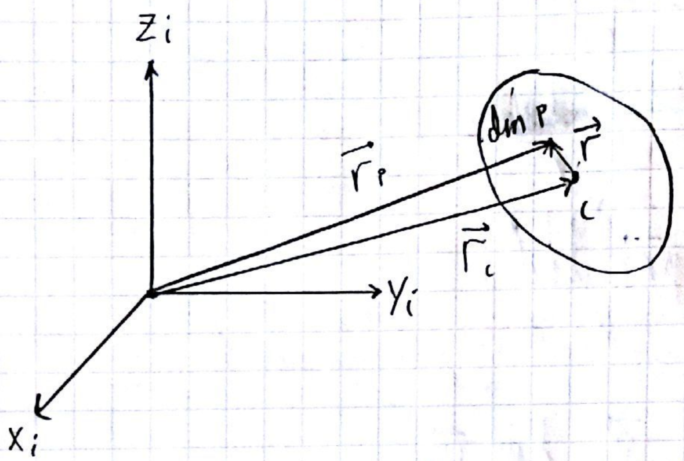
Newton-Eulers bevegelseslikning for stive legemer

Stivt legeme: samling av partikler, med fast innbyrdes posisjon, hver partikkel har masse dm.

07.04.16



$$m = \int_b dm$$

$$\vec{r}_c = \frac{\int_b \vec{r}_p \, dm}{m}$$

Nyttig identitet:

$$\int_b \vec{r} \, dm = \int_b \vec{r}_p \, dm - \int_b \vec{r}_c \, dm = m\vec{r}_c - m\vec{r}_c = 0$$

Newtons lov for en partikkel:

$$dm \cdot \vec{a}_p = \vec{f}_p = \vec{f}_{p,ext} + \vec{f}_{p,int}$$

$$\underbrace{\int_b \vec{a}_p \, dm}_{m\vec{a}_c} = \int_b \vec{f}_p = \underbrace{\int_b \vec{f}_{p,ext}}_{F_{bc}} + \underbrace{\int_b \vec{f}_{p,int}}_{0}$$

$$\boxed{m\vec{a}_c = F_{bc}}$$

$\vec{F}_{bc}$ = "resulterende kraft"

bev. mengde: $\vec{P}_c = m\vec{v}_c$

$$\boxed{\frac{{}^i d}{dt}\vec{P}_c = \vec{F}_{bc}}$$

Definer <u>spinn</u> for partikkel:

$$\vec{h}_p = \vec{r}_p \times \vec{P}_p \,, \quad \vec{P}_p = dm\,\vec{v}_p$$

Moment for partikkel:

$$\vec{T}_p = \vec{r}_p \times \vec{f}_p$$

$$\frac{{}^i d}{dt} h_p = \underbrace{\frac{{}^i d}{dt}\vec{r}_p \times \vec{P}_p}_{0} + \vec{r}_p \times \underbrace{\frac{{}^i d}{dt}\vec{P}_p}_{\vec{f}_p} = \vec{r}_p \times \vec{f}_p = \vec{T}_p$$

Integrerer over legeme

$$\int_b {}^i\frac{d}{dt}\vec{h}_p = \int_b \vec{\tau}_p$$

$${}^i\frac{d}{dt}\int_b \vec{r}_p \times \vec{v}_p = \int_b \vec{r}_p \times \vec{f}_p$$

$${}^i\frac{d}{dt}\int_b \vec{r}_p \times \vec{v}_p = {}^i\frac{d}{dt}\int_b (\vec{r}_c + \vec{r}) \times \vec{v}_p \, dm$$

$$= {}^i\frac{d}{dt}\int_b \vec{r} \times \vec{v}_p \, dm + \int_b \vec{r}_c \times {}^i\frac{d}{dt}\vec{p}_p + \underbrace{\int_b {}^i\frac{d}{dt}\vec{r}_c \times \vec{p}_p}$$

$$= {}^i\frac{d}{dt}\int_b \vec{r} \times \vec{v}_p \, dm + \int_b \vec{r}_c \times \vec{f}_p$$

$${}^i\frac{d}{dt}\vec{r}_c \times \int_b \vec{p}_p = \vec{v}_c \times dm\,\vec{v}_c = 0$$

$$\int_b \vec{r}_p \times \vec{f}_p = \int_b (\vec{r}_c + \vec{r}) \times \vec{f}_p = \int_b \vec{r}_c \times \vec{f}_p + \int_b \vec{r} \times \vec{f}_p$$

$$\Rightarrow \quad {}^i\frac{d}{dt}\underbrace{\int_b \vec{r} \times \vec{v}_p}_{\vec{h}_{b/c}} = \underbrace{\int_b \vec{r} \times \vec{f}_p}_{\vec{T}_{bc}}$$

$\vec{h}_{b/c}$: spinn om massesenter

$\vec{T}_{bc}$: "resulterende moment" om massesenter

Dvs.: Newton-Euler EoM om massesenter

${}^i\frac{d}{dt}\vec{p}_c = \vec{F}_{bc}$ (translasjon)	${}^i\frac{d}{dt}\vec{h}_{b/c} = \vec{T}_{bc}$ (rotasjon)

$$h_{b/c} = \int_b \vec{r} \times \vec{v}_p \, dm \quad , \text{ vct } \vec{v}_p = \vec{v}_c + \vec{\omega}_{ib} \times \vec{r}$$

$$= \int_b \vec{r} \times (\vec{v}_c + \vec{\omega}_{ib} \times \vec{r}) \, dm$$

$$= \int_b \vec{r} \, dm \times \vec{v}_c + \int_b \vec{r} \times (\omega_{ib} \times \vec{r}) \, dm$$

