Meta-classical Non-classical Logics

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February 13, 2023

Abstract

Recently, it has been proposed to understand a *logic* as containing not only a validity canon for inferences, but also a validity canon for metainferences of any finite level. Then, it has been shown that it is possible to construct infinite hierarchies of 'increasingly classical' logics—that is, logics that are classical at the level of inferences and of increasingly higher metainferences—all of which admit a transparent truth predicate. In this paper, we extend this line of investigation by taking a somehow different route. We explore logics that are *different* from classical logic at the level of inferences, but recover some important aspects of classical logic at every metainferential level. We dub such systems *meta-classical non-classical logics*. We argue that the systems presented deserve to be regarded as logics in their own right and, moreover, are potentially useful for the non-classical logician.

Keywords: *non-classical logics; metainferences; classical logic; mixed logics; paradoxes*

1 Introduction

Logical theories are explanations of what follows from what, that is, the relation of *logical consequence*. As is well known, classical logic has been gradually accepted as the best such explanation. Its predictive success, metatheoretical virtues, and multiple interrelations with set theory, arithmetic, and computer science are just some of the factors that seem to justify this stance. However, it is also well known that there are many alternative logics, which differ in the principles they declare valid. The elaboration of such non-classical logics is not only a theoretical exercise. There are multiple aspects of our inferential practices that seem to motivate them: vagueness, contingent futures, the quantum world, and semantic and set-theoretic paradoxes, just to mention some. Arguably, these elements provide good practical reasons for the development and study of non-classical logics.

The traditional conception of logical consequence takes this relation to go from sets of formulas to single formulas. In the last decades, however, several generalizations of this conception have been advanced.[1] In this paper, we focus on one particular generalization, which concerns the study of so-called *metainferences*. Intuitively, a metainference of level 1 (or metainference *simpliciter*) is as an inference between inferences. Then, a metainference of level 2 is an inference between metainferences of level 1, and so on for any $n > 2$. We focus on

[1] Thus, for instance, nowadays we have consequence relations that allow sets of formulas in their codomain [cf. 51], or allow collections that are not sets but perhaps multisets or sequences [cf. 41], or allow collections of things that are not necessarily formulas [cf. 13].

the generalization of logical consequence according to which this relation can take (not only collections of formulas, but also) collections of metainferences of arbitrary levels as its *relata*.

There is a sense in which the study of metainferences could be traced back to Gentzen's [34] pioneer works on sequent calculi.[2] However, the more recent interest in metainferences emerged within studies in truth, vagueness, and other paradoxical phenomena. First, they were used as a technical tool to characterize logics **ST**, **TS** (see below) and theories based upon them [e.g. 18, 45, 33]. As the debate progressed, though, metainferences started to attract philosophical attention. Among other things, they have been used to argue for or against various criteria of identity between logics [50, 42, 8], to show relevant similarities between some *prima facie* very different logical systems [24, 6, 17], to raise new insights about the notion of paraconsistency [5, 22], and to design refined versions of the collapse argument against logical pluralism [9].

One interesting application of metainferences has to do with the formulation of infinite hierarchies of 'increasingly classical' logical systems. In [40, 4, 7], the authors propose to understand a *logic* as including not only a validity canon for inferences, but also a validity canon for metainferences of any finite level. Then, they show how to define, for each level n, a logic that coincides with classical logic in inferences and metainferences up to level n, but differs with classical logic from that level upwards; notably, each of the logics in question can non-trivially accommodate a naive truth predicate. In this paper, we extend this line of investigation by taking a somehow different route. We define and explore various logics that are *different* from classical logic at the level of inferences, but recover some important aspects of classical logic at every metainferential level. We shall call such systems *meta-classical non-classical logics*. The systems that we present are based on the well-known validity notions for inferences **LP**, **K3** and **S3**. Some of our systems recover classical validities at all metainferential levels. Others recover some interesting proper subset of the classical validities. And yet others do not recover the metainferences that classical logic declares valid, but the ones that classical logic declares antivalid. We provide informal readings of the systems we present. We give an argument of why these systems should be considered *logics* in their own right. Lastly, we suggest that non-classical logicians might benefit from the systems we present here; mainly, our argument revolves around the well-known objection that non-classical logicians use classical logic in their metatheory, and thus incur in some kind of hypocrisy. We argue that our systems provide the non-classical logician with a novel and interesting kind of recapture result, which helps her to overcome this objection.

The structure of the paper is as follows. In Sections 2 and 3 we present the indispensable technical preliminaries. In Section 4 we make our technical exploration of what we called meta-classical non-classical logics. In Section 5 we address the more conceptual issues, such as the informal reading of our systems and their value for the non-classical logician.

