Exercise 9 - TTK4130 Modeling and Simulation

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Why on earth does this exercise even exist? What is this supposed to teach anyone except for an absolute bloody hatred for algebra?

1 Problem 1

$$\mathbf{R}_b^a = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{u}^b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{w}^a = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

1.1 a

We want to show that $\mathbf{R}_b^a$ s a rotational matrix by showing that it is a part of SO3. To show this it is enough to see that

$$\mathbf{R}_b^a (\mathbf{R}_b^a)^T = I = (\mathbf{R}_b^a)^T \mathbf{R}_b^a$$

and that

$$|\mathbf{R}_b^a| = 1.$$

1.2 b

$\mathbf{R}_b^a$ represents a rotation about the z-axis by $\psi = \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6}$.

1.3 c

$$\mathbf{R}_a^b = (\mathbf{R}_b^a)^{-1} = (\mathbf{R}_b^a)^T$$

$$\underline{\underline{\mathbf{R}_a^b = \begin{bmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}}}$$

1.4 d

$$\underline{\underline{\mathbf{u}^a = \mathbf{R}_b^a \mathbf{u}^b = \begin{bmatrix} \frac{\sqrt{3}}{2} - 1 \\ \frac{1}{2} + \sqrt{3} \\ 3 \end{bmatrix}}}$$

$$\underline{\underline{\mathbf{w}^b = \mathbf{R}_a^b \mathbf{w}^a = \begin{bmatrix} \frac{\sqrt{3}}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{\sqrt{3}}{2} \\ 2 \end{bmatrix}}}$$

1.5 e

i)

$$
\begin{aligned}
(\mathbf{u}^b)^T\mathbf{w}^b &= (\mathbf{R}_a^b\mathbf{u}^a)^T\mathbf{R}_a^b\mathbf{w}^a \\
&= (\mathbf{u}^a)^T(\mathbf{R}_a^b)^T\mathbf{R}_a^b\mathbf{w}^a \\
&= \underline{\underline{(\mathbf{u}^a)^T\mathbf{w}^a}}
\end{aligned}
$$

ii)

$$
\begin{aligned}
(\mathbf{u}^a)^\times\mathbf{w}^a &= \mathbf{u}^a \times \mathbf{w}^a \\
&= (\mathbf{R}_a^b\mathbf{u}^b) \times (\mathbf{R}_a^b\mathbf{w}^b) \\
&= \mathbf{R}_a^b(\mathbf{u}^b \times \mathbf{w}^b) \\
&= \underline{\underline{\mathbf{R}_a^b(\mathbf{u}^b)^\times\mathbf{w}^b}}
\end{aligned}
$$

1.6 f

Want to calculate $\mathbf{R}_b^a = \mathbf{R}_{z,\psi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\phi}$.

$$
\begin{aligned}
\mathbf{R}_{z,\psi} &= \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\mathbf{R}_{y,\theta} &= \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \\
\mathbf{R}_{x,\phi} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}
\end{aligned}
$$

$$
\underline{\underline{\mathbf{R}_b^a = \begin{bmatrix} c\psi c\theta & c\psi s\theta s\phi - s\psi c\phi & c\psi s\theta c\phi + s\psi s\phi \\ s\psi c\theta & s\psi s\theta s\phi + c\psi c\phi & s\psi s\theta c\phi - c\psi s\theta \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}}}
$$

1.7 g

For $\mathbf{R}_i$ to be in SO(3), they all must have determinant equal to 1. All the rows and columns must be unit vectors and must be orthogonal to each other. For $\mathbf{R}_1$, this means that

$$
\begin{aligned}
\mathbf{R}_1 &= \begin{bmatrix} r_{11} & r_{12} & r_{12} \\ r_{21} & -1 & r_{23} \\ r_{31} & r_{32} & 1 \end{bmatrix} \\
&= \begin{bmatrix} r_{11} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
|\mathbf{R}_1| &= r_{11}(-1)
\end{aligned}
$$

$$
\underline{\underline{\mathbf{R}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.}}
$$

For $\mathbf{R}_2$, this means that

$$
\begin{aligned}
\mathbf{R}_2 &= \begin{bmatrix} r_{11} & 1 & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{31} & 1 \end{bmatrix} \\
&= \begin{bmatrix} r_{11} & 1 & 0 \\ r_{21} & 0 & 0 \\ r_{31} & 0 & 1 \end{bmatrix} \\
|\mathbf{R}_2| &= -r_{21}
\end{aligned}
$$

$$
\underline{\underline{\mathbf{R}_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.}}
$$

Knowing that

$$
\mathbf{R}_3 = \begin{bmatrix} r_{11} & \frac{1}{\sqrt{3}} & r_{13} \\ r_{21} & r_{22} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & r_{32} & r_{33} \end{bmatrix}
$$

leaves us with

$$
\begin{aligned}
\sqrt{r_{11}^2 + \frac{1}{3} + r_{13}^2} &= 1 \\
\sqrt{r_{21}^2 + r_{22}^2 + \frac{1}{4}} &= 1 \\
\sqrt{\frac{1}{2} + r_{32}^2 + r_{33}^2} &= 1 \\
\frac{1}{\sqrt{3}} r_{11} + r_{21} r_{22} + \frac{1}{\sqrt{2}} r_{32} &= 0 \\
\frac{1}{\sqrt{3}} r_{13} - \frac{1}{2} r_{22} + r_{32} r_{33} &= 0 \\
r_{11} r_{13} - \frac{1}{2} r_{21} + \frac{1}{\sqrt{2}} r_{33} &= 0
\end{aligned}
$$

which I am in no way going to solve.

