^2}{1-\frac{3}{5}s+\frac{3}{20}s^2-\frac{1}{60}s^3}$	$\frac{1+\frac{1}{2}s+\frac{1}{10}s^2+\frac{1}{120}s^3}{1-\frac{1}{2}s+\frac{1}{10}s^2-\frac{1}{120}s^3}$

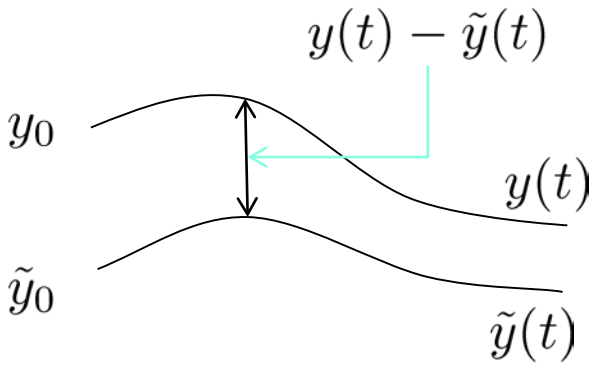


Gauss methods has $|R(jh\omega)| = 1$, i.e. no damping

- Stability functions for some implicit Runge-Kutta methods plotted as a function of $s = jh\omega$
- «Nyquist frequency» $h\omega = \pi$ indicated
- Dampens out high frequencies: «Rolloff» of -1 for Radau methods vs -2 for Lobatto IIIC methods

Nonlinear stability

- AN-stability
  - Stability for time-varying linear system
  - Implies A-stability



- B-stability:
  - Given "contracting" system $\dot{y} = f(y,t)$

    $$\|y(t) - \tilde{y}(t)\| \to 0 \text{ exponentially}$$

  - A Runge-Kutta method is B-stable if the solutions fulfill

    $$\|y_{n+1} - \tilde{y}_{n+1}\| \leq \|y_n - \tilde{y}_n\|$$

    for all contracting systems

  - B-stability implies AN-stability
  - "Difficult" to check (in general), but…

Nonlinear stability, cont'd

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^{\mathsf{T}} \end{array}$$

Algebraic stability:

- An (implicit) Runge-Kutta method is *algebraically stable* if
  - $b_i \geq 0, \quad i = 1, \ldots, \sigma$
  - $\mathbf{M} = \mathrm{diag}(\mathbf{b})\mathbf{A} + \mathbf{A}^{\mathsf{T}}\mathrm{diag}(\mathbf{b}) - \mathbf{b}\mathbf{b}^{\mathsf{T}} \geq 0$ (positive semidefinite)
- Easy to check

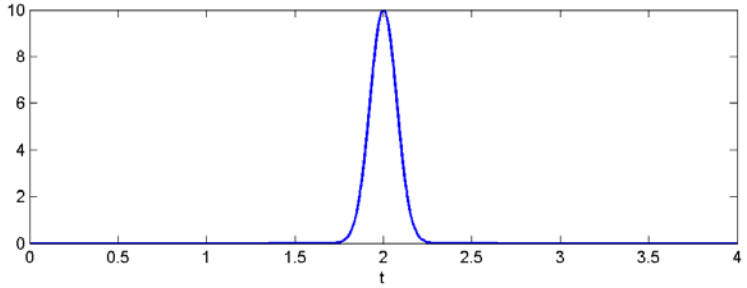
- The nonlinear stability concepts implies A-stability
  - Algebraic stability implies B-stability
  - B-stability implies AN-stability
  - AN-stability implies A-stability

- Interesting fact:
  - Trapezoid and Implicit Midpoint have same stability function (same linear stability), but only Implicit Midpoint is algebraically stable (and B-stable)

Automatic adjustment of steplength

- We have seen that accuracy depends on step length *h*
  - (e.g. A-stable methods: Always stable, but not necessarily accurate)
- How to choose step lengths?
  - Systems that are (close to) linear with eigenvalues in limited range:
    - "Easy" to choose *h* to have stability & desired accuracy everywhere
  - Systems that are (linear or nonlinear) and stiff or highly time-dependent:
    - How to choose *h*?
    - *h* too large: inaccurate (or even unstable) in some periods/regions
    - *h* too small: inefficient in some periods/regions
- Would it not be nice if we could specify what accuracy we want, and let the ODE solver choose appropriate step-lengths?

