Exam 2016 – review

Problem 1

Problem 1 (28 %)

Consider the following simulation method:

$$
\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(\mathbf{y}_n, t_n) \\
\mathbf{k}_2 &= \mathbf{f}(\mathbf{y}_n + \frac{h}{4}\mathbf{k}_1 + \frac{h}{4}\mathbf{k}_2, t_n + \frac{h}{2}) \\
\mathbf{k}_3 &= \mathbf{f}(\mathbf{y}_n + h\mathbf{k}_2, t_n + h) \\
\mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{h}{6}(\mathbf{k}_1 + 4\mathbf{k}_2 + \mathbf{k}_3)
\end{aligned}
$$

(2 %) (a) Is this method explicit or implicit? How many stages does it have?

(3 %) (b) Write up the Butcher array for this method.

(1 %) (c) How can you see from the Butcher array whether this is an implicit or explicit method?

(4 %) (d) Comment on how much work it is to implement and solve this algorithm, compared to a general three-stage implicit Runge-Kutta method.

(8 %) (e) Find the stability function for this method. *Hint*:

$$
\begin{pmatrix} a & 0 & 0 \\ d & b & 0 \\ 0 & e & c \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ -\frac{d}{ab} & \frac{1}{b} & 0 \\ \frac{de}{abc} & -\frac{e}{bc} & \frac{1}{c} \end{pmatrix}
$$

(4 %) (f) Is the method A-stable? L-stable? Justify the answer.

(6 %) (g) What is the order of the method? Justify the answer. Use the fact that for a Runge-Kutta method of order p, the stability function $R(s)$ approximates e^s with error $O(s^{p+1})$. (*Hint*: If you do long calculations/derivations, then you are probably attacking this the wrong (or at least not the most straightforward) way.)

Explicit Runge-Kutta (ERK) methods (L6)

- IVP: $\dot{y} = f(y,t), \quad y(0) = y_0$
- ERK:

$$
\begin{aligned}
k_1 &= f(y_n, t_n) \\
k_2 &= f(y_n + h a_{21} k_1, t_n + c_2 h) \\
k_3 &= f(y_n + h(a_{31} k_1 + a_{32} k_2), t_n + c_3 h) \\
&\vdots \\
k_\sigma &= f(y_n + h(a_{\sigma,1} k_1 + a_{\sigma,2} k_2 + \ldots + a_{\sigma,\sigma-1} k_{\sigma-1}), t_n + c_\sigma h) \\
y_{n+1} &= y_n + h(b_1 k_1 + b_2 k_2 + \ldots + b_\sigma k_\sigma)
\end{aligned}
$$

- Butcher array:

$$
\begin{array}{c|c}
\mathbf{c} & \mathbf{A} \\
\hline
 & \mathbf{b}^\mathsf{T}
\end{array}
\qquad
\begin{array}{c|ccccc}
0 & & & & & \\
c_2 & a_{21} & & & & \\
c_3 & a_{31} & a_{32} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
c_\sigma & a_{\sigma,1} & a_{\sigma,2} & \cdots & a_{\sigma,\sigma-1} & \\
\hline
 & b_1 & b_2 & \ldots & b_{\sigma-1} & b_\sigma
\end{array}
$$

Implicit Runge-Kutta (IRK) methods (L8)

- IVP: $\dot{y} = f(y,t), \quad y(0) = y_0$
- IRK:

$$
\begin{aligned}
k_1 &= f(y_n + h(a_{1,1}k_1 + a_{1,2}k_2 + \ldots + a_{1,\sigma}k_\sigma), t_n + c_1 h) \\
k_2 &= f(y_n + h(a_{2,1}k_1 + a_{2,2}k_2 + \ldots + a_{2,\sigma}k_\sigma), t_n + c_2 h) \\
k_3 &= f(y_n + h(a_{3,1}k_1 + a_{3,2}k_2 + \ldots + a_{3,\sigma}k_\sigma), t_n + c_3 h) \\
&\vdots \\
k_\sigma &= f(y_n + h(a_{\sigma,1}k_1 + a_{\sigma,2}k_2 + \ldots + a_{\sigma,\sigma}k_\sigma), t_n + c_\sigma h) \\
y_{n+1} &= y_n + h(b_1 k_1 + b_2 k_2 + \ldots + b_\sigma k_\sigma)
\end{aligned}
$$

- Butcher array:

$$
\begin{array}{c|c}
\mathbf{c} & \mathbf{A} \\
\hline
 & \mathbf{b}^\mathsf{T}
\end{array}
\qquad
\begin{array}{c|ccccc}
c_1 & a_{11} & a_{12} & \cdots & & a_{1,\sigma} \\
c_2 & a_{21} & a_{22} & \cdots & & a_{2,\sigma} \\
\vdots & \vdots & \vdots & \ddots & & \\
c_\sigma & a_{\sigma,1} & a_{\sigma,2} & \cdots & a_{\sigma,\sigma-1} & a_{\sigma,\sigma} \\
\hline
 & b_1 & b_2 & \ldots & b_{\sigma-1} & b_\sigma
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$$

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Recap: Test system, stability function (L6)

- One step method (typically: Runge-Kutta):

$$y_{n+1} = y_n + h\phi(y_n, t_n)$$

- Apply it to scalar test system:

$$\dot{y} = \lambda y$$

- We get:

$$y_{n+1} = R(h\lambda)y_n$$

  where $R(h\lambda)$ is stability function

- The method is stable (for test system!) if

$$|R(h\lambda)| \leq 1$$

Stability function for RK-methods (L8)

- Two alternative, equivalent expressions can be derived:
  - Either

$$R(h\lambda) = 1 + h\lambda \mathbf{b}^{\mathsf{T}} \left(\mathbf{I} - h\lambda \mathbf{A}\right)^{-1} \mathbf{1}$$

  - or

$$R(h\lambda) = \frac{\det\left[\mathbf{I} - h\lambda\left(\mathbf{A} - \mathbf{1}\mathbf{b}^{\mathsf{T}}\right)\right]}{\det\left[\mathbf{I} - h\lambda \mathbf{A}\right]}$$

- The latter can be simplified for ERK (since A is lower triangular):

$$R_E(h\lambda) = \det\left[\mathbf{I} - h\lambda\left(\mathbf{A} - \mathbf{1}\mathbf{b}^{\mathsf{T}}\right)\right]$$

- Two observations can be made
  1. $|R_E(h\lambda)|$ will tend to infinity when λ goes to infinity.
  2. $R_E(h\lambda)$ is a polynomial in $h\lambda$ of order less than or equal to σ.

