Exercise 7 - TTK4130 Modeling and Simulation

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1 Problem 1

$$\begin{aligned}
\frac{\partial p(x,t)}{\partial t} &= -cZ_0 \frac{\partial q(x,t)}{\partial x} \\
\frac{\partial q(x,t)}{\partial t} &= -\frac{c}{Z_0}\frac{\partial p(x,t)}{\partial x} - \frac{F(q(x,t))}{\rho_0} \\
c &= \sqrt{\frac{\beta}{\rho_0}} \\
Z_0 &= \frac{\rho_0 c}{A} = \frac{\sqrt{\rho_0 \beta}}{A} \\
F &= \rho_0 B q \\
B &= \frac{8\nu_0}{r_0^2}
\end{aligned}$$

Helmholtz mass balance

$$\frac{V}{\beta}\dot{p} = q \tag{1}$$

Helmholtz moment balance

$$h\rho_0 \dot{q} = -Ap \tag{2}$$

1.1 a

Using Equation 1, the mass balance for each volume can be set up as

$$\frac{V}{\beta}\dot{p}_i = q_{i-1} - q_i \Rightarrow \underline{\underline{\dot{p}_i = \frac{c^2\rho_0}{Ah}(q_{i-1} - q_i)}}.$$

Using Equation 2, the moment balance for each volume can be set up as

$$h\rho_0 \dot{q}_i = -A(p_{i-1} - p_i) - Fh \Rightarrow \underline{\underline{\dot{q}_{i-1} = \frac{A}{h\rho_0}(p_{i-1} - p_i) - Bq_{i-1}}}.$$

1.2 b

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\
\mathbf{y} &= \mathbf{Cx} + \mathbf{Du} \\
\mathbf{u} &= \begin{bmatrix} q_0 \\ q_N \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} p_1 \\ p_N \end{bmatrix} \\
\mathbf{x}^T &= \begin{bmatrix} p_1 & q_1 & p_2 & q_2 & \dots & q_{N-1} & p_N \end{bmatrix}
\end{aligned}$$

$$
\mathbf{A} = \begin{bmatrix}
0 & -\frac{c^2\rho_0}{Ah} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\frac{A}{h\rho_0} & -B & -\frac{A}{h\rho_0} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \frac{c^2\rho_0}{Ah} & 0 & -\frac{c^2\rho_0}{Ah} & 0 & \cdots & \cdots & 0 \\
0 & 0 & \frac{A}{h\rho_0} & -B & -\frac{A}{h\rho_0} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \frac{c^2\rho_0}{Ah} & 0 & -\frac{c^2\rho_0}{Ah} & 0 \\
0 & 0 & \cdots & \cdots & 0 & \frac{A}{h\rho_0} & -B & -\frac{A}{h\rho_0} \\
0 & 0 & \cdots & \cdots & \cdots & 0 & \frac{c^2\rho_0}{Ah} & 0
\end{bmatrix}
$$

$$
\mathbf{B} = \begin{bmatrix}
\frac{c^2\rho_0}{Ah} & 0 \\
0 & \vdots \\
\vdots & \vdots \\
\vdots & 0 \\
0 & -\frac{c^2\rho_0}{Ah}
\end{bmatrix}
$$

$$
\mathbf{C} = \begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 1
\end{bmatrix}
$$

$$
\mathbf{D} = 0
$$

The diagonal and the 1st and -1st superdiagonal of A alternate between two values. This makes it easy to implement in MATLAB.

1.3 d

The response is quite similar for the three values of N, as seen in Figure 1. When consulting the bode plots in figures 2 and 3, it is clear that the system is more oscillating for large values of N. The code for generating these plots can be seen in listings 1 and 2

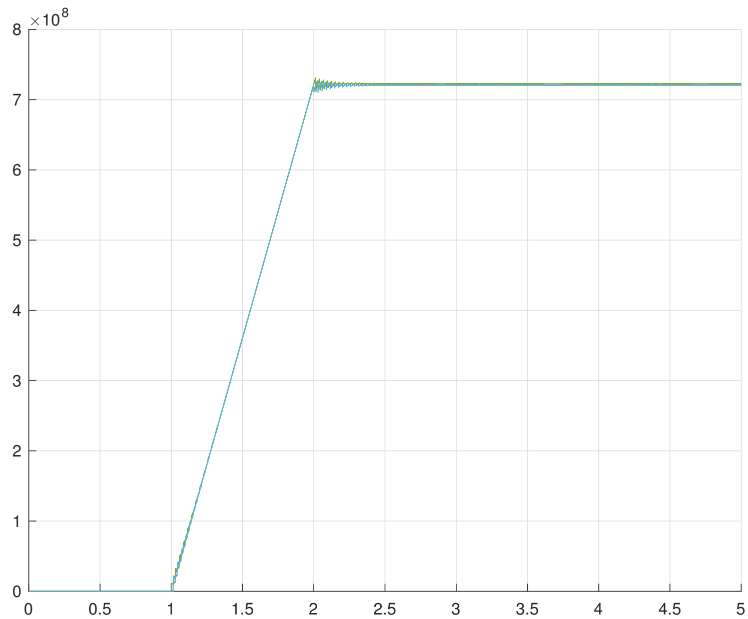

Figure 1: p_1 and p_N plotted for all values of N.

