Exercise 10

TTK4130 Modeling and Simulation

Problem 1 (Sliding stick (Exam 2010))

Consider a stick of length ℓ with uniformly distributed mass m. It has center of mass/gravity C, about which it has a moment of inertia I_z. The stick is in contact with a frictionless horizontal surface, and moves due to the influence of gravity. See Figure 1.

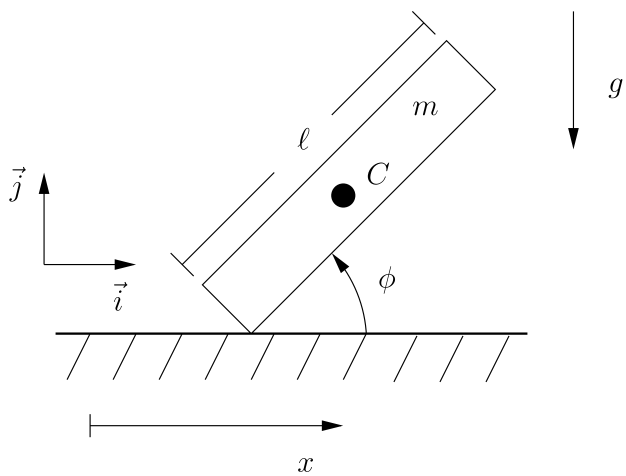

Figure 1: Stick sliding on frictionless surface

(a) Choose appropriate generalized coordinates (the figure should give you some hints). What are the corresponding generalized (actuator) forces?

Solution: A natural choice for generalized coordinates are the horizontal position of the center of mass/gravity (denoted x), and ϕ, the angle between the stick and the surface. An alternative to x could be the contact point between the stick and the surface.

There are no (generalized) actuator forces corresponding to these coordinates. (A candidate answering 'gravity' might get full score if he/she uses it correctly in the rest of the Problem.)

(b) What are the position, velocity, and angular velocity of the center of mass, as function of your chosen generalized coordinates (and/or their derivatives)?

Solution:

$$
\begin{aligned}
\vec{r}_c &= x\vec{i} + \frac{\ell}{2}\sin\phi\vec{j} \\
\vec{v}_c &= \dot{x}\vec{i} + \frac{\ell}{2}\dot{\phi}\cos\phi\vec{j} \\
\vec{w}_{ib} &= \dot{\phi}\vec{k}
\end{aligned}
$$

(Coordinate vectors also accepted for the position and velocity, scalar accepted for angular velocity.)

(c) Write up the kinetic and potential energy of the stick, as function of your chosen generalized coordinates (and/or their derivatives).

Solution: The kinetic energy for the rigid body is

$$\begin{aligned} T &= \frac{1}{2} m \vec{v}_c \cdot \vec{v}_c + \frac{1}{2} \vec{w}_{ib} \cdot \vec{M}_{b/c} \cdot \vec{w}_{ib} \\ &= \frac{1}{2} m \left(\dot{x}^2 + \frac{\ell^2}{4} \dot{\phi}^2 \cos^2 \phi \right) + \frac{1}{2} I_z \dot{\phi}^2. \end{aligned}$$

The potential energy due to gravity is

$$U = mg \frac{\ell}{2} \sin \phi.$$

(d) Derive the equations of motion for the stick.

Solution: It is probably easiest to use Lagrange's equation of motion. Define the Lagrangian $L = T - U$. Then the first equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0,$$

which reduces to

$$\ddot{x} = 0.$$

The second equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0$$

which gives

$$\left(\frac{m\ell^2}{4} \cos^2 \phi + I_z \right) \ddot{\phi} - \frac{m\ell^2}{4} \dot{\phi}^2 \cos \phi \sin \phi + mg \frac{\ell}{2} \cos \phi = 0.$$

Problem 2 (Double inverted pendulum)

The double inverted pendulum on a cart (DIPC) poses a The double inverted pendulum on a cart (DIPC) poses a challenging control problem. In a DIPC system, two rods are connected together on a moving cart as shown in Figure 2. The rod is located above the centre of mass of the cart. The length of the first rod is denoted by l_1 and the length of the second rod by l_2. The mass of the cart is denoted by m_0, its length by l_0 and its width by b_0. The height of the cart is denoted by h_0. Both rods have a mass, which are denoted by m_1 and m_2. All masses are assumed to be concentrated into the centre of mass. The moments of inertia are denoted by I_i. Furthermore, the force τ is acting on the cart.