2 Problem 2

2.1 a

$$
\begin{aligned}
\mathbf{A}_i &= Rot_{z,\theta_i} Trans_{x,d_i} Trans_{x,a_i} Rot_{x,\alpha_i} \\
&= \begin{bmatrix} \mathbf{R}_z(\theta_i) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & a_i \mathbf{i}_x \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & d_i \mathbf{i}_z \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_z(\theta_i) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \\
&= \underline{\underline{\begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}}}
\end{aligned}
$$

2.2 b

Manipilator A

$$
\underline{\underline{A_1 = \begin{bmatrix} cq_1 & -sq_1 & 0 & l_1 cq_1 \\ sq_1 & cq_1 & 0 & l_1 sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}}
$$

$$
\underline{\underline{A_2 = \begin{bmatrix} 1 & 0 & 0 & q_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}}
$$

Manipulator B

$$A_1 = \begin{bmatrix} cq_1 & -sq_1 & 0 & l_1cq_1 \\ sq_1 & cq_1 & 0 & l_1sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} cq_2 & -sq_2 & 0 & l_2cq_2 \\ sq_2 & cq_2 & 0 & l_2sq_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.3 c

Manipilator A

$$T_2^0 = A_1A_2 = \begin{bmatrix} cq_1 & -sq_1 & 0 & (q_2+l_1)cq_1 \\ sq_1 & cq_1 & 0 & (q_2+l_1)sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Manipilator B

$$T_2^0 = A_1A_2 = \begin{bmatrix} c(q_1+q_2) & -s(q_1+q_2) & 0 & l_2c(q_1+q_2)+l_1cq_1 \\ s(q_1+q_2) & c(q_1+q_2) & 0 & l_2s(q_1+q_2)+l_1sq_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.4 d

Why? Just how is this going to teach anyone anything other than hatred of algebra?

Manipilator A

$$T_2^0 = A_1A_2 = \begin{bmatrix} cq_1 - sq_1 + (q_2+l_1)cq_1 \\ sq_1 + cq_1 + (q_2+l_1)sq_1 \\ 1 \\ 1 \end{bmatrix}$$

Manipilator B

$$g^0 = T_2^0 g^2 = \begin{bmatrix} c(q_1+q_2) - s(q_1+q_2) + l_2c(q_1+q_2) + l_1cq_1 \\ s(q_1+q_2) + c(q_1+q_2) + l_2s(q_1+q_2) + l_1sq_1 \\ 1 \\ 1 \end{bmatrix}$$

3 Problem 3

$\mathbf{R} = \mathbf{R}_{\mathbf{k},\theta} = \mathbf{e}^{\times} + \cos\theta\mathbf{I} + \mathbf{k}\mathbf{k}^T(1-\cos\theta)$

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

3.1 a

Mikkel was not able to solve this, nor explain why the result was right. If even he can't explain me why the solution is as it is, there is no point in me typing the solution in here just to pretend I understood any of it.

3.2 b

Since $\mathbf{R}_{\mathbf{k},-\theta} = -\mathbf{e}^{\times} + \cos\theta\mathbf{I} + \mathbf{k}\mathbf{k}^T(1-\cos\theta)$, $\mathbf{R}_{\mathbf{k},\theta} - \mathbf{R}_{\mathbf{k},-\theta} = 2\mathbf{e}^{\times}$, and

$$\mathbf{R}_{\mathbf{k},\theta} - \mathbf{R}_{\mathbf{k},-\theta} = \begin{bmatrix} 0 & -2e_z & 2e_y \\ 2e_z & 0 & -2e_x \\ -2e_y & 2e_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & r_{12}-r_{21} & r_{13}-r_{31} \\ r_{21}-r_{12} & 0 & r_{23}-r_{32} \\ r_{31}-r_{13} & r_{32}-r_{23} & 0 \end{bmatrix}$$

we know that

$$\mathbf{e} = \frac{1}{2}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}.$$

3.3 c

Since $\mathbf{k}$ depends on $sin\theta$, and $\sin\theta$ is 2π periodic, the rotation will switch both rotational direction and the direction of the axis abiut which it is rotating. This will result in the exact same rotation.

3.4 d

```
theta =

    1.9999



k =

    0.6667
    0.3333
    0.6667
```

Figure 1: The output from running the code in Listing 1

Listing 1: Code for finding θ and $\mathbf{k}$ from a matrix $\mathbf{R}$ in MATLAB.

```

R = [0.2133 -0.2915 0.9325;
      0.9209 -0.2588 -0.2915
      0.3263 0.9209 0.2133]

e = 1/2*[R(3,2)-R(2,3); R(1,3)-R(3,1); R(2,1)-R(1,2)];
trace = R(1,1)+R(2,2)+R(3,3);
theta = acos((trace -1)/2)
```

```
k = e/sin(theta)
```