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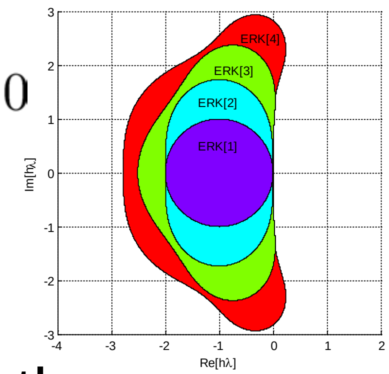
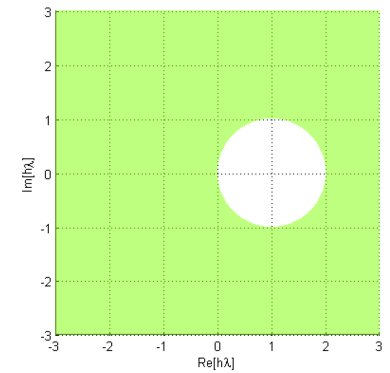
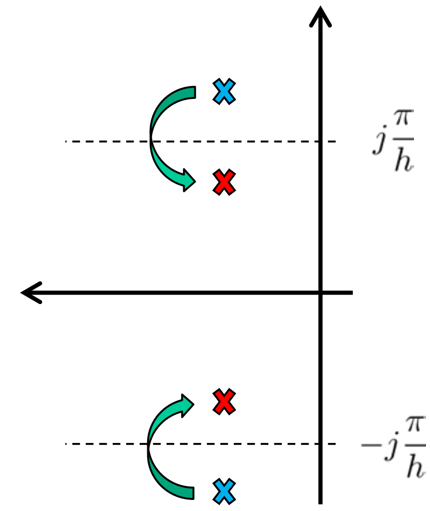
Linear stability: A- and L-stability (L10)

- A-stability: $|R(h\lambda)| \leq 1$ for all $\mathrm{Re}\,\lambda \leq 0$
  - Relevant (mostly) for **stiff** systems
  - No explicit methods are A-stable
  - Many implicit methods are A-stable





- Aliasing: Frequencies larger than the «Nyquist frequency» $\frac{\pi}{\mathrm{h}}$ are mapped to within the Nyquist frequency



- A method is L-stable if it is A-stable, and

$$|R(hj\omega)| \to 0 \text{ when } \omega \to \infty$$

  - Relevant for (stiff) systems with **oscillatory modes**
  - Dampens out fast frequencies
  - We often want L-stability in implicit methods, but not always:
    - We typically want to suppress dynamics that are faster than step length («stiff decay»)
    - However, we may want to not dampen oscillatory modes
    - We may want to **not** dissipate energy (numerically) in simulations

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Padé approximations to e^s (L9)

$k \backslash m$	0	1	2	3
0	$\frac{1}{1}$	$\frac{1+s}{1}$	$\frac{1+s+\frac{1}{2}s^2}{1}$	$\frac{1+s+\frac{1}{2}s^2+\frac{1}{6}s^3}{1}$
1	$\frac{1}{1-s}$	$\frac{1+\frac{1}{2}s}{1-\frac{1}{2}s}$	$\frac{1+\frac{2}{3}s+\frac{1}{6}s^2}{1-\frac{1}{3}s}$	$\frac{1+\frac{3}{4}s+\frac{1}{4}s^2+\frac{1}{24}s^3}{1-\frac{1}{4}s}$
2	$\frac{1}{1-s+\frac{1}{2}s^2}$	$\frac{1+\frac{1}{3}s}{1-\frac{2}{3}s+\frac{1}{6}s^2}$	$\frac{1+\frac{1}{2}s+\frac{1}{12}s^2}{1-\frac{1}{2}s+\frac{1}{12}s^2}$	$\frac{1+\frac{3}{5}s+\frac{3}{20}s^2+\frac{1}{60}s^3}{1-\frac{2}{5}s+\frac{1}{20}s^2}$
3	$\frac{1}{1-s+\frac{1}{2}s^2-\frac{1}{6}s^3}$	$\frac{1+\frac{1}{4}s}{1-\frac{3}{4}s+\frac{1}{4}s^2-\frac{1}{24}s^3}$	$\frac{1+\frac{2}{5}s+\frac{1}{20}s^2}{1-\frac{3}{5}s+\frac{3}{20}s^2-\frac{1}{60}s^3}$	$\frac{1+\frac{1}{2}s+\frac{1}{10}s^2+\frac{1}{120}s^3}{1-\frac{1}{2}s+\frac{1}{10}s^2-\frac{1}{120}s^3}$

L-stable L-stable A-stable

- $m = 0$: Explicit Runge-Kutta methods with $p = \sigma$
- $m = k$: Gauss, Lobatto IIIA/IIIB (incl. implicit mid-point, trapezoidal)
- $m = k+1$: Radau-methods (incl. implicit Euler)
- $m = k+2$: Lobatto IIIC

Problem 2

Problem 2 (32 %)

The double inverted pendulum on a cart (DIPC) poses a challenging control problem. In a DIPC system, two rods are connected together on a moving cart as shown in Figure 1. The length of the first rod is denoted by l_1 and the length of the second rod by l_2. The mass of the cart is denoted by m_0, its length by l_0 and its width by b_0. The height of the cart is denoted by h_0. Both rods have a mass, which are denoted by m_1 and m_2. All masses are assumed to be concentrated into the centre of mass. The moments of inertia are denoted by I_i. Furthermore, the force τ is acting on the cart.

