Underdetermined linear systems
a short story with some code

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Preface

Imagine a linear system of equations with more unknown variables than the number equations describing them. In a sense, you might think of such system as having unconstrained degrees of freedom. In fact, such system has got, in the most general case, infinitely many solutions. If we were interested in solving (or approximating) such system, we would be picking a linear operator that in some way, is the best fit to the available unknown variables. Thus, we would have to pose additional constraints into what kind of solutions are of interest to us. The question then remains: how do we choose from infinitely many solutions? This document presents several popular types of linear solutions to underdetermined systems.

The aim of this document is to raise the awareness into how different solutions work and into which solutions might be a good choice for certain applications. It is also a mathematical tutorial on the available methods, supported by some computational examples.

This document is still in preparation. Please feel free to contact me with any suggestions, corrections or comments.

Keywords

underdetermined linear systems, linear algebra, matrix decomposition, matrix approximation

Contents

1 Introduction

For an undertermined linear system (where matrix $\boldsymbol{A}$ is size $(n \times m)$ and $n < m$):

$$\boldsymbol{A}x = b \tag{1}$$

an infinite number of solutions exists. MATLAB implements various methods of computing possible solutions for x. In this paper we investigate the differences in the available solutions by analyzing their histograms. Four methods were selected: backslash \ operator, computing a pseudo-inverse `pinv()`, and two least-squares algorithms `lsqnonneg()` and `lsqr()`.

2 Matlab example

We focus on an example where matrix $\boldsymbol{A}$ is size (100×500) and a vector b is size (100×1); both are populated by normally distributed random numbers.

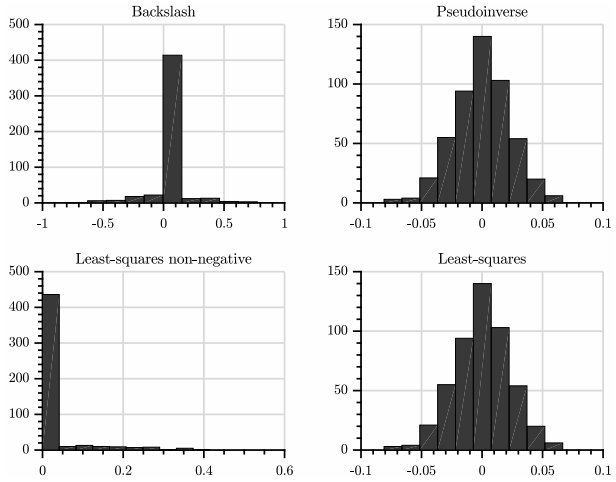

Figure 1: Histograms of four solutions to an undetermined linear system.

3 Moore-Penrose inverse (pseudoinverse)

4 Partial Least Squares (PLS) regression

5 Possible applications

References

[1] Nathan Kutz, *Data Driven Discovery of Dynamical Systems and PDEs*, an online lecture

[2] Gilbert Strang, *Introduction to Linear Algebra*, Fifth Edition, 2016