Lecture 16: Rigid body kinematics – Kinematic differential equations

- Brief recap of representations of rotation
  - Rotation matrices (6.4)
  - Euler angles (6.5)
    - 3-parameter representation of rotations
    - Roll-pitch-yaw
  - Angle-axis, Euler-parameters (6.6, 6.7)
    - 4-parameter representation of rotations
  - Angular velocity (6.8)

- Today:
  - Kinematic differential equations
  - Rigid body kinematics: Configuration

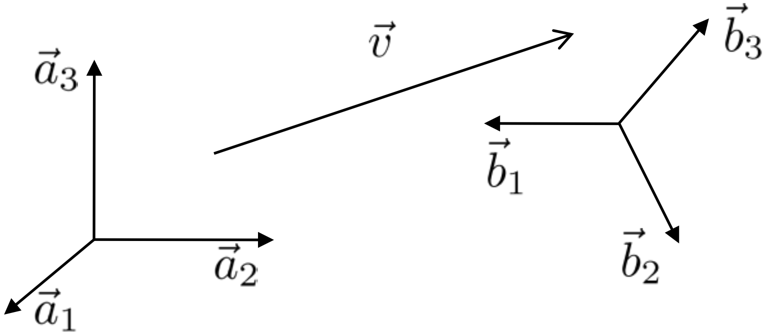
Book: Ch. 6.9, 6.12, 6.13

Rotation matrices

The rotation matrix from *a* to *b* $\mathbf{R}_b^a$ is used to

- **Transform** a coordinate vector from *b* to *a*

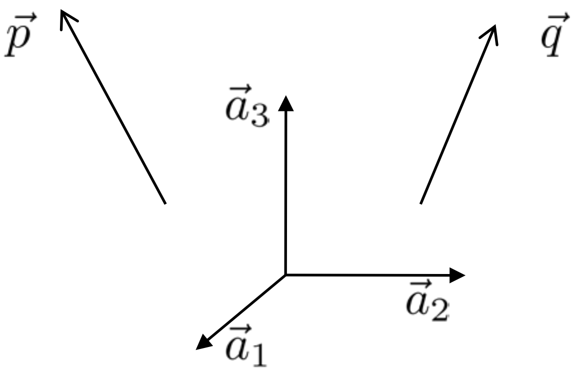
$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b$$



- **Rotate** a vector $\vec{p}$ to vector $\vec{q}$. If decomposed in *a*,

$$\mathbf{q}^a = \mathbf{R}_b^a \mathbf{p}^a$$

such that $\mathbf{q}^b = \mathbf{p}^a$.

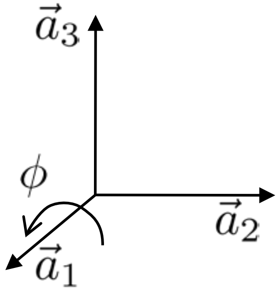


Simple rotations

- Simple rotation = rotation about an axis

- Example: Rotation matrix for rotation about *x*-axis:

$$\mathbf{R}_{x,\phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

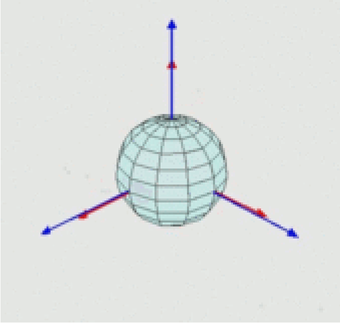


Representations of rotations

- Rotation matrix
  - Easy to use, but not to visualize (also over-parameterized, 9 parameters)

> Euler's Theorem:
>
> "Any two independent orthonormal coordinate frames can be related by a sequence of rotations (not more than three) about coordinate axes, where no two successive rotations may be about the same axis."