2 Stage Setting

Let $\mathcal{L}$ be a propositional language, identical to the set of its well-formed formulas, with a denumerable stock of variables $p, q, r, \ldots$ and logical constants $\bot$, $\neg$ and $\wedge$ with their usual arities and interpretations. We use letters $A, B, C, \ldots$ for arbitrary formulas of $\mathcal{L}$.

[2] This is because the usual reading of a sequent $\Gamma : \Delta$ is that Γ entails Δ. Thus, rules of sequent calculi can be taken to be (schematic) metainferences.

Definition 1. A *metainference of level 0* (or *inference*) is a pair $\langle\Gamma,\Delta\rangle$ where $\Gamma,\Delta\subseteq\mathcal{L}$. For $n>0$, a *metainference of level* n is a pair $\langle\Gamma,\Delta\rangle$, where Γ and Δ are sets of metainferences of level $n-1$.

We use letters $\phi,\psi,\ldots$ for arbitrary metainferences whose level is made clear by the context, and $\Gamma,\Delta,\ldots$ for sets thereof. By a metainference *simpliciter* we mean a metainference of level 1. We refer to metainferences of level n as *meta*$_n$*inferences*. For ease of notation, we write $\Gamma\Rightarrow^n\Delta$ to denote the meta$_n$inference $\langle\Gamma,\Delta\rangle$. Also, we sometimes exhibit metainferences in a rule-like fashion. Thus, for instance,

$$\frac{p\Rightarrow^0 r \qquad q\Rightarrow^0 r}{p\vee q\Rightarrow^0 r}$$

is a handy notation for the meta$_1$inference $p\Rightarrow^0 r, q\Rightarrow^0 r\Rightarrow^1 p\vee q\Rightarrow^0 r$. Lastly, $\mathrm{MInf}_n(\mathcal{L})$ is the set of all meta$_n$inferences.

For our purposes, it will suffice to focus on the Strong Kleene interpretations of $\mathcal{L}$:

Definition 2. The Strong Kleene algebra $\mathcal{K}3$ is the set $\{0,1/2,1\}$ together with the following operations $\dot\bot$, $\dot\neg$ and $\dot\wedge$, of arities 0, 1 and 2, respectively:

$$\dot\bot = 0$$
$$\dot\neg x = 1-x$$
$$x\dot\wedge y = \min(x,y)$$

A *strong Kleene interpretation of* $\mathcal{L}$ is a homomorfism $v:\mathcal{L}\to\mathcal{K}3$. The set of all such interpretations is called Val. If $\Gamma\subseteq\mathcal{L}$, we write $v(\Gamma)$ to denote the set $\{v(\gamma):\gamma\in\Gamma\}$.

We start from a very general characterization of what a notion of validity is:[3]

Definition 3. A *validity notion for meta*$_n$*inferences*, abbreviated VNM_n, is a function

$$\mathbf{V}: val\times\mathrm{MInf}_n\to\{1,0\}$$

where $val\subseteq Val$. We say that val is the *validity space* of $\mathbf{V}$.

Intuitively, $\mathbf{V}$ tells you which valuations in val satisfy which meta$_n$inferences. The expression $v\Vdash_{\mathbf{V}}\psi$ abbreviates $\mathbf{V}(v,\psi)=1$, and the expression $v\nVdash_{\mathbf{V}}\psi$ abbreviates $\mathbf{V}(v,\psi)=0$. If $\mathbf{V}$ has the valuation space val, we say that ψ is *valid* according to $\mathbf{V}$, written $\vDash_{\mathbf{V}}\psi$, just in case $v\Vdash_{\mathbf{V}}\psi$ for each $v\in val$. ψ is *invalid* according to $\mathbf{V}$ just in case $\nvDash_{\mathbf{V}}\psi$.

Throughout the paper, we focus on the so-called *local* approach to the validity of meta$_n$inferences—as opposed to its alternative, the *global* approach.[4,5] This means, roughly, that our notions of validity for meta$_n$inferences are defined by means of a universal statement of the following form: 'for every interpretation, if all the premises are satisfied, then at least one conclusion is satisfied'. We will frequently appeal to notions of validity that result from 'slicing' notions of an immediately inferior level. If $\mathbf{V}_1$ and $\mathbf{V}_2$ are VNM_ns, the *slice* of $\mathbf{V}_1$ and $\mathbf{V}_2$, denoted by $\langle\mathbf{V}_1,\mathbf{V}_2\rangle$, is the VNM_{n+1} defined by

[3] We draw the following definition from Scambler [50]

[4] See [24] for the distinction. A third, interesting option is called *absolute global validity*; it can be found in, e.g. [23, 37].

