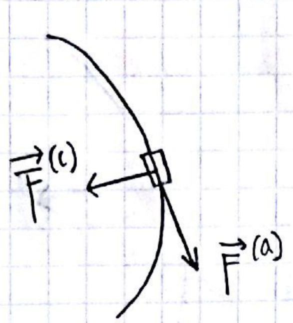
D'Alamberts prinsipp

14.04.16

To typer krefter:

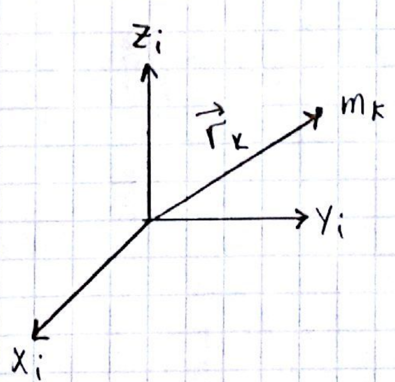
- Aktive krefter
- Føringskrefter

Eks. Tog



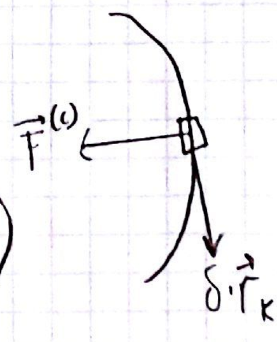
$$\vec{F}^{(r)} = \vec{F}^{(a)} + \vec{F}^{(c)}$$
$$= \vec{F} + \vec{F}^{(c)}$$

Anta system av partikler, hver med masse m_k og posisjon $\vec{r}_k$, $k = 1, \dots, N$



Definerer virtuell forskyvning, $\delta \vec{r}_k$

$$\vec{F}^{(c)} \cdot \delta \vec{r}_k = 0$$



(dvs. forskyvning konsistent med føringskraft)

Newtons lov:

$$m_K \frac{d^2}{dt^2}\vec{r}_K = \vec{F}_K^{(r)} = \vec{F} + \vec{F}^{(c)}$$

da er

$$\vec{F}^{(c)} \cdot \delta \vec{r}_K = \left[m_K \frac{d^2}{dt^2}\vec{r}_K - \vec{F} \right] \cdot \delta \vec{r}_K = 0$$

D'Alemberts prinsipp:

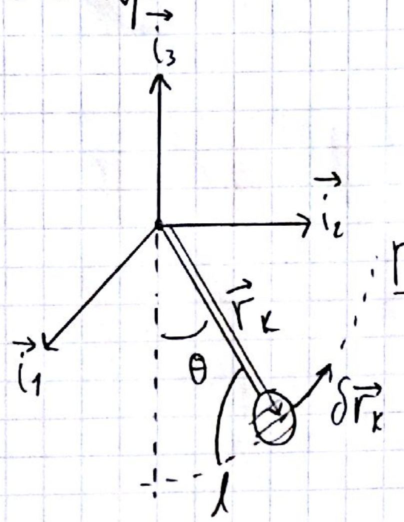
$$\boxed{\sum_{K=1}^{N} \delta \vec{r}_K \cdot \left(m_K \frac{d^2}{dt^2}\vec{r}_K - \vec{F} \right) = 0}$$

Generaliserte koordinater

$$\vec{r}_k = x_k \vec{i}_1 + y_k \vec{i}_2 + z_k \vec{i}_3$$

Pga føringskrefter så er de ikke uavhengig

Eksempel: Pendel



$$\underline{r}_k = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} = \begin{bmatrix} 0 \\ l\sin(\theta) \\ -l\cos(\theta) \end{bmatrix}$$

Generalisert koordinat θ

$$\vec{r}_k = \vec{r}_k(\theta)$$

Generelt:

$$\vec{r}_k = \vec{r}_k(q_1, q_2, q_3, \dots, q_n, t)$$
$$= \vec{r}_k(\underline{q}, t) \qquad q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$$

q: generalisert koordinat

$n < 3N$

Vi har $\delta \vec{r}_k = \sum_{i=1}^{n} \frac{\partial \vec{r}_k}{\partial q_i} \delta q_i$