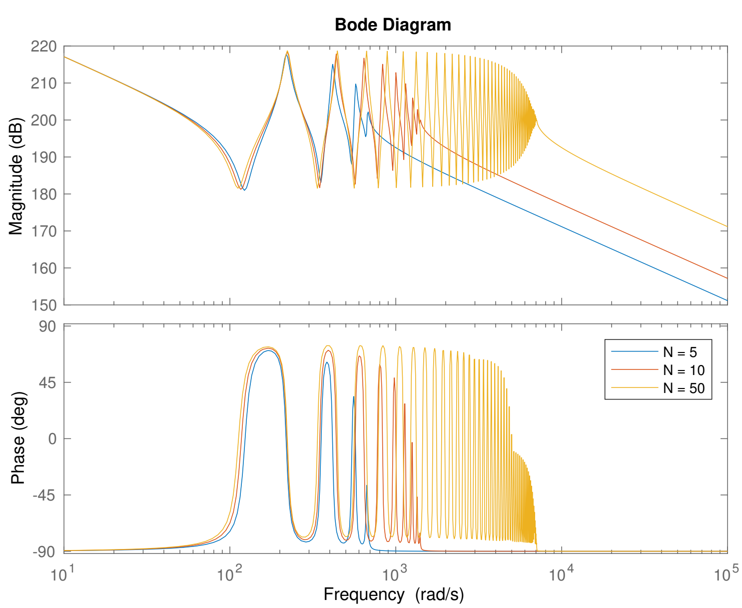

Figure 2: Bode plot of the response from q_0 to p_1 for all values of N.

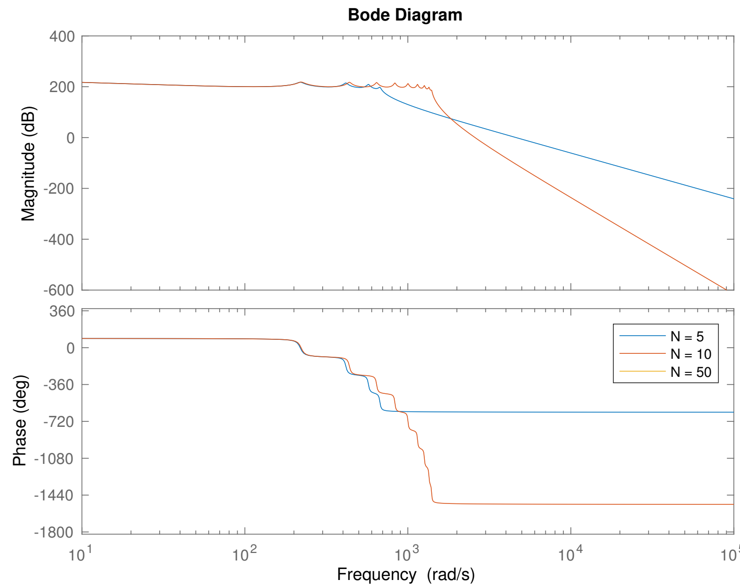

Figure 3: Bode plot of the response from q_0 to p_N for all values of N.

2 Problem 2

$$\frac{\partial q}{\partial t} = -\frac{A}{\rho_0}\frac{\partial p}{\partial x} + \nu_0(\frac{\partial^2 q}{\partial x^2} - \frac{8}{r_0^2}q). \tag{3}$$

2.1 a

Using that $\frac{\partial q_i}{\partial x} \simeq \frac{1}{h}(q_i - q_{i-1})$ for small h, twice, yields

$$\frac{\partial^2 q_i}{\partial x^2} \simeq \frac{1}{h}(\frac{\partial q_i}{\partial x} - \frac{\partial q_{i-1}}{\partial x}) \simeq \frac{1}{h}(\frac{1}{h}(q_i - q_{i-1}) - \frac{1}{h}(q_{i-1} - q_{i-2}))$$

$$\underline{\underline{\frac{\partial^2 q_i}{\partial x^2} = \frac{q_{i-2} - 2q_i + q_i}{h^2}}}$$