(a) Find the position of cart and the two rods.

Solution: The position of the cart is given as

$$r_0 = \begin{pmatrix} \theta_0 \\ 0 \end{pmatrix}, \tag{1}$$

the position of the centre of mass of the first rod is given by

$$r_1 = \begin{pmatrix} \theta_0 + \frac{1}{2} l_1 \sin \theta_1 \\ h_0 + \frac{1}{2} l_1 \cos \theta_1 \end{pmatrix} \tag{2}$$

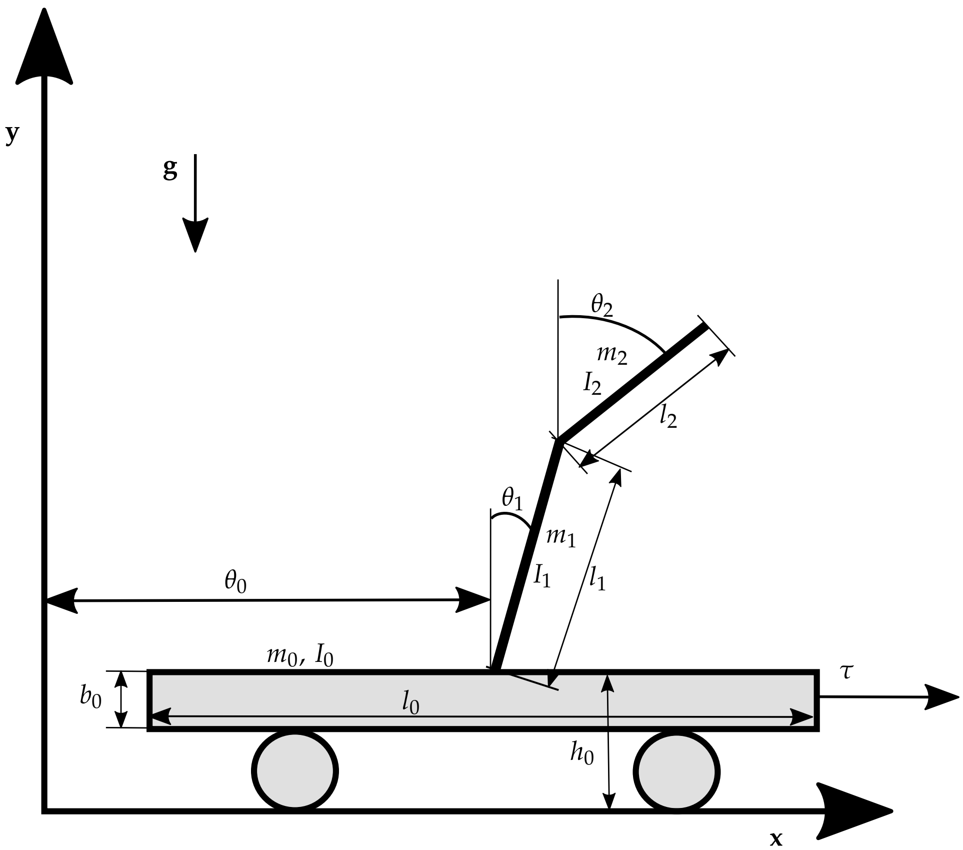

Figure 2: Double inverted pendulum on a cart

and the position of the second rod is given by

$$r_2 = \begin{pmatrix} \theta_0 + l_1 \sin\theta_1 + \frac{1}{2}\sin\theta_2 \\ h_0 + l_1\cos\theta_1 + \frac{1}{2}l_2\cos\theta_2 \end{pmatrix}. \tag{3}$$

If the origin of the coordinate system is chosen differently the terms can look slightly different. However, if the coordinate system is chosen differently then in Fig. 2, it has to be mentioned.