D'Alemberts prinsipp

$$\sum_{k=1}^{N} \delta \vec{r}_k \cdot \left(m_k \frac{d^2}{dt^2} \vec{r}_k - \vec{F}_k\right) = 0$$

$$\sum_{k=1}^{N} \sum_{i=1}^{n} \frac{\partial \vec{r}_k}{\partial q_i} \delta q_i \left(m_k \frac{d^2}{dt^2} \vec{r}_k - \vec{F}_k\right) = 0$$

$$\sum_{i=1}^{n} \delta q_i \sum_{k=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i} \left(m_k \frac{d^2}{dt^2} \vec{r}_k - \vec{F}_k\right) = 0$$

Hvis δq_i er k avhengig.

$$\boxed{\sum_{k=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i} \left(m_k \frac{d^2}{dt^2} \vec{r}_k - \vec{F}_k\right) = 0 \qquad i = 1, \ldots, n} \qquad (*)$$

7.8: Tilsv. for stive legemer

$$R_b = R_b(\underline{q}, t), \quad \vec{r}_c = \vec{r}_c(\underline{q}, t)$$

Lagrange EoM (for partikler)

Litt matematikk: $\qquad \vec{r}_k = \vec{r}_k(\underline{q}, t)$

$$\vec{v}_k = {}^{i}\frac{d}{dt}\vec{r}_k = \sum_{i=1}^{n} \frac{\partial \vec{r}_k}{\partial q_i}\cdot \dot{q}_i + \frac{\partial \vec{r}_k}{\partial t}$$

$$\frac{\partial \vec{v}_k}{\partial \dot{q}_i} = \frac{\partial \vec{r}_k}{\partial q_i}$$

$$\frac{\partial \vec{v}_k}{\partial q_i} = \frac{\partial}{\partial q_i}\,{}^{i}\frac{d}{dt}\vec{r}_k = {}^{i}\frac{d}{dt}\frac{\partial \vec{r}_k}{\partial q_i}$$

Kinetisk energi:

$$T = \sum_{k=1}^{N} \frac{1}{2} m_k \vec{v}_k \cdot \vec{v}_k$$

$$\frac{\partial T}{\partial q_i} = \sum_{k=1}^{N} m_k \vec{v}_k \frac{\partial \vec{v}_k}{\partial q_i} = \sum_{k=1}^{N} m_k \vec{v}_k \,{}^{i}\frac{d}{dt}\frac{\partial \vec{r}_k}{\partial q_i}$$

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_{k=1}^{N} m_k \vec{v}_k \frac{\partial \vec{v}_k}{\partial \dot{q}_i} = \sum_{k=1}^{N} m_k \vec{v}_k \cdot \frac{\partial \vec{r}_k}{\partial q_i}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} = \sum_{k=1}^{N} \frac{d}{dt}\left(m_k \vec{v}_k \cdot \frac{\partial \vec{r}_k}{\partial q_i}\right)$$

$$= \sum_{k=1}^{N} m_k \vec{a}_k \cdot \frac{d\vec{r}_k}{\partial q_i} + \overbrace{\sum_{k=1}^{N} m_k \vec{v}_k \cdot {}^{i}\frac{d}{dt}\frac{\partial \vec{r}_k}{\partial q_i}}^{\frac{\partial T}{\partial q_i}}$$

Sett inn i (*):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - \sum_{k=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i} \cdot \vec{F}_k = 0 \qquad i = 1, \dots, n$$

$$Q_i = \sum_{k=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i} \cdot \vec{F}_k$$ "generalisert kraft tilhørende q_i"

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i} \qquad i = 1, \dots, n$$

Anta potensialfelt $V(\underline{q})$ som gir kraft $-\frac{\partial V}{\partial q_i}$

<u>Eks.</u> Gravitasjon $V = mgh$, $-\frac{\partial V}{\partial h} = -mg$

Anta "generalisert aktuatorkraft"