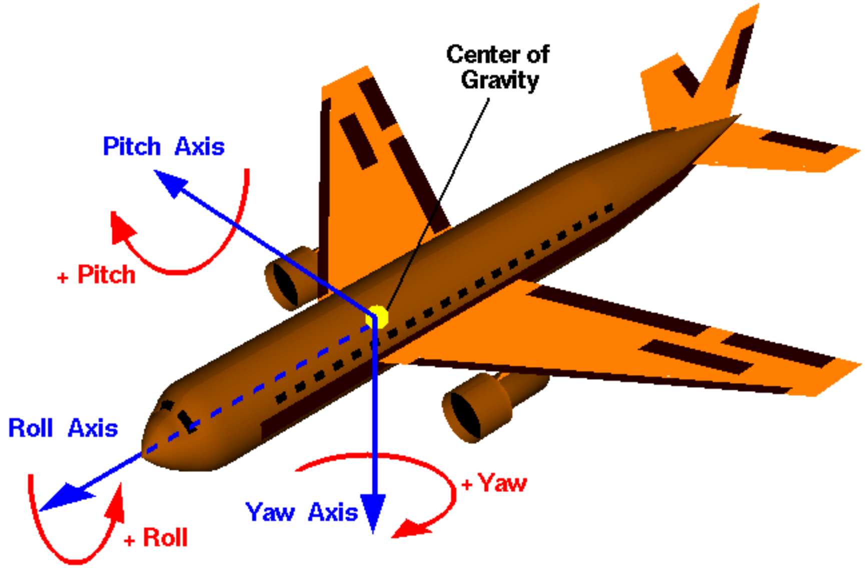
- Three rotations about axes are enough to specify any rotation
  - These representations are called Euler angles
    - 12 different combinations possible
    - Most common(?): Roll-pitch-yaw
  - Natural and (in many cases) simple to use, very much used
  - Problem: Singularity (more on this today)



Source: Wikipedia

- Angle-axis, Euler-parameters
  - 4-parameter representations of rotations
  - No singularity problems

Euler-angles: Roll-pitch-yaw

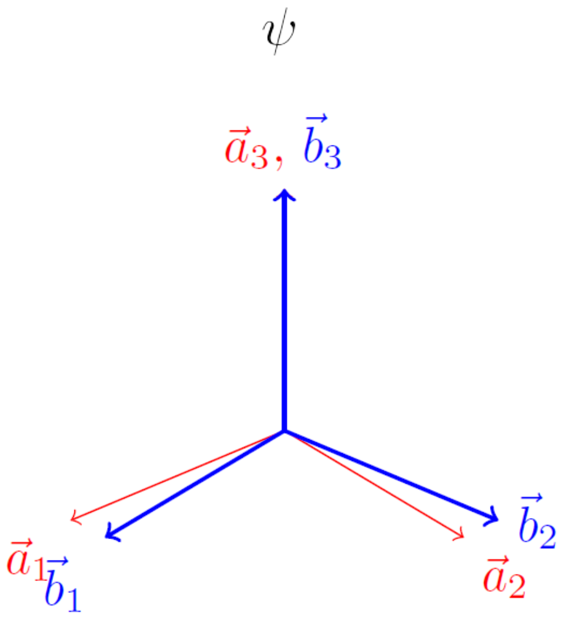


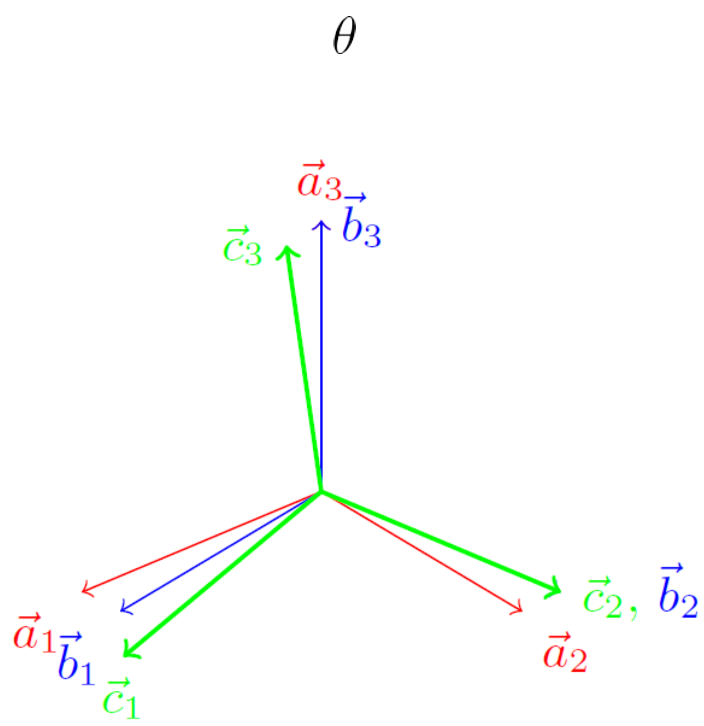
- Rotation ψ about *z*-axis, θ about (rotated) *y*-axis, ϕ about (rotated) *x*-axis

