Exercise 7
TTK4130 Modeling and Simulation

Problem 1 (Solving the transmission line PDEs)
In this exercise, we will simulate the transmission line partial differential equations (PDEs). The transmission line PDEs can be written (4.5.2 in the book)

$$\frac{\partial p(x,t)}{\partial t} = -cZ_0 \frac{\partial q(x,t)}{\partial x} \tag{1a}$$
$$\frac{\partial q(x,t)}{\partial t} = -\frac{c}{Z_0}\frac{\partial p(x,t)}{\partial x} - \frac{F(q(x,t))}{\rho_0}, \tag{1b}$$

where the sonic velocity c $[m/s]$ and the line impedance Z_0 $[kg/(m^4 s)]$ are defined as

$$c = \sqrt{\frac{\beta}{\rho_0}},$$
$$Z_0 = \frac{\rho_0 c}{A} = \frac{\sqrt{\rho_0 \beta}}{A}$$

and β is the fluid bulk modulus, which is the ratio of the pressure increase to relative decrease of volume $[N/m^2]$, ρ_0 is the fluid density $[kg/m^3]$, A is the cross sectional area $[m^2]$, and $F(q)$ is friction as a function of volume flow $[N]$.

The book outlines two different routes to simulating this model:

1. The first type of methods (4.5.10-4.5.16) is based on calculating rational transfer function approximations to the irrational transfer functions that results from Laplace-transformation of the PDE model. (Remember, irrational transfer functions cannot be simulated directly in (for example) Simulink).

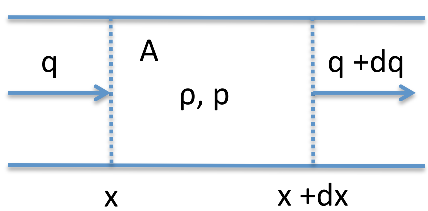

Figure 1: Volume element for hydraulic transmission line.

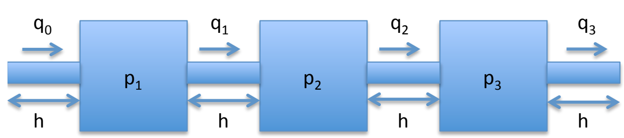

Figure 2: A chain of interconnected Helmholtz resonators ducts of length h representing a transmission line in admittance form.

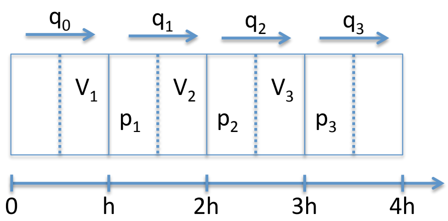

Figure 3: Spatial discretization of a transmission line with pressure inputs at both sides to get description in the form of a chain of Helmholtz resonators.

2. The second method is to discretize the spatial dimension into a set of "finite volumes", and use the Helmholtz resonator model inside each volume. This is explained in 4.6.

In this exercise, we will look into the second route. In addition to simulating the transmission line model, the objective is to give a glimpse into how general PDE models are solved using discretization methods (like difference methods, finite volume methods or finite element methods). The method used in this exercise is similar to the finite volume method, which is described in Chapter 15 in the book, but it is not necessary to read this before the exercise.

The "Finite Volume"-method can briefly be described by dividing the transmission line of length L into n smaller "volumes" with length $h = \frac{L}{n}$, and let the variables be spatially invariant within these volumes. Inside each volume, the time variation of the variables are given by a Helmholtz resonator model (4.6.2 and 4.6.3).

We will in this exercise assume that the flows at each end of the transmission line are the inputs, and we want to calculate the pressures in each end (that is, we will implement an impedance model version (4.6.5) of the PDE). In the first part of the exercise, we will assume a simple linear friction model in (1b), that is, $F = \rho_0 B q$ where $B = \frac{8\nu_0}{r_0^2}$.

(a) Set up the model for each volume element (hint: See Figure 4.17 and read Ch. 4.6.1-4.6.5).

