Reading Conclusions Conjunctively

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Abstract

In philosophical logic, mainstream multiple-conclusion systems induce a *conjunctive* reading of premises and a *disjunctive* reading of conclusions. In this paper, I explore an old but mostly ignored alternative approach, which is to work with systems that induce a conjunctive reading of *both* premises *and* conclusions. I analyse from a model- and a proof-theoretic perspective the system that results from applying this approach to classical logic. Then, I give reasons why the approach is of philosophical interest.

1 Introduction

Logical consequence is typically understood as a dyadic relation, say $\dashv\!\exists$, such that y is a logical consequence of x, written $x \dashv\!\exists\, y$, just in case the pair $\langle x, y\rangle$ constitutes an argument, and this argument is valid. What kind of things are denoted by variables x and y, it depends on what the constituents of arguments are supposed to be. The received wisdom in the philosophy of logic tells us that arguments from natural language can have several premises, but at least and at most one conclusion. Accordingly, traditional consequence relations go from collections of sentences, say $\Gamma, \Delta, ...$, to sentences of a given language, say $A, B, ...$ Thus, validity claims take the form $\Gamma \dashv\!\exists\, A$. These are what we call *single-conclusion* consequence relations.

In his seminal work on proof theory, Gentzen [22] presented his now well-known sequent calculus LK for classical logic. The usual informal reading of LK takes the sequent arrow '$\Rightarrow$' to denote logical consequence. Under this reading, consequence in LK goes from collections of sentences to collections of sentences; thus, it allows arguments with one, none or many conclusions.[1] With the later works of Carnap [4], Kneale [26], Scott [42] and Shoesmith and Smiley [43], among others, it has slowly become common to work with consequence relations of this sort. These are what we call *multiple-conclusion* consequence relations. Unless otherwise stated, we assume no cardinality constraints on the collections of premises and conclusions—they might well be infinite.

The mainstream is to follow Gentzen and work with consequence relations which (like LK) induce a conjunctive reading of multiple premises and a disjunctive reading of multiple conclusions. By this I mean that validity can be informally paraphrased thus:

> Γ entails Δ just in case the conjunction of the things in Γ entails the disjunction of the things in Δ

[1] In the article mentioned, Gentzen does not equate the sequent arrow with logical consequence but with the material conditional: he says that $A_1, ..., A_n \Rightarrow B_1, ..., B_n$ has "the same intuitive meaning" as the formula $A_1 \wedge ... \wedge A_n \to B_1 \vee ... \vee B_n$ (p. 290). Later, in [23], the author suggest that multiple conclusions represent the distinction into cases often found in mathematical reasoning; so, in $A_1, ..., A_n \Rightarrow B_1, ..., B_n$, conclusions $B_1, ..., B_n$ are cases which depend on assumptions $A_1, ..., A_n$. This second informal understanding of a sequent comes closer to the traditional reading of LK.

(where Γ and Δ are collections of the appropriate kind). Let us call this the *disjunctive* approach to multiple conclusions. Several philosophers have vindicated the use of multiple conclusions so understood. It has been argued that they solve Carnap's categoricity problem for classical propositional logic,[2] that they enable a proof-theoretic presentation of classical logic that abides by the standards of logical inferentialism,[3] and that they allow us to define certain logical constants that would be otherwise undefinable.[4] However, multiple conclusions also face various philosophical challenges. Some authors have claimed that they sneak in classicality by the back door, thus begging the question against rival theories.[5] Others, that they induce circular explanations of meaning that are ultimately incompatible with inferentialism.[6] But the most echoed complaint is, by far, that they do not have any clear correlate in our everyday reasoning:

> The vice of the idea of multiple conclusion arguments is that it seems completely foreign to the evidence of the arguments we see in practice. (Beall and Restall [1, p. 13].)

> The rarity, to the point of extinction, of naturally occurring multiple conclusion arguments has always been the reason why mainstream logicians have dismissed multiple-conclusion logic as little more than a curiosity. (Rumfitt [41, p. 79].)

Setting the controversy aside, I would like to notice that the disjunctive approach has at least two features that might be surprising to the unbiased eye of the neophyte. For the moment, suppose that collections related by consequence are sets—although nothing hinges on this. The first feature lies in the behaviour of the empty set, $\varnothing$. It is standard to assume that a disjunction that has no disjuncts is always false (for it never has a true disjunct); in other words, letting $\bigvee(\Sigma)$ be the disjunction of all the things in Σ, we have that $\bigvee(\varnothing)$ is a logical falsehood. Dually, a conjunction with no conjuncts is always true (for it never has a false conjunct); letting $\bigwedge(\Sigma)$ be the conjunction of all things in Σ, we have that $\bigwedge(\varnothing)$ is a logical truth. Given this, under the disjunctive approach $\varnothing$ is a kind of a cyclothymic character: it plays different inferential roles depending on where it appears in the argument. When it is the set of premises, it works as a logical truth, in the sense that it only entails logical truths; when it is the set of conclusions, it behaves as a logical falsehood, in the sense that it only follows from logical falsehoods.

The second feature concerns the notions of reflexivity and transitivity. These notions come from the theory of relations: a relation R is *reflexive* if and only if it is a dyadic relation on a set A such that, for every a in A, aRa. R is *transitive* if and only if it is a dyadic relation on a set A such that, for every a, b and c in A, if aRb and bRc then aRc. At least since the work of Tarski [45], it is a commonplace to say that logical consequence is both reflexive and transitive. Strictly speaking, however, *single*-conclusion consequence relations are neither, because they are not relations on a single set, and so they are not even the kind of thing that can have these properties. When we say that they are reflexive and/or transitive, by this we mean that they satisfy some principles resembling reflexivity and/or transitivity to a greater or lesser extent (for instance, the restrictions of these properties to sentences). Now, *multiple*-conclusion consequence relations *are* relations on a single set, and so they could be reflexive and transitive in principle. Under the disjunctive approach, however, they are usually not: on the one hand we typically have $\varnothing \not\succ \varnothing$, violating reflexivity; on the other, $\{p\} \succ \{p, q\}$ and $\{p, q\} \succ \{q\}$ but $\{p\} \not\succ \{q\}$, violating transitivity.

[2]See, for instance, Shoesmith and Smiley [43].

[3]In this line we find the works of Hacking [24], Read [34], Cook [10] and Restall [36].

[4]This has been recently maintained by Dicher [14].

[5]See Tennant [47].

[6]The objection is due to Dummett [15], and was recently elaborated by Steinberger [44].

At this point, a different approach to multiple conclusions enters the scene. It is to work with consequence relations that induce a conjunctive reading of both premises and conclusions:

> Γ entails Δ just in case the conjunction of the things in Γ entails the conjunction of the things in Δ

Let us call this the *conjunctive* approach to multiple conclusions. As we shall see below, it lacks the two surprising features we have just discussed: it makes the empty set behave always the same way and it is compatible with the consequence relation being reflexive and transitive. The approach has noble roots, as it can be traced back at least to the work of Bolzano [3]. Multiple conclusions under a conjunctive reading can often be found in abstract algebraic logic; the reason is that they enable notions of consequence that are *symmetric* in the sense of having relata of the same kind, and such notions of consequence can be generalised to wider classes of structures.[7] Also, Cintula and Paoli [7] have recently used multiple conclusions under a conjunctive reading to answer a philosophical challenge faced by non-contractive logics. The challenge is that, given any single-conclusion consequence relation $\dashv\!\!\!\varepsilon$, we expect being able to associate it with some closure operation Cn in the following way: $\Gamma \dashv\!\!\!\varepsilon A$ just in case $A \in Cn(\Gamma)$; alas, there is an impossibility result saying that this cannot be done if $\dashv\!\!\!\varepsilon$ is non-contractive.[8] In response, Cintula and Paoli show that, once we move to a multiple-conclusion framework where conclusions are read conjunctively, non-contractive consequence relations and closure operations can be matched in the expected way. This leads the authors to conclude that non-contractive consequence relations are "intrinsecally" or "esentially" multiple-conclusioned (p. 753).

In spite of the above, the conjunctive approach to multiple conclusions has gone largely unnoticed by philosophers; indeed, we find it near nowhere in the literature on philosophical logic. The purpose of this article is to give the first steps towards reversing this situation. My guiding questions are the following. *First*: What does a logic look like when we generalise it to multiple conclusions under a conjunctive reading? This is the topic of Sect. 2. I focus mostly on classical logic, though my procedures and results can be extended to a wide range of non-classical systems. I define a multiple-conclusion presentation of classical propositional logic where conclusions are read conjunctively, and analyse it both from a model- and a proof-theoretic perspective. *Second*: Can multiple conclusions under a conjunctive reading receive any motivation independent of what we already said in the last paragraph? This is the topic of Sect. 3. I give a positive answer: we have good reasons to think that multiple conclusions under a conjunctive reading can be found in natural language arguments; therefore, they are useful in modelling our everyday reasoning. *Lastly*: The disjunctive and the conjunctive approaches differ in their policies towards reflexivity and transitivity; does this difference have any philosophical significance? This is the topic of Sect. 4. Again, I answer positively. I argue that reflexivity and transitivity are tightly related to a number of aspects in which the conjunctive approach behaves better than its disjunctive counterpart; in particular, it allows a more uniform treatment of logical equivalence, and it gets along much better with some of our best accounts of logical consequence.

