Lecture 14: Rigid body kinematics – the rotation matrix

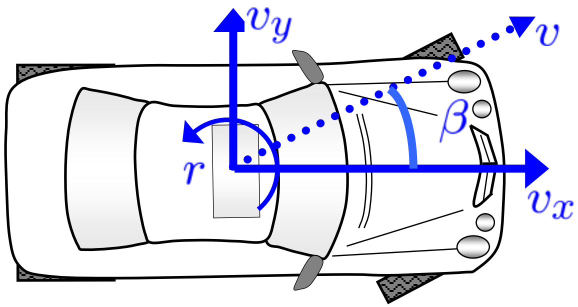
- What are rotation matrices used for?

- Rotation matrices
  - Composite rotations, simple rotations
  - Homogenous transformation matrices
- Euler angles
  - 3-parameter specification of rotations
  - Roll-pitch-yaw
- Angle-axis
  - 4-parameter specification of rotations

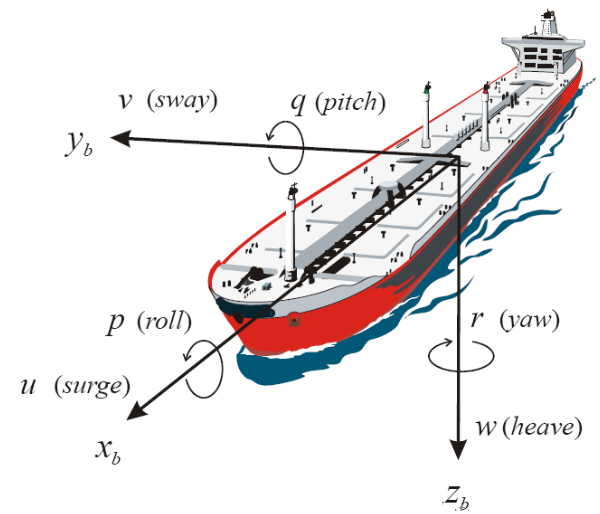
Book: Ch. 6.4, 6.5, 6.6

Why rotation matrices?

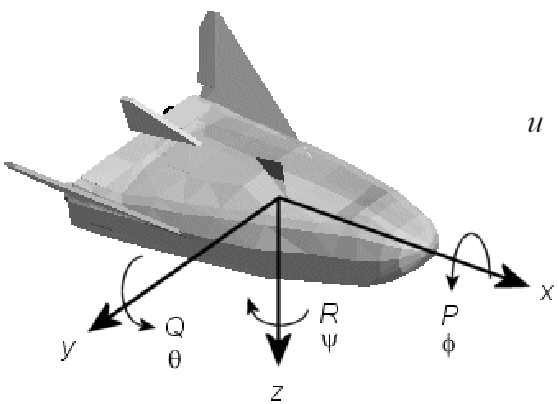
- Rotation matrices are used to describe rotations and orientations of rigid bodies
  - Road vehicles