1. Set up the mass balance equation.
2. Set up the moment balance equation.

Solution: The model becomes (see Ch. 4.6.5)

$$
\begin{aligned}
\dot{p}_i &= \frac{c^2\rho_0}{Ah}\left(q_{i-1} - q_i\right), \quad i = 1, \dots, N \\
\dot{q}_{i-1} &= \frac{A}{h\rho_0}\left(p_{i-1} - p_i\right) - Bq_{i-1}, \quad i = 2, \dots, N \\
q_0 &= q_{in}, \quad q_N = q_{out}
\end{aligned}
$$

(b) We will now write the model of all the volumes as a linear state space model where the boundary conditions (inputs) are volume flow into the first volume (q_0) and volume flow out of the last volume (q_N). That is, gather all variables in a state variable $\mathbf{x}$,

$$\mathbf{x}^\top = \begin{pmatrix} p_1 & q_1 & p_2 & q_2 & \cdots & q_{N-1} & p_N \end{pmatrix},$$

and write the model as

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\
\mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u},
\end{aligned}
$$

where the inputs and outputs are

$$\mathbf{u} = \begin{pmatrix} q_0 \\ q_N \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} p_1 \\ p_N \end{pmatrix}.$$

Set up **A**, **B**, **C** and **D** for n volumes. Can you see a pattern in **A**?

Solution: The matrices are

$$\mathbf{A} = \begin{pmatrix} 0 & -M & 0 & 0 & \dots & 0 & 0 \\ N & -B & -N & 0 & \dots & 0 & 0 \\ 0 & M & 0 & -M & \dots & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & M & 0 & -M & 0 \\ 0 & 0 & \dots & 0 & N & -B & -N \\ 0 & 0 & \dots & 0 & 0 & M & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} M & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -M \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where M and N are given as

$$M = \frac{c^2\rho_0}{Ah} = \frac{\beta}{Ah}, \quad N = \frac{A}{h\rho_0}.$$

Note the band pattern of **A** (the matrix is tridiagonal). Also note how the rows repeat: **A** can be written

$$\mathbf{A} = \begin{pmatrix} 0 \;\; -M \;\; 0 \;\; 0 & \dots & 0 \;\; 0 \;\; 0 \;\; 0 \\ \mathbf{A_1} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{A_1} \\ 0 \;\; 0 \;\; 0 \;\; 0 & \dots & 0 \;\; N \;\; -B \;\; -N \\ 0 \;\; 0 \;\; 0 \;\; 0 & \dots & 0 \;\; 0 \;\; M \;\; 0 \end{pmatrix}$$

where

$$\mathbf{A}_1 = \begin{pmatrix} N & -B & -N & 0 \\ 0 & M & 0 & -M \end{pmatrix}.$$

(c) We will now implement the model in Simulink. The parameters are

Parameter	Symbol	Value	Unit
Length	L	19.76	m
Bulk modulus	β	$1.7052 \cdot 10^9$	Pa
Density	ρ_0	870	kg/m^3
Pipe radius	r_0	$6.17 \cdot 10^{-3}$	m
Kinematic viscosity	ν_0	$8 \cdot 10^{-5}$	m^2/s

Use a state-space block in Simulink, with inputs and outputs as given in (b). Instead of inputting the matrices by hand in the state-space block, write just A under A, B under B, C under C, D under D and (if you want other initial conditions than 0) x0 under initial conditions. We will use a mask to specify these variables. This makes it also easier to vary the number of volumes.

Make a sub-system of the state-space block, as illustrated in Figure 4.

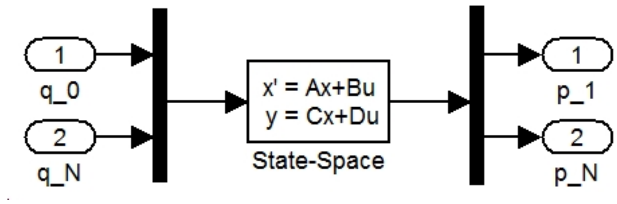

Figure 4: State-space model in Simulink

Right-click the subsystem, and choose Mask Subsystem. Add the parameters from the table above in addition to number of volumes (and possibly initial values) under Parameters, and the matlab code to calculate the **A**, **B**, **C** and **D**-matrices (and possibly $x0$) under Initialization.