(b) Find the kinetic energy of the system. $\left(\textit{Hint: } \cos(x-y) = \cos x \cos y + \sin x \sin y\right)$

Solution:

- The velocities can be calculated by derivation of the positions

$$v_0 = \begin{pmatrix} \dot\theta_0 \\ 0 \end{pmatrix}, \tag{4a}$$

$$v_1 = \begin{pmatrix} \dot\theta_0 + \frac{1}{2}l_1\dot\theta_1\cos\theta_1 \\ -\frac{1}{2}l_1\dot\theta_1\sin\theta_1 \end{pmatrix}, \tag{4b}$$

$$v_2 = \begin{pmatrix} \dot\theta_0 + l_1\dot\theta_1\cos\theta_1 + \frac{1}{2}l_2\dot\theta_2\cos\theta_2 \\ -l_1\dot\theta_1\sin\theta_1 - \frac{1}{2}l_2\dot\theta_2\sin\theta_2 \end{pmatrix}. \tag{4c}$$

- The kinetic energy is given by

$$T = \sum_i T_i, \tag{5}$$

with

$$T_i = \frac{1}{2} m_i v_i^T v_i + \frac{1}{2} I_i \omega_i^T \omega_i. \tag{6}$$

Consequently, the kinetic energy of each body part is given by

$$T_0 = \frac{1}{2} m_0 \dot{\theta}_0, \tag{7a}$$

$$T_1 = \frac{1}{2} m_1 \left(\dot{\theta}_0^2 + l_1 \dot{\theta}_0 \dot{\theta}_1 \cos\theta_1 + \frac{1}{4} l_1^2 \dot{\theta}_1^2 \right) + \frac{1}{2} I_1 \dot{\theta}_1^2, \tag{7b}$$

$$T_2 = \frac{1}{2} m_2 \left(\dot{\theta}_0^2 + l_1^2 \dot{\theta}_1^2 + \frac{1}{4} l_2^2 \dot{\theta}_2^2 + 2 l_1 \dot{\theta}_0 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_0 \dot{\theta}_2 \cos\theta_2 + l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right) + \frac{1}{2} I_2 \dot{\theta}_2^2, \tag{7c}$$

with

$$\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2. \tag{8}$$

(c) Find the potential energy of the system.

Solution: The potential energy is given by

$$U = \sum_i U_i \tag{9}$$

$$U_0 = m_0 g (h_0 - \frac{1}{2} b_0), \tag{10a}$$

$$U_1 = m_1 g (h_0 + \frac{1}{2} l_1 \cos\theta_1), \tag{10b}$$

$$U_2 = m_2 g (h_0 + l_1 \cos\theta_1 + \frac{1}{2} \cos\theta_2). \tag{10c}$$

If the origin of the coordinate system is chosen differently the terms can look slightly different. However, if the coordinate system is chosen differently then in Fig. 2, it has to be mentioned.

(d) Find the equation of motion of the system.

Solution: The Lagrange equation of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i \tag{11}$$

is used. L is defined as

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q}), \tag{12}$$

where T is the kinetic energy and U the potential energy of the system.

- The resulting Lagrangian is

$$
\begin{aligned}
L =& \frac{1}{2}\Big[(m_0 + m_1 + m_2)\,\dot{\theta}_0^2 + \left(\frac{1}{2}m_1 l_1^2 + m_2 l_1^2 + I_1\right)\dot{\theta}_1^2 + \\
& \left(\frac{1}{2}m_2 l_2^2 + I_2\right)\dot{\theta}_2^2 + (m_1 l_1 + 2m_2 l_1)\,\dot{\theta}_0\dot{\theta}_1\cos\theta_1 + \\
& m_2 l_2 \dot{\theta}_0\dot{\theta}_2\cos\theta_2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\Big] - \\
& \left(\frac{1}{2}m_1 + m_2\right) g l_1\cos\theta_1 - \frac{1}{2}m_2 g l_2\cos\theta_2 - (m_0+m_1+m_2)h_0 g.
\end{aligned} \tag{13}
$$