[7] See, for instance, Galatos and Tsinakis [21], Novak [29] and Cintula et. al. [6].
[8] The objection was raised by Ripley [38].

A disclaimer is in place. In this paper, I by no means want to defend that multiple conclusions *should* be read one way or another. The merits of the disjunctive and the conjunctive approaches will be assessed relative to the application at hand, and there is no hope of finding 'the one true way' in which multiple-conclusion consequence should be interpreted. The more modest goal of this paper is to point towards some interesting properties of a family of consequence relations that have been (rather unfairly, from my point of view) neglected in the philosophical literature.

2 Conjunctive Classical Logic

As mentioned, I take good old classical logic as my main test-case. The system that I present will be called **cCL**, for 'Conjunctive Classical Logic'. Sect. 2.1 addresses the model-theory of **cCL**. Sect. 2.2 addresses the proof-theory. Before going on, it pays to lay down some stipulations.

The language. For the purposes of this paper, it is enough to consider a propositional language. We use $\mathcal{L}$ both for the language and for the set of its formulas—hoping that context and the readers' generosity will suffice for disambiguation. We assume a denumerable set of variables $p, q, r, \ldots$. Our primitive logical constants will be $\bot$, $\wedge$, $\vee$ and $\rightarrow$, with their usual arities and interpretations; $\neg A$ will be defined as $A \rightarrow \bot$, and $\top$ as $\neg\bot$. Capital Latin letters $A, B, C, \ldots$ will stand for arbitrary sentences, and capital Greek letters $\Gamma, \Delta, \Sigma, \ldots$ for sets thereof. We use commas for set union and occasionally omit brackets on singleton sets; so e.g. Γ, A stands for $\Gamma \cup \{A\}$.

Terminology. As usual, I use symbols as their own names. I sometimes appeal to typical abbreviations as 'iff' for 'if and only if', etc. I keep using $\dashv$ as a neutral symbol for entailment (notice that it is *not* a connective, but a relation symbol). Lastly, when I talk about *disjunctive* and *conjunctive* multiple conclusions, I mean multiple conclusions under a disjunctive and conjunctive reading, respectively.

2.1 Model Theory

We start by defining the three logical systems that we will be mainly concerned with. The first one is single-conclusion classical logic, henceforth **CL**. The second one is the standard presentation of multiple-conclusion classical logic, which I call **dCL**, for 'Disjunctive Classical Logic'. Third and last, our main character, **cCL**.

Let $\mathcal{V}$ be the set of all classical (viz. Boolean bivalued) interpretations of $\mathcal{L}$. Given a particular interpretation v, we write $v[\Sigma]$ to denote the set $\{v(\sigma) : \sigma \in \Sigma\}$.

Definition 1. Relations $\models_{\mathbf{CL}} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ and $\models_{\mathbf{dCL}}, \models_{\mathbf{cCL}} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ are defined as follows

$$\begin{array}{llll} \Gamma \models_{\mathbf{CL}} & C & \text{iff} & \text{for each } v \in \mathcal{V}, \text{ if } v[\Gamma] \subseteq \{1\} \text{ then } v(C) = 1 \\ \Gamma \models_{\mathbf{dCL}} & \Delta & \text{iff} & \text{for each } v \in \mathcal{V}, \text{ if } v[\Gamma] \subseteq \{1\} \text{ then } v[\Delta] \not\subseteq \{0\} \\ \Gamma \models_{\mathbf{cCL}} & \Delta & \text{iff} & \text{for each } v \in \mathcal{V}, \text{ if } v[\Gamma] \subseteq \{1\} \text{ then } v[\Delta] \subseteq \{1\} \end{array}$$

So, reading 1 as 'true' and 0 as 'false', we can give the following informal paraphrases: (i) an argument is valid in **CL** just in case whenever the premises are all true the conclusion is true; (ii) an argument is valid in **dCL** just in case whenever the premises are all true, at least one of the conclusions is true; and (iii) an argument is valid in **cCL** just in case whenever the premises are all true the conclusions are all true.

Next, I describe **cCL** and compare it with the other two systems. Along the way, I record a number of simple but relevant facts, the proofs of which are left to the reader. To begin with, one set entails another in **cCL** just in case the former entails each of the sentences in the latter. In other words,

Fact 1. $\Gamma \models_{\mathbf{cCL}} \Delta$ if and only if $\Gamma \models_{\mathbf{cCL}} B$ for each $B \in \Delta$.

This justifies the idea that in **cCL** multiple conclusions should be read conjunctively.

In Sect. 1, we already anticipated two features that distinguish the disjunctive and the conjunctive approaches. One of them, remember, concerns the behaviour of the empty set, $\varnothing$. In **dCL**, $\varnothing$ works as per the disjunctive approach; this means that its behaviour varies depending on the place it has in the argument: when it is the set of premises, it behaves as a logical truth; when it is the set of conclusions, it behaves as a logical falsehood. Graphically, we have

$$\begin{array}{lcl} \varnothing \models_{\mathbf{dCL}} A & \text{iff} & \top \models_{\mathbf{dCL}} A \\ A \models_{\mathbf{dCL}} \varnothing & \text{iff} & A \models_{\mathbf{dCL}} \bot \end{array}$$

In **cCL**, on the contrary, $\varnothing$ is a more temperate character. Under the conjunctive approach, it is read as $\bigwedge(\varnothing)$ no matter what; thus, it always behaves as a logical truth:

$$\begin{array}{lcl} \varnothing \models_{\mathbf{cCL}} A & \text{iff} & \top \models_{\mathbf{cCL}} A \\ A \models_{\mathbf{cCL}} \varnothing & \text{iff} & A \models_{\mathbf{cCL}} \top \end{array}$$

Note that the last claim implies that $A \models_{\mathbf{cCL}} \varnothing$ for every A. And the claim generalises to arbitrary sets; that is, $\Gamma \models_{\mathbf{cCL}} \varnothing$ for any Γ.

The other feature that distinguished the disjunctive and the conjunctive approaches had to do with reflexivity and transitivity. **dCL** is neither reflexive nor transitive, while **cCL** is both:

Fact 2. The following properties hold for **cCL** and do not hold for **dCL**:

(i) $\Gamma \mathrel{-\!\!3} \Gamma$, for every Γ.

(ii) If $\Gamma \mathrel{-\!\!3} \Delta$ and $\Delta \mathrel{-\!\!3} \Sigma$, then $\Gamma \mathrel{-\!\!3} \Sigma$, for every Γ, Δ and Σ.

To exemplify the negative claims, we restate the counterexamples from before: (i) $\varnothing \not\models_{\mathbf{dCL}} \varnothing$, and (ii) $\{p\} \models_{\mathbf{dCL}} \{p, q\}$ and $\{p, q\} \models_{\mathbf{dCL}} \{q\}$, but $\{p\} \not\models_{\mathbf{dCL}} \{q\}$. Certainly, **dCL** satisfies some properties resembling reflexivity and transitivity; for instance, reflexivity restricted to non-empty sets, and transitivity as encoded by the property

(iii) If $\Gamma \mathrel{-\!\!3} A, \Delta$ and $\Sigma, A \mathrel{-\!\!3} \Pi$, then $\Gamma, \Sigma \mathrel{-\!\!3} \Delta, \Pi$, for every $\Gamma, \Sigma, \Delta, \Pi$

(This property is the model-theoretic counterpart of the sequent rule known as Cut.) Now, **cCL** satisfies (iii) as well. Indeed, Ripley [39] distinguishes other twelve properties resembling transitivity that **dCL** satisfies, and **cCL** satisfies them all. Hence, even when we focus on non-relation-theoretic variations of transitivity, **cCL** is not any less transitive than **dCL**.

Given that, as we have seen, $\Gamma \models_{\mathbf{cCL}} \varnothing$ for any Γ, the reader may perhaps wonder why **cCL** is not trivial. The answer concerns the property of monotonicity. A multiple-conclusion consequence relation $\mathrel{-\!\!3}$ is *monotone* iff it satisfies the properties

(iv) If $\Gamma \mathrel{-\!\!3} \Delta$, then $\Sigma, \Gamma \mathrel{-\!\!3} \Delta$, for every Γ, Δ, Σ

(v) If $\Gamma \mathrel{-\!\!3} \Delta$, then $\Gamma \mathrel{-\!\!3} \Delta, \Sigma$, for every Γ, Δ, Σ

(These properties are the model-theoretic counterparts of the sequent-rules of 'Left Weakening' (LW) and 'Right Weakening' (RW), respectively.) Monotonicity is often thought to encode the non-defeasible character of deductive reasoning. **dCL** satisfies both (iv) and (v), so it is monotone. **cCL** satisfies (iv) but not (v); to exemplify, $\{p\} \models_{\mathbf{cCL}} \{p\}$ but $\{p\} \not\models_{\mathbf{cCL}} \{p, q\}$. This explains why **cCL** is not trivial, even though $\Gamma \models_{\mathbf{cCL}} \varnothing$ for every Γ. The failure of (v) should not be taken as a deductive weakness of the system, however, or as evidence that it models defeasible reasoning only. First and foremost, under the conjunctive approach, (v) intuitively says that, whenever certain conjunction follows from our premises, adding some additional conjuncts delivers a conjunction that also follows. But this of course is not the case in general. Thus, (v) is not reasonable in this context, and it should not be taken to encode the non-defeasibility of deductive reasoning. Secondly, there are some limitative results concerning the properties of reflexivity, transitivity and monotonicity:

Fact 3. Let $\dashv\!\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$

(a) If $\dashv\!\vdash$ is reflexive and monotone, then it is trivial.