Example



- We want to simulate

$$\dot{y} = -0.6y + 10e^{-\frac{(t-2)^2}{2 \cdot 0.075^2}}, \quad y(0) = 0.5$$

- Matlab, using ode23

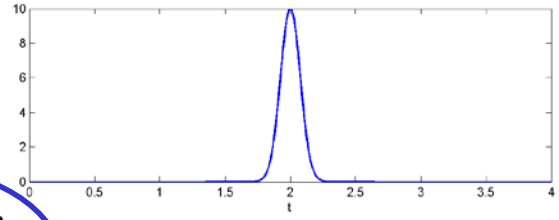
```
% f(t,y)
f = @(t,y) ( -0.6*y + 10*exp(-(t-2).^2/(2*(0.075^2))) );

% Set desired accuracy
options = odeset('RelTol',10^-3);

% Simulate
[t,y] = ode23(f, [0 4], 0.5, options);

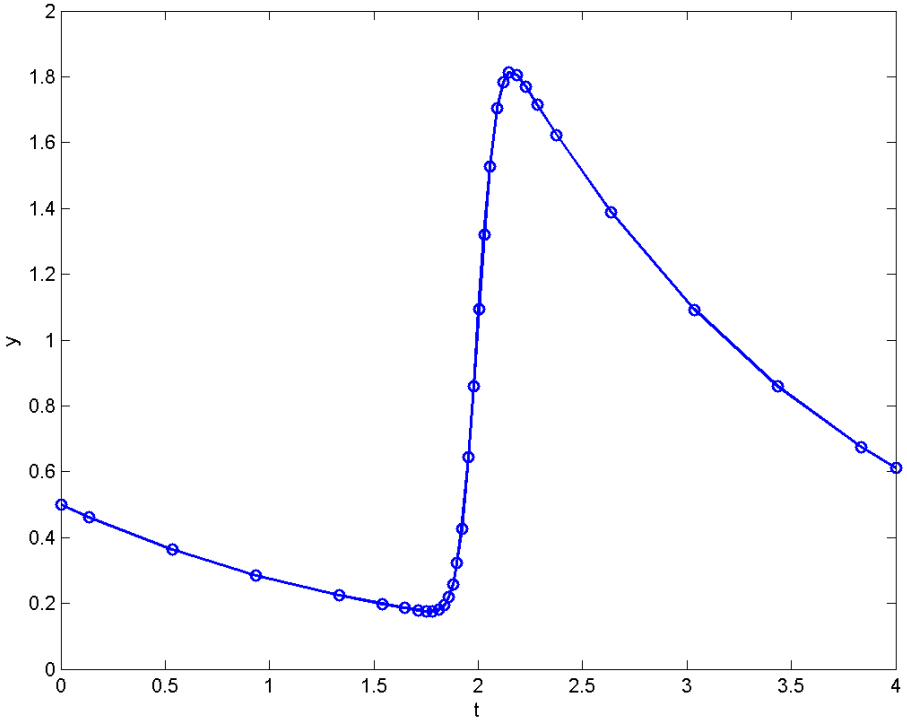
% Plot solution
plot(t,y,'-o');
```

Example, cont'd

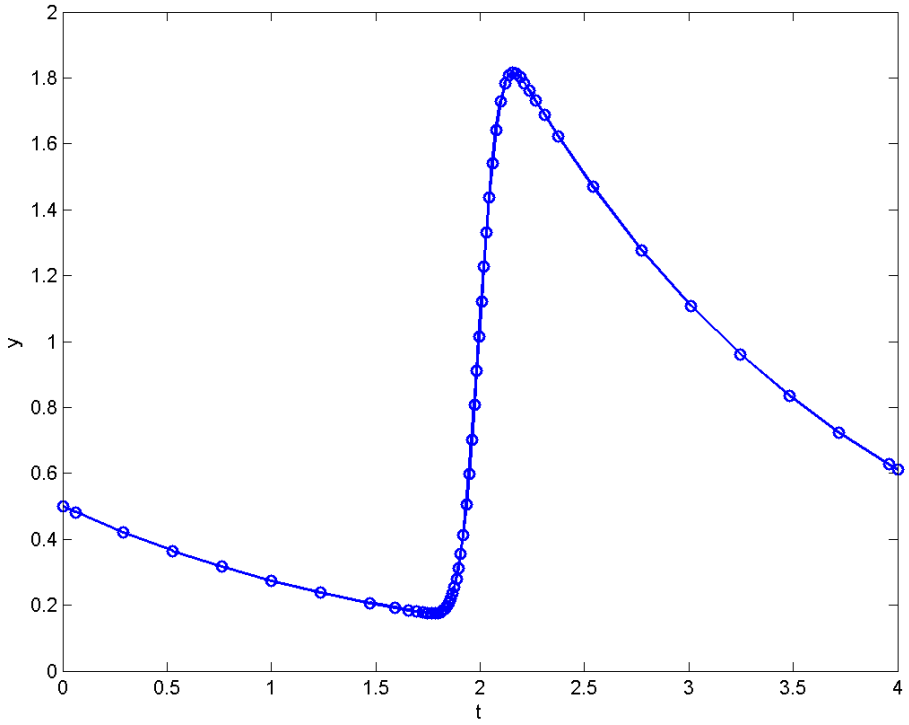


$$\dot{y} = -0.6y + 10e^{-\frac{(t-2)^2}{2 \cdot 0.075^2}}, \quad y(0) = 0.5$$

RelTol = 10^{-3}:

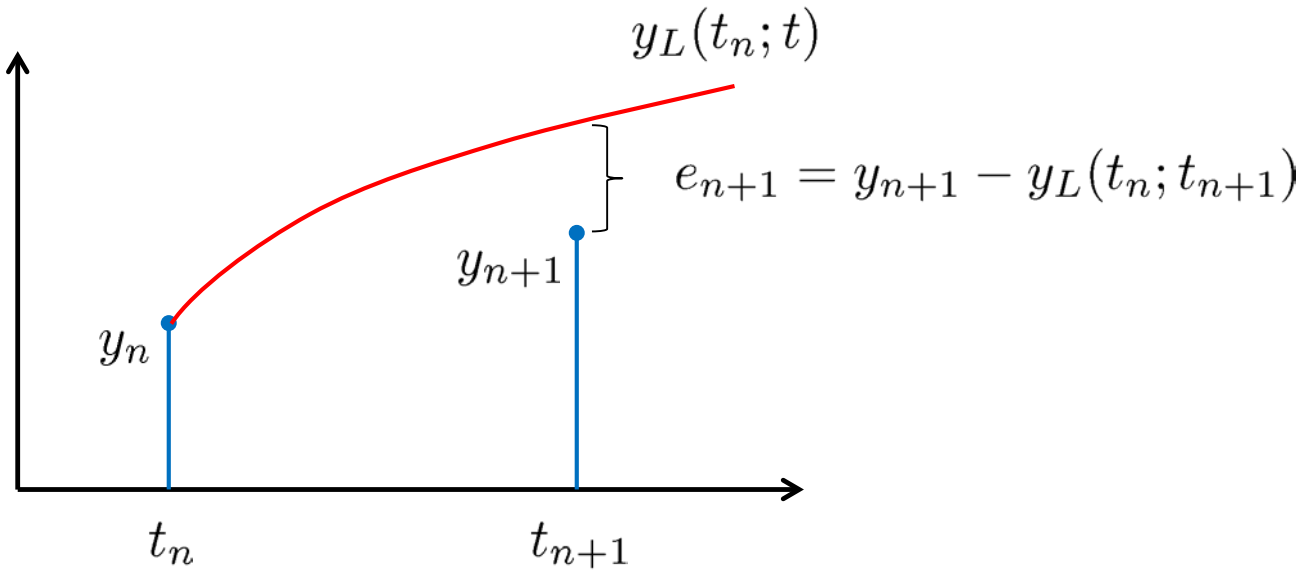


RelTol = 10^{-4}:



Estimation of local error

- Recall: Local error is error from y_n to y_{n+1}



Estimation of local error

- Starting at y_n,

Calculate y_{n+1} using (E)RK

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^{\mathsf{T}} \end{array}$$

with order *p*
local error $e_{n+1} = O(h^{p+1})$

Calculate $\hat{y}_{n+1}$ using (E)RK

$$\begin{array}{c|c} \hat{\mathbf{c}} & \hat{\mathbf{A}} \\ \hline & \hat{\mathbf{b}}^{\mathsf{T}} \end{array}$$

with order *p+1*
local error $\hat{e}_{n+1} = O(h^{p+2})$

- Local solution (per def.):

$$y_L(t_n; t_{n+1}) = y_{n+1} + e_{n+1}$$

$$y_L(t_n; t_{n+1}) = \hat{y}_{n+1} + \hat{e}_{n+1}$$

- Combine:

$$\hat{y}_{n+1} - y_{n+1} = e_{n+1} - \hat{e}_{n+1} \approx e_{n+1}$$

- Gives estimate of local error:

$$e_{n+1} \approx \hat{y}_{n+1} - y_{n+1}$$

RK4(5) Runge-Kutta-Fehlberg (1969)

- $\sigma = 6,\ p = 4,\ \hat{p} = 5$

0						
1/4	1/4					
3/8	3/32	9/32				
12/13	1932/2197	−7200/2197	7296/2197			
1	439/216	−8	3680/513	−845/4104		
1/2	-8/27	2	−3544/2565	1859/4104	−11/40	
	25/216	0	1408/2565	2197/4104	−1/5	0
	16/135	0	6656/12825	28561/56430	−9/50	2/55

- Issue: Why use y_{n+1} ($p = 4$) when we have calculated more accurate $\hat{y}_{n+1}$ ($\hat{p} = 5$)?
  - Use $\hat{y}_{n+1}$ instead: “local extrapolation”
  - Some numerical issues/optimizations concerning accuracy comes into play

Methods using local extrapolation

- Dormand-Prince 5(4) – DP5(4) (1980)

0							
1/5	1/5						
3/10	3/40	9/40					
4/5	44/45	−56/15	32/9				
8/9	19372/6561	−25360/2187	64448/6561	−212/729			
1	9017/3168	−355/33	46732/5247	49/176	−5103/18656		
1	35/384	0	500/1113	125/192	−2187/6784	11/84	
	5179/57600	0	7571/16695	393/640	−92097/339200	187/2100	1/40
	35/384	0	500/1113	125/192	−2187/6784	11/84	0

  - Implemented as ode45 in Matlab (and GNU Octave)
    - Freeware Fortran code Dopri5

- Bogacki-Shampine 2(3) – BS2(3) (1989)
  - ode23 in Matlab
  - Faster than ode45 if low accuracy demands

0				
1/2	1/2			
3/4	0	3/4		
1	2/9	1/3	4/9	
	2/9	1/3	4/9	0
	7/24	1/4	1/3	1/8

Use local error estimate to adjust step-size

- Local error estimate $e_{n+1} = (e_{1,n+1}, e_{2,n+1}, \ldots, e_{d,n+1})^\top$
- Measure of error: $\varepsilon_{n+1} = \|e_{n+1}\|_p$ (for instance $p = \infty$)