- The derivation of the Lagrangian with respect to $\dot{\mathbf{q}}$ is

$$
\begin{aligned}
&\frac{\partial}{\partial\dot{\mathbf{q}}}L(\mathbf{q},\dot{\mathbf{q}},t) = \\
&\begin{pmatrix}
(m_0+m_1+m_2)\dot{\theta}_0 + \frac{1}{2}\left((m_1+2m_2)l_1\dot{\theta}_1\cos\theta_1 + m_2 l_2\dot{\theta}_2\cos\theta_2\right) \\
(\frac{1}{2}m_1 l_1^2 + m_2 l_1^2 + I_1)\dot{\theta}_1 + \frac{1}{2}\left((m_1+2m_2)l_1\dot{\theta}_0\cos\theta_1 + m_2 l_1 l_2\dot{\theta}_2\cos(\theta_1-\theta_2)\right) \\
(\frac{1}{2}m_2 l_2^2 + I_2)\dot{\theta}_2 + \frac{1}{2}m_2 l_2\left(\dot{\theta}_0\cos\theta_2 + l_1\dot{\theta}_1\cos(\theta_1-\theta_2)\right)
\end{pmatrix}
\end{aligned} \tag{14}
$$

- The derivation of Eq. 14 with respect to time is

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\mathbf{q}}}\right) = \\
&\begin{pmatrix}
(\sum_i m_i)\ddot{\theta}_0 + \frac{1}{2}\left(l_1(m_1+2m_2)(\ddot{\theta}_1\cos\theta_1 - \dot{\theta}_1^2 sin\theta_1) + m_2 l_2(\ddot{\theta}_2\cos\theta_2 - \dot{\theta}_2^2\sin\theta_2)\right) \\
(\frac{1}{2}m_1 l_1^2 + m_2 l_1^2 + I_1)\ddot{\theta}_1 + \frac{1}{2}\Big(l_1(m_1+2m_2)(\ddot{\theta}_0\cos\theta_1 - \dot{\theta}_0\dot{\theta}_1\sin\theta_1) + ... \\
m_2 l_1 l_2\left[\ddot{\theta}_2\cos(\theta_1-\theta_2) - \dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + \dot{\theta}_2^2\sin(\theta_1-\theta_2)\right]\Big) \\
(\frac{1}{2}m_2 l_2^2 + I_2)\ddot{\theta}_2 + \frac{1}{2}\Big(m_2 l_2(\ddot{\theta}_0\cos\theta_2 - \dot{\theta}_0\dot{\theta}_2\sin\theta_2) + m_2 l_1 l_2\big[\ddot{\theta}_1\cos(\theta_1-\theta_2) - ... \\
\dot{\theta}_1^2\sin(\theta_1-\theta_2) + \dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)\big]\Big)
\end{pmatrix}
\end{aligned} \tag{15}
$$

- The derivation of the Lagrangian with respect to $\mathbf{q}$ is

$$
\frac{\partial}{\partial\mathbf{q}}L(\mathbf{q},\dot{\mathbf{q}},t) = \frac{1}{2}
\begin{pmatrix}
0 \\
-(m_1+2m_2)l_1\dot{\theta}_0\dot{\theta}_1\sin\theta_1 - m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + ... \\
(m_1+2m_2)g l_1\sin\theta_1 \\
-m_2 l_2\dot{\theta}_0\dot{\theta}_2\sin\theta_2 + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + m_2 g l_2\sin\theta_2
\end{pmatrix}. \tag{16}
$$

- The results (15) and (16) are put in Eq. 11, which gives

$$
\begin{aligned}
\tau = &\left(\sum m_i\right)\ddot{\theta}_0 + \frac{1}{2}\Big(l_1(m_1+2m_2 l_1)(\ddot{\theta}_1\cos\theta_1 - \dot{\theta}_1^2 sin\theta_1) + \\
&m_2 l_2(\ddot{\theta}_2\cos\theta_2 - \dot{\theta}_2^2\sin\theta_2)\Big),
\end{aligned} \tag{17a}
$$

$$
\begin{aligned}
0 = &\left(\frac{1}{2}m_1 l_1^2 + m_2 l_1^2 + I_1\right)\ddot{\theta}_1 + \frac{1}{2}\Big(l_1(m_1+2m_2)\ddot{\theta}_0\cos\theta_1 + \\
&m_2 l_1 l_2\left[\ddot{\theta}_2\cos(\theta_1-\theta_2) + \dot{\theta}_2^2\sin(\theta_1-\theta_2)\right] - (m_1+2m_2)g l_1\sin\theta_1\Big),
\end{aligned} \tag{17b}
$$

$$
\begin{aligned}
0 = &\left(\frac{1}{2}m_2 l_2^2 + I_2\right)\ddot{\theta}_2 + \\
&\frac{1}{2}\left(m_2 l_2\ddot{\theta}_0\cos\theta_2 + m_2 l_1 l_2\left[\ddot{\theta}_1\cos(\theta_1-\theta_2) - \dot{\theta}_1^2\sin(\theta_1-\theta_2)\right] - m_2 g l_2\sin\theta_2\right),
\end{aligned} \tag{17c}
$$

which are the equations of motion for the DIPC system.