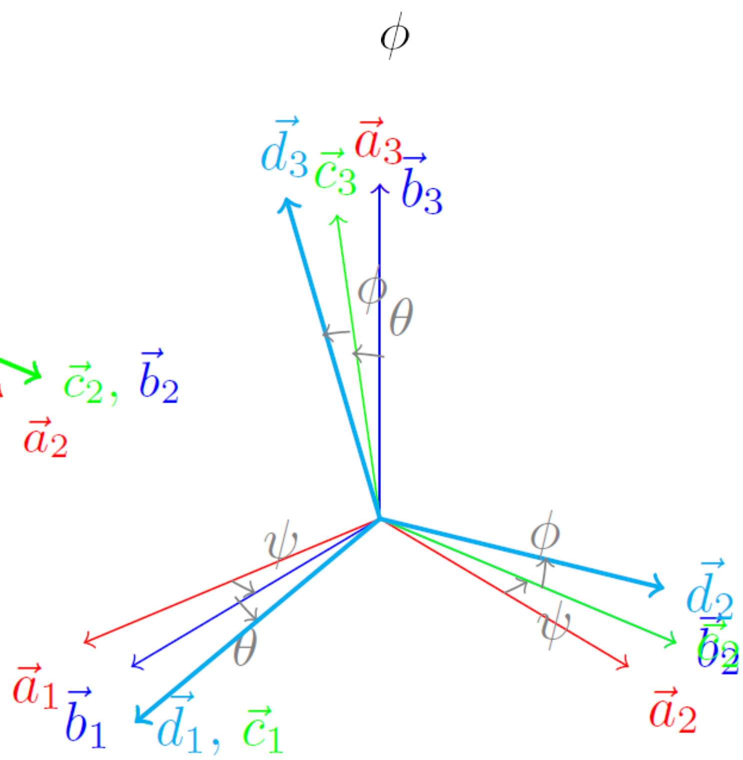
$$\mathbf{R}_b^a = \mathbf{R}_{z,\psi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\phi}$$

$$\mathbf{R}_b^a = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

Euler angles

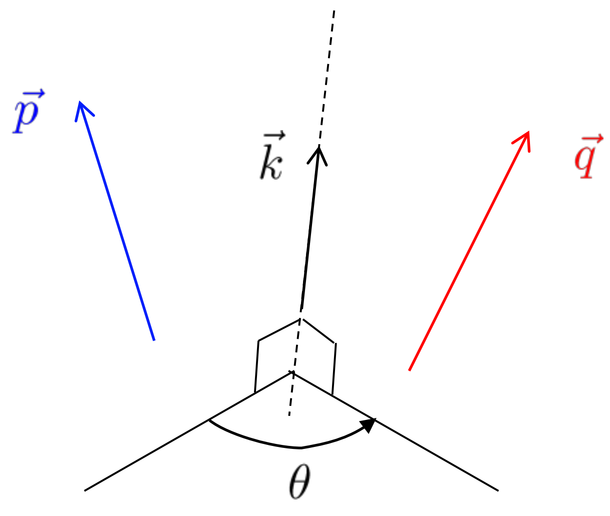






Angle-axis representation of rotations

All rotations can be represented as a simple rotation around an axis



- Angle-axis parameters:
  - Coordinate free: $\vec{k},\ \theta$

$$\vec{q} = \Big(\underbrace{\cos\theta\ \vec{I} + \sin\theta\ \vec{k}^{\times} + (1-\cos\theta)\,\vec{k}\vec{k}}_{\vec{R}_{\vec{k},\theta}}\Big)\cdot\vec{p}$$