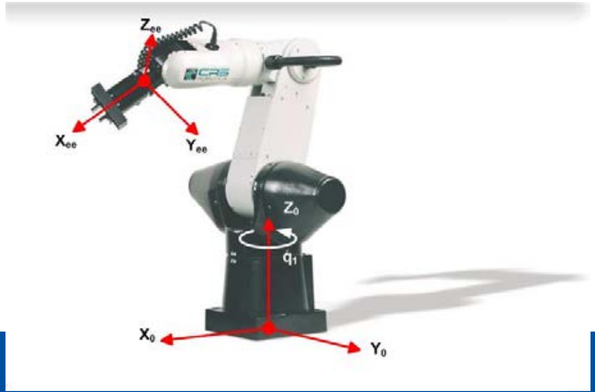
  - Marine vessels



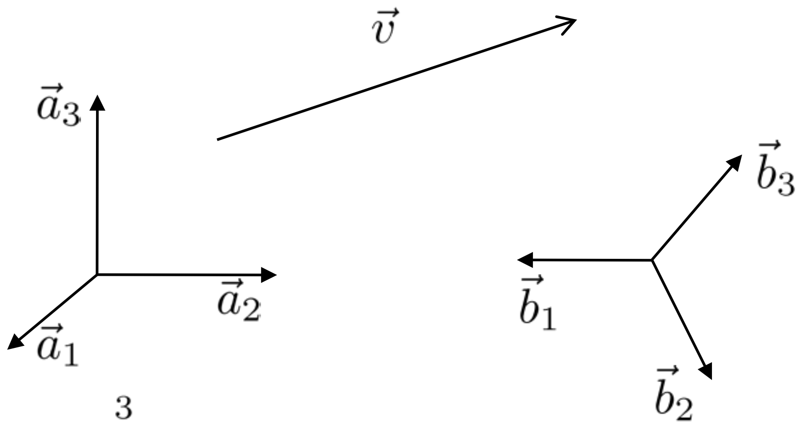
  - Airplanes, satellites



  - Robotics



Rotation matrices



- The vector $\vec{v}$ can be written as

$$\vec{v} = \sum_{j=1}^{3} v_j^a \vec{a}_j \qquad \text{or} \qquad \vec{v} = \sum_{j=1}^{3} v_j^b \vec{b}_j$$

- These must be the same:

$$\sum_{j=1}^{3} v_j^a \vec{a}_j = \sum_{j=1}^{3} v_j^b \vec{b}_j$$

- Scalar product with $\vec{a}_i$ on both sides:

$$\sum_{j=1}^{3} v_j^a \vec{a}_j \cdot \vec{a}_i = \sum_{j=1}^{3} v_j^b \vec{b}_j \cdot \vec{a}_i \quad \Rightarrow \quad v_i^a = \sum_{j=1}^{3} v_j^b \vec{a}_i \cdot \vec{b}_j$$

- Gives

$$\mathbf{v}^a = \begin{pmatrix} v_1^a \\ v_2^a \\ v_3^a \end{pmatrix} = \begin{pmatrix} \vec{a}_1 \cdot \vec{b}_1 & \vec{a}_1 \cdot \vec{b}_2 & \vec{a}_1 \cdot \vec{b}_3 \\ \vec{a}_2 \cdot \vec{b}_1 & \vec{a}_2 \cdot \vec{b}_2 & \vec{a}_2 \cdot \vec{b}_3 \\ \vec{a}_3 \cdot \vec{b}_1 & \vec{a}_3 \cdot \vec{b}_2 & \vec{a}_3 \cdot \vec{b}_3 \end{pmatrix} \begin{pmatrix} v_1^b \\ v_2^b \\ v_3^b \end{pmatrix} = \mathbf{R}_b^a \mathbf{v}^b$$

Rotation matrices, properties

- We have shown

$$\mathbf{v}^a = \begin{pmatrix} v_1^a \\ v_2^a \\ v_3^a \end{pmatrix} = \begin{pmatrix} \vec{a}_1 \cdot \vec{b}_1 & \vec{a}_1 \cdot \vec{b}_2 & \vec{a}_1 \cdot \vec{b}_3 \\ \vec{a}_2 \cdot \vec{b}_1 & \vec{a}_2 \cdot \vec{b}_2 & \vec{a}_2 \cdot \vec{b}_3 \\ \vec{a}_3 \cdot \vec{b}_1 & \vec{a}_3 \cdot \vec{b}_2 & \vec{a}_3 \cdot \vec{b}_3 \end{pmatrix} \begin{pmatrix} v_1^b \\ v_2^b \\ v_3^b \end{pmatrix} = \mathbf{R}_b^a \mathbf{v}^b$$