(e) The moment of inertia is defined as

$$I = \int_Q r^2 dm \tag{18}$$

where r is the distance from each point to the axis of rotation and Q is the entire mass. Derive the moment of inertia for the rectangular plate with the length l and height h for the centre of mass of the plate (Fig. 3). The axis of rotation is the z-axis (perpendicular to the plate). Assume that the plate has a constant width b and a homogeneous density ρ.

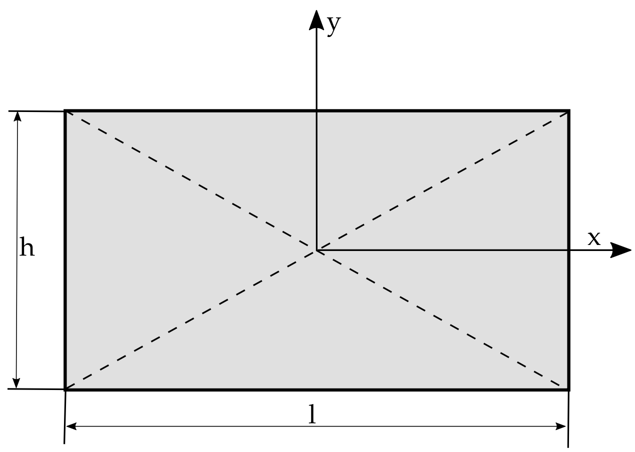

Figure 3: Rectangular plate

Solution: The distance r of each point on the plate to the z-axis is $y^2 + x^2$. Consequently, the moment of inertia is

$$I = \int (x^2 + y^2) dm. \tag{19}$$

In a next step the dm has to be replaced. We know that the infinitesimal mass is

$$dm = \rho dV = \rho b \cdot dA = \rho b \cdot dxdy, \tag{20}$$

where ρ is the density and dV is a infinitesimal volume, dA the infinitesimal area and dx and dy the infinitesimal lengths (Fig. 4).

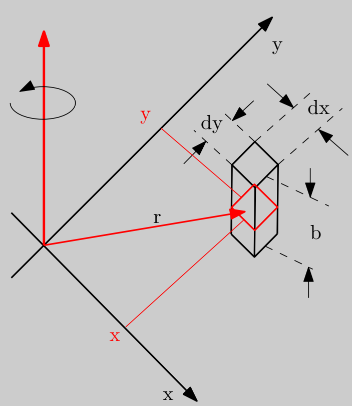

Figure 4: Infinitesimal volume of rectangular plate

Eq. 19 and 20 together with the borders of the integral, which are given by the dimensions of the plate, are

$$I = \rho b \int_{-h/2}^{h/2} \int_{-l/2}^{l/2} (x^2 + y^2) dx dy. \tag{21}$$

In the next step the integral has to be solved over the dimensions of the plate

$$\begin{aligned} I &= \rho b \int_{-h/2}^{h/2} \int_{-l/2}^{l/2} (x^2 + y^2) dx dy \\ &= \rho b \int_{-h/2}^{h/2} (\frac{1}{3}x^3 + y^2 x)\Big|_{-l/2}^{l/2} dy \\ &= \rho b \int_{-h/2}^{h/2} (\frac{1}{12}l^3 + y^2 l) dy \\ &= \rho b (\frac{1}{12}l^3 y + \frac{1}{3}y^3 l)\Big|_{-h/2}^{h/2} \\ &= \frac{1}{12}\rho b (l^3 h + h^3 l) \\ I &= \frac{1}{12} m (l^2 + h^2). \end{aligned} \tag{22}$$

I is the moment of inertia through the centre of mass.