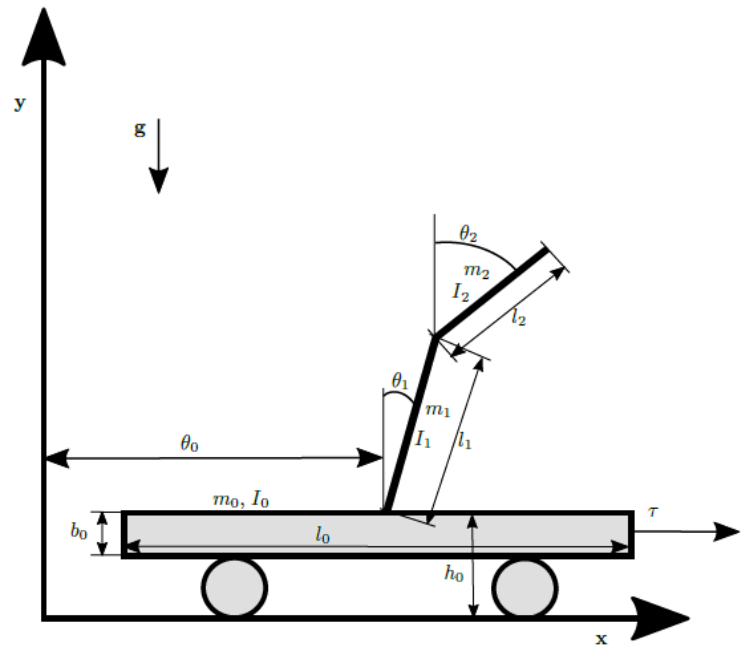

Figure 1: Double inverted pendulum on a cart

(4 %) (a) Choose generalized coordinates, and find the positions of the centers of mass for each of the three bodies (the cart and the two rods).

(10 %) (b) Find the kinetic energy of the system. (*Hint:* the following identity may simplify the expressions: $\cos(x-y) = \cos x \cos y + \sin x \sin y$.)

(4 %) (c) Find the potential energy of the system.

(14 %) (d) Find the equations of motion of the system.

Lagrange vs Newton-Euler

Newton-Euler

- Vectors
- Forces and moments
- Does not eliminate forces of constraints:
  - Obtains solutions for all forces and kinematic variables
  - ”Inefficient” (large DAE models)
- More general
  - Large systems can be handled, but for some configurations tricks are needed
  - Used in advanced modeling software

Lagrange

- Algebraic
- Energy
- Eliminates forces of constraints
  - Solutions only for generalized coordinates (and forces)
  - ”Efficient” (smaller ODE models)
- Less general
  - Need independent generalized coordinates
  - Difficult to automate for large/complex problems

Lagrange equations of motion (L20)

Generalized coordinates

- Find *n* generalized coordinates that parametrize ”degrees of freedom” (allowed motion).
  - That is, all positions are function of generalized coordinates

$$\vec{r}_k = \vec{r}_k(\mathbf{q}) \qquad \mathbf{q} = \begin{pmatrix} q_1 & q_2 & \dots & q_n \end{pmatrix}^\mathsf{T}$$

- Differentiate to find velocity

$$\vec{v}_k(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{r}_k(\mathbf{q}) = \sum_{i=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i}\dot{q}_i$$

  - For rigid bodies: velocity of center(s) of mass, and also angular velocity $\vec{\omega}_{ib}(\mathbf{q}, \dot{\mathbf{q}})$

- Find the generalized (actuator) forces τ_i associated with q_i
  - If q_i angle, then τ_i torque
  - If q_i displacement, then τ_i force

$$\tau_i = \sum_{i=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i} \cdot \vec{F}_k$$

- On coordinate form:

$$k = 1, \dots, N \text{ particles: } \mathbf{r}_k^i(\mathbf{q}), \quad \mathbf{v}_k^i(\mathbf{q}, \dot{\mathbf{q}})$$

$$k = 1, \dots, N \text{ rigid bodies: } \mathbf{r}_{ck}^i(\mathbf{q}), \quad \mathbf{v}_{ck}^b(\mathbf{q}, \dot{\mathbf{q}}), \quad \boldsymbol{\omega}_{ik}^b(\mathbf{q}, \dot{\mathbf{q}}), \quad \mathbf{M}_{k/c}^b$$

Lagrange equations of motion (L20)

Kinetic and potential energy

- Find kinetic energy:
  - N particles: $T = \sum_{k=1}^{N} \frac{1}{2} m_k \vec{v}_k \cdot \vec{v}_k$
  - Each rigid body (p. 273):

$$T = \int_b \frac{1}{2}\vec{v}_p \cdot \vec{v}_p \mathrm{d}m = \frac{1}{2} m \vec{v}_c \cdot \vec{v}_c + \frac{1}{2}\vec{\omega}_{ib} \cdot \vec{M}_{b/c} \cdot \vec{\omega}_{ib}$$

- On coordinate form:

$$N \text{ particles: } T = \sum T_k, \quad T_k(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2} m_k (\mathbf{v}_k^i)^\mathsf{T} \mathbf{v}_k^i = \frac{1}{2} m_k (\mathbf{v}_k^b)^\mathsf{T} \mathbf{v}_k^b$$

$$N \text{ rigid bodies: } T = \sum T_k, \quad T_k(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2} m_k (\mathbf{v}_{ck}^b)^\mathsf{T} \mathbf{v}_{ck}^b + \frac{1}{2} (\boldsymbol{\omega}_{ik}^b)^\mathsf{T} \mathbf{M}_{k/c}^b \boldsymbol{\omega}_{ik}^b$$

- Find (total) potential energy $U = U(\mathbf{q}) = \sum U_k(\mathbf{q})$
  - Gravity: $U_k(\mathbf{q}) = m_k g h(\mathbf{q})$
  - Spring: $U_k(\mathbf{q}) = \frac{1}{2} k x^2(\mathbf{q})$
  - ...

