Exercise 2 - TTK4130 Modeling and Simulation

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Problem 2

There are three criteria that ust be fullfilled by the rational, proper transfer funtion $H(s)$ for it to a positive real:

1. All the poles of $H(s)$ have $Re(\lambda_i) \leq 0$.
2. $Re[H(j\omega)] \geq 0 \; \forall \; \omega$ s.t. $j\omega$ is not a pole of $H(s)$.
3. If $j\omega_0$ is a pole of $H(s)$, it is simple and $Res_{s=j\omega_0}[H(s)] > 0$.

a

$$H_1(s) = \frac{1}{1+Ts}.$$

$H_1(s)$ has a pole at $-\frac{1}{T}$, which has a neagtive real part for $T > 0$.

$$Re[H_1(j\omega)] = Re[\frac{1}{1+Tj\omega}] = \frac{1}{1+\omega^2 T^2} \geq 0 \quad \forall \, \omega.$$

$\underline{\underline{H_1(s) \text{ is positive real}}}$

$$H_2(s) = \frac{s}{s^2+\omega_0^2}.$$

$H_2(s)$ has a pair of complex conjugated poles at $\pm j\omega_0$, which has a real part of zero. If we picture the phase plot of $H_2(j\omega)$, it will start out in $90°$ because of the zero in $\omega = 0$. The complex conjugate pole pair will cause the phase to fall by $180°$, making the phase end up at $-90°$. This means that $H_2(s)$ will always stay in the right half plane of the complex plane and we can conclude that $Re[H(j\omega)] \geq 0$.

$$Res_{s=j\omega_0}[H_2(s)] = \lim_{s \to j\omega_0} (s - j\omega_0)H_2(s) = \lim_{s \to j\omega_0} \frac{s}{s+j\omega_0} = \frac{1}{2} > 0.$$

$\underline{\underline{H_2(s) \text{ is positive real}}}$

b

$$H_3(s) = \frac{s+a}{(s+b)(s+c)}, \quad b, c > 0.$$

H_3 has two ploes in $-b$ and $-c$, both with negative real parts.
We have several cases in need of consideration to decide for which a $H_3(s)$ is positive real. The poles in $-b$ and $-c$ will contribute to the phase with $-180°$. If $a < 0$, $H_3(s)$ will start out in the left half plane, so that can be excluded. If $a > b, c$ the phase will fall below $-90°$ before a can pull it up again by $90°$. The only cases where $H_3(s)$ stays in the right half plane is if $a = 0$, thus stating out the phase plot in $90°$, or if $a < b + c$.
This can also be seen by calculating the real part of $H_3(s)$:

$$Re[H_3(j\omega)] = \frac{abc + \omega^2(b+c-a)}{(\omega^2+b^2)(\omega^2+c^2)}.$$

It is easy to see that this will only always be positive if the numerator stays positive for all ω, as the denominator will always be positive. Thsi only happens in the same cases as mentioned above, so
$\underline{\underline{H_3(s) \text{ is positive real for } 0 \leq a < b+c}}$

c

$$H_4(s) = \frac{s^2+a^2}{s(s^2+\omega_0^2)}, \quad a \geq 0.$$

The poles of $H_4(s)$ are at 0 and $\pm j\omega_0$, which are all at the imaginary axis with zero real part.

$$Re[H_4(j\omega)] = Re[\frac{a^2-\omega^2}{j\omega(\omega_0^2-\omega^2)}] = 0,$$

so criterion 2 is fullfilled.

$$Res_{s=0}[H_4(s)] = \lim_{s\to 0} s\frac{s^2+a^2}{s(s^2+\omega_0^2)} = \frac{a^2}{\omega_0^2} > 0 \quad \forall\, a \neq 0.$$

$$Res_{s=j\omega_0}[H_4(s)] = \lim_{s\to j\omega_0}(s-j\omega_0)\frac{s^2+a^2}{s(s^2+\omega_0^2)} = \frac{\omega_0^2-a^2}{2\omega_0^2} > 0, \quad a \in (-\omega_0, \omega_0)$$

$\underline{\underline{H_4(s) \text{ is positive real for } 0 < |a| < |\omega_0|}}$

d

$$T\dot{y} = -y + u \Rightarrow f(y,u) = \frac{1}{T}(-y+u)$$

The storage function must fulfill

$$\dot{V} = \frac{\partial V}{\partial y} f(y,u) = u^T y - g(y) \quad \forall\, u, \quad g(y) > 0.$$

Choose the storage function to be

$$V = \frac{1}{2}Ty^2$$

$$\dot{V} = Ty\dot{y} = y(-y+u)$$

$$\dot{V} = uy - y^2 \Rightarrow g(y) = y^2 > 0 \Rightarrow Passive$$

d

$$H(s) = \frac{(s+z_1)...(s+z_m)}{s(s+p_1)...(s+p_n)}, \quad Re(p_i) > 0,\ Re(z_i) > 0,\ n > m.$$

$H(s)$ is rational and proper. Criterion 1 is fulfilled, all the poles of $H(s)$ have negative real part, except for the one that is zero. The residual of the pole in zero is

$$Res_{s=0} H(s) = \frac{z_1...z_m}{p_1...p_n}.$$

Since complex poles and zeros of a transfer function always exist in complex conjugate pairs, and all p_i, z_i have positive real parts, the residual must be real and greater than zero.
This leaves only criterion 2, so

$$\underline{\underline{H(s) \text{ is positive real if and only if } Re[H(j\omega)] \geq 0}}$$