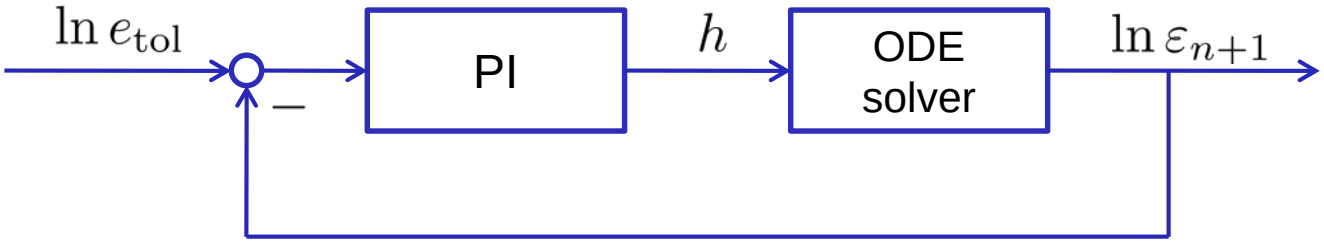
- Want error to be smaller than given tolerance e_{tol}:
  - Have: $\varepsilon_{n+1} \approx C h^{p+1}$
  - Want: $\varepsilon_{n+1} \approx e_{\text{tol}} \approx C h_{\text{new}}^{p+1}$

$$\frac{e_{\text{tol}}}{\varepsilon_{n+1}} \approx \left(\frac{h_{\text{new}}}{h}\right)^{p+1}$$

- Achieve this by choosing

$$h_{\text{new}} = h \left(\frac{e_{\text{tol}}}{\varepsilon_{n+1}}\right)^{\frac{1}{p+1}}$$

  - In practice, update somewhat smoother



How to choose e_{tol} in practice?

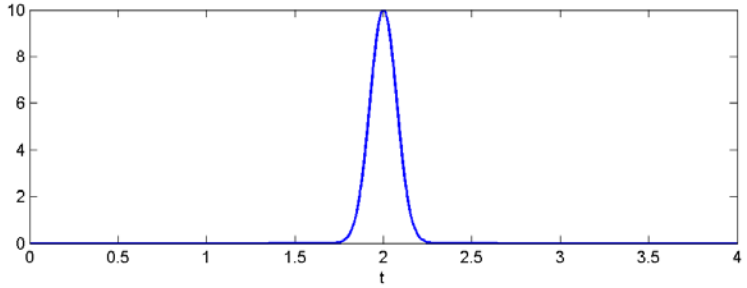
- Say you want to simulate a model of a chemical reaction, using SI units, and have:

State	Nominal values	Tolerances
Pressure	10^5 Pa	10 Pa
Concentration	0.01	10^{-6}

- To give the solver a single tolerance value, you have to scale your model!
  - (often a good idea also for other reasons)
  - In practice, a perfectly scaled model is difficult to achieve
- Alternatively, use solvers that implement relative tolerance (possibly in addition to absolute tolerance)
  - Matlab: $e_{\mathrm{tol},i} = \max\{r|y_i|, a_i\}$
  - CVode: $e_{\mathrm{tol},i} = r|y_i| + a_i$

r : RelTol (scalar) [10^{-3}]
a_i : AbsTol (vector) [10^{-6}]

