Lecture 17: Newton-Euler equations of motion

- Rigid body kinetics (Newton-Euler equations of motion)
  - Newton's law
  - Angular momentum
  - Inertia dyadic

Book: Ch. 7.3

What is rigid body dynamics?

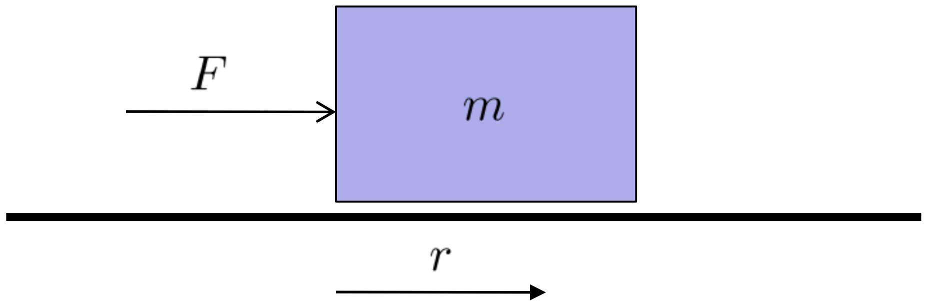
- Rigid body:
  - Wikipedia: “…a rigid body is an idealization of a solid body of finite size in which deformation is neglected.”

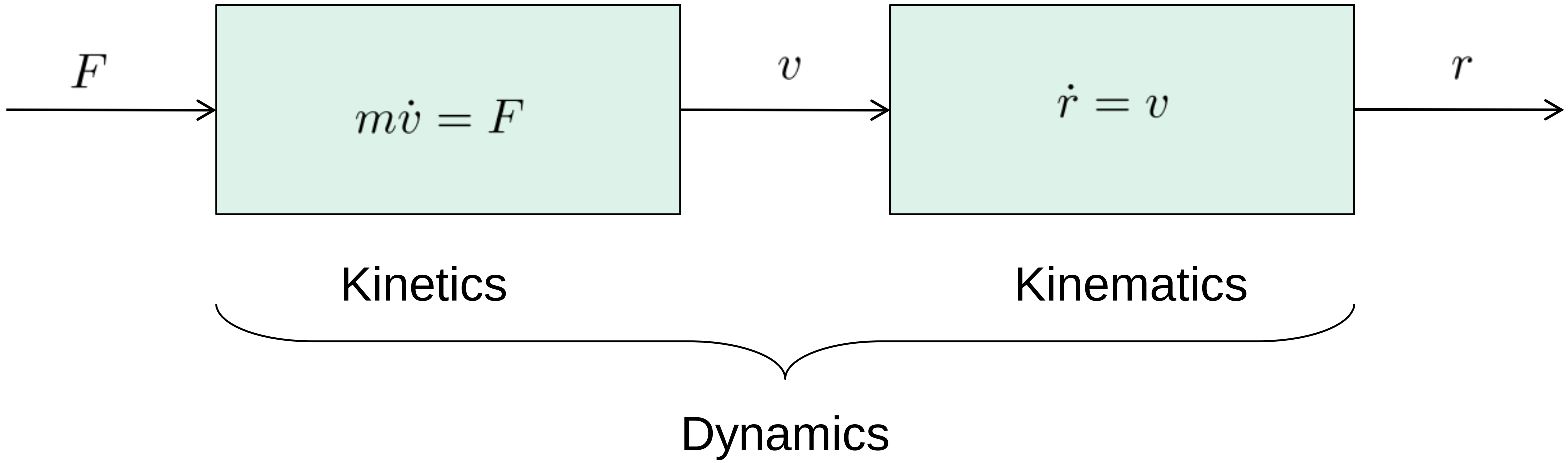
- Dynamics = Kinematics + Kinetics

- Kinematics
  - eb.com: “…branch of physics (…) concerned with the geometrically possible motion of a body or system of bodies without consideration of the forces involved (i.e., causes and effects of the motions).”
  - Book: Ch. 6
- Kinetics
  - eb.com: “…the effect of forces and torques on the motion of bodies having mass.”
  - Book: Ch. 7, 8.