(f) Use the parallel axes theorem to change the point of rotation from the centre of mass to point **A** given in Fig. 5.
(Help: If you were not able to solve the previous task, use as moment of inertia $I = \frac{1}{4}m(l^2 + h^2)$ *Please state explicitly, if you use this for further calculations.)*

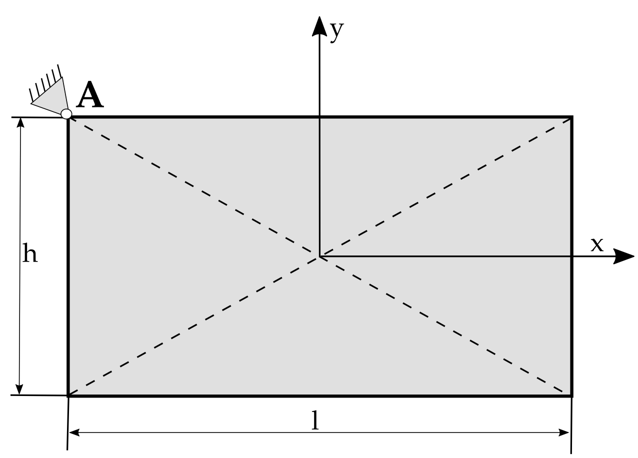

Figure 5: Rectangular plate with attachment on one corner

Solution: The parallel axes theorem states that the inertia I_A through some other axes is equal to the inertia through the centre of mass plus the mass multiplied with the distance of the new axes to the centre of mass squared

$$I_A = I + md^2, \tag{23}$$

where d is the distance of the new axes to the centre of mass. It can be easily seen that the distance of **A** to the centre of mass is

$$d = \sqrt{x^2 + y^2} = \sqrt{(-l/2)^2 + (h/2)^2}. \tag{24}$$

Consequently, the moment of inertia I_A is

$$\begin{aligned} I_A &= I + m((-l/2)^2 + (h/2)^2) \\ &= \frac{1}{12}m(l^2+h^2) + \frac{1}{4}m(l^2+h^2) \\ I_A &= \frac{1}{3}m(l^2+h^2) \end{aligned} \tag{25}$$

If the **hint** was use the result is

$$I_A = \frac{1}{2}m(l^2+h^2). \tag{26}$$

(g) Make an appropriate simplification for the moment of inertia of the rods used in the DIPC system. Justify your decision!

Solution: In case of the rod is the dimension of one axes much larger than the other dimension. It holds $l >> h$. Therefore, the moment of inertia can be simplified to

$$I_A = \frac{1}{3}ml^2 \;\; or \;\; I = \frac{1}{12}ml^2. \tag{27}$$

Problem 3 (Kinematic modeling of a quadrotor)

In this problem, and the next, we will develop a model for all degrees of freedom for a quadrotor, modeling the quadrotor as a rigid body. See Figure 6 for definition of coordinate systems, and a "free body diagram" with forces and moments acting on the quadrotor.

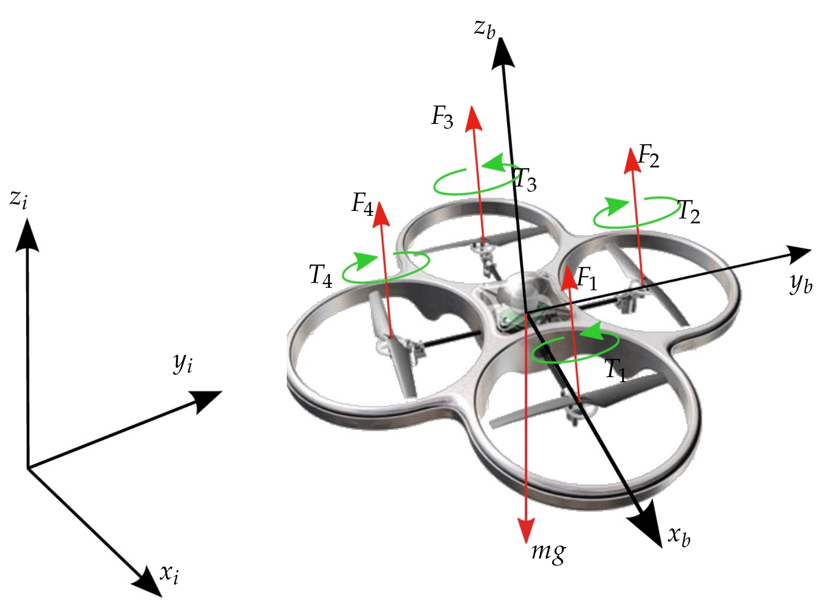

Figure 6: Coordinate systems and forces/moments.