Lagrange equations of motion (L20)

- Construct Lagrangian

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q})$$

- Find 2*n* partial derivatives (scalars)

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad \frac{\partial \mathcal{L}}{\partial q_i}$$

- Write up *n* equations of motion
  - That is, *n* 2nd order differential equations

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i}$$

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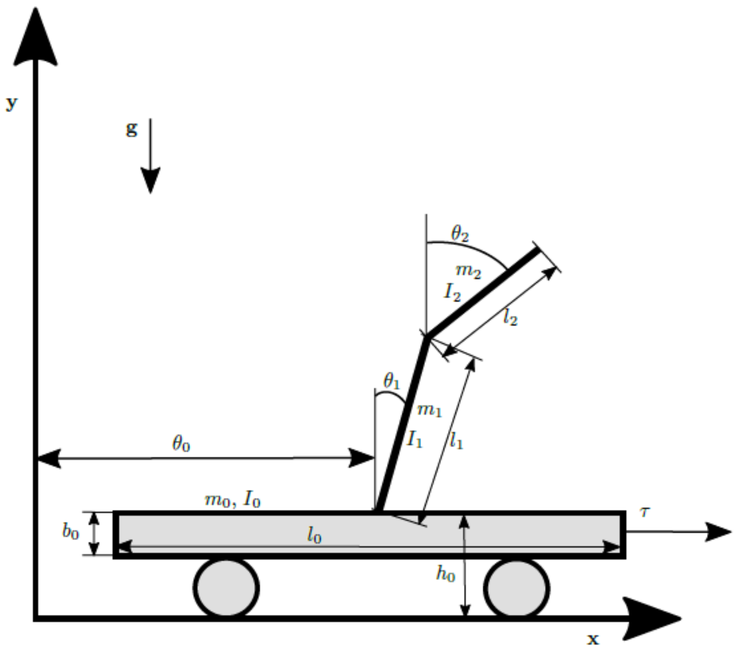

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Problem 2 d)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i$$

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q}),$$

$$\begin{aligned}
L = &\frac{1}{2}\Bigg[(m_0 + m_1 + m_2)\,\dot{\theta}_0^2 + \left(\frac{1}{4}m_1 l_1^2 + m_2 l_1^2 + I_1\right)\dot{\theta}_1^2 + \\
&\left(\frac{1}{4}m_2 l_2^2 + I_2\right)\dot{\theta}_2^2 + (m_1 l_1 + 2m_2 l_1)\,\dot{\theta}_0\dot{\theta}_1\cos\theta_1 + \\
&m_2 l_2 \dot{\theta}_0 \dot{\theta}_2 \cos\theta_2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\Bigg] - \\
&\left(\frac{1}{2}m_1 + m_2\right) g l_1 \cos\theta_1 - \frac{1}{2}m_2 g l_2 \cos\theta_2 - (m_0 + m_1 + m_2) h_0 g.
\end{aligned}$$

Problem 2d)

$$\tau = \left(\sum m_i\right)\ddot{\theta}_0 + \frac{1}{2}\Big(l_1(m_1 + 2m_2 l_1)(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 sin\theta_1) + \\ m_2 l_2(\ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2)\Big), \tag{1a}$$

$$0 = \left(\frac{1}{4}m_1 l_1^2 + m_2 l_1^2 + I_1\right)\ddot{\theta}_1 + \frac{1}{2}\Bigg(l_1(m_1 + 2m_2)\ddot{\theta}_0 \cos\theta_1 + \\ m_2 l_1 l_2\big[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\big] - (m_1 + 2m_2) g l_1 \sin\theta_1\Bigg), \tag{1b}$$

$$0 = \left(\frac{1}{4}m_2 l_2^2 + I_2\right)\ddot{\theta}_2 + \\ \frac{1}{2}\Big(m_2 l_2 \ddot{\theta}_0 \cos\theta_2 + m_2 l_1 l_2\big[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)\big] - m_2 g l_2 \sin\theta_2\Big), \tag{1c}$$

Problem 3

Problem 3 (16 %)

Figure 2 illustrates two coordinate frames in three dimensions. Note that all of the unit vectors shown and the dashed line segment (of 5 cm) are in the same plane, and unit vectors pointing into or out of the paper plane is not shown.

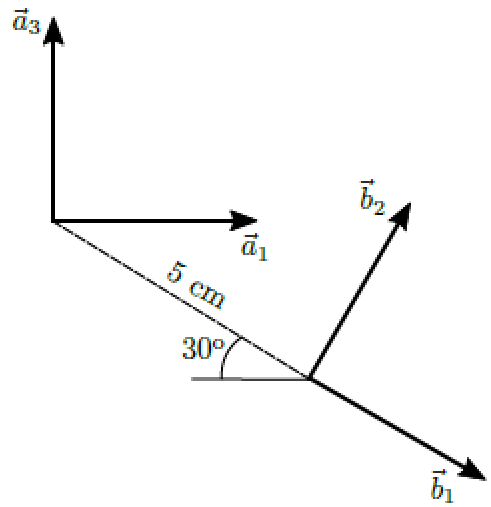


Figure 2: Two coordinate frames, rotated and translated.

(10 %) (a) What is $\mathbf{T}_b^a$, the homogenous transformation matrix representing the orientation and position of frame b relative to frame a?