(b) If $\dashv\!\vdash$ is transitive, monotone, and there are at least two sets Γ and Δ such that $\Gamma \dashv\!\vdash \varnothing$ and $\varnothing \dashv\!\vdash \Delta$, then $\dashv\!\vdash$ is trivial.

Thus, having all these properties at once was never a realistic goal to start with. **dCL** gives up both reflexivity and transitivity. **cCL** only gives up (one of the sides of) monotonicity.

Related to monotonicity is the property

(vi) If $\Gamma \dashv\!\vdash \Delta, \Sigma$, then $\Gamma \dashv\!\vdash \Delta$, for every Γ, Δ, Σ

(This is the counterpart of a sequent-rule sometimes called 'Right anti-Weakening' (RaW).) This property is satisfied by **cCL**, but not so by **dCL**. This is perfectly reasonable, given the informal readings that validity claims receive in these systems.

What we already said implies that **cCL** and **dCL** are contralogics of one another, that is, there are arguments that are valid in **cCL** but not in **dCL** and vice versa:

Fact 4.

(a) $\models_{\mathbf{cCL}} \not\subseteq \models_{\mathbf{dCL}}$ (e.g. $\{p\} \models_{\mathbf{cCL}} \varnothing$ but $\{p\} \not\models_{\mathbf{dCL}} \varnothing$)

(b) $\models_{\mathbf{dCL}} \not\subseteq \models_{\mathbf{cCL}}$ (e.g. $\{p\} \not\models_{\mathbf{cCL}} \{p, q\}$ but $\{p\} \models_{\mathbf{dCL}} \{p, q\}$)

Nevertheless, **dCL** and **cCL** are both what is known as *counterparts* of **CL**; this means that they coincide with **CL** in single-conclusion arguments:

Fact 5. $\Gamma \models_{\mathbf{CL}} C$ iff $\Gamma \models_{\mathbf{dCL}} \{C\}$ iff $\Gamma \models_{\mathbf{cCL}} \{C\}$

The last observation I would like to make about **cCL** is that there is a sense in which the system is compact:

Fact 6. $\Gamma \models_{\mathbf{cCL}} \Delta$ just in case, for each finite Δ' of Δ there is some finite subset Γ' of Γ such that $\Gamma' \models_{\mathbf{cCL}} \Delta'$.

(The result follows easily by Fact 1, Fact 5 and the compactness of **CL**.) Of course, we do not have in **cCL** a result analogous to the compactness of **dCL**, namely, that Γ entails Δ just in case there is some finite subset of Γ that entails Δ. For a counterexample, let AT be the set of all atomic sentences of $\mathcal{L}$. Clearly, AT $\models_{\mathbf{cCL}}$ AT, but there is no finite $\Gamma \subseteq$ AT such that $\Gamma \models_{\mathbf{cCL}}$ AT.

To close this subsection, I will give some hints of how the conjunctive approach can be applied to non-classical logics—the examples will be also useful later.

Example 1. The pattern behind Definition 1 can be applied to many other systems.[9] A case in point is the paraconsistent logic **PWK** (see [25]). It is based on the weak Kleene valuations of the language; these can described as valuations with range $\{1, 0, i\}$ which (i) behave classically when all the subformulas of the input formula receive a classical value, and (ii) deliver i in any other case. Let $\mathcal{K}$ be the set of all such valuations. Then,

$\Gamma \models_{\mathbf{PWK}} C$ iff, for every $v \in \mathcal{K}$, if $v[\Gamma] \subseteq \{1, i\}$, then $v(C) \in \{1, i\}$.

$\Gamma \models_{\mathbf{cPWK}} \Delta$ iff, for every $v \in \mathcal{K}$, if $v[\Gamma] \subseteq \{1, i\}$, then $v[\Delta] \subseteq \{1, i\}$.

Example 2. Sometimes, we may want our multiple conclusions to track the behaviour of some specific conjunction-like operation. To illustrate, we take an example by Cintula and Paoli (p. 754). They take Łukasiewicz's logic, **Ł**. It is defined on a language having just $\neg$ and $\rightarrow$ as primitives; additionally, $A \otimes B$ is defined as $\neg(A \rightarrow \neg B)$. The valuations have as range the closed real unit interval $[0, 1]$ and satisfy the conditions

$$v(\neg A) = 1 - v(A)$$
$$v(A \rightarrow B) = \min(1, 1 - v(A) + v(B))$$

Let $\mathcal{J}$ be the set of all such valuations. Then, **Ł** is defined thus:

$\Gamma \models_{\mathbf{Ł}} C$ iff, for every $v \in \mathcal{J}$, if $v[\Gamma] \subseteq \{1\}$ then $v(C) = 1$.

Łukasiewicz's logic can be used to define a system with conjunctive multiple conclusions which is non-contractive. Let $\Gamma_m, \Delta_m, \ldots$ be finite multisets of sentences. Also, if $\Sigma_m = [A_1, \ldots, A_n]$, let $\bigotimes \Sigma_m$ stand for the formula $A_1 \otimes \ldots \otimes A_n$. Hence,

$\Gamma_m \models_{\mathbf{mŁ}} \Delta_m$ iff $\varnothing \models_{\mathbf{Ł}} \bigotimes \Gamma_m \rightarrow \bigotimes \Delta_m$

(Label '**mŁ**' stands for 'multiset-Łukasiewicz'.) It is easy to check that $[p, p] \models_{\mathbf{mŁ}} [p, p]$ but $[p] \not\models_{\mathbf{mŁ}} [p, p]$.

2.2 Proof Theory

We will provide a sequent calculus for **cCL**. We define a *sequent* as a pair of finite sets of sentences of $\mathcal{L}$, and denote the sequent $\langle \Gamma, \Delta \rangle$ as $\Gamma \Rightarrow \Delta$. We start by presenting the sequent-rule counterparts of the various model-theoretic properties alluded to in the previous subsection:

$$\text{LW}\ \frac{\Gamma \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta} \qquad \text{Cut}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Sigma, A \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \qquad \text{RW}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \Sigma}$$

$$\text{Ref}\ \frac{}{\Gamma \Rightarrow \Gamma} \qquad \text{Tr}\ \frac{\Gamma \Rightarrow \Delta \qquad \Delta \Rightarrow \Sigma}{\Gamma \Rightarrow \Sigma} \qquad \text{RaW}\ \frac{\Gamma \Rightarrow \Delta, \Sigma}{\Gamma \Rightarrow \Delta}$$

We say that a sequent $\Gamma \Rightarrow \Delta$ is *valid* in **cCL** just in case $\Gamma \models_{\mathbf{cCL}} \Delta$. Also, a rule *preserves validity* in **cCL** just in case, for each of its instances, if the premise-sequents are all valid, the

[9] If we take single-conclusion systems that are non-reflexive or non-transitive for formulas (e.g. logics **TS** [20] and **ST** [9] couched in a sufficiently expressive language) then we will get multiple-conclusion counterparts which are non-reflexive and non-transitive for formulas as well. Which shows that, even though the conjunctive approach is compatible with the reflexivity and transitivity of consequence, it does not *impose* these properties on us.

conclusion-sequent is valid. It is easy to check that all of the above rules except for RW preserve validity in **cCL**.

In the quest for a calculus for **cCL**, our first step will be to exhibit a calculus for **CL**; It is the one given by Negri and von Plato [28, p. 114]. We write A/B for a formula that can be either A or B, and A^{at} for an arbitrary propositional variable.

Definition 2. The calculus $\mathcal{S}_{\mathbf{CL}}$ is determined by the following rules:

$$\text{Id-}at\ \frac{}{\Gamma, A^{at} \Rightarrow A^{at}}$$

$$\text{L}\vee\ \frac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \qquad\qquad \text{R}\vee\ \frac{\Gamma \Rightarrow A/B}{\Gamma \Rightarrow A \vee B}$$

$$\text{L}\wedge\ \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \qquad\qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\text{L}\to\ \frac{\Gamma, A \to B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C} \qquad\qquad \text{R}\to\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

$$\text{L}\bot\ \frac{}{\Gamma, \bot \Rightarrow A} \qquad\qquad \text{Gem-}at\ \frac{\Gamma, A^{at} \Rightarrow C \quad \Gamma, \neg A^{at} \Rightarrow C}{\Gamma \Rightarrow C}$$

Two remarks. First, the identity rule for arbitrary formulas,

$$\text{Id}\ \frac{}{\Gamma, A \Rightarrow A}$$

is derivable in $\mathcal{S}_{\mathbf{CL}}$. Second, we have the following result about how applications of excluded middle can be restricted in derivations:

Fact 7 (Negri and von Plato, p. 120)**.** If a sequent $\Gamma \Rightarrow C$ is derivable in $\mathcal{S}_{\mathbf{CL}}$, then it has a derivation where Gem-*at* is applied only on subformulas of C.