[5] For the reasons displayed in [8], we think that the local definition is superior in various respects. Because of the collapse result proven in [52], we think that not considering the global definition produces no significant conceptual loss.

$$v \Vdash_{\langle \mathbf{V}_1, \mathbf{V}_2 \rangle} \Gamma \Rightarrow^{n+1} \Delta \quad \text{iff} \quad (\text{if } v \Vdash_{\mathbf{V}_1} \gamma \text{ for each } \gamma \in \Gamma \text{ then } v \Vdash_{\mathbf{V}_2} \delta \text{ for some } \delta \in \Delta)$$

Intuitively, $\langle \mathbf{V}_1, \mathbf{V}_2 \rangle$ evaluates the premises of a meta$_n$inference according to $\mathbf{V}_1$, and the conclusions according to $\mathbf{V}_2$. The slice of a $\textsc{vnm}_n$ $\mathbf{V}$ and itself, viz. $\langle \mathbf{V}, \mathbf{V} \rangle$, is called the *lifting* of $\mathbf{V}$, and denoted by $\uparrow \mathbf{V}$.[6]

As we anticipated, in this paper, we understand a *logic* as comprising, at least, a validity notion for metainferences of each level n.[7] For concreteness, we stipulate:

Definition 4. A *logic* $\mathbf{L}$ is a sequence $\langle \mathbf{V}_0, \mathbf{V}_1, ... \rangle$ where each $\mathbf{V}_n$ is a $\textsc{vnm}_n$. A meta$_i$inference ψ is *valid* in $\mathbf{L}$, written $\vDash_{\mathbf{L}} \psi$, just in case ψ is valid according to the i-th validity notion in $\mathbf{L}$.

In the literature, when authors endorse a certain validity notion $\mathbf{V}$ for meta$_n$inferences, they usually implicitly assume that the validity notions for metainferences higher than n are to be obtained by repeatedly lifting $\mathbf{V}$. Thus, let $\mathbf{V} = \langle \mathbf{V}_1, ..., \mathbf{V}_n \rangle$ be a sequence such that for each $1 \leq i \leq n$, $\mathbf{V}_i$ is a $\textsc{vnm}_i$. We define the *default logic* of $\mathbf{V}$, denoted by $\widehat{\mathbf{V}}$, as the logic $\langle \mathbf{V}_1, ..., \mathbf{V}_n, \uparrow \mathbf{V}_n, \uparrow\uparrow \mathbf{V}_n, ... \rangle$. Notice that $\widehat{\cdot}$ is an operator that takes finite sequences containing exactly one $\textsc{vnm}_i$ for each i up to some n, and delivers logics, that is, infinite sequences containing exactly one $\textsc{vnm}_i$ for each $i \in \mathbb{N}$.

3 Basic Characters

There is a number of characters that will play an important role throughout the play; we introduce them now. To begin with, we will work with six basic validity notions for *inferences*: **CL** corresponds to classical logic, **LP** to the logic of paradox [1, 43], **K3** to the strong Kleene logic [38], **S3** to the intersection of the last two [29], **ST** to the strict-tolerant logic [16] and **TS** to the tolerant-strict logic [33]. Except for **S3**, the remaining $\textsc{vnm}_0$s mentioned are all what following Chemla et. al. [15] we shall call *mixed* validity notions: they can be characterized in terms of two subsets X and Y of $\{1, 1/2, 0\}$, called *standards*, by means of the general schema:

$$v \Vdash_{XY} \Gamma \Rightarrow \Delta \qquad \text{iff} \qquad (\text{if } \forall \gamma \in \Gamma [v(\gamma) \in X] \text{ then } \exists \delta \in \Delta [v(\delta) \in Y])$$

S3, in turn, is what we call an *intersective-mixed* validity notion: it can be obtained by intersecting mixed validity notions.[8] Let Val_2 be the set of the bivalent interpretations of the language, viz. $Val_2 = \{v \in Val \mid v : \mathcal{L} \to \{1, 0\}\}$. Also, let $S = \{1\}$ and $T = \{1, 1/2\}$; S stands for 'Strict' and T for 'Tolerant':

Definition 5. The $\textsc{vnm}_0$s **LP**, **K3**, **S3**, **ST** and **TS** have domain $Val \times \text{MInf}_0$. The $\textsc{vnm}_0$ **CL** has domain $Val_2 \times \text{MInf}_0$. Let $v \in Val$, $v_2 \in Val_2$, and $X, Y \in \{S, T\}$:

$$\begin{aligned} v \Vdash_{\mathbf{LP}} \Gamma \Rightarrow^0 \Delta \qquad &\text{iff} \qquad (\text{if } \forall \gamma \in \Gamma [v(\gamma) \in T] \text{ then } \exists \delta \in \Delta [v(\delta) \in T]) \\ v \Vdash_{\mathbf{K3}} \Gamma \Rightarrow^0 \Delta \qquad &\text{iff} \qquad (\text{if } \forall \gamma \in \Gamma [v(\gamma) \in S] \text{ then } \exists \delta \in \Delta [v(\delta) \in S]) \end{aligned}$$

[6] Our notion of lifting is similar, but not identical, to that of Ripley [47]. Ripley's notion does not apply to validity notions but to to what the author calls counterexample relations.

