

Conduction in a rod with internal heat production

This is a derivation of the temperature distribution function $T(x)$ for a steady-state heat conduction in a straight rod of length L . We assume that the internal heat production Q_p is present in every point inside the rod volume (which may simulate heating of an electrical wire). The rod is perfectly insulated along its length and it loses heat only through its endpoints which in a steady-state case are kept at a fixed temperature T_0 .

We will take for the control volume a slice dx from the rod.

The energy balance for the rod element:

$$\frac{dE}{dt} = E_{in} - E_{out} + E_{production} \quad (1)$$

(note here that E_{in} , E_{out} and $E_{production}$ are energies per unit time and so have the units of $\frac{J}{s}$)

The heat flow is described by the Fourier's law:

$$\phi = \lambda A \left(- \frac{dT}{dx} \right) \quad (2)$$

Hence:

$$E_{in} = \lambda A \left(- \frac{dT}{dx} \right)_x \quad (3)$$

$$E_{out} = \lambda A \left(- \frac{dT}{dx} \right)_{x+dx} \quad (4)$$

The energy per unit time coming from the production can be written as Q_p (which is in the units of $\frac{W}{m^3}$) multiplied by the volume of the slice dx :

$$E_{production} = Q_p A dx \quad (5)$$

In the steady-state $\frac{dE}{dt} = 0$ and the energy balance becomes:

$$\lambda A \left(- \frac{dT}{dx} \right)_x - \lambda A \left(- \frac{dT}{dx} \right)_{x+dx} + Q_p A dx = 0 \quad (6)$$

Simplifying the above energy balance we get:

$$\frac{\left(\frac{dT}{dx} \right)_{x+dx} - \left(\frac{dT}{dx} \right)_x}{dx} = - \frac{Q_p}{\lambda}$$

If we now substitute some function $f(x) = \frac{dT}{dx}$ we notice that we have:

$$\frac{f(x+dx) - f(x)}{dx} = - \frac{Q_p}{\lambda}$$

in other words:

$$\frac{df(x)}{dx} = - \frac{Q_p}{\lambda} \quad (7)$$

With the above substitution, the differential equation that we are about to solve becomes:

$$\frac{d^2 T}{dx^2} = -\frac{Q_p}{\lambda} \quad (8)$$

The solution to the above differential equation is:

$$T(x) = -\frac{Q}{2\lambda}(x^2 - Lx) + T_0 \quad (9)$$

It is interesting to note here that the solution does not depend on the cross-sectional surface area of the rod.

Computational example

As a computational example we will draw the graph of the temperature distribution in a copper rod 200m long. We take that the thermal conductivity for this rod is $400 \frac{W}{m \cdot ^\circ C}$. The internal heat production is 20W.

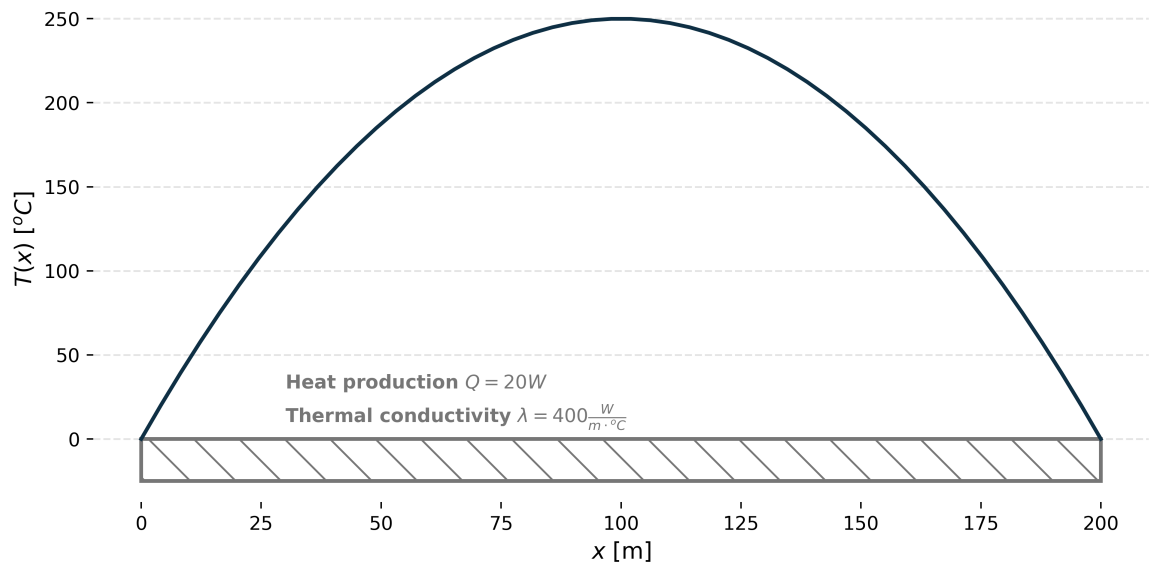


Figure 1: Temperature distribution in a rod with internal heat production of 20W

This material was created by or adapted from material posted on the DelftX website, delftx.tudelft.nl, and created by TU Delft faculty members Robert Mudde, Professor of Multiphase Flow at Chemical Engineering and Peter Hamersma, Associate professor in the Dept. of Chemical Engineering, 2015. DelftX is not responsible for any changes made to the original materials posted on its website and any such changes are the sole responsibility of Kamila Zdybał.

This material is subjected to copyright by Delft University of Technology and is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.

<https://creativecommons.org/licenses/by-nc-sa/4.0/>

This document was prepared as part of the course
The Basics of Transport Phenomena from Delft University,
available on edX.org as DelftX: TP101x.

Copyright © K. Zdybał, 2018

For more projects similar to this one

visit me on GitHub: @camillejr

camillejr.github.io/science-docs/

To contact me personally drop me a line at:

kamilazdybal@gmail.com

Conduction in a rod with internal heat production

version 1.0

Typeset with L^AT_EX

This work is licensed under the Creative Commons
Attribution-NonCommercial-ShareAlike 4.0 International
(CC BY-NC-SA 4.0) license.