  - With coordinates: $\mathbf{k}^a,\ \theta$

$$\mathbf{R}^a_b = \mathbf{R}_{\mathbf{k},\theta} = \cos\theta\ \mathbf{I} + \sin\theta\ (\mathbf{k}^a)^{\times} + (1-\cos\theta)\,\mathbf{k}^a(\mathbf{k}^a)^{\mathsf{T}}$$

Euler parameters

- Euler parameters are closely related to angle-axis:
  - Coordinate-free:

$$\eta = \cos\frac{\theta}{2}$$
$$\vec{\epsilon} = \vec{k}\sin\frac{\theta}{2}$$

With coordinates:

$$\eta = \cos\frac{\theta}{2}$$
$$\boldsymbol{\epsilon} = \mathbf{k}\sin\frac{\theta}{2}$$

- Rotation matrix (on coordinate form):

$$\mathbf{R}(\eta, \boldsymbol{\epsilon}) = \mathbf{I} + 2\eta\boldsymbol{\epsilon}^\times + 2\boldsymbol{\epsilon}^\times\boldsymbol{\epsilon}^\times$$

- Much used, since:
  - Compact, singularity-free representation of orientation
  - No trigonometric terms in expression for rotation matrix
  - $\eta^2 + \vec{\epsilon}\cdot\vec{\epsilon} = 1$: Easy to normalize (avoid roundoff errors)
    - Rotation matrices may tend to become non-orthogonal when simulated
  - Euler parameters are *(unit) quaternions*:
    - Quaternions are generalized complex numbers
    - Can use algebra of quaternions for calculations and analysis

Derivatives of rotations

- Derivative of position $\mathbf{r}$ is velocity, $\dot{\mathbf{r}} = \mathbf{v}$.
- Derivative of rotation matrix $\mathbf{R}_b^a$ is $\dot{\mathbf{R}}_b^a$. What is this?
- Seems natural that a concept of angular velocity should be involved, but how?
  - (Tuesday, repeated next slide)
- What are derivatives of representations of rotations?
  - Derivatives of Euler angles? Euler parameters?
  - These are the kinematic differential equations (today's main topic)

Angular velocity

- The rotation matrix is orthogonal:

$$\mathbf{R}_b^a \left(\mathbf{R}_b^a\right)^\mathsf{T} = \mathbf{I}$$

- Differentiate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{R}_b^a \left(\mathbf{R}_b^a\right)^\mathsf{T}\right] = \dot{\mathbf{R}}_b^a \left(\mathbf{R}_b^a\right)^\mathsf{T} + \mathbf{R}_b^a \left(\dot{\mathbf{R}}_b^a\right)^\mathsf{T} = \mathbf{0}$$

- If we define $\mathbf{S} = \dot{\mathbf{R}}_b^a \left(\mathbf{R}_b^a\right)^\mathsf{T}$, this says that $\mathbf{S} + \mathbf{S}^\mathsf{T} = \mathbf{0}$ which means that $\mathbf{S}$ is skew symmetric.

$$\mathbf{S} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \left(\boldsymbol{\omega}_{ab}^a\right)^\times$$

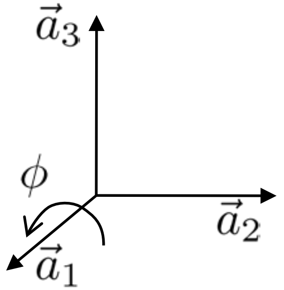
- The vector $\boldsymbol{\omega}_{ab}^a$ defined by $\left(\boldsymbol{\omega}_{ab}^a\right)^\times = \dot{\mathbf{R}}_b^a \left(\mathbf{R}_b^a\right)^\mathsf{T}$ is the angular velocity of frame *b* relative to frame *a* (decomposed in *a*)

- The equation $\dot{\mathbf{R}}_b^a = \left(\boldsymbol{\omega}_{ab}^a\right)^\times \mathbf{R}_b^a$ is the kinematic differential equation for rotation matrices