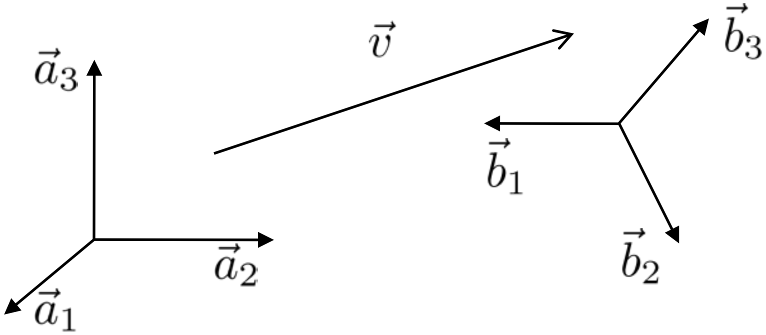
(6 %) (b) What is $\mathbf{T}_a^b$?

Rotation matrices (L15)

The <u>rotation matrix from *a* to *b*</u> $\mathbf{R}_b^a$ is used to

- Transform a coordinate vector from *b* to *a*

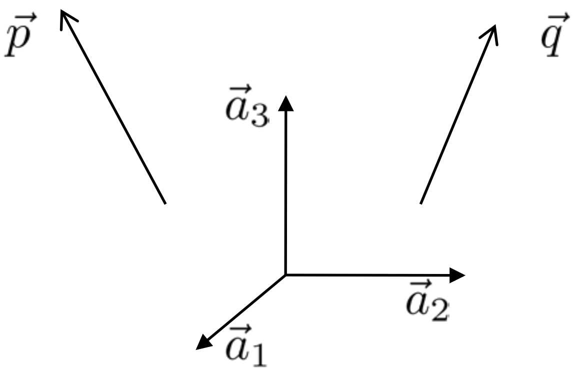
$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b$$



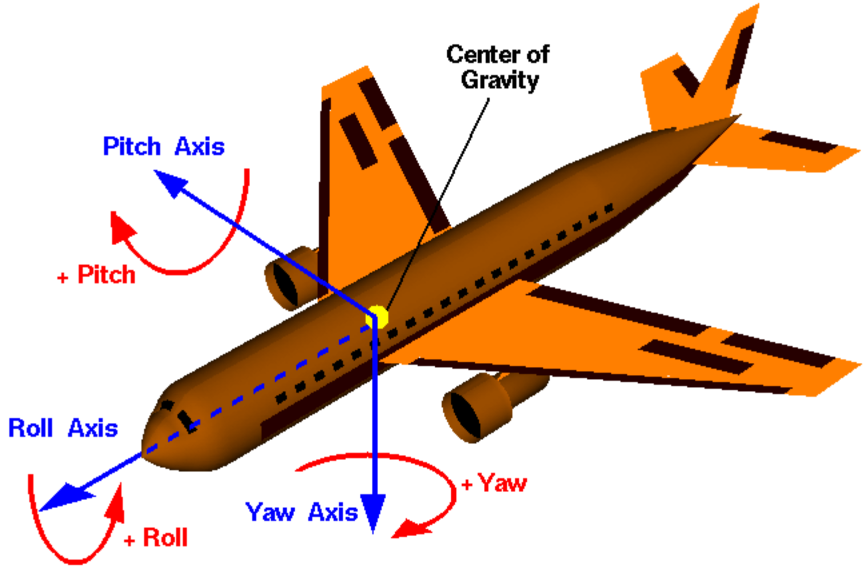
- Rotate a vector $\vec{p}$ to vector $\vec{q}$. If decomposed in *a*,

$$\mathbf{q}^a = \mathbf{R}_b^a \mathbf{p}^a$$

such that $\mathbf{q}^b = \mathbf{p}^a$.



Euler-angles: Roll-pitch-yaw (L15)

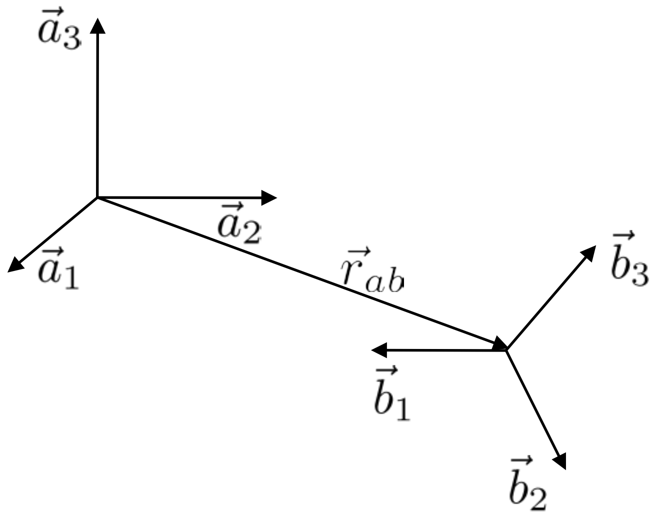


- Rotation ψ about *z*-axis, θ about (rotated) *y*-axis, ϕ about (rotated) *x*-axis

$$\mathbf{R}_b^a = \mathbf{R}_{z,\psi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\phi}$$

$$\mathbf{R}_b^a = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

Homogenous transformation matrix (L14)



$$\mathbf{T}_b^a = \begin{pmatrix} \mathbf{R}_b^a & \mathbf{r}_{ab}^a \\ \mathbf{0}^T & 1 \end{pmatrix} \in SE(3)$$

$$(\mathbf{T}_b^a)^{-1} = \mathbf{T}_a^b = \begin{pmatrix} (\mathbf{R}_b^a)^T & -(\mathbf{R}_b^a)^T \mathbf{r}_{ab}^a \\ \mathbf{0}^T & 1 \end{pmatrix}$$

Problem 3

Problem 3 (16 %)
Figure 2 illustrates two coordinate frames in three dimensions. Note that all of the unit vectors shown and the dashed line segment (of 5 cm) are in the same plane, and unit vectors pointing into or out of the paper plane is not shown.