[7] We say 'at least' but not 'at most' because authors in [4] consider an even more stringent definition, according to which a logic comprises, in addition, notions of antivalidity and contingency for metainferences of each level. We remain neutral with respect to this latter approach.

[8] As the authors show in [15], **S3** is *not* mixed, that is, it cannot be defined by means of two standards and the above mentioned general schema.

$$
\begin{array}{lll}
v \Vdash_{\mathbf{ST}} \Gamma \Rightarrow^0 \Delta & \text{iff} & (\text{if } \forall \gamma \in \Gamma[v(\gamma) \in S] \text{ then } \exists \delta \in \Delta[v(\delta) \in T]) \\
v \Vdash_{\mathbf{TS}} \Gamma \Rightarrow^0 \Delta & \text{iff} & (\text{if } \forall \gamma \in \Gamma[v(\gamma) \in T] \text{ then } \exists \delta \in \Delta[v(\delta) \in S]) \\
v \Vdash_{\mathbf{S3}} \Gamma \Rightarrow^0 \Delta & \text{iff} & (v \Vdash_{\mathbf{LP}} \Gamma \Rightarrow^0 \Delta \text{ and } v \Vdash_{\mathbf{K3}} \Gamma \Rightarrow^0 \Delta) \\
v_2 \Vdash_{\mathbf{CL}} \Gamma \Rightarrow^0 \Delta & \text{iff} & (\text{if } \forall \gamma \in \Gamma[v(\gamma) \in X] \text{ then } \exists \delta \in \Delta[v(\delta) \in Y])
\end{array}
$$

We say a few words about the default logics of these validity notions, in case the reader is not acquainted with them. Logic $\widehat{\mathbf{LP}}$ is paraconsistent; it validates the laws known as Pseudo Modus Ponens and Pseudo Explosion,[9]

$$\varnothing \Rightarrow^0 (A \wedge (A \to B)) \to B \qquad \text{(PMP)} \qquad\qquad \varnothing \Rightarrow^0 (A \wedge \neg A) \to B \qquad \text{(PEx)}$$

but it invalidates the principles of Modus Ponens and Explosion

$$A, A \to B \Rightarrow^0 B \qquad \text{(MP)} \qquad\qquad A, \neg A \Rightarrow^0 B \qquad \text{(Ex)}$$

as well as Meta Modus Ponens and Meta Explosion:

$$\frac{\varnothing \Rightarrow^0 A \qquad \varnothing \Rightarrow^0 A \to B}{\varnothing \Rightarrow^0 B} \;\text{(MMP)} \qquad\qquad \frac{\varnothing \Rightarrow^0 A \qquad \varnothing \Rightarrow^0 \neg A}{\varnothing \Rightarrow^0 B} \;\text{(MEx)}$$

$\widehat{\mathbf{K3}}$ is paracomplete; it validates each one of the principles just stated, as well as reflexivity and conditional contraposition as encoded by the inferences

$$A \Rightarrow^0 A \qquad \text{(R)} \qquad\qquad A \to B \Rightarrow^0 \neg B \to \neg A \qquad \text{(CC)}$$

but it invalidates the associated laws, which for uniformity we call Pseudo Reflexivity and Pseudo Conditional Contraposition:

$$\varnothing \Rightarrow^0 A \to A \qquad \text{(PR)} \qquad\qquad \varnothing \Rightarrow^0 (A \to B) \to (\neg B \to \neg A) \qquad \text{(PCC)}$$

Logics $\widehat{\mathbf{LP}}$ and $\widehat{\mathbf{K3}}$ are dual, which means that an inference $\Gamma \Rightarrow^0 \Delta$ is valid in $\widehat{\mathbf{LP}}$ just in case the inference $\{\neg\delta : \delta \in \Delta\} \Rightarrow^0 \{\neg\gamma : \gamma \in \Gamma\}$ is valid in $\widehat{\mathbf{K3}}$. Logic $\widehat{\mathbf{S3}}$ is, as anticipated, the intersection of $\widehat{\mathbf{LP}}$ and $\widehat{\mathbf{K3}}$ at every metainferential level. All these systems are similar in that they are *structural*, this meaning that they validate each structural principle of classical logic.[10] In contrast, systems $\widehat{\mathbf{ST}}$ and $\widehat{\mathbf{TS}}$ are *substructural*, that is, they invalidate some classically valid structural principles. $\widehat{\mathbf{ST}}$ validates R, but invalidates transitivity as encoded by the rule

$$\frac{\Gamma, A \Rightarrow^0 \Delta \qquad \Pi \Rightarrow^0 A, \Sigma}{\Gamma, \Pi \Rightarrow^0 \Delta, \Sigma} \qquad \text{(Cut)}$$

