Lecture 15: Rigid body kinematics – Rotations, angular velocity

Representations of rotation

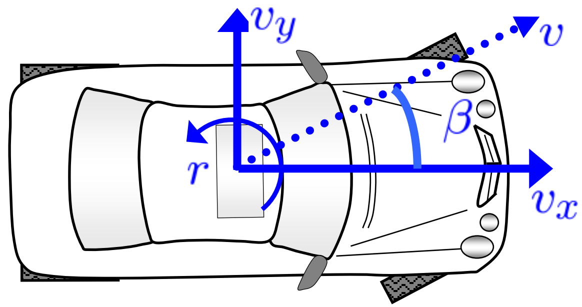
- Rotation matrices
- Euler angles
- 3-parameter specification of rotations
  - Roll-pitch-yaw
- Angle-axis, Euler-parameters
  - 4-parameter specification of rotations

- Angular velocity

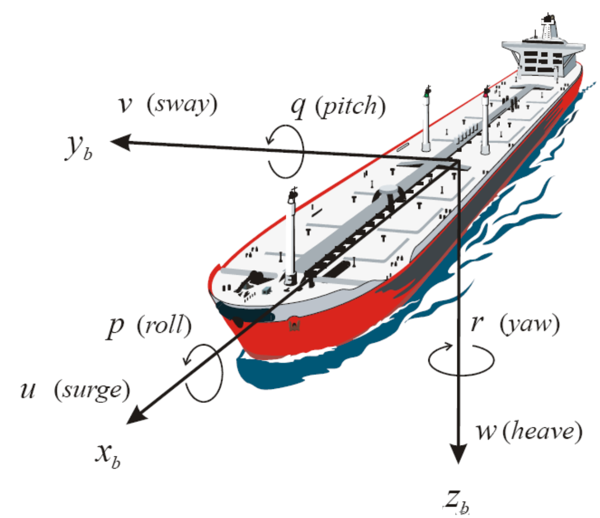
Book: Ch. 6.6, 6.7, 6.8

Why rotation matrices?

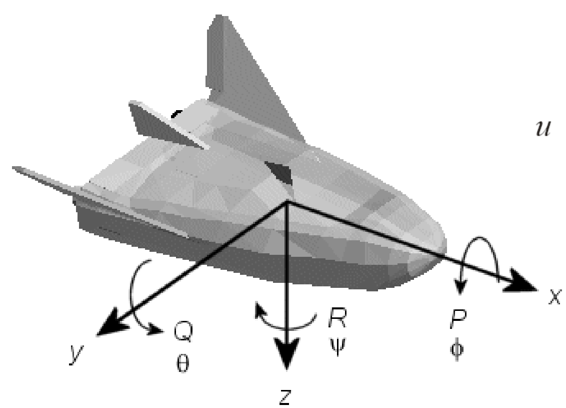
- Rotation matrices are used to describe rotations and orientations of rigid bodies
  - Road vehicles