But Gem-*at* is the only elimination rule of the calculus; so, it follows that if a sequent is derivable in $\mathcal{S}_{\mathbf{CL}}$, then it has a derivation with the subformula property—that is, a derivation whose formulas are all subformulas of formulas in the end-sequent.

Now, to obtain a calculus for **cCL**, we just add a pair of rules to handle multiple conclusions:

Definition 3. The calculus $\mathcal{S}_{\mathbf{cCL}}$ results from $\mathcal{S}_{\mathbf{CL}}$ by adding the rules

$$\text{R}\varnothing\ \frac{}{\Gamma \Rightarrow \varnothing} \qquad\qquad \text{SM}\ \frac{\Gamma_1 \Rightarrow \Delta_1\ \ ...\ \ \Gamma_n \Rightarrow \Delta_n}{\Gamma_1, ..., \Gamma_n \Rightarrow \Delta_1, ...\Delta_n}$$

Some comments. First, the intuitive readings of rules R∅ and SM are quite straightforward: the former says that the empty set follows from any set whatsoever, and the latter says that, if several arguments are valid, the union of their premises entails the union their conclusions. Second, rules LW and Ref are easily derivable in the system:[10]

$$\frac{\dfrac{}{A_1 \Rightarrow A_1} \qquad \dots \qquad \dfrac{}{A_n \Rightarrow A_n}}{A_1, ..., A_n \Rightarrow A_1, ..., A_n} \qquad\qquad \frac{\Gamma \Rightarrow \Delta \qquad \dfrac{}{\Sigma \Rightarrow \varnothing}}{\Gamma, \Sigma \Rightarrow \Delta}$$

(The case of Ref where Γ is $\varnothing$ is just an instance of R∅.) Third, rule L→ is a bit awkward-looking, because it contains two occurrences of the formulas $A \to B$. In presence of LW, however, this rule can be replaced by the more natural

[10]Intuitively, a rule $\mathcal{R}$ is *admissible* in a sequent calculus $\mathcal{S}$ just in case the set of $\mathcal{S}$-provable sequents is closed under it. $\mathcal{R}$ *derivable* in $\mathcal{S}$ just in case it is admissible in $\mathcal{S}$ and in every calculus $\mathcal{S}'$ that results by adding to $\mathcal{S}$ some additional rules. (See e.g. Da Ré [12] for the precise definitions.)

$$\text{L}{\to}^* \frac{\Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{\Gamma, A \to B \Rightarrow C}$$

since both rules are interderivable. Lastly, the typical rules for negation,

$$\text{L}\neg \frac{\Gamma \Rightarrow A}{\Gamma, \neg A \Rightarrow \bot} \qquad\qquad \text{R}\neg \frac{\Gamma, A \Rightarrow \bot}{\Gamma \Rightarrow \neg A}$$

are easily derivable using L$\to^*$, L$\bot$ and R$\to$.

Having said that, we are ready to define proof-theoretic consequence:

Definition 4. $\Gamma \vdash_{\mathcal{S}_{\mathbf{cCL}}} \Delta$ just in case, for each finite subset Δ' of Δ there is a finite subset Γ' of Γ such that $\Gamma' \Rightarrow \Delta'$ is provable in $\mathcal{S}_{\mathbf{cCL}}$.

Here we have slightly departed from the definitions that are usual in the literature. Those definitions state that Δ is a syntactic consequence of Γ just in case for *some* finite subset Δ' of Δ there is a finite subset Γ' of Γ such that the sequent $\Gamma' \Rightarrow \Delta'$ is provable. But a definition of this sort delivers a notion of consequence that is unsound with respect to **cCL** (it validates, e.g. the argument that goes from $\{p\}$ to $\{p, q\}$). In contrast, Definition 4 delivers a notion that is sound and complete:

Theorem 8. $\Gamma \models_{\mathbf{cCL}} \Delta$ *if and only if* $\Gamma \vdash_{\mathcal{S}_{\mathbf{cCL}}} \Delta$

Proof. We leave soundness as an exercise, and prove completeness. So, suppose $\Gamma \models_{\mathbf{cCL}} \Delta$. First, assume $\Delta = \varnothing$. For any finite subset Γ' of Γ, the sequent $\Gamma' \Rightarrow \varnothing$ is an instance of R$\varnothing$. So, a fortiori, $\Gamma' \Rightarrow \varnothing$ is derivable for *some* finite subset Γ' of Γ. Hence, $\Gamma \vdash_{\mathcal{S}_{\mathrm{CCL}}} \Delta$. Now assume $\Delta \neq \varnothing$. For any $\delta \in \Delta$, $\Gamma \models_{\mathbf{cCL}} \delta$ (by Fact 1), and thus $\Gamma \models_{\mathbf{CL}} \delta$ (by Fact 5), and thus $\Gamma \vdash_{\mathcal{S}_{\mathbf{CL}}} \delta$ (by completeness of $\mathcal{S}_{\mathbf{CL}}$), and finally $\Gamma \vdash_{\mathcal{S}_{\mathbf{cCL}}} \delta$ (because $\mathcal{S}_{\mathbf{CL}}$ is a subsystem of $\mathcal{S}_{\mathbf{cCL}}$). So, let Δ' be any finite subset of Δ, and suppose its elements are $A_1, ..., A_n$. By the above we know that there is a sequence $\mathcal{D}_1, ..., \mathcal{D}_n$ such that, for each A_i in Δ, $\mathcal{D}_i$ is a derivation in $\mathcal{S}_{\mathbf{cCL}}$ of the sequent $\Gamma_i \Rightarrow A_i$, where Γ_i is some finite subset of Γ. Hence, consider the following schematic proof tree:

$$\text{SM} \frac{\begin{matrix}\mathcal{D}_1 \\ \vdots \\ \Gamma_1 \Rightarrow A_1\end{matrix} \qquad \begin{matrix}\cdots \\ \vdots \\ \cdots\end{matrix} \qquad \begin{matrix}\mathcal{D}_n \\ \vdots \\ \Gamma_n \Rightarrow A_n\end{matrix}}{\Gamma_1, ..., \Gamma_n \Rightarrow A_1, ..., A_n}$$

Since each Γ_i is finite, also the union of all the Γ_is in finite. Hence, we have a derivation in $\mathcal{S}_{\mathbf{cCL}}$ of the sequent $\Gamma' \Rightarrow \Delta'$, for some finite subset Γ' of Γ. But our initial Δ' was arbitrary. Therefore, $\Gamma \vdash_{\mathcal{S}_{\mathbf{CCL}}} \Delta$. $\square$

The adequacy result for $\mathcal{S}_{\mathbf{cCL}}$, together with Facts 1 and 7 easily imply

Corollary 9. If a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathcal{S}_{\mathbf{cCL}}$, then it has a derivation where Gem-*at* is applied only on subformulas of formulas occurring in Δ.

But again, Gem-*at* is the only elimination rule of the calculus; so, it follows that if a sequent is derivable in $\mathcal{S}_{\mathbf{cCL}}$, then it has a derivation with the subformula property.

Lastly, given that Cut, RaW and Tr are validity preserving in **cCL**, we have:

Corollary 10. Rules Cut, RaW and Tr are admissible in $\mathcal{S}_{\mathbf{cCL}}$.

A word on this result. One important application of Cut-free sequent calculi is to enable purely syntactic proofs of consistency. Now, proofs of this sort typically rely on the fact that sequent $\varnothing \Rightarrow \varnothing$ is invalid. But the sequent in question is valid in **cCL**. So, one may perhaps think that, in a framework with conjunctive multiple conclusions, purely syntactic proofs of consistency are not available. But the impression is misleading. The consistency of $\mathcal{S}_{\mathbf{cCL}}$ can be proven by purely syntactic means. First, one notes that $\mathcal{S}_{\mathbf{cCL}}$ is consistent just in case sequent $\Rightarrow \bot$ is not derivable. Then, one shows that no derivation of $\Rightarrow \bot$ can exist. Intuitively, this is because $\Rightarrow \bot$ can only be obtained using Gem-*at* or SM in one the following ways:

$$\text{Gem-}at\,\frac{\begin{array}{c}\vdots\\ p \Rightarrow \bot\end{array} \qquad \begin{array}{c}\vdots\\ \neg p \Rightarrow \bot\end{array}}{\Rightarrow \bot} \qquad \text{SM}\,\frac{\begin{array}{c}\vdots\\ \Rightarrow \bot\end{array} \qquad \begin{array}{c}\vdots\\ \Rightarrow \bot\end{array}}{\Rightarrow \bot} \qquad \text{SM}\,\frac{\begin{array}{c}\vdots\\ \Rightarrow \bot\end{array} \qquad \overline{\Rightarrow}\,\text{R}\varnothing}{\Rightarrow \bot}$$

But in each case, at least one of the premises can only be obtained using Gem-*at* or SM again, which leads to an infinite regress. Thus, purely syntactic proofs of consistency can still be obtained in our framework (although in a slightly different way).