(a) To specify the orientation of the quadrotor, the Z-X-Y Euler angles are sometimes used. These are specified by first a rotation α about the (inertial) z-axis, then β about the intermediate (rotated) x-axis, and finally γ about the body y-axis. Write up an expression for the rotation matrix $\mathbf{R}_b^i = \mathbf{R}_b^i(\boldsymbol{\phi})$ as a function of the Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)^\mathsf{T}$.

Solution: The rotation matrix is found by writing up the simple rotations in the order they appear:

$$\begin{aligned}
\mathbf{R}_b^i &= \mathbf{R}_{z,\alpha}\mathbf{R}_{x,\beta}\mathbf{R}_{y,\gamma} \\
&= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\beta\sin\alpha & \cos\alpha\sin\gamma + \cos\gamma\sin\alpha\sin\beta \\ \cos\gamma\sin\alpha + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta & \sin\alpha\sin\gamma - \cos\alpha\cos\gamma\sin\beta \\ -\cos\beta\sin\gamma & \sin\beta & \cos\beta\cos\gamma \end{pmatrix}
\end{aligned}$$

These Euler angles are used for instance in Vijay Kumar's lab at University of Pennsylvania, and Raffaello D'Andrea's lab at ETH. Other groups use other conventions.

(b) Find the kinematic differential equations for this choice of Euler angles. Assume that the angular velocity is given in body-frame. (It is not necessary to perform a matrix inversion for full score.)

Solution: The answer depends on whether the angular velocity is given in the inertial or body system. The latter is more natural, and is assumed in this problem.

The total angular velocity is the sum of the angular velocities of each rotation (6.269), but we need to transform the angular velocities to a common coordinate system when summing. In this case this common system is the body system:

$$\begin{aligned}
\boldsymbol{\omega}_{ib}^b &= \mathbf{R}_{y,-\gamma}\mathbf{R}_{x,-\beta}\begin{pmatrix} 0 \\ 0 \\ \dot{\alpha} \end{pmatrix} + \mathbf{R}_{y,-\gamma}\begin{pmatrix} \dot{\beta} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{\gamma} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sin\gamma\cos\beta\dot{\alpha} + \cos\gamma\dot{\beta} \\ \sin\beta\dot{\alpha} + \dot{\gamma} \\ \cos\gamma\cos\beta\dot{\alpha} + \sin\gamma\dot{\beta} \end{pmatrix} \\
&= \begin{pmatrix} -\sin\gamma\cos\beta & \cos\gamma & 0 \\ \sin\beta & 0 & 1 \\ \cos\gamma\cos\beta & \sin\gamma & 0 \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} \\
&= \mathbf{E}_b(\phi)\dot{\boldsymbol{\phi}}
\end{aligned}$$

where $\dot{\boldsymbol{\phi}} = (\dot{\alpha}, \dot{\beta}, \dot{\gamma})^{\mathsf{T}}$ and

$$\mathbf{E}_b(\boldsymbol{\phi}) = \begin{pmatrix} -\sin\gamma\cos\beta & \cos\gamma & 0 \\ \sin\beta & 0 & 1 \\ \cos\gamma\cos\beta & \sin\gamma & 0 \end{pmatrix}.$$

The kinematic differential equations are then

$$\dot{\boldsymbol{\phi}} = \mathbf{E}_b^{-1}(\boldsymbol{\phi})\boldsymbol{\omega}_{ib}^b.$$

Compare (6.316) for the choice of Euler angles used in the book (roll-pitch-yaw Euler angles).

Not asked for: The inverse of $\mathbf{E}_b(\boldsymbol{\phi})$ is

$$\mathbf{E}_b^{-1}(\boldsymbol{\phi}) = \frac{1}{\cos\beta}\begin{pmatrix} -\sin\gamma & 0 & \cos\gamma \\ \cos\gamma\cos\beta & 0 & \sin\gamma\cos\beta \\ \sin\beta\sin\gamma & \cos\beta & -\cos\gamma\sin\beta \end{pmatrix}$$

and we see that we have a singularity for $\beta = \pi/2 + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.

