The tensor necessity
– a short story about momentum transport in fluids

Kamila Zdybał

Université libre de Bruxelles, kamila.zdybal@ulb.ac.be
kamilazdybal.github.io/science-docs, kamila.zdybal@gmail.com

Please feel free to contact me with any suggestions, corrections or comments.

Preface

At first encounter, tensors can seem like strange mathematical objects. It can be challenging to grasp their meaning and their relevance might not be immediately obvious. At the same time, tensors are indispensable when studying fluid dynamics. So what's with the tensors and why do we need them?

In this document, I would like to convince you that we necessarily do need tensors in fluid dynamics! I will motivate their usefulness by discussing momentum transport in the Couette flow. Hopefully, by the end of this document, you will find that tensors can be very useful mathematical objects that make our life easier. Join me on the journey!

Keywords

tensor, momentum transport, transport phenomena, fluid dynamics, Couette flow

Viscous momentum flux tensor

To build our intuition around tensor quantities we will begin our journey with an illustrative example. We will look at momentum transport in the Couette flow – the flow of fluid between two parallel plates. The top plate is stationary and the bottom plate moves in the positive x-direction with the velocity $\mathbf{u}$, as presented in Figure 1.

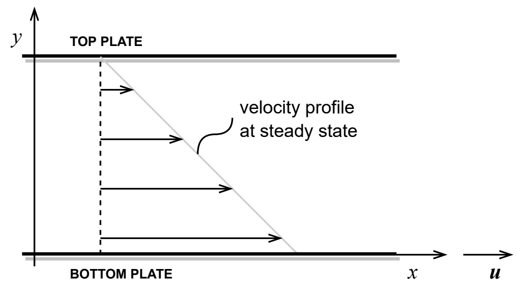


Figure 1: The Couette flow is the flow of fluid between two parallel plates. The bottom plate moves in the positive x-direction with the velocity $\mathbf{u}$ and the top plate is stationary. At steady state, the fully developed velocity profile between plates is a linear function of y.

Suppose that we start with an initial situation when both plates are stationary. At some moment in time, we begin sliding the bottom plate, reaching the velocity $\mathbf{u}$ after a while. You may already expect that the moment we started moving the bottom plate something interesting starts to happen to the fluid in-between – it begins moving as well. As the flow develops, the moving bottom plate successively "drags" fluid particles in the layer directly above it. Once those fluid particles are set in motion in the positive x-direction, that moving layer of fluid "drags" another layer directly on top of it. This "dragging" progresses upward in the positive y-direction, until at the top stationary plate the fluid is stagnant again to match the zero velocity of the top plate. After a sufficient amount of time, the fluid motion becomes steady. At steady state, the velocity profile between the two plates is fully developed and does not change in time anymore. The steady state velocity profile in the Couette flow is a linear function of y (this profile is plotted schematically in Figure 1). This is something that we will not prove here but you can give it a try as an exercise!

If we assume that the fluid flow between the two plates is laminar, we may assume that the fluid flows in thin layers, or *laminates*, stacked one on top of the other. Knowing the velocity profile, we know exactly what the velocity of each laminate is. In other words, since we know $\mathbf{u}(y)$, we can compute the velocity at any position y, denoted as $\mathbf{u}|_y$. For the flow described in Figure 1, a laminate at a lower y-coordinate will have a larger velocity than a laminate at a larger y-coordinate. This holds in our case, where we have assumed that the bottom plate is moving and the fluid velocity decreases to zero once we reach the top plate. Similar reasoning can be made assuming that the top plate is moving with the velocity $\mathbf{u}$, and the bottom plate is stationary. In Figure 2, we have drawn three such laminates, each having thickness dy.

Knowing the velocities at any position y, we also know the momentum carried by each laminate. We will in fact consider a specific momentum – momentum per unit volume – which can be computed as $\rho\mathbf{u}$, where ρ is the fluid density. Let's look at the situation from the perspective of one of the laminates whose y-coordinate is simply denoted y. This laminate is marked in gray in Figure 2. Its specific momentum is $\rho\mathbf{u}|_y$. It gains momentum from the faster moving laminate directly below it, whose specific momentum is $\rho\mathbf{u}|_{y-dy}$. It also looses momentum to the

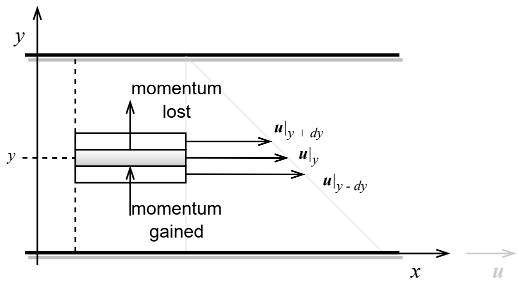


Figure 2: Transporting momentum between fluid *laminates* from the perspective of one of the laminates at position y (marked in gray).