As the reader may expect, the way in which we defined calculus $\mathcal{S}_{\mathbf{cCL}}$ can serve as a heuristics for finding sound and complete sequent calculi for other logics with conjunctive multiple conclusions. More precisely, given one such logic, we first look for a sequent calculus for its single-conclusion fragment and then extended that calculus with rules R$\varnothing$ and SM. One easy example of this strategy is the following: let $\mathcal{S}_{\mathbf{IL}}$ be the calculus that results from $\mathcal{S}_{\mathbf{CL}}$ by removing Gem-*at*. $\mathcal{S}_{\mathbf{IL}}$ is sound and complete with respect to intuitionistic logic. Then, extending this calculus with rules R$\varnothing$ and SM will deliver a system $\mathcal{S}_{\mathbf{cIL}}$ which is sound and complete with respect to intuitionistic logic with conjunctive multiple conclusions (I let the reader fill in the details).

3 Multiple Conclusions in Nature

Arguably, one of the most important applications of logical systems is to describe and/or prescribe the way in which we deductively reason and/or ought to reason in natural language; in other words, to model our everyday reasoning. In this section I claim that the conjunctive approach to multiple conclusions is useful in this respect.

As hinted in Sect. 1, most authors take sides with tradition and claim that arguments in natural language have exactly one conclusion [1, 40, 41, 47]. Quoting Steinberger,

> It seems difficult to deny that multiple-conclusion systems constitute a departure from our ordinary forms of inference and argument. Arguments in 'real life' always lead to a unique conclusion. [44, p. 340].

Some have challenged this attitude, and argued that in natural language we sometimes find arguments with multiple conclusions [36, 43] or, at least, logical constants whose formalisation requires a multiple conclusion framework [14]. Both sides in the debate, however, share a key implicit assumption, namely, that multiple conclusions are to be read disjunctively.[11] I drop that assumption, and argue that, at least when conclusions are read conjunctively, there are arguments in natural language whose most natural and simple formalisation involves multiple conclusions.

[11] *A note to avoid confusion.* According to Steinberger [44], those who favour the so-called bilateralist reading of multiple-conclusion consequence (where $\Gamma \Rightarrow \Delta$ is read "It is incoherent to accept everything in Γ and deny everything in Δ") do not have a disjunctive reading of conclusions. But they do have such a reading in our usage of words, because they work with systems where validity *can* be paraphrased as per the disjunctive approach.

My starting observation is simple. In English, it makes perfect sense to say things as these:

(1) Such and such predictions follow from such and such hypothesis.
(2) What you said entails the following set of statements: $A_1, A_2, ..., A_n$.
(3) This theoretical standpoint has a series of undesirable consequences

It is clear that, were these fragments to be formalised using multiple conclusions, those conclusions should be read conjunctively. Hence, I next consider some objections against the idea that they can be plausibly formalised using multiple conclusions.

The most immediate objection is that the above fragments can be formalised just as well using a conjunction as the only conclusion; therefore, multiple conclusions are dispensable. For starters, it should be noted that, if this objection is convincing, then an analogous one applies to multiple conclusions under the disjunctive reading: they can be explained away by means of disjunctions. Setting this aside, there are several answers to the objection.

The first one is that, sometimes, to formalise using sets is more faithful to the speaker's intention than to formalise using a conjunction. Identity of conjunctions is sensitive to the order of their elements, while identity of sets is not. Besides, in many cases, the identity of an argument does not seem to be sensitive to the order in which the conclusions occur. For instance, let us precisify (3) as follows:

(3*) Hard determinism has a series of undesirable consequences, namely, the nonexistence of moral responsibility, the lack of individual autonomy, and a depressing fatalism.

and compare this with

(3**) Hard determinism has a series of undesirable consequences, namely, the nonexistence of moral responsibility, a depressing fatalism, and the lack of individual autonomy.

Intuitively, these passages express not just two arguments that are equivalent,[12] but one and the same argument. If we want to honour this intuition, then we should formalise them using a set of conclusions. While non-decisive, the point should not be very controversial: it is for similar considerations that, when we face an argument with prima facie many premises, we formalise it using a set of premises rather than a conjunction thereof.

Secondly, it is true that, in our setting (viz. **cCL**), conjunction and multiple conclusions are equivalent in the following sense:

$$\Gamma \mathrel{\text{-}\!\!\ni} A, B, \Delta \quad \text{iff} \quad \Gamma \mathrel{\text{-}\!\!\ni} A \wedge B, \Delta$$

But this happens because **cCL** is classical. In many non-classical systems, the equivalence will break. For instance, it may break in systems where conjunction is somehow non-standard, but validity is defined as preservation of designated value. One case in point is logic **cPWK** from *Example 1*. Of course, we have $\{p \wedge q\} \mathrel{\text{-}\!\!\ni} \{p \wedge q\}$ in this logic. But any valuation v such that $v(p) = i$ and $v(q) = 0$ shows that $\{p \wedge q\} \not\mathrel{\text{-}\!\!\ni} \{p, q\}$. The point is that the behaviour of conjunctive multiple conclusions cannot be taken to be *in general* reducible to mere conjunctions.

Thirdly, infinite collections of conclusions are not reducible to ordinary conjunctions. And we sometimes use arguments with prima facie infinite conclusions, as when we say

[12] By saying that two arguments are equivalent we mean, roughly, that they stand or fall together: one of them is valid (sound) just in case the other is.

(4) The Peano axioms entail the truths of arithmetic
(5) This theory of truth entails all sentences of the language.

Perhaps, the objector could insist by appealing to infinite conjunctions. But the formulation of infinite conjunctions requires set-theoretical vocabulary anyway; hence, it does not contribute to economise on expressive resources. For instance, let us try to formalise (5). Let T be the relevant theory, and $\mathcal{L}$ our language. If we allow ourselves of multiple conclusions, we can write

$$\mathrm{T} \models \{A : A \in \mathcal{L}\}$$

If we use single conclusions, we have to write

$$\mathrm{T} \models \bigwedge\{A : A \in \mathcal{L}\}$$

In both cases we will need the machinery of set-theory. Thus, why don't we allow multiple conclusions from the outset?[13]

Fourth and last. If the above reasons do not convince the reader, then they may also lack good reasons to admit multiple *premises*. Now that our reading of multiple premises and conclusions is similar, reasons that justify the former tend also to justify the latter, and vice versa. Also arguments with prima facie many premises could be formalised with a conjunction as the only premise. If the reader is consequent, they should opt for such a formalisation. But I doubt that this would be pleasing for them.

The second possible objection to my proposal runs as follows. If an argument in natural language appears to have multiple conclusions, then it is just an abbreviation of multiple different arguments, one for each of the apparent conclusions. In particular, when a speaker asserts that Γ entails Δ, and Δ is understood conjunctively, what the speaker means is the universal statement "Γ entails C for each C in Δ". Accordingly, a multiple-conclusion sequent $\Gamma \Rightarrow \Delta$ does not express an argument, but the collection of single-conclusion sequents $\{\Gamma \Rightarrow C : C \in \Delta\}$, each of which expresses an argument. This picture builds on some suggestions by Cintula and Paoli [7].

This objection is more plausible, but its significance is rather limited. To begin with, if it is convincing, then again there is an analogous objection that arguably affects multiple conclusions under a disjunctive reading; indeed, Cintula and Paoli's original argument is meant to give an eliminativist account of those. Very roughly, the assertion that Γ entails Δ, where Δ is read disjunctively, can be understood as expressing the universal statement according to which each C in Δ follows from Γ together with the negations of the remaining things in Δ. Regardless of this, the objection is again contestable.

First, it is true that, in our setting, the following 'reduction' holds:

$$\Gamma \dashv\!\varepsilon\, \Delta \quad \text{iff} \quad \Gamma \dashv\!\varepsilon\, \delta \text{ for each } \delta \in \Delta$$

But again, whether such result is available will depend on the details of the framework. For instance, plausible reformulations of this reduction may fail in systems where the relata of logical consequence are not sets, but some other forms of sentence aggregation, such as multisets or sequences. One case in point is logic **mŁ** from *Example 2*. For this logic, the corresponding reduction would arguably look as follows:

[13]My last two answers develop some considerations made by Shoesmith and Smiley in their defence of disjunctive multiple conclusions [43, p. 2].

$$\Gamma_m \multimap \Delta_m \quad \text{iff} \quad \Gamma_m \multimap [\delta] \text{ for each } \delta \in |\Delta|$$

where $|\Delta_m|$ is the set of all formulas that appear at least once in Δ_m. But this fails in **mŁ**, as witnessed by the fact that $[p] \multimap [p]$ but $[p] \not\multimap [p,p]$ (for the counterexample, take any v such that $v(p) = 0.9$). The point is, now, that conjunctive multiple conclusions cannot be in general taken to abbreviate classes of single-conclusion arguments.

But even for those systems where the reduction *can* be done, it is hasty to conclude that conjunctive multiple conclusions are just dispensable. First and foremost, when a speaker utters, for instance, "Such and such predictions follow from such and such hypothesis", or even "This theory entails that one", they do not make explicit use of any quantifiers—and they are not aware of making any implicit use of quantifiers either. Hence, the most literal and simple way to model their claim is by means of two collections of statements, say Γ and Δ, of which they are saying that $\Gamma \multimap \Delta$. If we relinquish from multiple conclusions, and model the speaker's utterance by means of a universal quantification over a certain class of single-conclusion arguments, we make a non-literal (or at least, a *less* literal) reading of what the speaker have said. Of course, non-literal readings can have their advantages sometimes. But, all other things being equal, the more literal reading is to be preferred.