Example



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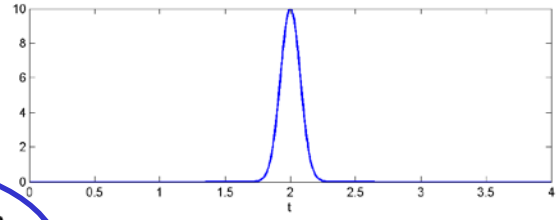
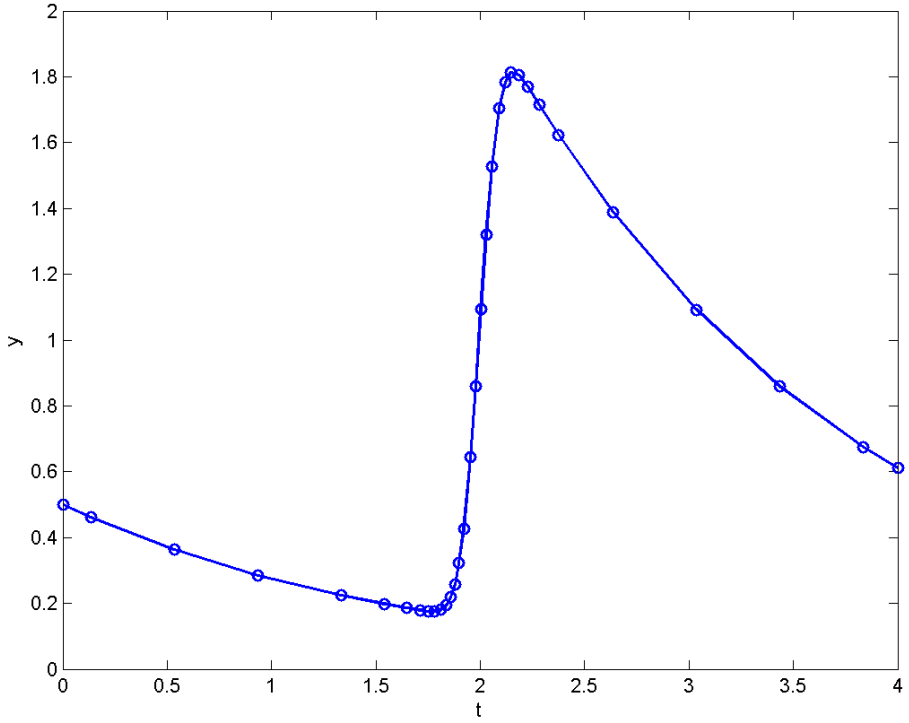
% Plot solution
plot(t,y,'-o');
```

Example, cont'd

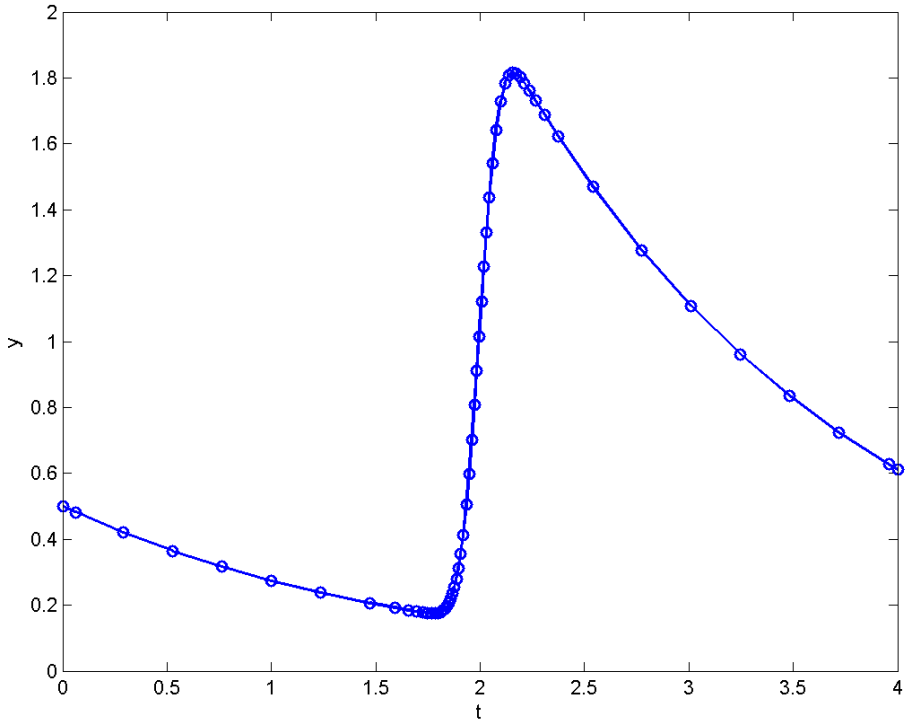


$$\dot{y} = -0.6y + 10e^{-\frac{(t-2)^2}{2 \cdot 0.075^2}}, \quad y(0) = 0.5$$

RelTol = 10^{-3}:



RelTol = 10^{-4}:

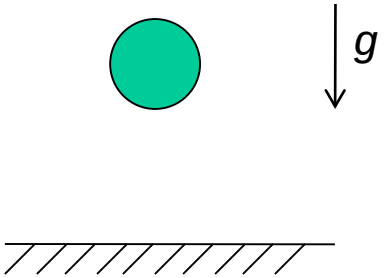


Event-detection and interpolated solutions

Event example: Bouncing ball

- Newton's law:

$$m\ddot{x} = -mg$$



- State-space:

$$\dot{x} = v$$

$$\dot{v} = -g$$

- Implementation of derivative

```
function dy = f_bb(t,y)
dy = zeros(2,1);    % column vector
g = 9.81;           % gravity
dy(1) = y(2);       % derivative of height
dy(2) = -g;         % derivative of velocity
```

- What if we hit the ground?