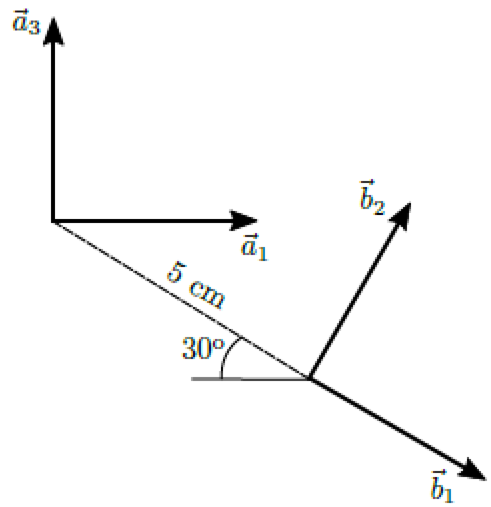
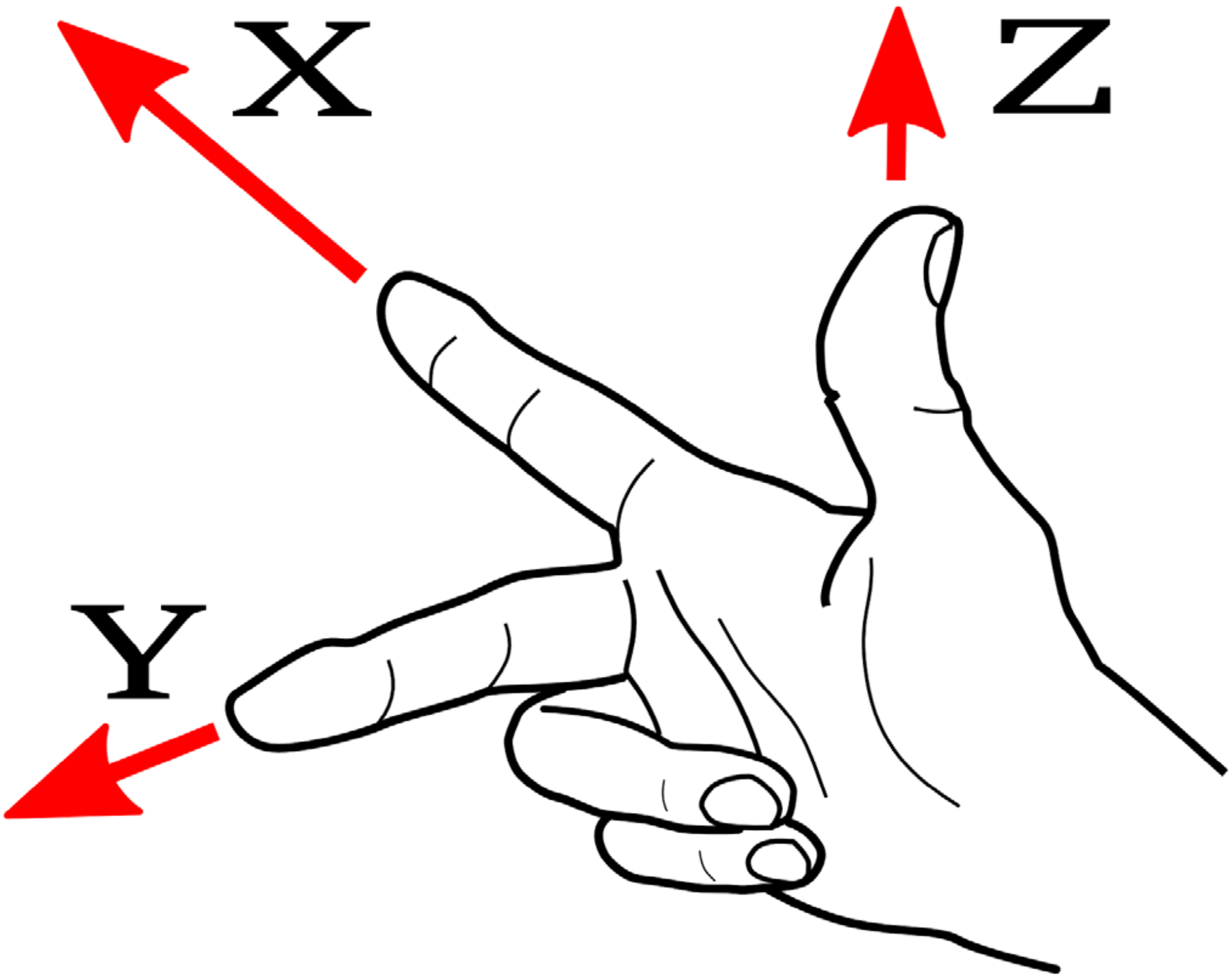


Figure 2: Two coordinate frames, rotated and translated.

(10 %) (a) What is $\mathbf{T}_b^a$, the homogenous transformation matrix representing the orientation and position of frame b relative to frame a?

(6 %) (b) What is $\mathbf{T}_a^b$?

One of the Possibilities



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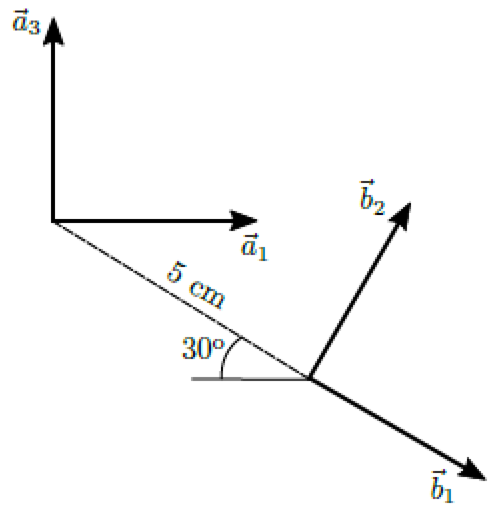