For another thing, the reduction makes essential use of certain metatheoretical expressions such as the indicative biconditional and the universal quantifier. Since these expressions pertain to the metalanguage, they are preformal in that their usage is not regimented. Multiple conclusions allow us to dispense with these expressions and thus give a more rigorous account of the logic regulating fragments like (1) to (5)—and the interactions between these fragments. This gain in rigour arguably brings about some epistemic gains. For instance, it enables a proof-theoretic decision procedure for the validity of these fragments, and it allows a better analysis of the structural properties that these pieces of reasoning display. These epistemic gains, I take it, provide further reasons to formalise fragments (1) to (5) by means of multiple conclusions.

I conclude that, at least when conclusions are read conjunctively, we have good reasons to admit that there are multiple conclusions in natural language. This makes conjunctive multiple conclusions useful in the enterprise of modelling our everyday reasoning.

To be clear, I have not argued that in natural language there are no arguments with *disjunctive* multiple conclusions. There may well be. For instance, one could try to mimic the reasoning from this section by appealing to examples like "These facts entail the following set of possible scenarios", "What you said leaves the following possibilities open", and so on. But following this line of thought escapes the subject of this paper.

4 On Reflexivity and Transitivity

One of the major differences between the conjunctive and the disjunctive approaches has to do with their policies towards the properties of reflexivity and transitivity: the disjunctive approach induces failures of these properties, while the conjunctive one does not. In this section we will see that this difference is tightly related to a number of aspects in which the conjunctive approach seems to be more satisfactory than the disjunctive one. In Sect. 4.1, I claim that it allows a more natural generalisation of logical equivalence from sentences to collections thereof. In Sects. 4.2 to 4.4, I claim that it gets along better with some of our best accounts of logical consequence, namely, the ones based on preservation, on content-inclusion, and on existence of a proof.

4.1 Generalising Logical Equivalence

The relation of logical equivalence is typically assumed to hold between sentences. But when we work in a multiple-conclusion framework, the relata of logical consequence are *collections* of sentences. Hence, it makes sense to ask what logical equivalence looks like when we generalise it to collections as well.

In the single-conclusion framework, two sentences A and B are said to be *logically equivalent* just in case they mutually entail each other, in symbols $A \leftrightarrows B$. So, let us extend this stipulation to the multiple-conclusion framework: two sets Γ and Δ are logically equivalent just in case $\Gamma \leftrightarrows \Delta$. Under this modest assumption, the disjunctive approach has some consequences that strike me as highly counter-intuitive:

> *Case 1:* Set $\{A, B\}$ is logically equivalent to $\{A \vee B\}$, but also to $\{A \wedge B\}$! (In other words, $\{A, B\} \dashv\vDash_{\mathbf{dCL}} \{A \vee B\}$ and $\{A, B\} \dashv\vDash_{\mathbf{dCL}} \{A \wedge B\}$.) How can the same set be logically equivalent to sentences that have different truth conditions (or, more precisely, to sets that do not have the same models)?[14]
>
> *Case 2:* The empty set, $\varnothing$, is logically equivalent to $\{A, \neg A\}$, but not to itself! (In other words, $\varnothing \dashv\vDash_{\mathbf{dCL}} \{A, \neg A\}$ but $\varnothing \not\dashv\not\vDash_{\mathbf{dCL}} \varnothing$.) If something entails an inconsistent set, and classical logic is explosive, why does it not entail everything and, in particular, why does it not entail itself?

Of course, these questions have clear technical answers. As for *Case 1*, sets $\{A, B\}$ and $\{A \vee B\}$ are logically equivalent because, although they do not have the same models, every model of the latter assigns value 1 to at least one sentence in the former and vice versa; this is all we need to have validity in **dCL**. As for *Case 2*, $\varnothing$ is not equivalent to itself because, although every model assigns 1 to each of its sentences, no model assigns 1 to at least one of them; this precludes **dCL** validity. But notice that these answers *assume* a disjunctive reading of multiple conclusions; hence, they just beg the question in favour of this reading. Needless to say, none of the counter-intuitive examples affects the conjunctive approach: in **cCL**, $\varnothing \leftrightarrows \varnothing$ and $\{A, B\} \not\leftrightarrows \{A \vee B\}$.

Examples 1 and 2 have quite a bit in common. Indeed, they can both be explained by a single fact about **dCL**, namely, that the system invalidates the following principle:

$$\text{Ax-1}\ \frac{\Gamma \leftrightarrows \Sigma \qquad \Delta \leftrightarrows \Sigma}{\Gamma \leftrightarrows \Delta}$$

The informal reading is: 'If two things are logically equivalent to a third, they are logically equivalent to one another'. Label 'Ax-1' honours the clear similarity with the first axiom of Euclid. In *Case 1*, Γ is $\{A \vee B\}$, Δ is $\{A \wedge B\}$, and Σ is $\{A, B\}$. In *Case 2*, Γ and Δ are both $\varnothing$, and Σ is $\{A, \neg A\}$. Both examples show that Ax-1 does not hold in **dCL**.

I submit that Ax-1 throws light upon an important aspect in which **cCL** is more akin to **CL** than **dCL** is. In the single-conclusion framework, logical equivalence is defined for *sentences*, while in the multiple-conclusion framework, it is defined for *sets* of sentences. Thus, in the cases of both **CL** and **cCL**, logical equivalence is a genuine *equivalence* relation, that is, it is reflexive, transitive and symmetric. This is not so, however, in the case of **dCL**; the explanation is that Ax-1 is a necessary condition for a relation $\leftrightarrows$ to be an equivalence relation. The upshot is that there is

[14] It is noteworthy that, as a special case of this example, we have that $\{A, \neg A\}$ is logically equivalent both to $\{A \wedge \neg A\}$ and to $\{A \vee \neg A\}$, that is, both to a set without models and to a set without counter-models.

a sense in which the conjunctive reading of multiple conclusions is more faithful to the spirit of single-conclusion consequence than the disjunctive reading is.

At this point, the sympathiser of the disjunctive approach could object. The argument would go more or less like this. When we focus on sentences, the relations of 'having the same models' and 'entailing each other' are coextensive: A has the same models as B just in case $A \leftfishtail\!\rightfishtail B$. As a consequence, logical equivalence can be defined in terms of any of these relations, and the results will be the same. When we focus on sets, however, things are different. The relations mentioned now come apart: there are pairs of sets that have the same models, but do not entail each other (e.g. $\varnothing$ and $\varnothing$) and there are also pairs of sets that entail each other in spite of not having the same models (e.g. $\{A, B\}$ and $\{A \vee B\}$). Thus, logical equivalence cannot adjust to both relations at once. We must choose. And we have just seen that defining logical equivalence in terms of mutual entailment runs into troubles. Hence, we should define it in terms of sameness of models. This keeps the counter-intuitive examples at bay: now, $\{A, B\}$ and $\{A \vee B\}$ are not logically equivalent, but $\varnothing$ and $\varnothing$ are.

The objection is relevant, since it provides a coherent picture where multiple conclusions are read disjunctively and, yet, undesirable consequences are avoided. However, I do not think that the position depicted is ultimately satisfactory; the reason is that it incurs in theoretical costs that can be avoided. Under the conjunctive approach, the relations of 'having the same models' and 'entailing each other' are coextensive *both* for sentences and for sets. Also, the approach is not prone to the counter-intuitive consequences that threaten the disjunctive reading. But then, why pay the cost that the disjunctive reading supposes? Why break the symmetry between the notions of logical equivalence for sentences and for sets, if this is not indispensable to save the data? I dot not see good answers in favour of the disjunctive approach.

4.2 Validity as Preservation

One of the (if not *the*) most established analysis of logical consequence in the literature tells us that an argument is valid just in case it preserves certain property from premises to conclusion(s). The property that is assumed to be preserved varies across logical systems and philosophical views; it can be, e.g. truth or satisfaction, assertability, constructive provability or even evidence. For concreteness, in what follows I assume the relevant property to be truth. Not much hinges on this, however. My argument is quite general, and it aims to apply to most (if not all) explanations of logical consequence as preservation.

The starting point I would like to make is that the idea of 'truth preservation' suggests that there is a pair of entities such that the first 'transfers' its truth to the second or, alternatively, the second 'inherits' the truth of the first. Of course, this is metaphoric. But that should not be a problem, since the very talk of 'truth preservation' is metaphoric as well. I am just positing further informal conditions that should intuitively hold for the metaphor to make sense. Our guiding question, then, will be the following: *What are the entities between which truth is preserved in valid arguments?* Let $\rightfishtail$ be any logical consequence relation. We will consider three options: that $\rightfishtail$ stands for consequence in **CL**, in **cCL**, and in **dCL**.

If $\rightfishtail$ stands for consequence in **CL**, things seem quite straightforward. First, we stipulate that a set of sentences is *true* just in case all of its sentences are true. Then, we note that, under the usual reading of the semantics for classical logic (where 1 stands for 'true' and 0 for 'false'), the following obtains:

(Set-Fmla) $\Gamma \dashv C$ just in case, whenever Γ is true, C is true

Thus, we are justified in giving the next answer to our question: in valid arguments, truth is preserved *between the set of premises and the conclusion.* Label 'Set-Fmla' stands for 'Set-Formula truth preservation'.