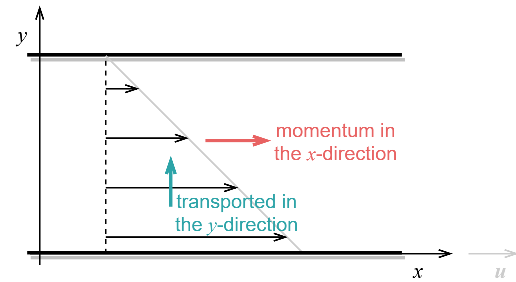
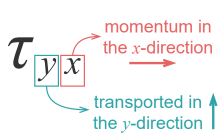


Figure 3: In the Couette flow, the momentum in the x direction is transported in the y direction. The transported momentum is called the *momentum flux*.

slower moving laminate directly above it, whose specific momentum is $\rho \mathbf{u}|_{y+dy}$. Such *transport* (loosing and gaining) of momentum is in practice possible due to molecular collisions between fluid particles. Every such collision is an opportunity to exchange momentum. When molecules from one laminate collide with molecules from another laminate, momentum can be transported in the positive y-direction. The "dragging", that we talked about before, is thus achieved by the momentum transport. The faster moving laminate gives some of its momentum to the slower moving one.

An interesting concept now emerges. Momentum is a vector quantity that has a direction of the velocity vector. In our example of the Couette flow, momentum of every laminate has the direction parallel to the x-axis (in accordance with the velocity profile). However, the transport of that momentum is happening in the direction parallel to the y-axis: from the laminates at the lower y-coordinate to the laminates at the larger y-coordinate. This is conceptually presented in Figure 3.



This transported momentum (also called the *momentum flux*) is denoted with a symbol τ_{yx}. Notice now that this quantity, τ_{yx}, carries information about two directions. The first one tells us which direction is the momentum vector pointing at (x direction). The second one tells us in which direction that momentum is being transported (y direction). Now that's a strange object! It's neither a scalar nor a vector, since we need to "attach" two directions to τ_{yx} in order to give complete information about this particular momentum flux. We call such object a tensor quantity – just a fancy mathematical name!

Tensors are objects that allow us to associate multiple directions to a physical quantity. Hence, they can also be thought of as a generalization of the concept of a vector, which is a quantity associated with a single direction. Or even a generalization of a scalar, which is a quantity with zero directions associated with it! A scalar would then be called a *zeroth-order tensor* – a magnitude with no directions associated with it. A vector is characterized by the magnitude and one direction – we call that a *first-order tensor*. In the case of the Couette flow, we can define the tensor quantity τ_{yx}, representing the momentum flux. It would be called a *second-order tensor* because it carries information about two directions. In general, tensors can be characterized by the magnitude and zero, or one, or multiple directions attached to it. Of course, we need to create such notation that we know what each direction means physically. In our example, we used the indices yx. The first index, y, keeps track of the transport direction and the second, x, keeps track of the direction of the momentum vector. Keep in mind, that in the case of zero or one direction we already have convenient names for such quantities: *scalars* and *vectors*. Therefore, often people only bother callings something a tensor when it's a quantity with at least two directions associated with it.

Hopefully, by now you can appreciate that tensors are very useful tools in fluid dynamics, where vector quantities (such as momentum) can be transported in many directions! As a final thought of this section, let's take a step back and look at fluid dynamics from a broader perspective. The reason why fluid dynamics is so populated with tensor quantities, is the fact that it is all about transport and describing fluid in motion. When fluid is physically transported through space, it carries with it all the other physical quantities that it possesses. Some of those properties are vector quantities, carried by a flow velocity – another vector quantity.