Problem 4 (Complete dynamic model of a quadrotor)

In this problem, we will continue to develop the complete dynamic model of the quadrotor by modeling the kinetics. The forces and moments acting on the quadrotor are illustrated in Figure 6. The body system has origin in the center of mass, and the quadrotor has mass m and an inertia matrix $\mathbf{M}^b_{b/c}$. Note that the moments T_i due to rotation of the rotors give moments acting about the z_b-axis, and that the rotor forces F_i will give cause to moments about the x_b and y_b axis, with "arm" (distance from center of mass to rotor) L for all rotors. Note also that T_i has a "sign" defined in the figure, due to the default direction of rotation of the rotors.

(a) Why is it natural to use the Newton-Euler equations of motions as starting point, rather than the Lagrange equations of motion?

Solution: We will model the quadcopter in all ("six") degrees of freedom, therefore there are no "forces of constraints" to eliminate.

(b) Write up expressions for the force and torque vectors acting on the center of mass, $\mathbf{F}^b_{bc}$ and $\mathbf{T}^b_{bc}$, decomposed in the body system, as function of the forces and torques defined in Figure 6.

Solution:

$$\mathbf{F}^b_{bc} = \mathbf{R}^b_i(\boldsymbol{\phi}) \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^4 F_i \end{pmatrix}, \quad \mathbf{T}^b_{bc} = \begin{pmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ T_1 - T_2 + T_3 - T_4 \end{pmatrix}$$

(The very observant will have noticed that the sign of T_i in the figure is wrong when you consider the propeller configuration used on this particular quadcopter.)

(c) What are the equations of motion of the quadrotor, on vector form? The components of vectors equations in the answer should amount to 12 first-order differential equations, including the answer from Problem 2(b).

Solution: There are (at least) two different correct answers here, depending on whether the velocity is expressed in body-fixed or inertial coordinates (and whether angular velocity is expressed in body-fixed or inertial coordinates, see Problem 2(b)).

First, the force balance. With velocity in inertial coordinates (which perhaps is simplest and most natural in this case), we can write down

$$\begin{aligned} m\dot{\mathbf{v}}^i_c &= \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} + \mathbf{R}^i_b(\boldsymbol{\phi}) \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^4 F_i \end{pmatrix} \\ \dot{\mathbf{r}}^i_c &= \mathbf{v}^i_c \end{aligned}$$

where the latter equation is a kinematic differential equation.

Alternatively, with velocity in body-fixed coordinates, we get (since $\dot{\mathbf{v}}^i_c = \mathbf{R}^i_b(\boldsymbol{\phi}) \left(\dot{\mathbf{v}}^b_c + (\boldsymbol{\omega}^b_{ib})^\times \mathbf{v}^b_c \right)$)

$$\begin{aligned} m\dot{\mathbf{v}}^b_c &= -m(\boldsymbol{\omega}^b_{ib})^\times \mathbf{v}^b_c + \mathbf{R}^b_i(\boldsymbol{\phi}) \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^4 F_i \end{pmatrix} \\ \dot{\mathbf{r}}^i_c &= \mathbf{R}^i_b(\boldsymbol{\phi}) \mathbf{v}^b_c. \end{aligned}$$

Second, the torque balance (Euler's equation) is

$$\mathbf{M}^b_{b/c} \dot{\boldsymbol{\omega}}^b_{ib} = \mathbf{T}^b_{bc} - \boldsymbol{\omega}^b_{ib} \times \left(\mathbf{M}^b_{b/c} \boldsymbol{\omega}^b_{ib} \right).$$

Together with

$$\dot{\boldsymbol{\phi}} = \mathbf{E}_b^{-1}(\boldsymbol{\phi})\boldsymbol{\omega}_{ib}^b$$

from Problem 2(b), this specifies 12 ODEs.

(Problems 3 and 4 are based on the article: Daniel Mellinger, Nathan Michael and Vijay Kumar, *Trajectory generation and control for precise aggressive maneuvers with quadrotors*, The International Journal of Robotics Research 31(5):664–674, 2012.)