If $\dashv$ stands for consequence in **cCL**, no additional complications seem to arise. We stick to the above stipulation, and note that the following obtains:

(Set-Set) $\Gamma \dashv \Delta$ just in case, whenever Γ is true, Δ is true

Thus, we are justified in giving the next answer to our question: in valid arguments, truth is preserved *between the set of premises and the set of conclusions.* The meaning of label 'Set-Set' is the one to be expected, namely 'Set-Set truth preservation'.

When $\dashv$ stands for consequence in **dCL**, however, things are way less obvious. To begin with, the idea that truth is preserved between sets seems bound to failure. The reason is that there seems to be no reasonable stipulation of what it means that a set of sentences is 'true' such that Set-Set obtains. Suppose that we stick to the stipulation we entertained so far: a set of sentences is true just in case all of its sentences are true. Then, Set-Set is false because $\{p\} \models_{\mathbf{dCL}} \{p, q\}$ but it is not the case that whenever al the sentences in $\{p\}$ are true, all the sentences in $\{p, q\}$ are true. Suppose, alternatively, that a set of sentences is true just in case at least one of its sentences is true. Then, Set-Set is false because $\varnothing \not\models_{\mathbf{dCL}} \{p \wedge \neg p\}$ even though, whenever some sentence in $\varnothing$ is true, some sentence in $\{p \wedge \neg p\}$ is true (namely, never). Maybe, one could try some more intricate stipulations; for instance, one could say something like this: "A set is true *in the premises of an argument* just in case all of its sentences are true, and a set is true *in the conclusions of an argument* just in case at least one of its sentences is true'. But this, I take it, makes no philosophical sense at all. In general, I see no stipulation that satisfies Set-Set without being horribly ambiguous or context-dependent.

Let us discard, then, the idea that in valid arguments truth is preserved between sets. Another option that could come to mind is that in valid arguments truth is preserved between the set of premises and *some sentence* in the set of conclusions. The problem with this proposal is that, for it to be justified, the following should obtain:

(Set-Set*) $\Gamma \dashv \Delta$ just in case there is a C in Δ such that, whenever Γ is true, C is true

Yet, this fact fails spectacularly in **dCL**; for instance, we have that $\varnothing \models_{\mathbf{dCL}} \{p, \neg p\}$ but it is neither the case that p is always true, nor that $\neg p$ is always true. Indeed, Set-Set* fails for all the reasonable logical systems I know of.

The last, and most plausible answer that I could come up with runs as follows: in valid arguments, truth is preserved between the appropriate *sentence-translations* of the set of premises and the set of conclusions. Let us define the *premise-sentence-translation* of a set Σ, denoted by $\tau_p(\Sigma)$, as $\bigwedge(\Sigma \cup \{\top\})$. Let us also define the *conclusion-sentence-translation* of a set Σ, denoted by $\tau_c(\Sigma)$, as $\bigvee(\Sigma \cup \{\bot\})$. Then, the justifying fact for this position would be the following:

(Set-Set**) $\Gamma \dashv \Delta$ just in case whenever $\tau_p(\Gamma)$ is true, $\tau_c(\Delta)$ is true.

Of course, this obtains for **dCL** as well as many non-classical systems. The problem I see with this proposal is that, if we take seriously the idea that the relata of logical consequence are sets and, moreover, we assume that logical consequence is to be explained in terms of truth preservation, then it seems odd, at the very least, that the relation of truth preservation does not have sets anywhere among its relata. In other words, under this proposal, sets can be arguably understood as mere abbreviations: the genuine relata of logical consequence are not the sets anymore, but the sentences they abbreviate. But if this is the case, then the sympathiser of disjunctive multiple conclusions has lost multiple conclusions (as well as multiple premises) along the way.[15]

I conclude that the conjunctive reading allows a simpler and more reasonable specification of what are the entities between which truth is preserved in valid arguments. Arguably, the *reason* for this has to do with the properties of reflexivity and transitivity. The very notion of preservation seems to support these properties: any object $\mathfrak{a}$ preserves its own features, and for any objects $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$, if $\mathfrak{b}$ preservers a certain feature P of $\mathfrak{a}$, and $\mathfrak{c}$ preserves feature P of $\mathfrak{b}$, then $\mathfrak{c}$ preserves feature P of $\mathfrak{a}$. Since the disjunctive approach violates reflexivity and transitivity, it cannot account for this plausible fact about the notion of preservation—but the conjunctive approach can.

Notice that I nowhere appealed to specificities of the notion of *truth*. Hence, the above line of reasoning applies just as well to any other property that one may think that is preserved in valid arguments.

4.3 Validity as Content Inclusion

Another venerable explanation of logical consequence maintains that an argument is valid just in case the content of the conclusion is included in the content of its premises. The account can be traced back to Aristotle and passes through Sextus Empiricus and most prominently Kant.[16] In the early 20th century, some logicians such as Carnap [5] and Popper [31] thought that classical logic can be characterised in terms of content inclusion. As the discussion proceeded, however, a broad consensus was reached that this is not the case. The reason, in a nutshell, is that classical logic overgenerates valid arguments. In particular, it validates some arguments that allow the occurrence in the conclusion of a *subject matter* that was not present in the premises, and this is deemed incompatible with the idea that the content of the premises includes that of the conclusion. The point was famously made by Parry [30]:

> If a system contains the assertion that two points determine a straight line, does the theorem necessarily follow that either two points determine a straight line or the moon is made of green cheese? No, for the system may contain no terms from which 'moon,' etc., can be defined.

This is why, in the last decades, the literature on logics of content inclusion focuses mostly on non-classical systems. Accordingly, I shall not restrict my attention to classical logic in this subsection. Rather, I will make some general considerations that are relevant for (the multiple-conclusion counterparts of) many systems.

Logics of content inclusion are usually developed in a propositional language and a single conclusion framework. It is standard to impose on them a syntactic restriction that Parry called the *proscriptive principle:*

(PP) $\Gamma \dashv\!\!\exists\, A$ only if $Var(A) \subseteq Var(\Gamma)$

where $Var(\Gamma)$ is the set of propositional variables occurring in Γ, and likewise for A. The idea is that PP warrants that no novel subject matter appears in the conclusion of a valid argument,

[15] One may perhaps understand the argument of this subsection as an elaboration of the complaint made by Gareth Evans (quoted in [43, 44]). One reading of Evans complaint is that, once disjunctive multiple conclusions are properly understood, they are nothing more than single-conclusions in disguise.

[16] See Ferguson [16] for a nice summary.

and thus avoids the kind of problems that affected classical logic. A question arising is how we should extend PP to a consequence relation that intends to capture content inclusion in a multiple-conclusion framework.

I submit that, if the relata of logical consequence are assumed to be sets, then a natural answer to this question is:

(PP*) $\Gamma \dashv\!\!\!\vdash \Delta$ only if $Var(\Delta) \subseteq Var(\Gamma)$

This warrants that no conclusion introduces a subject matter that is absent in the premises, and thus allows us to read $\Gamma \dashv\!\!\!\vdash \Delta$ as saying that the content of Δ is included in that of Γ. Under this generalisation of PP, however, the disjunctive reading of conclusions runs into troubles. Arguably, the relation of content-inclusion is like that of preservation in that it supports reflexivity and transitivity: any content $\mathfrak{a}$ is included in itself,[17] and for any contents $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$, if $\mathfrak{a}$ is included in $\mathfrak{b}$ and $\mathfrak{b}$ is included in $\mathfrak{c}$, then $\mathfrak{a}$ is included in $\mathfrak{c}$. But the disjunctive approach will induce counterexamples to this. First, for any plausible choice of $\dashv\!\!\!\vdash$ we will have $\top \not\dashv\!\!\!\vdash \bot$, which, by the stipulation that $\bigvee(\varnothing)$ amounts to $\bot$ and $\bigwedge(\varnothing)$ amounts to $\top$, implies $\varnothing \not\dashv\!\!\!\vdash \varnothing$. Second, for many plausible choices of $\dashv\!\!\!\vdash$ we will have $\{p \vee q\} \dashv\!\!\!\vdash \{p, q\}$ and $\{p, q\} \dashv\!\!\!\vdash \{p \wedge q\}$ (these validities do not violate PP*); but of course we will also have $\{p \vee q\} \not\dashv\!\!\!\vdash \{p \wedge q\}$. For the conjunctive reading of conclusions, on the other hand, similar issues do not arise. For any reasonable choice of $\dashv\!\!\!\vdash$ we will have $\varnothing \dashv\!\!\!\vdash \varnothing$, avoiding the first problem, and $\{p \vee q\} \not\dashv\!\!\!\vdash \{p, q\}$, avoiding the second one. Notice also that since now $\varnothing$ behaves as $\top$ everywhere, we will also have that $\Gamma \dashv\!\!\!\vdash \varnothing$ for any Γ; this is coherent with a plausible interpretation of $\varnothing$, which is that it lacks any content.