Solution: The system that contains the state-space model can look something like in Figure 5.

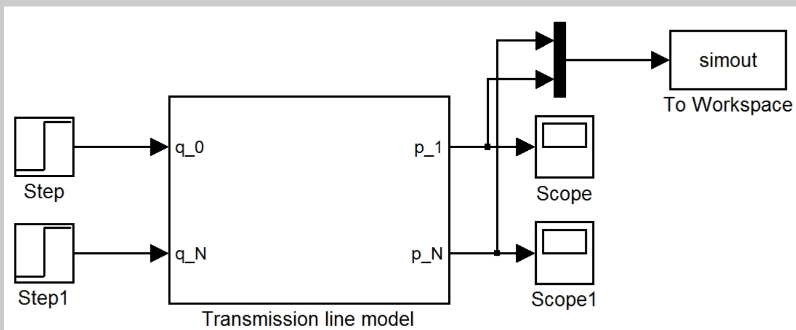

Figure 5: Simulink model that simulates the transmission line model

The code that is entered into the Initialization-part of the mask, is

```
area = r0^2*pi;
h = L/n;
M = beta/(area*h);
N = area/(h*rho0);
B = 8*nu0/r0^2;
x0 = zeros(2*n-1,1);
mat = [M 0 -M 0; 0 N -B -N];
A = [0 -M zeros(1,(2*n-3)); N -B -N zeros(1,(2*n-4))];
x0(1:2) = [p0; q0];
for i = 1:1:n-2
 addmat = [zeros(2,2*i-1) mat zeros(2,2*(n-2-i))];
 A = [A;addmat];
 x0((2*(i+1)-1):2*(i+1)) = [p0; q0];
end
x0(2*n-1) = p0;
A = [A; zeros(1, (2*n-3)) M 0];
B = [M 0 ; zeros((2*n-3),2); 0 -M];
C = [1 zeros(1, 2*n-2); zeros(1, 2*n-2) 1];
D = zeros(2,2);
```

The name of the parameters must of course match the ones entered in the parameter list.

(d) Simulate the system over 5 seconds using the inputs

$$q_0 = \begin{cases} 0 & t < 1s \\ 0,001 & t \geq 1s \end{cases}, \qquad q_N = \begin{cases} 0 & t < 2s \\ 0,001 & t \geq 2s \end{cases},$$

for $n = 5, 10$ and 50. Tip: Use the To Workspace-block to make comparing plots in Matlab. Comment.

Make Bode-plots of transfer function from q_0 to pressures on both sides (p_1 and p_N) for the same ns. Use same method as on earlier exercises to make bode-plots. Comment on plots.

Solution: The simulations are shown in Figure 6. When the inflow increases without outflow increasing, the pressure increases. Pressure stops increasing ones outflow matches inflow. In addition to these large effects, there are oscillations corresponding to pressure waves going back and forth in the transmission line. We see that using few elements, tend to smooth out the pressure waves.