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Problem 3a)

$$\mathbf{T}_b^a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 & 0 & 0 \\ -1/2 & \sqrt{3}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 5cm \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Problem 3b)

$$\mathbf{T}_a^b = (\mathbf{T}_b^a)^{-1} = \begin{pmatrix} (\mathbf{R}_b^a)^T & -(\mathbf{R}_b^a)^T \mathbf{r}_{ab}^a \\ \mathbf{0}^T & 1 \end{pmatrix}$$

$$(\mathbf{R}_b^a)^T = \begin{pmatrix} \sqrt{3}/2 & 0 & -1/2 \\ 1/2 & 0 & \sqrt{3}/2 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\mathbf{r}_a^b = -\begin{pmatrix} \sqrt{3}/2 & 0 & -1/2 \\ 1/2 & 0 & \sqrt{3}/2 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \cdot 5cm \\ 0 \\ -1/2 \cdot 5cm \end{pmatrix} = -\begin{pmatrix} (3/4 + 1/4) \cdot 5cm \\ (\sqrt{3}/4 - \sqrt{3}/4) \cdot 5cm \\ 0 \end{pmatrix}$$

$$= -\begin{pmatrix} 5cm \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{T}_a^b = \begin{pmatrix} \sqrt{3}/2 & 0 & -1/2 & -5cm \\ 1/2 & 0 & \sqrt{3}/2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 4

Problem 4 (24 %)

Heat exchangers are basic unit processes which are found in almost every plant in the chemical process industries. As the name suggests, heat exchangers are used for energy (heat) exchange between a hot and a cold fluid stream (the hot stream heats the cold stream).

Heat exchangers are constructed in various ways, for example to maximize the energy transfer. Often they are considered as distributed systems since the temperatures will vary along the stream lines inside the heat exchanger. However, in the first part of this task we will develop a simple heat-exchanger model based on a very simple geometry (which can be an approximation for more complex geometries) and where we assume the temperatures at the hot and cold side (T_h and T_c) are spatially constant (that is, we assume the temperatures are averaged/lumped).

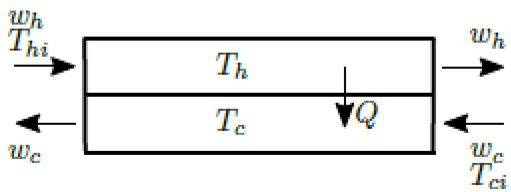

Figure 3: Simple heat exchanger

Consider Figure 3. The the hot and cold mass flow rates w_j, $j = \{h, c\}$ are assumed constant, such that the mass on each side are constant. The inlet temperatures are T_{hi} on the hot side, and T_{ci} on the cold side. The heat transfer from hot to cold side is

$$Q = UA\,(T_h - T_c)$$

where U is a heat transfer coefficient and A is the "effective contact area". The volumes of the hot and cold side are denoted V_h and V_c, respectively.

Problem 4

(10 %) (a) Derive a model for the temperatures T_h and T_c. Assume that the densities ρ_j and specific heats c_{pj}, $j = \{h, c\}$ are constant. Assume incompressible liquids and constant pressure, such that specific internal energy and specific enthalpy are equal.

(10 %) (b) The model found above, can be written

$$\begin{pmatrix} \dot{T}_h \\ \dot{T}_c \end{pmatrix} = \begin{pmatrix} -a_1 - k_1 & k_1 \\ k_2 & -a_2 - k_2 \end{pmatrix} \begin{pmatrix} T_h \\ T_c \end{pmatrix} + \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} T_{hi} \\ T_{ci} \end{pmatrix}$$

where a_j and k_j, $j = \{h, c\}$ are appropriate positive constants (found in (a)). Let $\mathbf{x} = (x_1, x_2)^\mathsf{T} = (T_h, T_c)^\mathsf{T}$. Show that this model is passive from $\mathbf{u} = (u_1, u_2)^\mathsf{T} = (T_{hi}, T_{ci})^\mathsf{T}$ to $\mathbf{y} = (y_1, y_2)^\mathsf{T} = \left(\frac{a_1}{k_1} x_1, \frac{a_2}{k_2} x_2\right)^\mathsf{T}$, using the storage function

$$V(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\mathsf{T} P \mathbf{x}, \quad P = \begin{pmatrix} \frac{1}{k_1} & 0 \\ 0 & \frac{1}{k_2} \end{pmatrix}.$$

A hint that may or may not be useful, is a special case of Gershgorin's Theorem:

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(2 %) (c) Make a sketch of the temperature profiles in the hot and cold stream of a counter-flow heat exchanger (in principle like the one shown in Figure 3, where the hot and cold fluid enter on different ends). Draw them both in the same diagram, with position along x-axis and temperature along y-axis. Mark T_{ci} and T_{hi} in the sketch.

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Energy

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_c} \rho e \mathrm{d}V = -\iint_{\partial V_c} \rho e \vec{v} \cdot \vec{n} \mathrm{d}A + \dot{Q} - \dot{W}$$

- The energy of a fluid of mass *m*, moving with a velocity *v* at a height *z* in a gravitational field:

$$E = \underbrace{U}_{\substack{\text{internal} \\ \text{energy}}} + \underbrace{\frac{1}{2}mv^2}_{\substack{\text{kinetic} \\ \text{energy}}} + \underbrace{mgz}_{\substack{\text{potential} \\ \text{energy}}}$$

- Specific energy:

$$e = u + \frac{1}{2}v^2 + gz$$

Enthalpy

- The energy balance can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_c} \rho e \mathrm{d}V = -\iint\limits_{\partial V_c} \rho \left(e + \frac{p}{\rho}\right) \vec{v} \cdot \vec{n} \mathrm{d}A - \dot{W}_s + \dot{Q}$$

  where the first term on the RHS is convection and flow work
- Define **enthalpy** as

$$h = u + \frac{p}{\rho}$$

- Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_c} \rho \left(u + \frac{1}{2}v^2 + gz\right) \mathrm{d}V = -\iint\limits_{\partial V_c} \rho \left(h + \frac{1}{2}v^2 + gz\right) \vec{v} \cdot \vec{n} \mathrm{d}A - \dot{W}_s + \dot{Q}$$