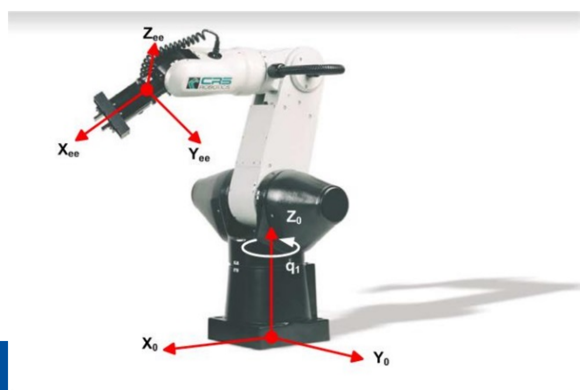
  - Marine vessels



  - Airplanes, satellites



  - Robotics

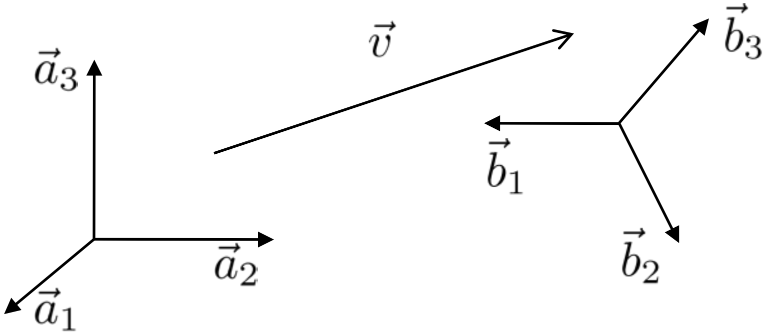


Rotation matrices

The <u>rotation matrix from *a* to *b*</u> $\mathbf{R}_b^a$ is used to

- Transform a coordinate vector from *b* to *a*

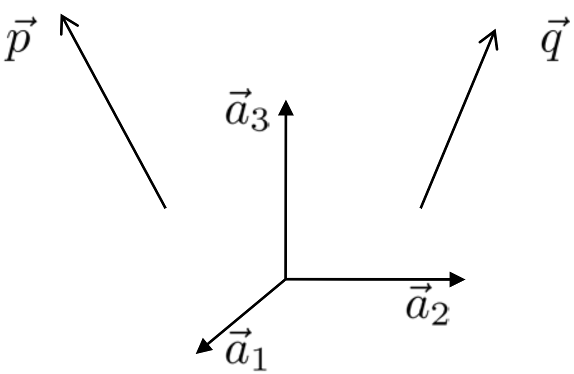
$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b$$



- Rotate a vector $\vec{p}$ to vector $\vec{q}$. If decomposed in *a*,

$$\mathbf{q}^a = \mathbf{R}_b^a \mathbf{p}^a$$

such that $\mathbf{q}^b = \mathbf{p}^a$.



Representations of rotations

- Rotation matrix
  - Simple, but over-parameterized (9 parameters)

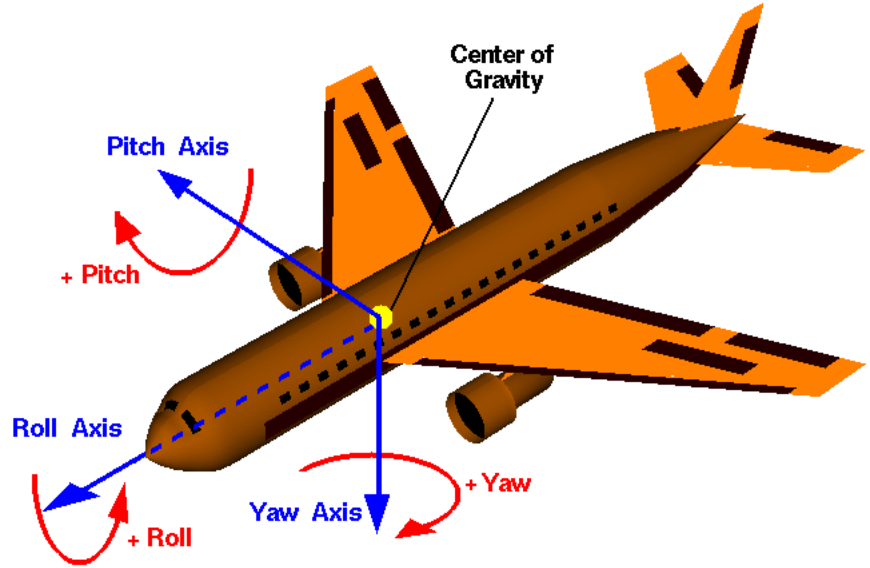
Euler's Theorem:

"Any two independent orthonormal coordinate frames can be related by a sequence of rotations (not more than three) about coordinate axes, where no two successive rotations may be about the same axis."

- Three rotations about axes are enough to specify any rotation
  - These representations are called Euler angles
    - 12 different combinations possible
    - Most common: Roll-pitch-yaw
  - Natural and (in many cases) simple to use, very much used
  - Problem: Singularity (more on this later)

- Angle-axis, Euler-parameters
  - 4-parameters are used
  - No singularity problems

Euler-angles: Roll-pitch-yaw



- Rotation ψ about *z*-axis, θ about (rotated) *y*-axis, ϕ about (rotated) *x*-axis