- Switching *a* and *b*, we obtain

$$\mathbf{v}^b = \begin{pmatrix} v_1^b \\ v_2^b \\ v_3^b \end{pmatrix} = \begin{pmatrix} \vec{b}_1 \cdot \vec{a}_1 & \vec{b}_1 \cdot \vec{a}_2 & \vec{b}_1 \cdot \vec{a}_3 \\ \vec{b}_2 \cdot \vec{a}_1 & \vec{b}_2 \cdot \vec{a}_2 & \vec{b}_2 \cdot \vec{a}_3 \\ \vec{b}_3 \cdot \vec{a}_1 & \vec{b}_3 \cdot \vec{a}_2 & \vec{b}_3 \cdot \vec{a}_3 \end{pmatrix} \begin{pmatrix} v_1^a \\ v_2^a \\ v_3^a \end{pmatrix} = \mathbf{R}_a^b \mathbf{v}^a$$

- We see that $\mathbf{R}_a^b = \left(\mathbf{R}_b^a\right)^\mathsf{T}$

- From $\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b = \mathbf{R}_b^a \mathbf{R}_a^b \mathbf{v}^a$, we see that $\mathbf{R}_b^a \mathbf{R}_a^b = \mathbf{I}$

$$\boxed{\mathbf{R}_a^b = \left(\mathbf{R}_b^a\right)^\mathsf{T} = \left(\mathbf{R}_b^a\right)^{-1}}$$

The set of rotation matrices

For a matrix $\mathbf{R}$ to be a rotation matrix:

- The matrix must be orthogonal:

$$\mathbf{R}\mathbf{R}^{\mathsf{T}} = \mathbf{I}$$

- The determinant must be one

$$\det \mathbf{R} = 1$$

- The set of these matrices has a name: SO(3), or Special Orthogonal group of order 3:

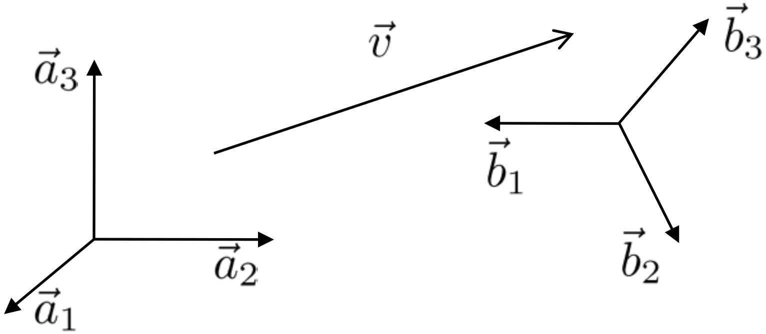
$$\mathrm{SO}(3) = \{\mathbf{R} \in \mathbb{R}^{3\times 3} \mid \mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I},\ \det \mathbf{R} = 1\}$$

Rotation matrices

The rotation matrix from *a* to *b* $\mathbf{R}_b^a$ is used to

- **Transform** a coordinate vector from *b* to *a*

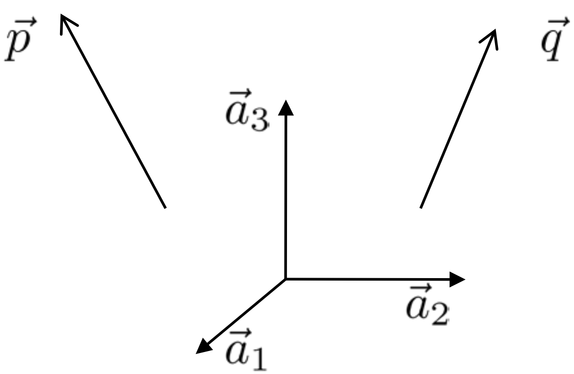
$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b$$



- **Rotate** a vector $\vec{p}$ to vector $\vec{q}$. If decomposed in *a*,

$$\mathbf{q}^a = \mathbf{R}_b^a \mathbf{p}^a$$

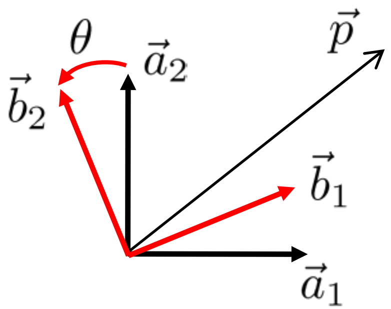
such that $\mathbf{q}^b = \mathbf{p}^a$.

