Bruxelles, June-December 2020

The tensor necessity

– a short story about momentum transport in fluids

Kamila Zdybał

Université libre de Bruxelles, kamila.zdybal@ulb.ac.be
kamilazdybal.github.io/science-docs, kamila.zdybal@gmail.com

Preface

At first encounter, tensors can seem like strange mathematical objects. It can be challenging to grasp their meaning and their relevance might not be immediately obvious. At the same time, tensors are indispensable when studying fluid dynamics. So what's with the tensors and why do we need them?

In this document we would like to convince you that we necessarily do need tensors in fluid dynamics! We will motivate their usefulness by discussing momentum transport in the Couette flow. Hopefully by the end of this document you will find that tensors can be very useful mathematical objects that make our life easier. Join us on the journey!

Please feel free to contact me with any suggestions, corrections or comments.

Keywords

tensor, momentum transport, transport phenomena, fluid dynamics, Couette flow

Viscous momentum flux tensor

To build our intuition around tensor quantities we will begin our journey with an illustrative example. We will look at the momentum transport in the Couette flow – the flow of fluid between two parallel plates. The top plate is stationary and the bottom plate moves in the positive x-direction with the velocity $\mathbf{u}$, as presented in Figure 1.

Suppose that we start with an initial situation when both plates are stationary. At some moment in time, we begin moving the bottom plate, reaching the velocity $\mathbf{u}$ after a while. You may already expect that the moment we started moving the bottom plate something interesting starts to happen to the fluid in-between – it begins moving as well. As the flow develops, the moving bottom plate successively "drags" fluid particles in the layer directly above that plate. Once those fluid particles are set in motion in the positive x-direction, that moving layer of fluid "drags" another layer directly on top of it. This "dragging" progresses upward in the positive y-direction, until at the top stationary plate the fluid is stagnant again to match the zero velocity of the top plate. After a sufficient amount of time, the fluid motion becomes steady. At steady state, the velocity profile between the two plates is fully developed and does not change in time anymore. The steady state velocity profile in the Couette flow is a linear function of y (this profile is plotted schematically in Figure 1). This is something that we will not prove here but you can give it a try as an exercise!

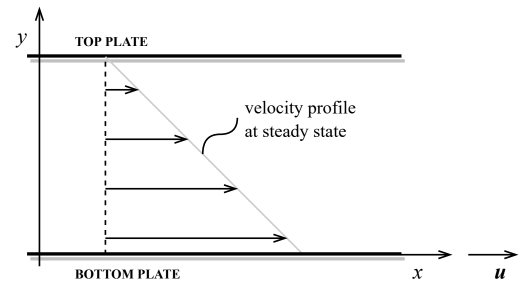
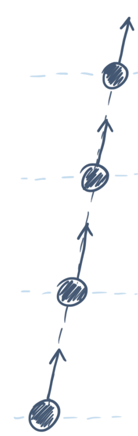


Figure 1: The Couette flow is the flow of fluid between two parallel plates. The bottom plate moves in the positive x-direction with the velocity $\mathbf{u}$ and the top plate is stationary. At steady state, the fully developed velocity profile between plates is a linear function of y.



If we assume that the fluid flow between the two plates is laminar we may assume that the fluid flows in thin layers, or *laminates*, stacked one on top of the other. Knowing the velocity profile, we know exactly what the velocity of each laminate is. In other words, since we know $\mathbf{u}(y)$, we can compute the velocity at any position y. For the flow described in Figure 1, the laminate at the lower y-coordinate will have a larger velocity than the laminate at the larger y-coordinate. This holds in our case where we assumed that the bottom plate is moving and the fluid velocity decreases to zero once we reach the top plate. In Figure 2, we have drawn three such laminates, each having thickness dy. Similar reasoning can be made assuming that the top plate is moving with the velocity $\mathbf{u}$, and the bottom plate is stationary.

Knowing the velocities at any position y, we also know the momentum carried by each laminate. We will in fact consider the specific momentum – the momentum per unit volume. Let's look at the situation from the perspective of one of the laminates whose y-coordinate is simply denoted y. This laminate is marked in gray in Figure 2. Its specific momentum is $\rho \mathbf{u}|_y$, where ρ is the density of the fluid. It gains momentum from the faster moving laminate directly below it, whose specific momentum is $\rho \mathbf{u}|_{y-dy}$. It also looses momentum to the slower moving laminate directly above it, whose specific momentum is $\rho \mathbf{u}|_{y+dy}$. Such *transport* of momentum is in practice possible due to molecular collisions. Every collision is an opportunity to exchange momentum. When molecules from one laminate collide with molecules from another laminate, momentum can be transported in the positive y-direction. The "dragging", that we talked about before, is thus achieved by the momentum transport. The faster moving laminate gives some of its momentum to the slower moving one.