In contrast, $\widehat{\mathbf{TS}}$ validates Cut but invalidates R. In a language without the means to express any semantic values (e.g. a language like $\mathcal{L}$ but without the constant $\perp$), $\widehat{\mathbf{TS}}$ has no valid inferences at all; $\widehat{\mathbf{ST}}$, in contrast, has the same valid inferences that classical logic.

[9] Here and throughout the paper we present logical laws as inferences without premises. But this choice is not very substantial: our results and arguments would still run if we understood laws as plain sentences.

[10] Roughly, a principle is *structural* just in case no logical constants feature in its formulation. If a principle is not structural, it is *operational* (see [41] for more on this distinction).

Two notions of validity for metainferences will be of particular interest to us. One is $\langle \mathbf{ST}, \mathbf{ST} \rangle$. In [6], the authors show that this $\textsc{vnm}_1$ is *modulo* translation coextensive with the $\textsc{vnm}_0$ **LP**. More precisely, let $\tau : \mathrm{MInf}_0(\mathcal{L}) \to \mathcal{L}$ be a function defined as follows

$$\tau(\Gamma \Rightarrow^0 \Delta) = \begin{cases} \bigwedge(\Gamma) \to \bigvee(\Delta) & \text{if } \ \Gamma, \Delta \neq \varnothing \\ \bigvee(\Delta) & \text{if } \ \Gamma = \varnothing, \Delta \neq \varnothing \\ \neg \bigwedge(\Gamma) & \text{if } \ \Gamma \neq \varnothing, \Delta = \varnothing \\ \bot & \text{if } \ \Gamma = \Delta = \varnothing \end{cases}$$

where $\bigvee(\Sigma)$ and $\bigwedge(\Sigma)$ are the disjunction and the conjunction, respectively, of all the sentences in Σ. Then, a metainference $\Gamma \Rightarrow^1 \Delta$ is valid in $\langle \mathbf{ST}, \mathbf{ST} \rangle$ just in case the inference $\{\tau(\gamma) : \gamma \in \Gamma\} \Rightarrow^0 \{\tau(\delta) : \delta \in \Delta\}$ is valid in **LP**. Works [6, 24] (partly) rely on this result to argue that logic $\widehat{\mathbf{ST}}$ is in relevant respects similar to $\widehat{\mathbf{LP}}$.

The other $\textsc{vnm}_1$ that will be of interest to us is $\langle \mathbf{TS}, \mathbf{ST} \rangle$. The authors in [40, 8] show that it validates the same metainferences as classical logic. Indeed, they introduce the following construction:

Definition 6. For $n \geq 0$, let $\mathbf{ST}_n$ and $\mathbf{TS}_n$ be the $\textsc{vsm}_n$s defined as follows:

$$\begin{array}{c} \mathbf{ST}_0 = \mathbf{ST} \\ \mathbf{ST}_{n+1} = \langle \mathbf{TS}_n, \mathbf{ST}_n \rangle \end{array} \qquad\qquad \begin{array}{c} \mathbf{TS}_0 = \mathbf{TS} \\ \mathbf{TS}_{n+1} = \langle \mathbf{ST}_n, \mathbf{TS}_n \rangle \end{array}$$

and let $\overrightarrow{\mathbf{ST}_n}$ denote the sequence $\langle \mathbf{ST}_0, \mathbf{ST}_1, ..., \mathbf{ST}_n \rangle$

(So, e.g. $\overrightarrow{\mathbf{ST}_1} = \langle \mathbf{ST}_0, \mathbf{ST}_1 \rangle = \langle \mathbf{ST}, \langle \mathbf{TS}, \mathbf{ST} \rangle \rangle$.) The authors show that, for each $n \geq 0$, the default logic of $\overrightarrow{\mathbf{ST}_n}$ coincides with classical logic $\widehat{\mathbf{CL}}$ up to and including the n-th metainferential level, but diverges from there upwards. (So, e.g. $\widehat{\overrightarrow{\mathbf{ST}_1}}$ coincides with $\widehat{\mathbf{CL}}$ up to and including meta$_1$inferences, but diverges at meta$_n$inferential levels with $n \geq 2$.) This suggests the idea of taking the infinite sequence of all the $\mathbf{ST}_i$s:

Definition 7. Logic $\mathbf{ST}_\omega$ is given by the sequence $\langle \mathbf{ST}_0, \mathbf{ST}_1, ..., \mathbf{ST}_n, ... \rangle$

The resulting system is coextensive with classical logic at *all metainferential levels*: for $n \geq 0$, a metainference $\Gamma \Rightarrow^n \Delta$ is valid in $\widehat{\mathbf{CL}}$ just in case it is valid in $\mathbf{ST}_\omega$.

