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Underdetermined linear systems
a short story with some code

Kamila Zdybał

Université libre de Bruxelles, kamila.zdybal@ulb.ac.be
camillejr.github.io/science-docs, kamila.zdybal@gmail.com

Preface

This document is still in preparation. Please feel free to contact me with any suggestions, corrections or comments.

Keywords

underdetermined linear systems, linear algebra, matrix decomposition, matrix approximation

Contents

1 Introduction

For an undertermined linear system (where matrix $\boldsymbol{A}$ is size $(n \times m)$ and $n < m$):

$$\boldsymbol{A}x = b \tag{1}$$

an infinite number of solutions exists. MATLAB implements various methods of computing possible solutions for x. In this paper we investigate the differences in the available solutions by analyzing their histograms. Four methods were selected: backslash \ operator, computing a pseudo-inverse `pinv()`, and two least-squares algorithms `lsqnonneg()` and `lsqr()`.

2 Matlab example

We focus on an example where matrix $\boldsymbol{A}$ is size (100×500) and a vector b is size (100×1); both are populated by normally distributed random numbers.

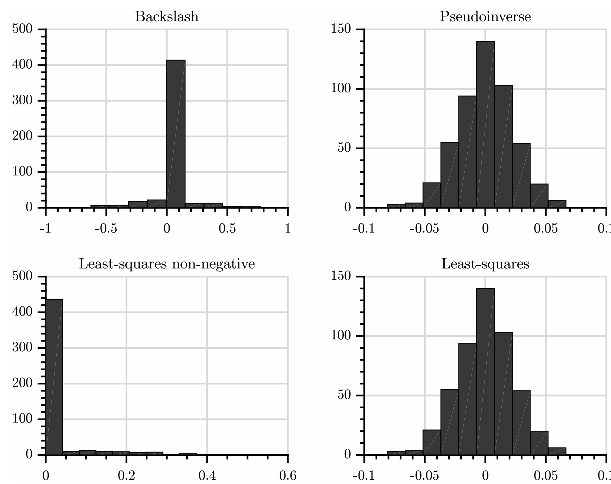

Figure 1: Histograms of four solutions to an undetermined linear system.

3 Possible applications

4 Low-rank approximations

We perform Principal Component Analysis (PCA) on three generated matrices of size (10×6): a random matrix which is populated by random numbers 0-1, and a semi-structured and structured matrices which are populated by the user and have some and increasing level of structure that was judged visually. These matrices are presented in Fig. 2.

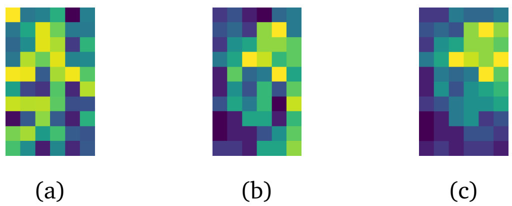

Figure 2: Original matrices: (a) random matrix, (b) semi-structured matrix, (c) structured matrix.

The judgment on the level of imposed ”structure” is quite objective but the aim was to group the elements of high value in one region of the matrix and elements of low value in other region of the matrix.

The level of ”structure” imposed in a matrix can be also seen from the eigenvalue distribution in Fig. 3.

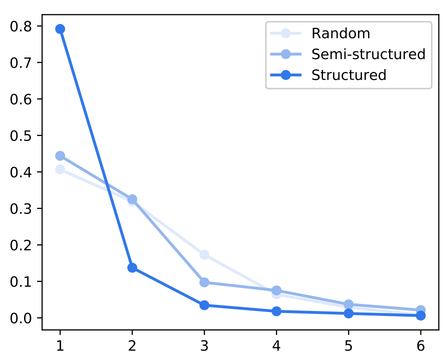

Figure 3: Eigenvalues distribution.

We may reconstruct the original data matrices using a certain number of PC-scores and corresponding PCs. Using the Matlab notation we may write the approximation as:

$$\boldsymbol{D}_{app} = \text{PC-scores}(:, 1:q) \cdot \text{PCs}(1:q, :) \tag{2}$$

Below we can see a rank-1 approximation of the original matrices using the 1^{st} Principal Component found by PCA. The vectors (10×1) represent the PC-scores and the vectors (1×6) represent the PCs.

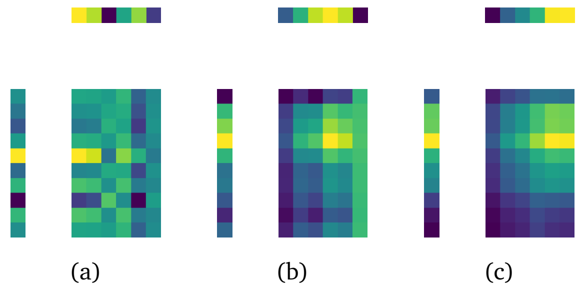

Figure 4: PCA reconstruction with 1^{st} Principal Component of: (a) random matrix, (b) semi-structured matrix, (c) structured matrix.

In Fig. 5 we present a rank-2 approximation, where we took two PCs. Again, the matrix (10×2) represent the PC-scores and the vectors (2×6) represent the PCs.

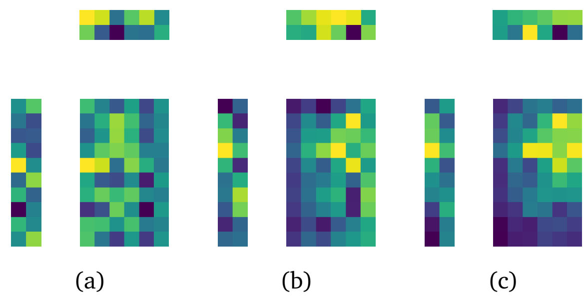

Figure 5: PCA reconstruction with 2^{nd} Principal Component of: (a) random matrix, (b) semi-structured matrix, (c) structured matrix.

References

[1] Nathan Kutz, *Data Driven Discovery of Dynamical Systems and PDEs*, an online lecture

[2] Gilbert Strang, *Introduction to Linear Algebra*, Fifth Edition, 2016