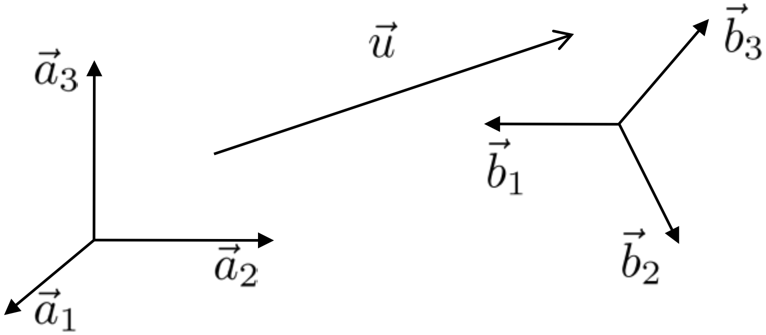
Remark: Sometimes “dynamics” is used for “kinetics” only

Simplest scalar case





Differentiations of vectors (6.8.5, 6.8.6)



- Coordinate representation:

$$\mathbf{u}^a = \mathbf{R}_b^a \mathbf{u}^b$$

- Differentiation:

$$\dot{\mathbf{u}}^a = \mathbf{R}_b^a \dot{\mathbf{u}}^b + \dot{\mathbf{R}}_b^a \mathbf{u}^b$$

$$\dot{\mathbf{R}}_b^a = \mathbf{R}_b^a \left(\boldsymbol{\omega}_{ab}^b\right)^\times$$

$$\boxed{\dot{\mathbf{u}}^a = \mathbf{R}_b^a \left[\dot{\mathbf{u}}^b + \left(\boldsymbol{\omega}_{ab}^b\right)^\times \mathbf{u}^b\right]}$$

Note! Generally,

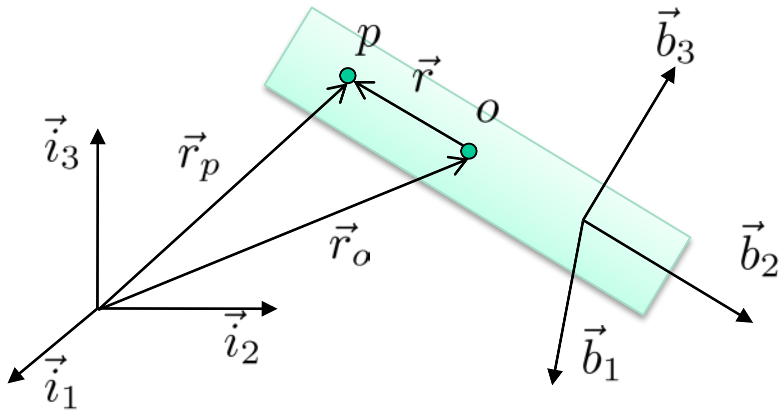
$$\dot{\mathbf{u}}^a \neq \mathbf{R}_b^a \dot{\mathbf{u}}^b$$

- On vector form:

$$\boxed{\frac{{}^a\mathrm{d}}{\mathrm{d}t}\vec{u} = \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{u} + \vec{\omega}_{ab} \times \vec{u}}$$

Rigid body kinematics

- Velocities and accelerations (Ch. 6.12)



$$\vec{v}_o := \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r}_o, \quad \vec{v}_p := \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r}_p$$

$$\vec{a}_o := \frac{{}^i\mathrm{d}^2}{\mathrm{d}t^2}\vec{r}_o, \quad \vec{a}_p := \frac{{}^i\mathrm{d}^2}{\mathrm{d}t^2}\vec{r}_p$$

$$\vec{\alpha}_{ib} := \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{\omega}_{ib} = \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{\omega}_{ib}$$

$$\begin{aligned}\vec{v}_p &= \vec{v}_o + \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r} \\ &= \vec{v}_o + \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{r} + \vec{\omega}_{ib} \times \vec{r} \\ &= \vec{v}_o + \vec{\omega}_{ib} \times \vec{r}, \quad \vec{r} \text{ fixed.}\end{aligned}$$

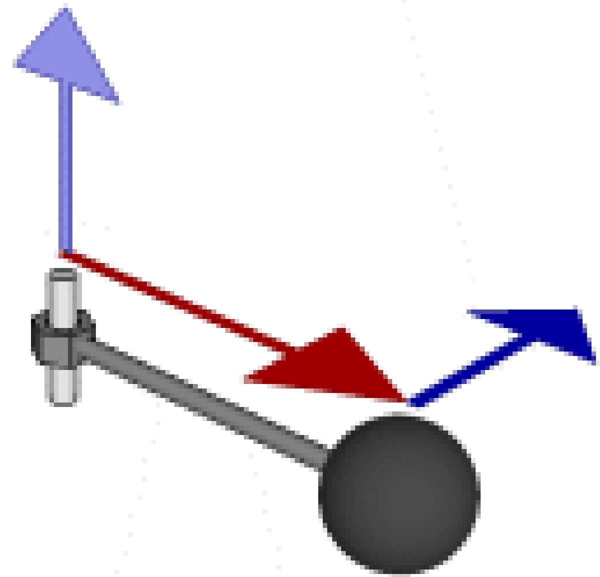
$$\boxed{\vec{a}_p = \vec{a}_o + \frac{{}^b\mathrm{d}^2}{\mathrm{d}t^2}\vec{r} + 2\vec{\omega}_{ib} \times \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{r} + \vec{\alpha}_{ib} \times \vec{r} + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r})}$$