Enough preambles. We can tackle our proposal.

4 Meta-classical Non-Classical Logics

As we anticipated, we shall take **LP**, **K3** and **S3** as the basic validity notions for inferences upon which we define what we called meta-classical non-classical logics. There are various alternative ways to proceed in order to obtain logics of this sort. We shall consider three of them.

The first proposal can be intuitively described as follows: first, choose your preferred non-classical validity notion for inferences (viz. $\textsc{vnm}_0$); then, at each metainferential level $n > 0$, take as much classical logic as you can get in Strong Kleene models. The resulting logics are the following:

Definition 8.

Logic **mcLP** is given by the sequence $\langle \mathbf{LP}, \mathbf{ST}_1, ..., \mathbf{ST}_n, ...\rangle$
Logic **mcK3** is given by the sequence $\langle \mathbf{K3}, \mathbf{ST}_1, ..., \mathbf{ST}_n, ...\rangle$
Logic **mcS3** is given by the sequence $\langle \mathbf{S3}, \mathbf{ST}_1, ..., \mathbf{ST}_n, ...\rangle$

where '**mc**' stands for 'meta-classical'. Notice, then, that each of these logics is exactly like $\mathbf{ST}_\omega$ except in that it replaces **ST** with some other $\textsc{vnm}_0$. The behavior of these logics at the level of *inferences* is exactly like the the behavior of the corresponding default logics, viz. $\widehat{\mathbf{LP}}$, $\widehat{\mathbf{K3}}$ and $\widehat{\mathbf{S3}}$. Thus, for instance, **mcLP** invalidates MP but not PR, **mcK3** invalidates PR but not MP, and **mcS3** invalidates both principles. However, default logics and **mc**-logics diverge from the first metainferential level upwards. Default logics invalidate many classically valid meta$_n$inferences; for example, $\widehat{\mathbf{LP}}$ invalidates MMP and MEx as already stated, and $\widehat{\mathbf{K3}}$ invalidates Contraposition and Hypothetical Proof:

$$\frac{A \Rightarrow^0 C}{\neg C \Rightarrow^0 \neg A} \quad \text{(C)} \qquad\qquad \frac{A \Rightarrow^0 C}{\varnothing \Rightarrow^0 A \to C} \quad \text{(HP)}$$

In contrast, **mc**-logics are coextensive with classical logic $\widehat{\mathbf{CL}}$ at every level $n \geq 1$:

Fact 9. *For $n \geq 1$, A meta$_n$inference is valid in $\widehat{\mathbf{CL}}$ just in case it is valid in* **mcLP**, **mcK3** *and* **mcS3**.

The result is originally proven in [8] (Theorem [4.12]). In $\widehat{\mathbf{CL}}$, the fact that $\Gamma \Rightarrow^n \Delta$ is valid implies that $\varnothing \Rightarrow^{n+1} \Gamma \Rightarrow^n \Delta$ is valid. Thus, from the above it follows that

Fact 10. *For $n \geq 0$, $\Gamma \Rightarrow^n \Delta$ is valid in $\widehat{\mathbf{CL}}$ just in case $\varnothing \Rightarrow^{n+1} \Gamma \Rightarrow^n \Delta$ is valid in* **mcLP**, **mcK3** *and* **mcS3**.

Let us say that $\varnothing \Rightarrow^{n+1} \Gamma \Rightarrow^n \Delta$ is the *pseudo-metavariant* of $\Gamma \Rightarrow^n \Delta$. Then, another way to state Fact 10 is by saying that a meta$_n$inference with $n \geq 0$ is valid in $\widehat{\mathbf{CL}}$ just in case its pseudo-metavariant is valid in the **mc**-logics. The case in which $n = 0$ gives us that **mc**-logics validate meta$_1$inferential principles such as

$$\frac{\varnothing}{A, A \to B \Rightarrow^0 B} \quad \text{(MP*)} \qquad\qquad \frac{\varnothing}{\varnothing \Rightarrow^0 A \to A} \quad \text{(PR*)}$$

More in general, they validate all and only the pseudo-metavariants of inferences that are valid in classical logic. This suggests that there is a sense in which **mc**-logics recover, in the metainferential level, the full *inferential* power of classical logic.