$$= -\int_b \vec{r} \times (\vec{r} \times \vec{\omega}_{ib}) \, dm$$

$$= \underbrace{-\int_b \vec{r}^{\times} \cdot \vec{r}^{\times} \, dm}_{\vec{M}_{b/c}} \cdot \vec{\omega}_{ib}$$

$$\frac{{}^i d}{dt} \vec{h}_{b/c} = \frac{{}^i d}{dt} (\vec{M}_{b/c} \cdot \vec{\omega}_{ib}) = \frac{{}^b d}{dt} (\vec{M}_{b/c} \cdot \vec{\omega}_{ib}) + \vec{\omega}_{ib} \times (\vec{M}_{b/c} \cdot \vec{\omega}_{ib})$$

$$= \vec{M}_{b/c} \vec{\alpha}_{ib} + \vec{\omega}_{ib} \times (\vec{M}_{b/c} \cdot \omega_{ib})$$

Treghetsdyaden

$$\vec{M}_{b/c} = -\int_b \vec{r}^\times \cdot \vec{r}^\times dm \overset{Ex.86}{=} \int_b (\vec{r}\cdot\vec{r}\,\mathrm{I} - \vec{r}\vec{r})\, dm$$

Dekomponert i b:

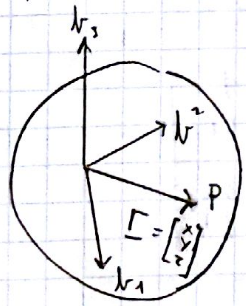
$$\vec{M}_{b/c} = \sum_{i=1}^{3}\sum_{j=1}^{3} m_{ij}^b \cdot \vec{b}_i \cdot \vec{b}_j \,, \quad \vec{m}_{ij}^b \text{ konstante}$$

Matrise form:

$$M_{b/c} = \begin{bmatrix} m_{11}^b & m_{12}^b & \cdots \\ m_{21}^b & \ddots & \\ \vdots & & \ddots \end{bmatrix}$$

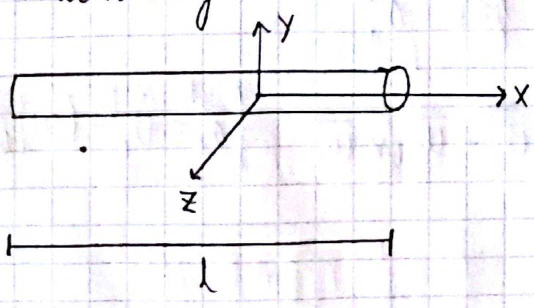
$$M_{b/c}^b = \int_b \left[(r^b)^T r^b \mathrm{I} - r^b (r^b)^T \right] dm$$

$$\underline{r}^b = \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \quad M_{b/c}^b = \int_b \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dm$$



$$M_{b/c}^b : \text{konstant}, \quad M_{b/c}^i = R_b^i M_{b/c}^b R_i^b : \text{ ikke konstant}$$

<u>Eks.</u> Slank bjelke



$$dm = \frac{m}{l} dx$$

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \frac{m}{l} dx = \frac{m}{l}\left[\frac{1}{3}x^3\right] = \frac{m}{3l}\left[\frac{l^3}{8} + \frac{l^3}{8}\right]$$

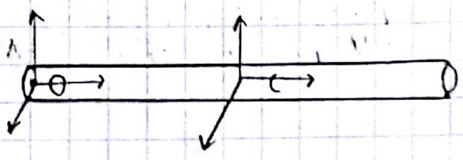
$$= \frac{ml^2}{12} \Rightarrow M_{b/c}^b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{ml^2}{12} & 0 \\ 0 & 0 & \frac{ml^2}{12} \end{bmatrix}$$

$$M_{b/c}^{b} = \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{bmatrix}$$

Parallell akse-teoremet

$$M_{b/o} = M_{b/c} - m\left(\underline{r}_g^b\right)^{\times}\left(\underline{r}_g^b\right)^{\times} = M_{b/c} + m\left[\left(\left(\underline{r}_g^b\right)^T \underline{r}_g^b\right) I - \underline{r}_g^b\left(\underline{r}_g^b\right)^T\right]$$

<u>Eks.</u>



$$\underline{r}_g^b = \begin{bmatrix} -\frac{l}{2} \\ 0 \\ 0 \end{bmatrix}$$

$$M_{b/o} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{ml^2}{12} & 0 \\ 0 & 0 & \frac{ml^2}{12} \end{bmatrix} + m\left[\frac{l^2}{4} I - \begin{bmatrix} \frac{l^2}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{ml^2}{3} & 0 \\ 0 & 0 & \frac{ml^2}{3} \end{bmatrix}$$

Newton-Euler EoM koordinatform

$$\begin{bmatrix} mI & 0 \\ 0 & M_{b/c}^b \end{bmatrix} \begin{bmatrix} \underline{a}_c^b \\ \underline{\alpha}_{ib}^b \end{bmatrix} + \begin{bmatrix} 0 \\ \left(\underline{\omega}_{ib}^b\right)^{\times} M_{b/c}^b \underline{\omega}_{ib}^b \end{bmatrix} = \begin{bmatrix} \underline{F}_{bc}^b \\ \underline{T}_{bc}^b \end{bmatrix}$$

→ ofte: $\dot{\underline{v}}_c^b$ istedenfor $\underline{a}_c^b$

$$\vec{a}_c = {}^i\frac{d}{dt}\vec{v}_c = {}^b\frac{d}{dt}\vec{v}_c + \vec{\omega}_{ib} \times \vec{v}_c \qquad \left[\underline{a}_c^b = \dot{\underline{v}}_c^b + \left(\underline{\omega}_{ib}^b\right)^{\times} \underline{v}_c^b\right]$$