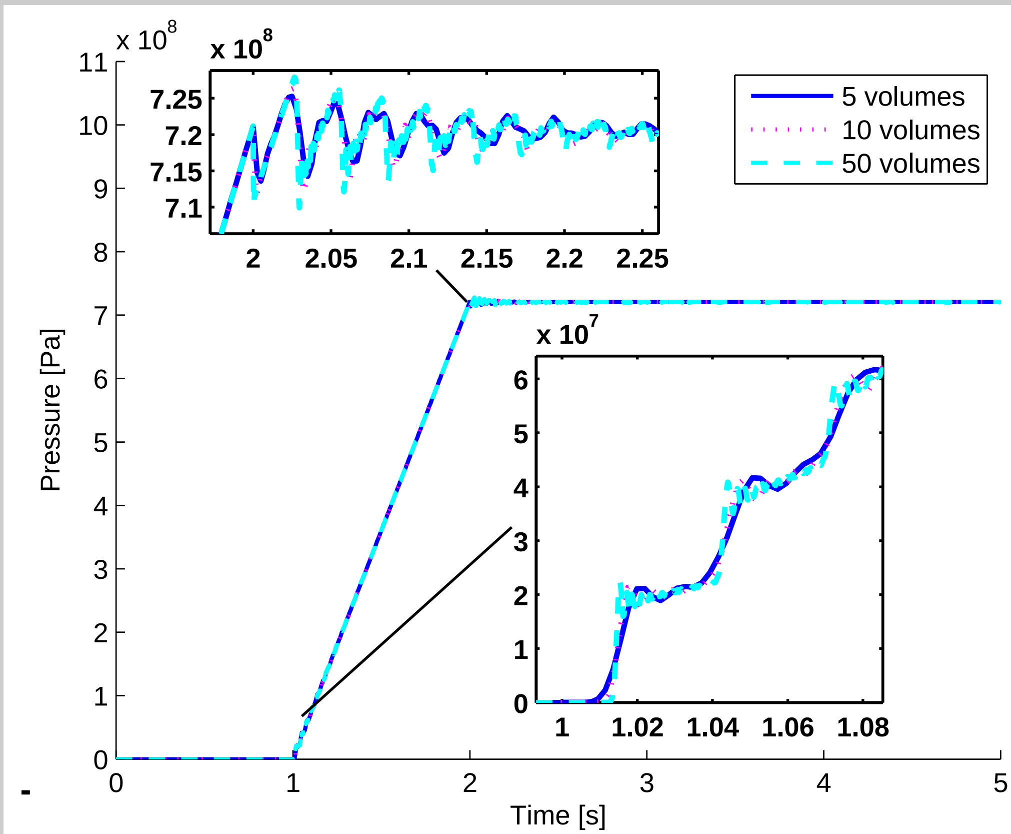

Figure 6: Simulation with zoom.

The Bode-plots are shown in Figure 7 and 8.

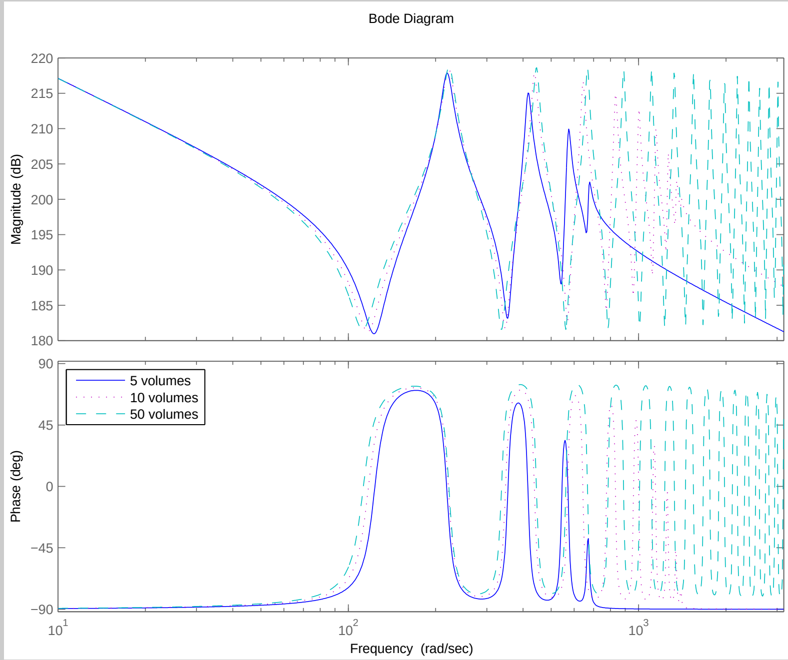

Figure 7: Frequency response from q_0 to p_1.