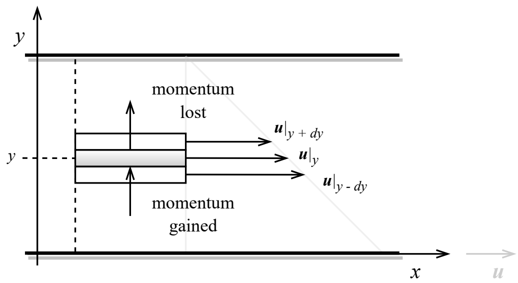


Figure 2: Transporting momentum between fluid *laminates* from the perspective of one of the laminates at position y (marked in gray).

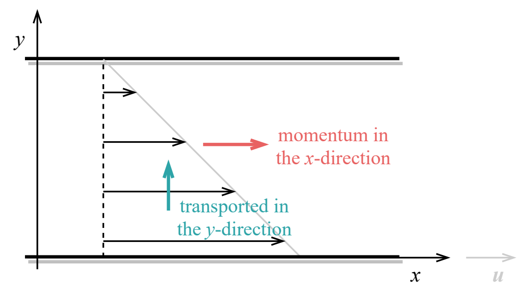
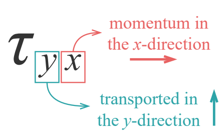


Figure 3: In the Couette flow, the momentum in the x direction is transported in the y direction. The transported momentum is called the *momentum flux*.

An interesting concept now emerges. Momentum is a vector quantity that has a direction of the velocity vector. In our example of the Couette flow, momentum of every laminate has the direction parallel to the x-axis (in accordance with the velocity profile). However, the transport of that momentum is happening in the direction perpendicular to the x-axis: from the laminates at the lower y-coordinate to the laminates at the larger y-coordinate. This is conceptually presented in Figure 3.



This transported momentum (also called the *momentum flux*) is denoted with a symbol τ_{yx}. Notice now that this quantity, τ_{yx}, carries information about two directions. The first one, tells us which direction is the momentum vector pointing at. The second one, tells us in which direction that momentum is being transported. Now that's a strange object! It's neither a scalar nor a vector, since we need to "attach" two directions to τ_{yx} in order to give complete information about this particular momentum flux. We call such object a tensor quantity.

Tensors are objects that allow us to associate multiple directions to a physical quantity. Hence, they can also be thought of as a generalization of the concept of a vector. A vector is characterized by the magnitude and one direction – we call that a *first-order tensor*. In the case of the Couette flow, we can define the tensor quantity $\boldsymbol{\tau}$, representing the momentum flux. It would be called a *second-order tensor* because it carries information about two directions. In general, tensors can be characterized by the magnitude and one or multiple directions attached to it. Of course, we need to create such notation that we know what each direction means physically. In our example, we used the indices $_{yx}$. The first index keeps track of the transport direction and the second keeps track of the direction of the momentum vector.

Tensors as matrices

Hopefully, by now you can appreciate that tensors are very useful tools in fluid dynamics, where vector quantities (such as momentum) can be transported in many directions! If we considered a different system than the Couette flow, perhaps there would be momentum in the x-direction transported in the z-direction. Or, momentum in the y-direction transported in the x-direction[1].

In a three-dimensional world, the momentum vector can be pointing in any of the three directions x, y or z, and can be transported in any of the three directions x, y or z. Tensors allow us to keep track of all those directions as well. If we wanted to account for all these possibilities in three-dimensions we would need $3 \times 3 = 9$ quantities τ_{ij}. For convenience, we often write out the elements of a tensor in a form of a table that resembles a matrix. Unlike in a scalar matrix, this table has additional information attached to every element – the directions represented. Those directions are agreed upon and often we can omit specifying them and just write out the magnitudes in a form of a matrix. But we should keep in mind that the directions are still there, even if at the "backstage" of that matrix.

Figure 4 shows an example of the most general second-order tensor that could represent viscous momentum flux in the three-dimensional fluid flow. We noted the directions that each element is keeping track of. Once we write down a specific τ_{ij}, its directions can be understood through the indices i and j. In the case of the Couette flow, we only needed one of those elements, τ_{yx}, since there was only one direction for momentum and only one direction in which that momentum was transported.

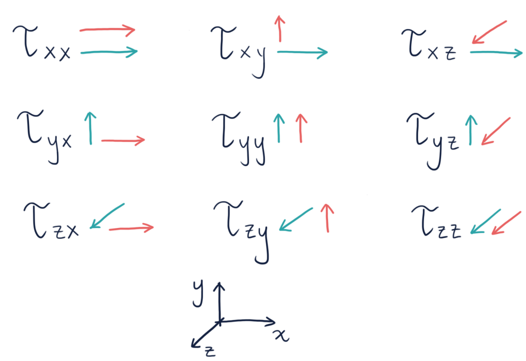


Figure 4: Second-order tensor from the three-dimensional world. It can represent viscous momentum flux in a three-dimensional fluid flow.

References

[1] R.B. Bird, W.E.Stewart, E.N. Lightfoot, *Transport Phenomena*, John Wiley & Sons, Inc., 2001

[1] Can you think of a real situation when momentum in the x-direction is transported in the x-direction?