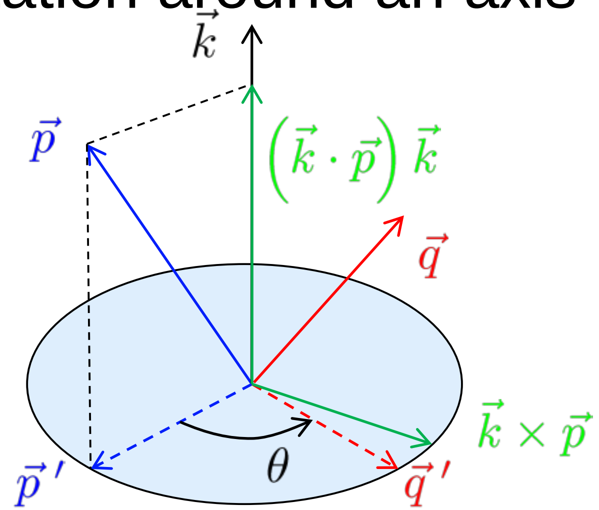
$$\mathbf{R}_b^a = \mathbf{R}_{z,\psi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\phi}$$

$$\mathbf{R}_b^a = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

Rotation of vectors based on angle-axis representation

- Angle-axis: All rotations can be represented as a simple rotation around an axis

Somewhat different derivation of the rotation dyadic. Compare p. 228 in book.



$$\vec{p}\,' = \vec{p} - \left(\vec{k}\cdot\vec{p}\right)\vec{k}$$

$$\vec{q}\,' = \vec{q} - \left(\vec{k}\cdot\vec{q}\right)\vec{k} = \vec{q} - \left(\vec{k}\cdot\vec{p}\right)\vec{k}$$

$$\vec{q}\,' = \cos\theta\,\vec{p}\,' + \sin\theta\,\vec{k}\times\vec{p}$$

$$\vec{q} - \left(\vec{k}\cdot\vec{p}\right)\vec{k} = \cos\theta\,\left(\vec{p} - \left(\vec{k}\cdot\vec{p}\right)\vec{k}\right) + \sin\theta\,\vec{k}\times\vec{p}$$

$$\boxed{\vec{q} = \cos\theta\,\vec{p} + \sin\theta\,\vec{k}\times\vec{p} + (1-\cos\theta)\left(\vec{k}\cdot\vec{p}\right)\vec{k}}$$

Angle-axis rotation dyadic, rotation matrix

- Rotation θ about an axis $\vec{k}$

$$\vec{q} = \cos\theta\,\vec{p} + \sin\theta\,\vec{k}\times\vec{p} + (1-\cos\theta)\,\vec{k}\left(\vec{k}\cdot\vec{p}\right)$$

- Angle-axis rotation by a dyadic

$$\vec{q} = \Big(\underbrace{\cos\theta\,\vec{I} + \sin\theta\,\vec{k}^{\times} + (1-\cos\theta)\,\vec{k}\vec{k}}_{\vec{R}_{\vec{k},\theta}}\Big)\cdot\vec{p}$$

$$\vec{q} = \vec{R}_{\vec{k},\theta}\cdot\vec{p}$$

- Angle-axis rotation matrix

$$\boxed{\mathbf{R}_b^a = \mathbf{R}_{\mathbf{k},\theta} = \cos\theta\,\mathbf{I} + \sin\theta\,(\mathbf{k}^a)^{\times} + (1-\cos\theta)\,\mathbf{k}^a(\mathbf{k}^a)^{\mathsf{T}}}$$

- Alternative expression (using $\mathbf{k}^a = \mathbf{k}$ and $\mathbf{k}^{\times}\mathbf{k}^{\times} = \mathbf{k}(\mathbf{k})^{\mathsf{T}} - \mathbf{I}$):

$$\boxed{\mathbf{R}_b^a = \mathbf{R}_{\mathbf{k},\theta} = \mathbf{I} + \sin\theta\,\mathbf{k}^{\times} + (1-\cos\theta)\,\mathbf{k}^{\times}\mathbf{k}^{\times}}$$

Use of Euler parameters

- ABB robots use Euler parameters (quaternions) internally in the robot control program
  - and Euler angles “externally”

- In Modelica.multibody, one can use either rotation matrices or Euler parameters (quaternions)

- Euler parameters (quaternions) often used in “advanced control” of robots, satellites, etc.

Angular velocity