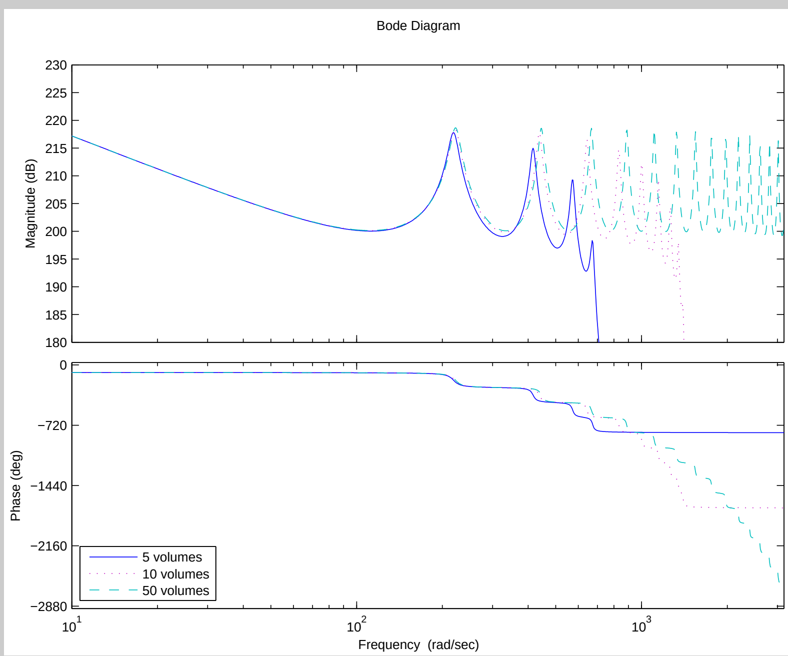

Figure 8: Frequency response from q_0 to p_N.

There is no delay (other than a phase-shift) from inflow to pressure in the first volume element, while there is a strong delay from inflow to pressure in the last volume element. The number of resonances depend on the number of elements in the model.

Problem 2 (Using another friction model)

We will now investigate another friction model (linear friction and diffusion) for the same system. The partial differential equation for the volume flow is now

$$\frac{\partial q}{\partial t} = -\frac{A}{\rho_0}\frac{\partial p}{\partial x} + \nu_0\left(\frac{\partial^2 q}{\partial x^2} - \frac{8}{r_0^2}q\right). \tag{2}$$

(a) Show that a reasonable discretization (difference approximation) of $\frac{\partial^2 q}{\partial x^2}$ is

$$\frac{\partial^2 q_i}{\partial x^2} = \frac{q_{i-2} - 2q_{i-1} + q_i}{h^2}.$$

Hint: Use forward and backward Euler approximations to the spatial derivative, twice.

Solution:

$$\begin{aligned}
\frac{\partial^2 q_i}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial q_i}{\partial x} \\
&\approx \frac{1}{h}\left(\frac{\partial q_i}{\partial x} - \frac{\partial q_{i-1}}{\partial x}\right) \\
&\approx \frac{1}{h}\left(\frac{1}{h}(q_i - q_{i-1}) - \frac{1}{h}(q_{i-1} - q_{i-2})\right) \\
&= \frac{q_{i-2} - 2q_{i-1} + q_i}{h^2}.
\end{aligned}$$

(b) Using this, discretize (2) in a similar manner as in Problem 1. Write down a linear state-space model for the discretized system.