$$\boxed{\vec{a}_p = \vec{a}_o + \vec{\alpha}_{ib} \times \vec{r} + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r}), \quad \vec{r} \text{ fixed.}}$$

Torque, and linear/angular momentum

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$



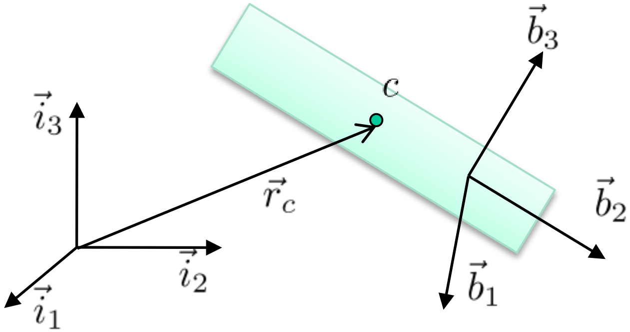
Source: Wikipedia

- Book:
  - Torque: $\vec{N}, \vec{T}$
  - Angular momentum: $\vec{h}$

Newton-Euler EoM

- Referenced to center of mass (CoM):

$$\vec{F}_{bc} = m\vec{a}_c$$
$$\vec{T}_{bc} = \vec{M}_{b/c} \cdot \vec{\alpha}_{ib} + \vec{\omega}_{ib} \times \left(\vec{M}_{b/c} \cdot \vec{\omega}_{ib}\right)$$



- Sometimes convenient to have them referenced to other point *o*:
  - Forces and moments in *o*:

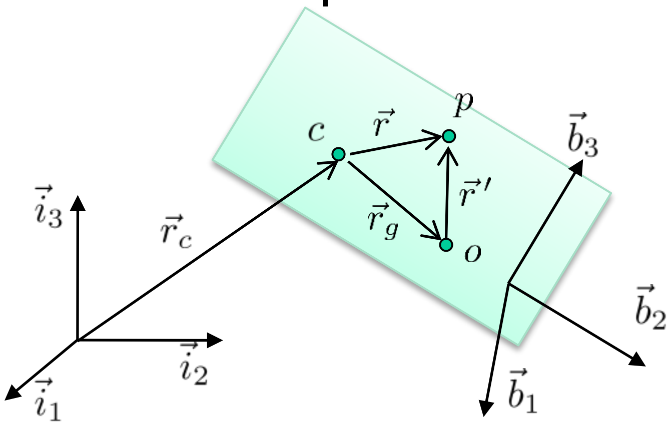
$$\vec{F}_{bo} = \vec{F}_{bc}$$
$$\vec{T}_{bo} = \vec{T}_{bc} + \vec{r}_g \times \vec{F}_{bc}$$

  - Use

$$\vec{a}_c = \vec{a}_o + \vec{\alpha}_{ib} \times \vec{r}_g + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r}_g)$$

  - Define

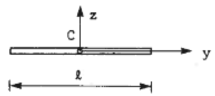
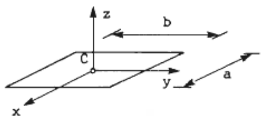
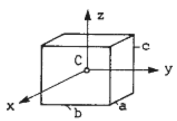
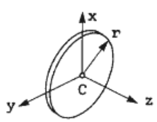
$$\vec{M}_{b/o} := -\int_b (\vec{r}\,')^\times (\vec{r}\,')^\times \mathrm{d}m$$