```
if (y(1) <=0),      % Check if we hit the ground
   y(2) = - 0.8*y(2);      % 80% elastic
end
```

Bouncing ball: Euler implementation

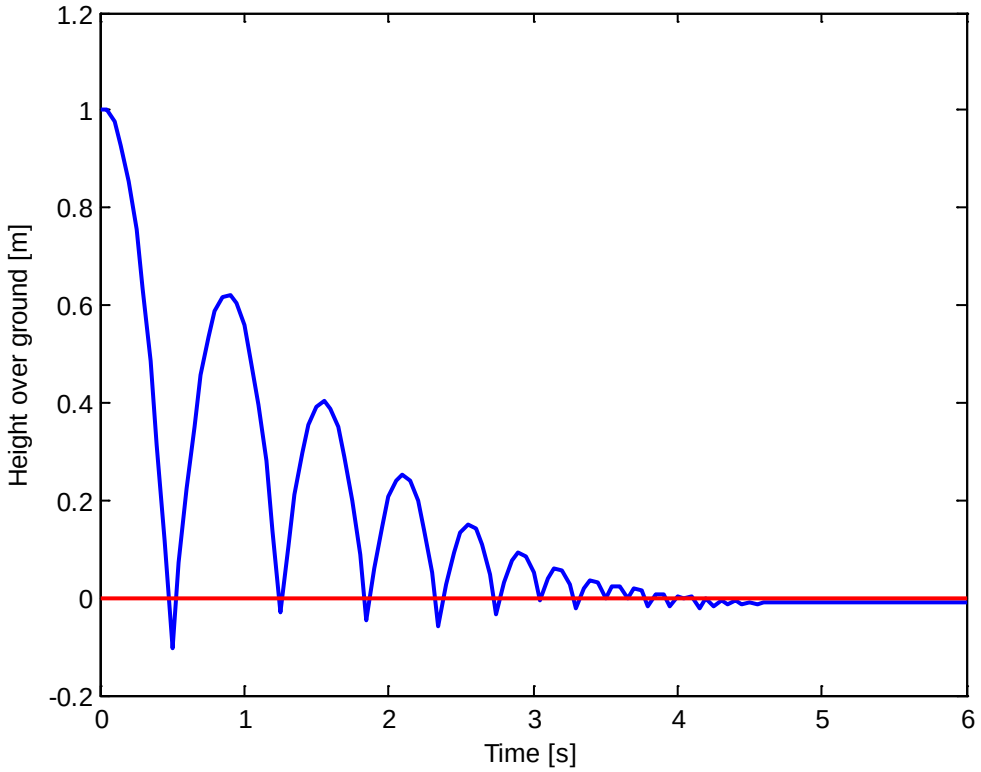
```
t_0 = 0; t_end = 6; h = 0.05; timespan = t_0:h:t_end;
y = zeros(2,length(timespan)+1); % Allocate space

x_0 = 1;          % Initial height above ground [m]
v_0 = 0;          % Initial velocity [m/s]
y(:,1) = [x_0; v_0];

for i = 1:length(timespan),
    t = timespan(i);
    y(:,i+1) = y(:,i) + h*f_bb(t,y(:,i)); % Euler integration

    if (y(1,i+1) <=0),              % Check if we hit the ground
        y(2,i+1) = - 0.8*y(2,i);    % 80% elastic
    end
end
```

Bouncing ball solved using Euler



ODE-solver with event detection

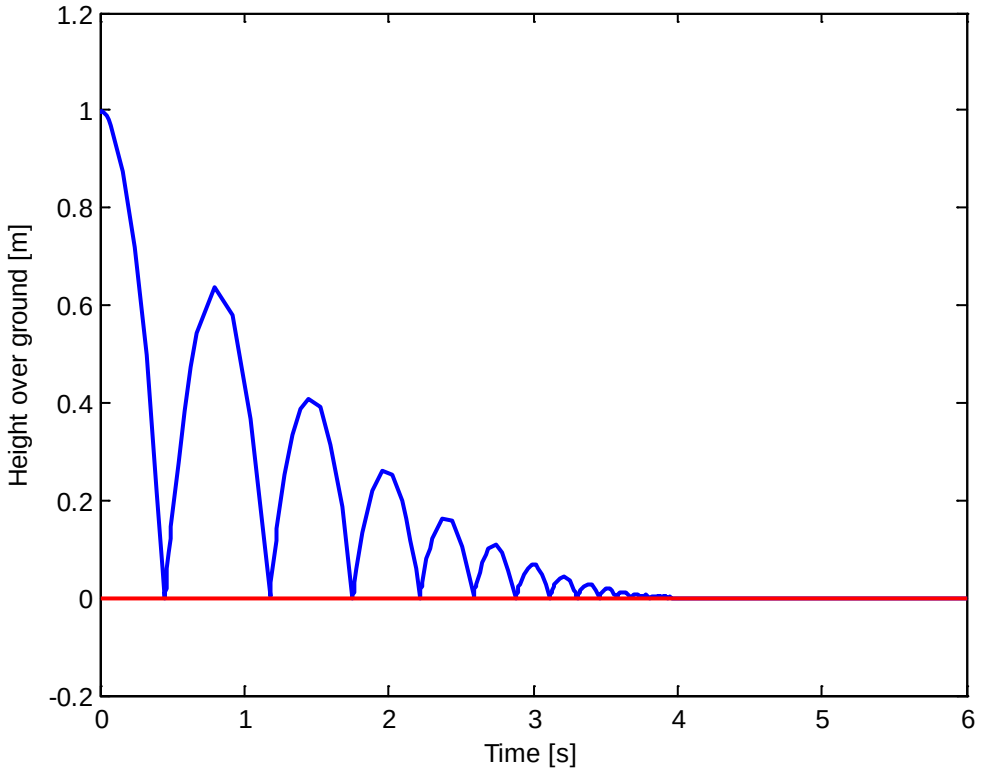
- Event-function