Angular velocity of simple rotations

- Rotation about *x*-axis:

$$\mathbf{R}_{x,\phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$



- We calculate $(\boldsymbol{\omega}_{ab}^{a})^{\times} = \dot{\mathbf{R}}_b^a \left(\mathbf{R}_b^a\right)^{\mathsf{T}}$:

$$\dot{\mathbf{R}}_{x,\phi}\left(\mathbf{R}_{x,\phi}\right)^{\mathsf{T}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & -\cos\phi \\ 0 & \cos\phi & -\sin\phi \end{pmatrix} \dot{\phi} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\phi} \\ 0 & \dot{\phi} & 0 \end{pmatrix}$$

- That is:

$$\boldsymbol{\omega}_x = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix}$$

- Similar for rotations around *y*- and *z*-axis: $\boldsymbol{\omega}_y = \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix}, \quad \boldsymbol{\omega}_z = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$

- Angle-axis representations (constant axis):

$$\boldsymbol{\omega}_{ab}^{a} = \dot{\theta}\mathbf{k}^{a}$$

Composite rotations

- Given
  - composite rotation $\mathbf{R}_d^a = \mathbf{R}_b^a \mathbf{R}_c^b \mathbf{R}_d^c$, and
  - individual angular velocities $\boldsymbol{\omega}_{ab}^a$, $\boldsymbol{\omega}_{bc}^b$, and $\boldsymbol{\omega}_{cd}^c$

  How to calculate the composite angular velocity $\boldsymbol{\omega}_{ad}^a$?

- It can be shown (easy, see book p. 241) that

$$\vec{\omega}_{ad} = \vec{\omega}_{ab} + \vec{\omega}_{bc} + \vec{\omega}_{cd}$$

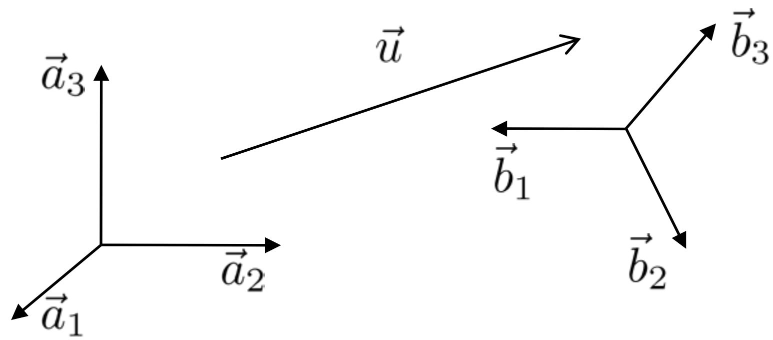
- On coordinate form:

$$\boldsymbol{\omega}_{ad}^a = \boldsymbol{\omega}_{ab}^a + \boldsymbol{\omega}_{bc}^a + \boldsymbol{\omega}_{cd}^a$$

- So:

$$\boldsymbol{\omega}_{ad}^a = \boldsymbol{\omega}_{ab}^a + \mathbf{R}_b^a \boldsymbol{\omega}_{bc}^b + \mathbf{R}_b^a \mathbf{R}_c^b \boldsymbol{\omega}_{cd}^c$$

Differentiation of vectors (6.8.5, 6.8.6)



- Coordinate representation:

$$\mathbf{u}^a = \mathbf{R}_b^a \mathbf{u}^b$$

- Differentiation:

$$\dot{\mathbf{u}}^a = \mathbf{R}_b^a \dot{\mathbf{u}}^b + \dot{\mathbf{R}}_b^a \mathbf{u}^b$$

$$\dot{\mathbf{R}}_b^a = \mathbf{R}_b^a \left(\boldsymbol{\omega}_{ab}^b\right)^\times$$

$$\boxed{\dot{\mathbf{u}}^a = \mathbf{R}_b^a \left[\dot{\mathbf{u}}^b + \left(\boldsymbol{\omega}_{ab}^b\right)^\times \mathbf{u}^b\right]}$$

Note! Generally,

$$\dot{\mathbf{u}}^a \neq \mathbf{R}_b^a \dot{\mathbf{u}}^b$$

- On vector form:

$$\boxed{\frac{{}^a\mathrm{d}}{\mathrm{d}t}\vec{u} = \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{u} + \vec{\omega}_{ab} \times \vec{u}}$$

Kahoot

- https://play.kahoot.it/#/k/4152faff-75ee-49ea-bb9e-b4c79dd85785