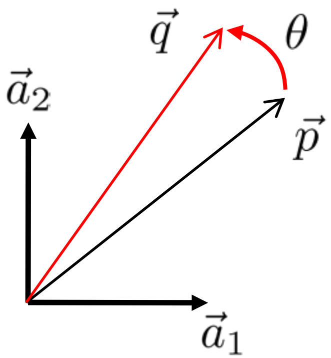


Rotation vs transformation (same, again)

- A coordinate vector may change either as a result of a rotation of a coordinate system (a **coordinate transformation**) or a rotation of the vector itself (a **rotation**).

- That is, a rotation from *a* to *b* can be interpreted in two ways:





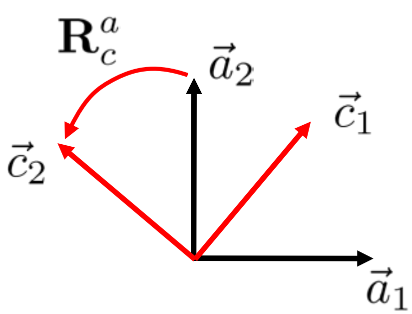
$\mathbf{p}^b = \mathbf{R}^b_a \mathbf{p}^a$ (or $\mathbf{p}^a = \mathbf{R}^a_b \mathbf{p}^b$)

$\mathbf{q}^a = \mathbf{R}^a_b \mathbf{p}^a$ such that $\mathbf{q}^b = \mathbf{p}^a$

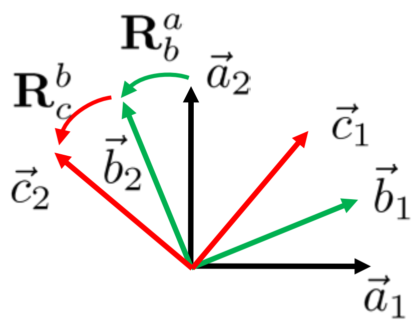
- That is, the matrix $\mathbf{R}^a_b$ rotates from *a* to *b*, but transforms from *b* to *a*!

- (Sometimes these two interpretations of the rotations originating from a rotation matrix are called passive vs active transformations, or alias vs alibi transformations)