τ_i (tilhørende q_i)

$$\Rightarrow \quad Q_i = -\frac{\partial V}{\partial q_i} + \tau_i$$

Lagrange EoM:

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = \tau_i, \quad i = 1, \dots, n} \longrightarrow$$

Definer Lagrangian

$$\mathcal{L}(\underline{q}, \underline{\dot{q}}, t) = T(\underline{q}, \underline{\dot{q}}, t) - U(\underline{q})$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i \,, \quad i = 1, \dots, n}$$

<u>NB</u>: Gjelder for stive legemer

$$T(\underline{q}, \underline{\dot{q}}) = \frac{1}{2} m \vec{v}_c \cdot \vec{v}_c + \frac{1}{2} \vec{\omega}_{ib} \cdot \vec{M}_{b/c} \cdot \vec{\omega}_{ib}$$

$$\vec{v}_c = \vec{v}_c(\underline{q}, \underline{\dot{q}}), \quad \vec{\omega}_{ib} = \vec{\omega}_{ib}(\underline{q}, \underline{\dot{q}})$$

$$U(\underline{q}) = m g h(\underline{q})$$

$$\mathcal{L} = T(\underline{q}, \underline{\dot{q}}) - U(\underline{q})$$

Tolkning av generalisert kraft

$$p = \sum_{k=1}^{N} \vec{v}_k \cdot \vec{F}_k = \sum_{k=1}^{N} \sum_{i=1}^{n} \frac{\partial \vec{r}_k}{\partial q_i} \cdot \dot{q}_i \cdot \vec{F}_k$$

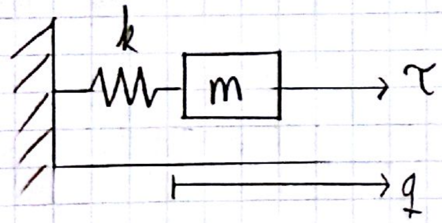
$$= \sum_{i=1}^{n} \dot{q}_i \underbrace{\sum_{k=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i} \cdot \vec{F}_k}_{Q_i}$$

$$= \sum_{i=1}^{n} \dot{q}_i Q_i$$

q_i posisjon : Q_i kraft

q_i vinkel : Q_i moment

__Eks.__



$$\mathcal{L} = T - V$$

$$= \frac{1}{2} m \dot{q} - \frac{1}{2} k q^2$$

$$T = \frac{1}{2} m \dot{q}^2 \quad , \quad V = \frac{1}{2} k q^2$$

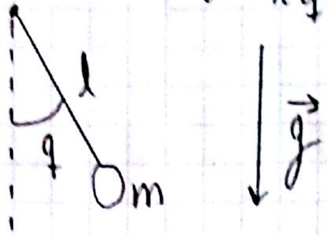
$$\frac{\partial \mathcal{L}}{\partial q} = m \dot{q} \qquad \frac{\partial \mathcal{L}}{\partial q} = -kq$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial q}\right) - \frac{\partial \mathcal{L}}{\partial q} = \tau$$

$$\boxed{m \ddot{q} + k q = \tau}$$

Eks.

$v = l\dot{q}$



$$\underline{r} = \begin{bmatrix} l\sin(q) \\ -l\cos(q) \end{bmatrix}$$

$$\bar{v} = \frac{d}{dt}\underline{r} = \begin{bmatrix} l\dot{q}\cos(q) \\ l\dot{q}\sin(q) \end{bmatrix}$$

$$v^2 = \underline{v}^T\underline{v} = l^2\dot{q}^2\cos^2(q) + l^2\dot{q}^2\sin^2(q)$$

$$v^2 = l^2\dot{q}^2$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{q}^2$$

$$U = mgl(1-\cos q)$$

$$\mathcal{L} = T - U = \frac{1}{2}ml^2\dot{q}^2 - mgl(1-\cos(q))$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = ml^2\dot{q} \quad , \quad \frac{\partial \mathcal{L}}{\partial q} = -mgl\sin q$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = ml^2\ddot{q} + mgl\sin(q) = 0$$

$$\boxed{\ddot{q} + \frac{g}{l}\sin q = 0}$$