However, **mc**-logics exhibit some putative drawbacks. One may intuitively expect that the supporter of a logic gives, for any level n, some explanation of why her system has this or that validity notion for meta$_n$inferences. The explanation should tell us what is the *link* between metainferences of level n and metainferences of level n+/–1. Otherwise, the talk about 'metainferences' could be seen as unjustified, and the system could be regarded not as one logic—in the more philosophical sense of this notion—but as a sequence of different formalisms not even related with one another. In the case of **mc**-logics, we have given no such explanation yet. The fact that these systems recover all the meta$_n$inferences with $n \geq 1$ valid in $\widehat{\mathbf{CL}}$ constitutes, at best, an instrumentalist justification of their legitimacy. Those who expect a more robust explanation of what counts as a logic, will probably not be happy with **mc**-logics as they stand.

That's why we move to our second proposal. In a nutshell, it consists in saying that a sequence of validity notions, one for each metainferential level n, is a *logic* in the philosophical sense only if all the validity notions involved are *modulo* translation coextensive with one another. The idea is that uniformity under translation indicates that the notions of validity at play are, in a relevant sense, the same. In the end, a logic might be seen as characterized by only one validity notion—which can be conveniently applied to different kinds of syntactic objects. Next, we make the proposal more precise.

For starters, we take our function τ from Section 3 and stipulate the following:

$$\text{For any } n > 0,\ \tau(\Gamma \Rightarrow^n \Delta) = \{\tau(\gamma) : \gamma \in \Gamma\} \Rightarrow^{n-1} \{\tau(\delta) : \delta \in \Delta\}$$

Thus, our translation procedure admits inputs from any metainferential level. Now, we define uniformity under translation:

Definition 11. A logic $\mathbf{L}$ is *uniform under translation* just in case, for each $n > 0$, a meta$_n$inference $\Gamma \Rightarrow^n \Delta$ is valid in $\mathbf{L}$ if and only if the meta$_{n-1}$inference $\tau(\Gamma \Rightarrow^n \Delta)$ is valid in $\mathbf{L}$.

To illustrate, we give a couple of examples of systems that are logics in the technical sense of Definition 4, but are not uniform under translation—and thus, do not qualify as logics in the philosophical sense, according to this proposal. For one example, take $\widehat{\mathbf{ST}}$; the system invalidates MMP but validates MP, which is its translation. For another (less obvious) case, take $\widehat{\mathbf{LP}}$. The system invalidates the metainference MP*, but it validates the inference $\varnothing \Rightarrow^0 (A \wedge (A \to B)) \to B$, which is its translation.

One example of a system that *is* a logic on this proposal is given (for everyone's relief) by good old classical logic $\widehat{\mathbf{CL}}$. Another example is given by the system that we introduce next, which we call **uLP**, for 'uniform **LP**':

Definition 12. **uLP** is the logic $\langle \mathbf{LP}, \langle \mathbf{ST}_0, \mathbf{ST}_0 \rangle, \langle \mathbf{ST}_1, \mathbf{ST}_1 \rangle, \ldots \rangle$

That is, one takes **LP** as one's canon of valid inference, and then, for each level $n \geq 1$, one takes $\langle \mathbf{ST}_{n-1}, \mathbf{ST}_{n-1} \rangle$ as one's canon of valid meta$_n$inference.

Fact 13. *Logic* **uLP** *is uniform under translation*

Proof. The result can be found in [8] (Theorem 4.16). □

System **uLP** differs from **mcLP** in that it does not validate every classical validity at every metainferential level; for instance, meta$_1$inferential principles MMP and Cut are both invalid in the system. Still, there is a strong sense in which **uLP** is meta-classical, namely, it satisfies a result analogous of Fact 10:

Fact 14. *For* $n \geq 0$*,* $\Gamma \Rightarrow^n \Delta$ *is valid in* $\widehat{\mathbf{CL}}$ *just in case* $\varnothing \Rightarrow^{n+1} \Gamma \Rightarrow^n \Delta$ *is valid in* **uLP**.

Hence, **uLP** validates all and only the pseudo-metavariants of meta$_n$inferences valid in classical logic. Again, the case in which $n = 0$ tells us that the system recaptures the full inferential power of classical logic at the metainferential level.