Composite rotations



$$\mathbf{v}^a = \mathbf{R}_c^a \mathbf{v}^c$$

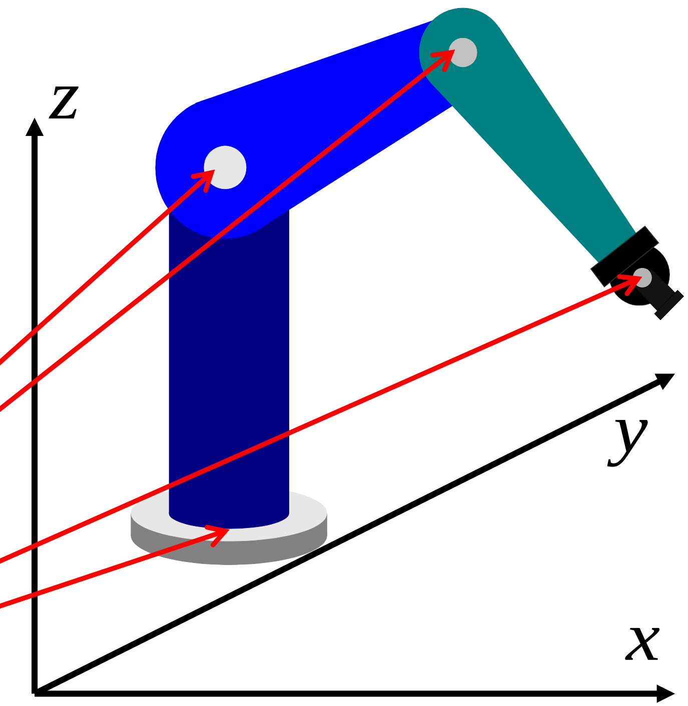


$$\mathbf{v}^b = \mathbf{R}_c^b \mathbf{v}^c$$
$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b = \mathbf{R}_b^a \mathbf{R}_c^b \mathbf{v}^c$$

$$\boxed{\mathbf{R}_c^a = \mathbf{R}_b^a \mathbf{R}_c^b}$$

(and $\mathbf{R}_d^a = \mathbf{R}_b^a \mathbf{R}_c^b \mathbf{R}_d^c$, etc.)