One might perhaps object that, in a way, PP* begs the question in favour of the conjunctive approach. If the relation $\dashv\!\!\!\vdash$ is assumed to induce the disjunctive approach, then a different generalisation of the proscriptive principle should be imposed on it. For instance, Ciuni et. al. [8] work with multiple-conclusion logics of content inclusion that induce the disjunctive approach and satisfy the restriction

(PP**) $\Gamma \dashv\!\!\!\vdash \Delta$ only if $Var(\Delta') \subseteq Var(\Gamma)$ for some $\Delta' \subseteq \Delta$

This warrants that there is a subset of the conclusions that does not introduce a subject matter that is absent in the premises; thus, it allows us to read $\Gamma \dashv\!\!\!\vdash \Delta$ as saying that there is a subset of Δ whose content is included in that of Γ. However, I think that PP** is not a sufficiently demanding generalisation of PP. It allows the reappearance, at the structural level, of the kind of phenomena that motivated the abandonment of classical logic in the first place. For many choices of $\dashv\!\!\!\vdash$ (for instance, the systems studied by Ciuni et al.) we will have validities such as $\{p\} \dashv\!\!\!\vdash \{p, q\}$. And I do not see why this is more innocuous than $\{p\} \dashv\!\!\!\vdash \{p \vee q\}$. If, following Parry's example, "Either two points determine a straight line or the moon is made of green cheese" does not follow from "Two points determine a straight line" because the language of geometry might not even have the means to talk about the moon, cheese and so on, then, by parity of reasoning, a set comprising the statements "Two points determine a straight line" and "The moon is made of green cheese" should not follow from the former of these two statements alone, for the very same reasons. In a way, PP** allows us to extend logics of content inclusion to a multiple-conclusion framework, but at the cost of giving up content inclusion.

[17] Of course, I am working with a *non-strict* notion of content inclusion here. Strict notions of content inclusion would justify non-reflexivity, and even require irreflexivity.

4.4 Validity as Existence of a Proof

The philosophical standpoint known as *logical inferentialism* maintains that the meaning of logical constants is determined by the rules that govern their behaviour. These rules are assumed to be sound without further justification. Then, an argument is said to be valid if and only if there is a proof that goes from the premises to the conclusion(s) and only uses sound rules of inference.[18]

Steinberger [44] already provided a battery of reasons to think that typical multiple-conclusion systems (viz. systems inducing a disjunctive reading of conclusions) are not compatible with logical inferentialism. Here, however, I will rehearse an independent argument put forward by Fiore [17], which bears on the properties of reflexivity and transitivity. While there is no space to present the argument in full here, I offer a brief sketch of how it goes.

We focus on the metalinguistic comma that is used to aggregate premises and/or conclusions. In a nutshell, we present an analogy between the comma as it behaves in typical multiple-conclusion systems and Prior's infamous connective TONK. As is well-known, Prior [33] presents TONK as an alleged counterexample to logical inferentialism; the idea is that the constant is meaningless or somehow illegitimate, and thus it is not the case that any set of rules determines a meaningful or legitimate constant. The analogy we present shows that TONK and the comma have much in common; indeed, the latter can be understood as nothing more a structural incarnation of the former. Arguably, then, whatever philosophical story one has to tell about TONK, there are good reasons to tell a similar story about the comma, and viceversa.

The first and most noticeable similarity between TONK and the comma stems from the rules governing these expressions. TONK can be characterised by means of the sequent rules[19]

$$\textsc{tonk-L}\,\frac{A \Rightarrow B}{C \,\textsc{tonk}\, A \Rightarrow B} \qquad\qquad \textsc{tonk-R}\,\frac{A \Rightarrow B}{A \Rightarrow B \,\textsc{tonk}\, C}$$

The comma, in turn, can be characterised by means of the rules of left and right weakening, which for the sake of the analogy we relabel as follows:

$$\textsc{Set-L}\,\frac{\Gamma \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta} \qquad\qquad \textsc{Set-R}\,\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \Sigma}$$

It is apparent that these two pairs of rules are formally identical: each rule for the comma results by taking the corresponding rule for TONK and uniformly replacing arbitrary formulas with sets and TONK with the comma. Both expressions are introduced as conjunctions on the left-hand side of the turnstile, and as disjunctions on the right-hand side—we could say that they are ambiguous in a similar way.

A second important similarity concerns the pathological character that both TONK and the comma display. It is well-known that TONK does not get along with systems that are reflexive and transitive. Take a calculus $\mathcal{S}$ whose only rules are Id and Tr, and extend it with TONK-L and TONK-R. Then you get triviality, as witnessed by the derivation

$$\frac{\dfrac{\dfrac{}{A \Rightarrow A}}{A \Rightarrow A \,\textsc{tonk}\, B} \qquad \dfrac{\dfrac{}{B \Rightarrow B}}{A \,\textsc{tonk}\, B \Rightarrow B}}{A \Rightarrow B}$$

[18] I refer the reader to Murzi and Steinberger [27] for a gentle overview of inferentialism in general, and logical inferentialism in particular.

[19] These are the usual sequent rules for TONK (see e.g. [18, 37]) with the only difference that we restrict them to a single-conclusion and single-premise framework.

Now, the comma does not get along with reflexivity and transitivity either. Take system $\mathcal{S}$ again, and now extend it with Set-L and Set-R. Then you get almost triviality: letting $A \in \Gamma$ and $B \in \Delta$, we have the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}}{\Gamma \Rightarrow A}}{\Gamma \Rightarrow \Gamma, \Delta} \qquad \dfrac{\dfrac{\dfrac{}{B \Rightarrow B}}{B \Rightarrow \Delta}}{\Gamma, \Delta \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

That is, we prove $\Gamma \Rightarrow \Delta$ for any non-empty Γ and Δ. Thus, if being incompatible with reflexivity and transitivity is symptom of decease, then both TONK and the comma should receive treatment.

The last parallel we will highlight here is that, indeed, TONK and the comma have been treated likewise in the literature. A few attempts have been made to design logical systems where TONK is admissible without triviality. Cook, for instance [11], defined a non-transitive but reflexive system where the rules of TONK can be conservatively added. Fjellstad, however [18], convincingly argued that a system for TONK should be both non-transitive and non-reflexive; the main reason is that, in a sequent calculus containing an axiom of reflexivity, the rules of TONK fail to uniquely define a connective—which undermines the idea that the calculus admits the addition of *the* connective TONK, as opposed to a family of connectives.[20] Now, of course, reflexivity and transitivity are the key structural properties that fail in typical multiple-conclusion systems. Then, we could say that, since Gentzen, our sequent calculi avoid triviality by means of the same kind of trick that we do when we want to get away with TONK.

As we announced, the upshot of the analogy is that TONK and the comma are beasts of the same blood, and indeed, the latter can be seen as a structural incarnation of the former. The philosophical moral is that, whatever story we may have to tell about TONK, we should arguably tell a similar story about the comma, and viceversa. In particular: some inferentialist follow the trace of Belnap [2] and think that TONK is unacceptable only *relative* to certain background assumptions about the notion of logical consequence.[21] Those who follow this path *may* reject TONK and welcome multiple conclusions at the same time, as long as they claim that our notion of consequence is transitive and reflexive for formulas but not for sets. Many other inferentialist, however, follow the traces of Prawitz [32] and Dummett [15], and think that TONK is unacceptable in an inherent or *absolute* sense—the reason being that its rules are not in harmony.[22] Those who follow this path will have a much harder time justifying why the comma of typical multiple-conclusion systems should not also be regarded as unacceptable. Absent some such justification, they seem forced to part ways with typical multiple conclusions.

It goes without saying that conjunctive multiple conclusions are not subject to the kind of analogy we discussed, for they are governed by entirely different patterns of inference.

4.5 Takeaway

When we talk about the reflexivity and transitivity of logical consequence, we usually have in mind properties such as Id or Cut. In this section we have seen that proper reflexivity and transitivity can be of philosophical significance; indeed, they seem to be tightly related to some of our most entrenched ways of thinking about logical consequence.

[20] Roughly, a connective is uniquely defined in a calculus just in case it is intersubstitutable in inference without loss of validity with any other connective that has formally identical rules. See Belnap [2] for the precise definition.

[21] See, for instance, Cook [11], Ripley [37] and Dicher [13].

[22] See, for instance, Read [35], Tennant [48] and Francez [19].

5 Closing Remarks

In this paper, I explored some technical and philosophical aspects of an approach to multiple conclusions that has applications in abstract algebraic logic, has recently been shown to be useful in the study and development of certain non-classical systems (namely, the non-contractive ones), and yet, has gone largely unnoticed by the philosophical community. I defined and analysed a presentation of multiple-conclusion classical logic where conclusions are read conjunctively. I argued that we can find arguments with conjunctive multiple conclusions in natural language. Lastly, I claimed that the fact that the disjunctive and the conjunctive approaches have different policies towards reflexivity and transitivity has philosophical consequences; in particular, it is related to various aspects in which the conjunctive reading seems to behave in a more satisfactory way than the disjunctive one. I hope that the previous pages awaken the reader's curiosity about the structure, informal reading and explanation of our claims of logical consequence. After all, following Tarski [46], "In considerations of a general theoretical nature, the proper concept of consequence must be placed in the foreground".

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