2.2 b

Discretizing in a similar manner as in Problem 1

$$\underline{\underline{\dot{p}_i = \frac{c^2\rho_0}{Ah}(q_{i-1} - q_i)}}$$

$$q_{i-1} = -\frac{A}{h\rho_0}(p_i - p_{i-1}) + \nu_0(\frac{\partial^2 q_i}{\partial x^2} - \frac{8}{r_0^2}q_{i-1})$$

$$\underline{\underline{q_{i-1} = \frac{A}{h\rho_0}(p_{i-1} - p_i) - (\frac{2\nu_0}{h^2} + \frac{8}{r_0^2})q_{i-1} + \frac{\nu_0}{h^2}(q_{i-2} + q_i)}}$$

$$\mathbf{A} = \begin{bmatrix} 0 & -\frac{c^2\rho_0}{Ah} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \frac{A}{h\rho_0} & -(\frac{2\nu_0}{h^2} + \frac{8}{r_0^2}) & -\frac{A}{h\rho_0} & \frac{\nu_0}{h^2} & 0 & \cdots & \cdots & 0 \\ 0 & \frac{c^2\rho_0}{Ah} & 0 & -\frac{c^2\rho_0}{Ah} & 0 & \cdots & \cdots & 0 \\ 0 & \frac{\nu_0}{h^2} & \frac{A}{h\rho_0} & -(\frac{2\nu_0}{h^2} + \frac{8}{r_0^2}) & -\frac{A}{h\rho_0} & \frac{\nu_0}{h^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{c^2\rho_0}{Ah} & 0 & -\frac{c^2\rho_0}{Ah} & 0 \\ 0 & 0 & \cdots & 0 & \frac{\nu_0}{h^2} & \frac{A}{h\rho_0} & -(\frac{2\nu_0}{h^2} + \frac{8}{r_0^2}) & -\frac{A}{h\rho_0} \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & \frac{c^2\rho_0}{Ah} & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \frac{c^2\rho_0}{Ah} & 0 \\ \frac{\nu_0}{h^2} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & \frac{\nu_0}{h^2} \\ 0 & -\frac{c^2\rho_0}{Ah} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

$$\mathbf{D} = 0$$

3 Problem 3

3.1 a

Since x_1 and x_2 appear as time derivatives in the DAE, they are differential variables. x_3 is an algebraic variable, as its derivative does not appear.

3.2 b

x_1 is dependent on x_3 (or the other way around), and x_2 s dependent on x_1. This means that there is only one independent variable in the system, and therefore the system has 1 degree of freedom.

3.3 c

To transform the DAE to an ODE we have to differentiate twice:

$$\frac{d}{dt}(x_1 - u) = \dot{x}_1 - \dot{u}$$
$$x_3 = \dot{u}$$
$$\frac{d^2}{dt^2}(x_1 - u) = \dot{x}_3 - \ddot{u}$$
$$\dot{x}_3 = \ddot{u}$$

Therefore, the system is of index 2.

3.4 d

$$0 = x_3 - \dot{u}$$
$$\dot{x}_2 = x_1$$
$$0 = x_1 - u$$

Listing 1: Code for initialising a system with N volumes.

```
L = 19.76;
beta = 1.7052*10^9;
rho = 870;
r = 6.17*10^(-3);
nu = 8*10^(-5);
area = pi*r^2;
c = sqrt(beta/rho);
h = L/N;
Z_0 = rho*c/area;
b = 8*nu/r^2;


x_0 = zeros(N+N-1,1);
tsim = 5;

A = zeros(N+N-1,N+N-1);
B = zeros(N+N-1,2);
C = zeros(2,N+N-1);
D = zeros(2,2);

C(1,1) = 1;
C(2,N +N-1) = 1;


B(1,1) = c^2*rho/(area*h);
B(N+N-1,2) = -c^2*rho/(area*h);
```

```

v = repmat([0 -b], 1,N-1);
v(1,N + N -1) = 0;

A = diag(v);

v = repmat([-c^2*rho/(area*h) -area/(rho*h)], 1,N-1);

A = A + diag(v,1);

v = repmat([area/(rho*h) c^2*rho/(area*h)], 1,N-1);

A = A + diag(v,-1);
```

Listing 2: Code for running and plotting a system with N volumes.

```
clear; clc;
N = 5;

ModSim_ex7_1c_init;

sim('ModSim_ex7_1c_model')

figure(1)
hold on; grid on;
plot(p_1.Time, p_1.Data, p_N.Time, p_N.Data);

sys5 = ss(A,B,C,D);


N = 10;

ModSim_ex7_1c_init;

sim('ModSim_ex7_1c_model')

figure(1)
plot(p_1.Time, p_1.Data, p_N.Time, p_N.Data);

sys10 = ss(A,B,C,D);



N = 50;

ModSim_ex7_1c_init;
```

```

sim('ModSim_ex7_1c_model')

figure(1)
plot(p_1.Time, p_1.Data, p_N.Time, p_N.Data);
print -depsc ex7_1c

sys50 = ss(A,B,C,D);

figure(2)
bode(sys5(1,1),sys10(1,1),sys50(1,1))
legend('N = 5','N = 10','N = 50')
print -depsc ex7_1c_bode_p1

figure(3)
bode(sys5(1,2),sys10(1,2),sys50(1,2))
legend('N = 5','N = 10','N = 50')
print -depsc ex7_1c_bode_pN
```