Solution: The model becomes

$$\begin{aligned}
\dot{p}_i &= \frac{c^2\rho_0}{Ah}\left(q_{i-1} - q_i\right), \quad i = 1, \ldots, N \\
\dot{q}_{i-1} &= \frac{A}{h\rho_0}\left(p_{i-1} - p_i\right) - \left(\frac{2\nu_0}{h^2} + \frac{8\nu_0}{r_0^2}\right)q_{i-1} + \frac{\nu_0}{h^2}\left(q_{i-2} + q_i\right), \quad i = 2, \ldots, N \\
q_0 &= q_{in}, \quad q_N = q_{out}
\end{aligned}$$

This can be written on state-space form,

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}_v\mathbf{x} + \mathbf{B}_v\mathbf{u}, \\
\mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u},
\end{aligned}$$

where $\mathbf{A}_v$ is given as

$$\mathbf{A}_v = \begin{pmatrix} \begin{matrix} 0 & -M & 0 & 0 \\ N & -d & -N & k \end{matrix} & \cdots & \mathbf{0} \\ \begin{matrix} & \mathbf{0} & \mathbf{A}_2 & \end{matrix} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{A}_2 \quad \mathbf{0} \\ \mathbf{0} & \cdots & \begin{matrix} M & 0 & -M & 0 \\ k & N & -d & -N \\ 0 & 0 & M & 0 \end{matrix} \end{pmatrix}$$

$$\mathbf{A}_2 = \begin{pmatrix} M & 0 & -M & 0 & 0 \\ k & N & -d & -N & k \end{pmatrix}$$

$$M = \frac{\beta}{Ah},\ N = \frac{A}{h\rho_0},\ d = \frac{2\nu_0}{h^2} + \frac{8\nu_0}{r_0^2},\ k = \frac{\nu_0}{h^2},\ \mathbf{u} = \begin{pmatrix} q_0 \\ q_N \end{pmatrix}$$

and $\mathbf{B}_v$ as

$$\mathbf{B}_v = \begin{pmatrix} M & 0 \\ k & 0 \\ \vdots & \vdots \\ 0 & k \\ 0 & -M \end{pmatrix}$$

and **C** and **D** as in Problem 1.

Problem 3 (Index analysis of DAE-systems)

Consider the following DAE-system

$$\dot{x}_1 = x_3 \tag{3}$$
$$\dot{x}_2 = x_1 \tag{4}$$
$$0 = x_1 - u, \tag{5}$$

where u is the input to the system.

(a) Determine the differential and algebraic variables.

Solution: Based on (3)-(5) it can be determined that x_1 and x_2 are differential variables and x_3 is an algebraic variable since the time derivative of x_1 and x_2 is present in the considered system while x_3 does not occur in differential form. It occurs only in algebraic form.

(b) Find the degrees of freedom of the considered system.

Solution: To determine the number of degree of freedom (DOF) we need to find the number of independent variables that define the system's configuration. In (a), it was shown that we have 3 equations, 2 differentiable variable, 1 algebraic variable and 1 input variable. Based on these information and that x_2 is dependent on x_1 it can be concluded that system has 1 DOF the input would be a parameter the DOF of the system would be 0.

(c) Determine the index of the DAE-system.

Solution: We need to differentiate the algebraic equation to find a differential equation for the algebraic variable since $\frac{\partial g(x,y)}{\partial y}$ is singular

$$\frac{\partial g(x,y)}{\partial y} = 0, \tag{6}$$

which indicate that it is not a index 1 system.

In order to find the differential index the algebraic equation has to be differentiated. The derivative of the algebraic equation becomes

$$\begin{aligned} 0 &= \dot{x}_1 - \dot{u} \\ &= x_3 - \dot{u}. \end{aligned} \tag{7}$$

Now (7) can be solved for x_3 by taking the time derivative, which become

$$\dot{x}_3 = \ddot{u} \tag{8}$$

The differential index of the system (3)-(5) is 2, since we have to differentiate once to reach index 1 system and twice to transform the DAE system into an ODE system.

(d) Set up the system such that it has index 1 by using the maximum amount of algebraic equations and minimal amount of differential equations.

Solution: By using the maximum amount of algebraic equations and minimal amount of differential equations the system becomes

$$0 = x_3 - \dot{u} \tag{9}$$
$$\dot{x}_2 = x_1 \tag{10}$$
$$0 = x_1 - u, \tag{11}$$

which satisfy the condition of having a index 1.