Problem 4a)

$$\frac{d}{dt}(u\rho_h V_h) = w_h h(T_{hi}) - w_h h(T_h) - Q$$

Internal energy and enthalpy

- Specific heat capacities:

$$c_v := \left.\frac{\partial u}{\partial T}\right|_{\text{constant volume}} \qquad c_p := \left.\frac{\partial h}{\partial T}\right|_{\text{constant pressure}}$$

  (found in tables for different fluids, often assumed constant)

- If assumed constant, implies that energy and enthalpy is (linear) function of temperature only:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = c_v \frac{\mathrm{d}T}{\mathrm{d}t} \qquad u(T_2) - u(T_1) = c_v\,(T_2 - T_1)$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = c_p \frac{\mathrm{d}T}{\mathrm{d}t} \qquad h(T_2) - h(T_1) = c_p\,(T_2 - T_1)$$

- For ideal gases:

$$c_v = c_p + R$$

- For incompressible fluids (often assumed for liquids):

$$c_v = c_p$$

Problem 4a)

$$\frac{d}{dt}(u\rho_h V_h) = w_h h(T_{hi}) - w_h h(T_h) - Q$$

$$\frac{d}{dt}(\rho_h V_h c_{ph} T_h) = w_h c_{ph}(T_{hi} - T_h) - UA(T_h - T_c)$$

$$\frac{d}{dt}(\rho_c V_c c_{pc} T_c) = w_h c_{pc}(T_{ci} - T_c) + UA(T_h - T_c)$$

$$\frac{d}{dt}T_h = \frac{w_h}{\rho_h V_h}(T_{hi} - T_h) - \frac{UA}{\rho_h V_h c_{ph}(T_h - T_c)}$$

$$\frac{d}{dt}T_c = \frac{w_c}{\rho_c V_c}(T_{ci} - T_c) + \frac{UA}{\rho_c V_c c_{pc}(T_h - T_c)}$$

- Mass balance!

$$\left.\begin{aligned}\dot{m}_h &= w_h - w_h = 0\\ \dot{m}_c &= w_c - w_c = 0\end{aligned}\right\} \text{ or } \quad \dot{m} = w_h - w_h + w_c - w_c$$

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Passivity (L4)

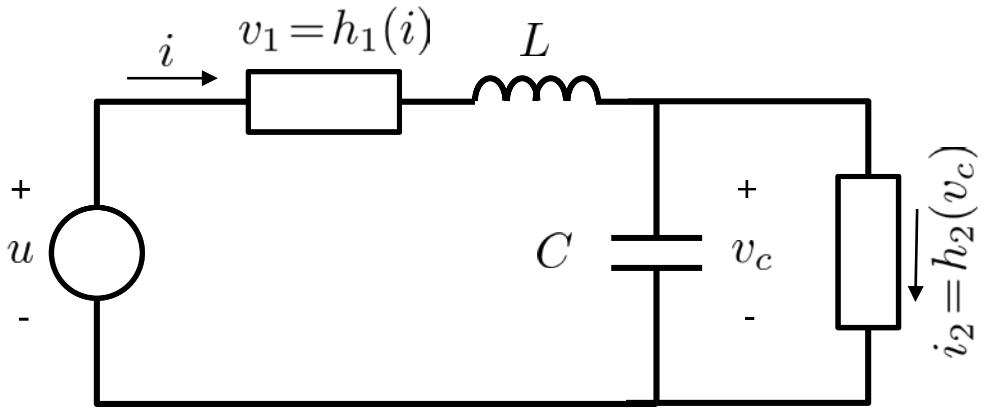
- A system with input *u* and output *y* is passive if

$$\int_0^t y(\tau)u(\tau)d\tau \geq -E_0$$

  for all $t \geq 0$, for all input trajectories.

- If the product yu has power as unit, then if (for all *u*)
  - $\int_0^t y(\tau)u(\tau)d\tau \geq 0$: Energy is absorbed within the system, nothing delivered to the outside
  - $\int_0^t y(\tau)u(\tau)d\tau \geq -E_0$: Some energy can be delivered to the outside, limited (typically) by the initial energy in the system.
  - $\int_0^t y(\tau)u(\tau)d\tau \to -\infty$: There is an inexhaustible energy source in the system. Not passive!

Example storage functions (L4)



- States: $x_1 = i, \quad x_2 = v_c$
- Model (Kirchoff's laws):

$$L\dot{x}_1 = u - h_1(x_1) - x_2$$
$$C\dot{x}_2 = x_1 - h_2(x_2)$$

- Output&input: $y = i, \quad u = u$
- Nonlinear resistors fulfilling

$$x_i h_i(x_i) \geq 0$$

- Storage (energy) function:

$$V(\mathbf{x}) = \frac{1}{2}Lx_1^2 + \frac{1}{2}Cx_2^2$$

- Differentiate:

$$\begin{aligned}\dot{V} &= Lx_1\dot{x}_1 + Cx_2\dot{x}_2 \\ &= x_1(u - h_1(x_1) - x_2) + x_2(x_1 - h_2(x_2)) \\ &= yu - x_1h_1(x_1) - x_2h(x_2)\end{aligned}$$

- Passive!

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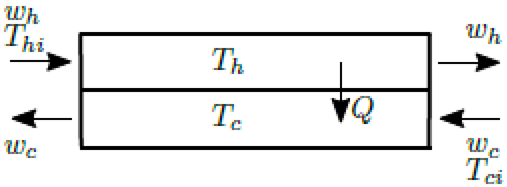

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Good
Luck
in your
exams