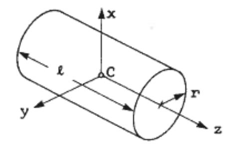
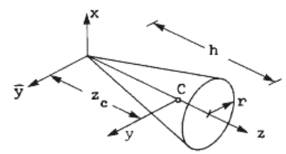
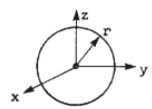


$$\vec{F}_{bo} = m\left(\vec{a}_o + \vec{\alpha}_{ib} \times \vec{r}_g + \vec{\omega}_{ib} \times (\vec{\omega}_{ib} \times \vec{r}_g)\right)$$
$$\vec{T}_{bo} = \vec{r}_g \times \vec{a}_o + \vec{M}_{b/o} \cdot \vec{\alpha}_{ib} + \vec{\omega}_{ib} \times \left(\vec{M}_{b/o} \cdot \vec{\omega}_{ib}\right)$$

  - Useful when CoM changes – no need to recalculate inertia matrix – still need to know CoM

Finding moments of inertia

Homogen slank stav 	$I_z = \frac{1}{12} m l^2$ $I_{\bar{z}} = \frac{1}{3} m l^2$
Tynn rektangulær plate 	$I_z = \frac{1}{12} m\ (a^2 + b^2)$ $I_x = \frac{1}{12} m b^2$ $I_y = \frac{1}{12} m a^2$
Rektangulært prisme 	$I_z = \frac{1}{12} m\ (a^2 + b^2)$
Tynn sirkulær skive 	$I_z = \frac{1}{2} m r^2$ $I_x = I_y = \frac{1}{4} m r^2$

Sirkulær sylinder 	$I_z = \frac{1}{2} m r^2$ $I_x = I_y = \frac{1}{12} m (3r^2 + l^2)$
Tynt sylinderskall	$I_z = m r^2$ $I_x = I_y = \frac{1}{2} m r^2 + \frac{1}{12} m l^2$
Rett sirkulær kjegle 	$I_z = \frac{1}{10} m r^2$ $I_y = \frac{3}{20} m r^2 + \frac{3}{80} m h^2$ $I_{\bar{y}} = \frac{3}{20} m r^2 + \frac{3}{5} m h^2$ $z_c = 3h/4$
Kule 	$I_C = \frac{2}{5} m r^2$
Kuleskall	$I_C = \frac{2}{3} m r^2$

From F. Irgens, Dynamikk

- http://en.wikipedia.org/wiki/List_of_moment_of_inertia_tensors
- For other/general rigid bodies (vessels/planes/etc.), computer programs can find moments of inertia

	Kinematics	Kinetics
	Derivatives of position and orientation as function of velocity and angular velocity	Derivatives of velocity and angular velocity as function of applied forces and torques

	Kinematics	Kinetics
	Derivatives of position and orientation as function of velocity and angular velocity	Derivatives of velocity and angular velocity as function of applied forces and torques
Translation	1D: $\dot{r} = v$ 3D: $\dot{\mathbf{r}}_c^i = \mathbf{v}_c^i$ Note! By definition $\vec{v}_c := \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r}_c$ $\dot{\mathbf{r}}_c^i = \mathbf{v}_c^i = \mathbf{R}_b^i \mathbf{v}_c^b$	1D: $m\dot{v} = F$ 3D: $m\dot{\mathbf{v}}_c^i = \mathbf{F}_{bc}^i$ Usually convenient to have forces and velocities in body system: $m\left(\dot{\mathbf{v}}_c^b + \left(\boldsymbol{\omega}_{ib}^b\right)^\times \mathbf{v}_c^b\right) = \mathbf{F}_{bc}^b$
Rotation/ orientation	1D: $\dot{\theta} = \omega$ 3D: Depends on parameterization Rotation matrix: $\dot{\mathbf{R}}_b^i = \mathbf{R}_b^i \left(\boldsymbol{\omega}_{ib}^b\right)^\times$ Euler angles: $\dot{\boldsymbol{\phi}} = \mathbf{E}_d^{-1}(\boldsymbol{\phi})\boldsymbol{\omega}_{ib}^b$ Euler parameters: $\dot{\eta} = -\frac{1}{2}\boldsymbol{\epsilon}^\top \boldsymbol{\omega}_{ib}^b$ $\dot{\boldsymbol{\epsilon}} = \frac{1}{2}\left(\eta \mathbf{I} + \boldsymbol{\epsilon}^\times\right)\boldsymbol{\omega}_{ib}^b$	1D: $J\dot{\omega} = T$ 3D: $\mathbf{M}_{b/c}^b \dot{\boldsymbol{\omega}}_{ib}^b + \left(\boldsymbol{\omega}_{ib}^b\right)^\times \mathbf{M}_{b/c}^b \boldsymbol{\omega}_{ib}^b = \mathbf{T}_{bc}^b$