Kinematics in robotics



- Forward kinematics
  - Given joint variables

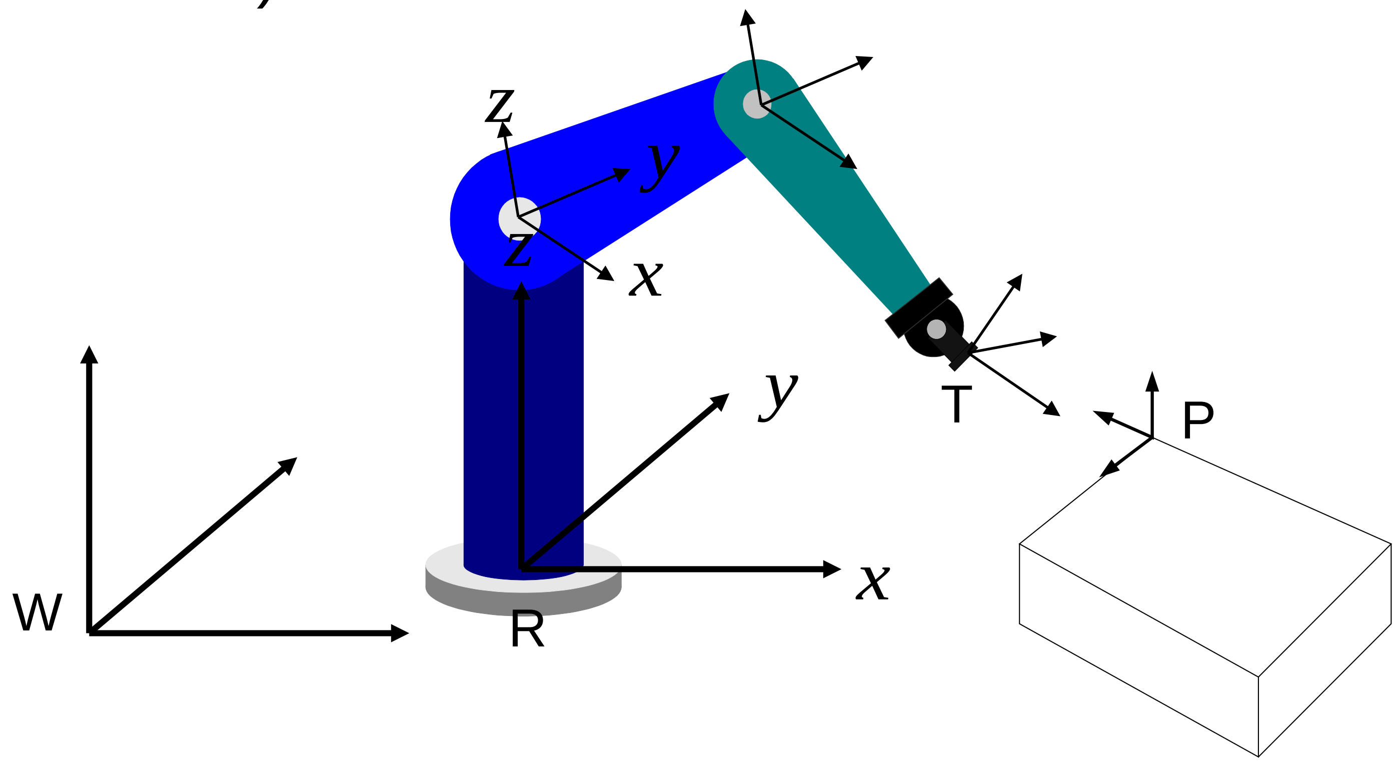
$$q = (q_1, q_2, q_3, \ldots, q_n)$$

  - What are end-effector position and orientation?

- Inverse kinematics
  - Given (desired) end-effector position and orientation.
  - What are the corresponding joint variables?

Coordinate systems in robotics

- World frame
- Joint frame
- Tool (end-effector) frame



Representations of rotations

- Rotation matrix
  - Simple, but over-parameterized (9 parameters)

Euler's Theorem:

"Any two independent orthonormal coordinate frames can be related by a sequence of rotations (not more than three) about coordinate axes, where no two successive rotations may be about the same axis."

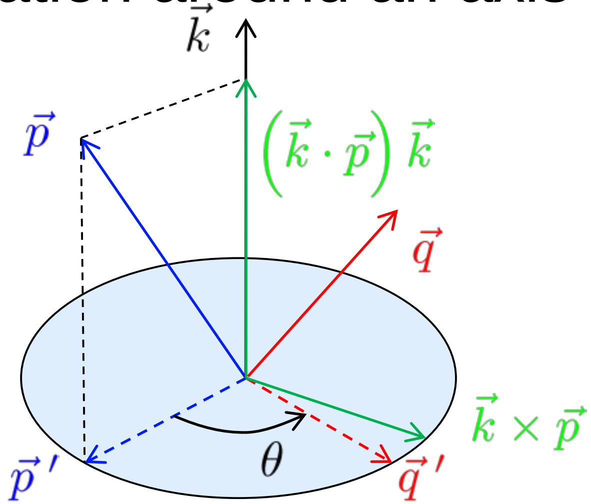
- Three rotations about axes are enough to specify any rotation
  - These representations are called Euler angles
    - 12 different combinations possible
    - Most common: Roll-pitch-yaw
  - Natural and (in many cases) simple to use, very much used
  - Problem: Singularity (more on this later)

- Angle-axis, Euler-parameters
  - 4-parameters are used
  - No singularity problems

Rotation of vectors based on angle-axis representation

- Angle-axis: All rotations can be represented as a simple rotation around an axis

Somewhat different derivation of the rotation dyadic. Compare p. 228 in book.



$$\vec{p}\,' = \vec{p} - \left(\vec{k}\cdot\vec{p}\right)\vec{k}$$

$$\vec{q}\,' = \vec{q} - \left(\vec{k}\cdot\vec{q}\right)\vec{k} = \vec{q} - \left(\vec{k}\cdot\vec{p}\right)\vec{k}$$

$$\vec{q}\,' = \cos\theta\,\vec{p}\,' + \sin\theta\,\vec{k}\times\vec{p}$$

$$\vec{q} - \left(\vec{k}\cdot\vec{p}\right)\vec{k} = \cos\theta\,\left(\vec{p} - \left(\vec{k}\cdot\vec{p}\right)\vec{k}\right) + \sin\theta\,\vec{k}\times\vec{p}$$

$$\boxed{\vec{q} = \cos\theta\,\vec{p} + \sin\theta\,\vec{k}\times\vec{p} + (1-\cos\theta)\left(\vec{k}\cdot\vec{p}\right)\vec{k}}$$

Kahoot

- https://play.kahoot.it/#/k/8c1f768d-76cf-40e4-8163-ea279354e62a