Practice exercises on the Fourier Transform

A. Periods and frequencies

1. Consider the following signals (assume t is in seconds):

   (a) A single sine wave:
   $$f_1(t) = \sin(2\pi \cdot 3\,t).$$

   (b) The sum of two sine waves with different frequencies:
   $$f_2(t) = \sin(2\pi \cdot 3\,t) + \frac{1}{2}\sin(2\pi \cdot 7\,t).$$

   Questions:

   - What frequencies (in Hz) are present in f_1 and f_2?
   - What is the period of f_1 and f_2?

B. Discrete Fourier Analysis

1. Let
   $$f(t) = \cos(2\pi t) + \tfrac{1}{2}\cos(4\pi t).$$

   Questions:

   - Which frequencies are present?
   - Which Fourier coefficients are non-zero?
   - Sketch the corresponding spectrum.

2. Plot the signal $f(t) = \sin(2\pi t)$ at $N = 8, 16, 32, ...$ points over one period.

   Questions:

   - Compute the Discrete Fourier Transform numerically.
   - How does the spectrum change as N increases?

C. Definitions

1. Fourier transforms (a decomposition of a discrete (f_n) or continuous ($f(t)$) signal into waves:
   $$\hat{f}_k = \sum_{n=0}^{N-1} f_n e^{-2\pi ikn/N}, \quad \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt.$$

   Inverse transforms:
   $$f_n = \frac{1}{N}\sum_{k=0}^{N-1} \hat{f}_k\, e^{2\pi ikn/N}, \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega.$$

Questions:

- What represents "frequency" in each case?
- What changes conceptually when going from the sum to the integral?

D. Simple Fourier Transforms

1. Compute the Fourier transform of $f(t) = \sin(\omega_0 t)$.

   Questions:

   - Where is the *support* of the spectrum? That is, of all the frequencies, which are the nonzero ones? What are their values?

2. Repeat for
$$f(t) = \sin(\omega_1 t) + \sin(\omega_2 t).$$

3. Consider the Gaussian
$$f(t) = e^{-t^2/2\sigma^2}.$$

   Questions:

   - Calculate the Fourier transform $\hat{f}(\omega)$ as a function of ω. What is the shape of its Fourier transform?
   - Make sure you understand what the Fourier coefficients give you in terms of the
   - How does it change as σ increases?

E. Spectral Density

1. In general, the Fourier coefficients are *complex numbers*. Given complex Fourier coefficients $\hat{f}(\omega)$ - essentially a function over the frequencies - we can have access to what a function looks like in frequency space (what you did before in the previous exercises, but in a systematic way instead of just counting the frequencies every time).

   We are interested in seeing how much weight every frequency has in a given signal. As in general $\hat{f}(\omega)$ is a complex function (look at the definition of the transform), the quantity that shows the weight of frequency ω in $f(t)$ is not just the coefficient $f(\omega)$ but rather
$$|\hat{f}(\omega)|^2.$$

   This is called the spectral energy. For a signal $f(t)$ it is literally: how much power (or weight) is in each frequency. How would it look for a pure sine wave?

2. Compare
$$f_1(t) = \cos(\omega t), \quad f_2(t) = \cos(\omega t + \phi).$$

   Questions:

   - Plot the two signals and compare their spectral densities.
   - Compare their spectral densities $|\hat{f}_1(\omega)|$ and $|\hat{f}_2(\omega)|$. What information is lost?

E. Random signals

1. Try to construct some random signals from scratch. First, choose a power spectrum, for example

$$P(\omega) = |\omega|^{-2}.$$

2. For each *positive* frequency $\omega > 0$, sample a complex Fourier coefficient

$$\hat{f}(\omega) = a(\omega) + i\, b(\omega),$$

with

$$a(\omega) \sim \mathcal{N}\left(0, \frac{P(\omega)}{2}\right), \qquad b(\omega) \sim \mathcal{N}\left(0, \frac{P(\omega)}{2}\right).$$

3. For each negative frequency, define the coefficient by

$$\hat{f}(-\omega) = \hat{f}^*(\omega).$$

4. For the zero-frequency mode, set $\hat{f}(0) = 0$.

5. After these steps, you have a full set of complex Fourier coefficients $\hat{f}(\omega)$. Use the Fourier expansion to reconstruct a signal $f(t)$ and plot it.

Questions:

1. Why was the relation $\hat{f}(-\omega) = \hat{f}^*(\omega)$ imposed?

2. Plot $|\hat{f}(\omega)|^2$. How is it related to $P(\omega) = |\omega|^{-2}$? How would you get from the former to the latter. The quantity $P(\omega)$ is called the *power spectrum*. This is an extremely important concept, so we should make sure that you are familiar with it.

3. What you have created here is a Gaussian random field (signal). What is the main difference with a non Gaussian one?

Now that you understand the concept of frequency and spectral density (or spectrum) of a given signal, you can go again to the beginning of the Jupyter Notebook and read a bit the 1-d examples.

Finally, try to see how this is generalised to images instead of 1-d signals. Instead of ω you have a vector $\mathbf{k} = (k_x, k_y)$ which is the frequency *for each dimension*. For images we have two dimensions, and so the "frequency" is a vector - we call it a *wavevector*.

Please try to convince yourself about what the relationship between *scales* and *frequency* or wavenumber is. Larger wavelength waves will capture the 'larger' features in an image, and the finer details will be captured by waves of high frequency. High frequencies $\leftrightarrow$ small length scales, low frequencies $\leftrightarrow$ high length scales, and vice versa.

I hope it's more clear, and please do not hesitate to experiment on your own, and to send me any questions you might have :)