```
function [value,isterminal,direction] = g_bb(t,y)
value = y(1);        % Value of event-function
isterminal = 1;      % Should we stop at event? (1/0)
direction = -1;      % Event at negative to positive (1),
                     % positive to negative (-1), or both (0)
```

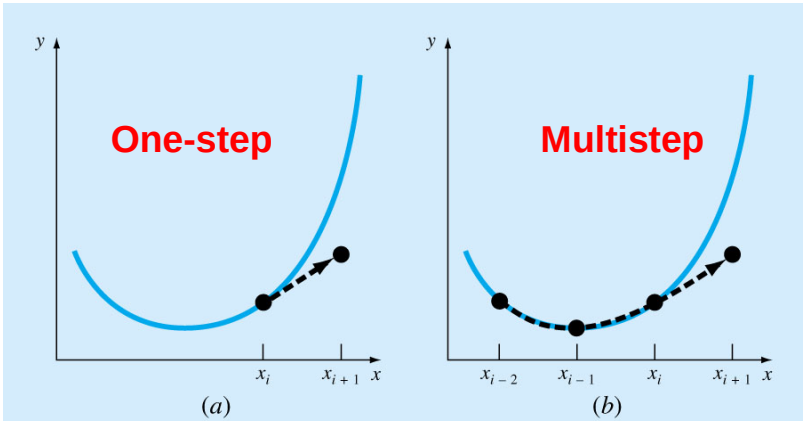
- Simulate using ode45 with event detection:

```
options = odeset('events','g_bb');          % Set event-function
t_curr = t_0; x_curr = x_0; v_curr = v_0;
while (t_curr < t_end),
    [ttemp,ytemp] = ode45(@f_bb,[t_curr t_end],[x_curr; v_curr],options);
    t_curr = ttemp(end);
    if t_curr<t_end, % Not simulated to end yet, therefore it is an event
        x_curr = 0;
        v_curr = -.8*ytemp(end,2); % 80% elastic
    end
    t = [t;ttemp]; y = [y;ytemp]; % Add to solution
end
```

Bouncing ball solved using event-function



Multi-step methods



- One-step

$$y_{n+1} = y_n + h\phi(y_n, t_n)$$

- Multi-step

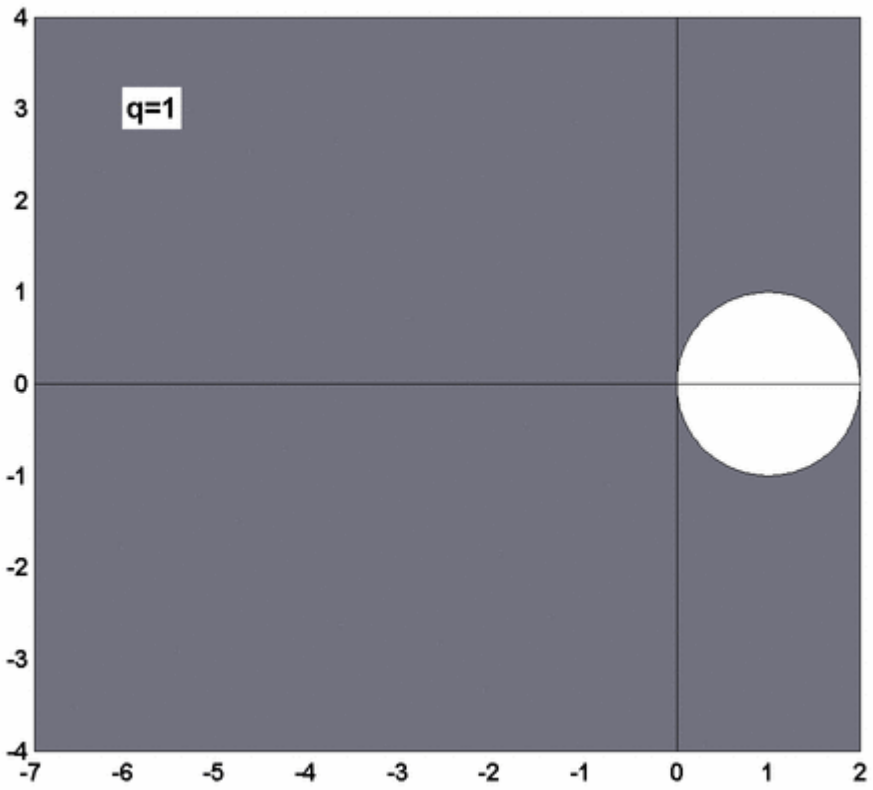
$$y_{n+1} = \alpha_1 y_n + \alpha_2 y_{n-1} + \ldots + h\left(\beta_0 f(y_{n+1}, t_{n+1}) + \beta_1 f(y_n, t_n) + \beta_2 f(y_{n-1}, t_{n-1}) + \ldots\right)$$

- Derived by fitting polynomials to previous steps

- Multi-step methods:
  - Adams-Bashforth (Explicit – $\beta_0 = 0$)
  - Adams-Moulton (Implicit)
    - PECE (Adams-Bashforth-Moulton)
  - Backward Differentiation Formula (BDF) (Implicit)
    - Numerical Differential Formula (NDF)

- Same stability concepts as for one-step RK methods apply

Stability region for Adams-Moulton



Differential-Algebraic Equations (DAE)

- ODE: $\dot{y} = f(y), \quad y(0) = y_0$
- DAE: $\dot{y} = f(y, z), \quad y(0) = y_0$
  $0 = g(y, z)$

- Advanced simulation tools (like those based on Modelica) in general generate DAEs

- If $\frac{\partial g(y,z)}{\partial z}$ is invertible [DAE is index 1] then
  $$0 = g(y, z) \text{ can be solved to } z = z(y)$$
  either symbolically (by hand/computer) or numerically

- Then the DAE can be written as ODE:
  $$\dot{y} = f(y, z(y)) = \tilde{f}(y)$$
  and ODE solvers (ERK, IRK, BDF, ...) can be used

Problems if *g* is not invertible

- Not all differential variables are state variables of the system
- The initial values y_0 of the differential variables cannot be chosen freely, but are constrained.
- The constraints are “not visible” and not explicitly given with the systems equations
- The initialization of the simulation is not without further ado possible

→ Before the simulation an analysis of the properties of the DAE system has to be performed

Differential-Algebraic equations (DAE), II

- Some higher-order index systems ($\frac{\partial g(y,z)}{\partial z}$ not invertible) can reduced to index 1 [index reduction] by using certain tricks
  - And thereby be transformed to ODE system

- Not always possible *nor desirable* to solve DAE as ODE
  - Some numerical solvers can solve (low index) DAE problems directly (especially *implicit* solvers, who must solve nonlinear equations anyway)
  - See book for examples (IRK: 14.12.1, BDF: 14.12.2)

- DAE Software
  - Matlab: ode15s, ode23t
  - DASSL/DASPK
  - Sundials IDA
  - And others...

Software

ODE solvers:

- Numerical implementations with control over accuracy (variable time-step solvers)

(ODE Integrators: Fixed step solvers)

Matlab ODE-solvers – nonstiff systems

- **ode45** is based on an explicit Runge-Kutta (4,5) formula, the Dormand-Prince pair. It is a one-step solver – in computing y(tn), it needs only the solution at the immediately preceding time point, y(tn-1). In general, **ode45** is the best function to apply as a first try for most problems.
- **ode23** is an implementation of an explicit Runge-Kutta (2,3) pair of Bogacki and Shampine. It may be more efficient than ode45 at crude tolerances and in the presence of moderate stiffness. Like **ode45**, **ode23** is a one-step solver.
- **ode113** is a variable order Adams-Bashforth-Moulton PECE solver. It may be more efficient than **ode45** at stringent tolerances and when the ODE file function is particularly expensive to evaluate. **ode113** is a multistep solver — it normally needs the solutions at several preceding time points to compute the current solution.

Matlab ODE-solvers – stiff systems

- **ode15s** is a variable order solver based on the numerical differentiation formulas (NDFs). Optionally, it uses the backward differentiation formulas (BDFs) that are usually less efficient. Like **ode113**, **ode15s** is a multistep solver. Try **ode15s** when **ode45** fails, or is very inefficient, and you suspect that the problem is stiff, or when solving a differential-algebraic problem.

- **ode23s** is based on a modified Rosenbrock formula of order 2. Because it is a one-step solver, it may be more efficient than **ode15s** at crude tolerances. It can solve some kinds of stiff problems for which **ode15s** is not effective.
- **ode23t** is an implementation of the trapezoidal rule using a "free" interpolant. Use this solver if the problem is only moderately stiff and you need a solution without numerical damping. **ode23t** can solve DAEs.
- **ode23tb** is an implementation of TR-BDF2, an implicit Runge-Kutta formula with a first stage that is a trapezoidal rule step and a second stage that is a backward differentiation formula of order two. By construction, the same iteration matrix is used in evaluating both stages. Like **ode23s**, this solver may be more efficient than **ode15s** at crude tolerances.

When to use?

Solver	Problem Type	Order of Accuracy	When to Use
ode45	Nonstiff	Medium	Most of the time. This should be the first solver you try.
ode23	Nonstiff	Low	For problems with crude error tolerances or for solving moderately stiff problems.
ode113	Nonstiff	Low to high	For problems with stringent error tolerances or for solving computationally intensive problems.
ode15s	Stiff	Low to medium	If ode45 is slow because the problem is stiff.
ode23s	Stiff	Low	If using crude error tolerances to solve stiff systems and the mass matrix is constant.
ode23t	Moderately Stiff	Low	For moderately stiff problems if you need a solution without numerical damping.
ode23tb	Stiff	Low	If using crude error tolerances to solve stiff systems.

Other ODE packages

- Sundials, https://computation.llnl.gov/casc/sundials/main.html
  - SUNDIALS (SUite of Nonlinear and DIfferential/ALgebraic equation Solvers) consists of C-code for the following five solvers:
    - **CVODE solves initial value problems for ordinary differential equation (ODE) systems.**
    - CVODES solves ODE systems and includes sensitivity analysis capabilities (forward and adjoint).
    - IDA solves initial value problems for differential-algebraic equation (DAE) systems.
    - IDAS solves DAE systems and includes sensitivity analysis capabilities (forward and adjoint).
    - KINSOL solves nonlinear algebraic systems.

- CVODE
  - Non-stiff systems: Adams-Moulton
  - Stiff systems: BDF
  - With event detection («rootfinding»)

Other solvers: DASSL

- Differential Algebraic System Solver
  - http://engineering.ucsb.edu/~cse/software.html
  - Based on BDF
  - Fortran implementation
  - With event detection («rootfinding»)

- Developed in the eighties, widely used
- Default solver in many Modelica packages
  - Dymola, OpenModelica, ...

- Extensions
  - DASPK 2.0: Large-scale systems
  - DASPK 3.1: With sensitivy analysis

Kahoot

- https://play.kahoot.it/#/k/5199a4d4-e54b-4f4b-81ea-8c8f1c3170e7