Tensors as matrices

If we considered a different system than the Couette flow, perhaps there would be momentum in the x-direction transported in the z-direction. Or, momentum in the y-direction transported in the x-direction[1]. In a three-dimensional world, the momentum vector can be pointing in any of the three directions x, y or z, and can be transported in any of the three directions x, y or z. Tensors allow us to keep track of all those directions as well. If we wanted to account for all these possibilities in three-dimensions we would need $3 \times 3 = 9$ quantities τ_{ij}. For convenience, we often write out the elements of a tensor in a form of a table that resembles a matrix. Unlike in a scalar matrix, this table has additional information attached to every element – the directions represented. Those directions are

[1] Can you think of a real situation when momentum in the x-direction is transported in the x-direction?

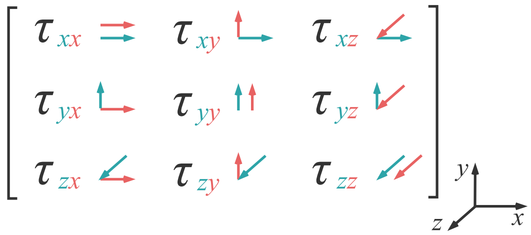


Figure 4: Second-order tensor from a three-dimensional world. It can represent viscous momentum flux in a three-dimensional fluid flow.

agreed upon and often we can omit specifying them and just write out the magnitudes in a form of a matrix. But we should keep in mind that the directions are still there, even if at the ”backstage” of that matrix.

Figure 4 shows an example of the most general second-order tensor that could represent viscous momentum flux in a three-dimensional fluid flow. We note the directions that each element is keeping track of. Once we write down a specific τ_{ij}, its directions can be understood through the indices i and j. In the case of the Couette flow, we only needed one of those elements, τ_{yx}, since there was only one direction for momentum and only one direction in which that momentum was transported.

Operations on tensors

Divergence reduces the tensor rank by one

Taking a divergence of a tensor of rank m results in a tensor of rank $m-1$. Below, we illustrate that on an example where $m = 2$, but the principle extends to any rank m. Let $\mathbf{\Phi}$ be a tensor of rank two. Let's compute the divergence of $\mathbf{\Phi}$:

$$\nabla \cdot \mathbf{\Phi} = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right] \cdot \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} & \cdots & \Phi_{1,n} \\ \Phi_{2,1} & \Phi_{2,2} & \cdots & \Phi_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n,1} & \Phi_{n,2} & \cdots & \Phi_{n,n} \end{bmatrix}.$$

The above matrix multiplication results in a vector (tensor of rank one):

$$\mathbf{V} = \left[V_1, V_2, \cdots, V_n \right],$$

where

$$V_j = \sum_{i=1}^{n} \frac{\partial \Phi_{i,j}}{\partial x_i}.$$

By taking a divergence of a tensor of rank two, $\mathbf{\Phi}$, we've obtained a tensor of rank one – a vector $\mathbf{V}$. This example illustrates the ”mechanism” by which the divergence operator reduces rank by one. You can observe that divergence is like taking a dot (scalar) product. For a dot product of two vectors the intuition is perhaps even stronger: a dot product of two vectors results in a scalar (tensor of rank zero).

Gradient increases the tensor rank by one

Taking a gradient of a tensor of rank m results in a tensor of rank $m+1$. We present an illustrative example for this as well. Let $\mathbf{V}$ be a vector (a tensor of rank one). Let's compute the gradient of $\mathbf{V}$:

$$\nabla \mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} \cdot \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right].$$

The above matrix multiplication results in a tensor of rank two:

$$\begin{bmatrix} \frac{\partial V_1}{\partial x_1} & \frac{\partial V_1}{\partial x_2} & \cdots & \frac{\partial V_1}{\partial x_n} \\ \frac{\partial V_2}{\partial x_1} & \frac{\partial V_2}{\partial x_2} & \cdots & \frac{\partial V_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial V_n}{\partial x_1} & \frac{\partial V_n}{\partial x_2} & \cdots & \frac{\partial V_n}{\partial x_n} \end{bmatrix}.$$

Thus, by taking a gradient of a tensor of rank one, $\mathbf{V}$, we've obtained a tensor of rank two, $\mathbf{\Phi}$. This mechanism can also be understood by noting that gradient of a scalar (tensor of rank zero) results in a vector (tensor of rank one).

References

[1] R.B. Bird, W.E.Stewart, E.N. Lightfoot, *Transport Phenomena*, John Wiley & Sons, Inc., 2001