The strategy that we employed to define **uLP** cannot be straightforwardly transposed to the case of **K3**. The resulting logic would look like this:

Definition 15. $\mathbf{uK3}^*$ is the logic $\langle \mathbf{K3}, \langle \mathbf{ST}_0, \mathbf{ST}_0 \rangle, \langle \mathbf{ST}_1, \mathbf{ST}_1 \rangle, ... \rangle$

That is, $\mathbf{uK3}^*$ is exactly like **uLP**, except in that it takes **K3** as the canon of valid inference. It is easy to check the system is not uniform under translation. For instance, the metainference

$$\frac{\varnothing \Rightarrow^0 A \qquad A \Rightarrow^0 \varnothing}{\varnothing \Rightarrow^0 \varnothing}$$

is invalid in $\langle \mathbf{ST}_0, \mathbf{ST}_0 \rangle$, but its translation is the inference

$$A, \neg A \Rightarrow^0 \bot$$

which is valid in **K3**.

This does not mean that the case for **K3** is hopeless, as the following logic that selects **K3** as its inferential standard is in fact uniform under translation.

Definition 16. **uK3** is the logic $\langle \mathbf{K3}, \langle \mathbf{TS}_0, \mathbf{TS}_0 \rangle, \langle \mathbf{TS}_1, \mathbf{TS}_1 \rangle, ... \rangle$

That is, one takes **K3** as the canon of valid inference, and then, for each level $n \geq 1$, one takes $\langle \mathbf{TS}_{n-1}, \mathbf{TS}_{n-1} \rangle$ as the canon of valid meta$_n$inference.

Fact 17. *Logic* **uK3** *is uniform under translation*

Proof. The fact follows from definitions and results in [19], Section 8 (Definition 5, Theorem 5 and Corollary 8). There, the authors prove that, for each $0 \leq n$, the n-th element in **uK3** is the *lowering* of the $n+1$-th element; this means that the valid meta$_{n+1}$inferences of the logic correspond, *via* translation, with the valid meta$_n$inferences. □

At first, one might think that **uK3** does not qualify as what we called a meta-classical non-classical logic. The reason is that it does not satisfy a result analogous to Facts 10 and 14. For instance, the system invalidates PR as well as all the higher-level variants of this principle, which are given by the meta$_n$inference

$$\varnothing \Rightarrow^n \varnothing \Rightarrow^{n-1} ... \varnothing \Rightarrow^0 A \to A$$

for each $n > 0$. However, **uK3** also recovers important aspects of classical logic. To show this, we appeal to no notion of *antivalidity*, introduced by Chris Scambler [50]:

Definition 18. Let $\mathbf{V}$ be a $\textsc{vnm}_n$, with $n \geq 0$. A meta$_n$inference $\Gamma \Rightarrow^n \Delta$ is *antivalid* according to $\mathbf{V}$ just in case, for every relevant interpretation v, $v \nVdash_{\mathbf{V}} \Gamma \Rightarrow^n \Delta$.

Hence, a meta$_n$inference is antivalid just in case it is never satisfied by a valuation.[11] Following Barrio et. al. in [4], we can understand the antivalidities of a logic as the meta$_n$inferences that the logic *rejects*:

[11] Keep in mind that the property of not being satisfied is defined using a conjunction: every premise satisfies the standard for premises, *and* no conclusion satisfies the standard for conclusions. This makes the difference between our notion of antivalidity and the property studied under the same name by Cobreros et. al. [20].

> Antivalidities are formulas, inferences, metainferences, etc, that should be rejected no matter what, in any context. And this is not what happens with every invalid inference. Inductive reasoning, for example, is classically invalid. Nevertheless, we should not always reject it (...) Where is the limit to what can be embraced? A quick—and straightforward—answer is: antivalidities

Scambler notes that, while classical logic has many antivalid inferences (e.g. $A \vee \neg A \Rightarrow^0 A \wedge \neg A$), **ST** has none. And the same difference extends to higher levels: at every n, there are many meta$_n$inferences that are antivalid in classical logic, but none that are antivalid in $\mathbf{ST}_n$. So, the author argues that, while $\mathbf{ST}_\omega$ provides (at best) a *positive* characterization of classical logic, it does not provide a *negative* one. The author shows that, to obtain a negative characterization, we have to appeal to another logic:

Definition 19. Logic $\mathbf{TS}_\omega$ is given by the sequence $\langle \mathbf{TS}_0, \mathbf{TS}_1, ..., \mathbf{TS}_n, ... \rangle$

Fact 20. $\mathbf{TS}_\omega$ *has the same antivalid meta*$_n$*inferences as classical logic at every* $n \geq 0$.

Proof. The result can be found in [50] (Lemma 26). □

On the other hand, at every level n, $\mathbf{TS}_\omega$ has no valid meta$_n$inferences where the constant $\bot$ does not occur. Thus, it certainly falls short of a positive characterization of classical logic. In this sense, $\mathbf{ST}_\omega$ and $\mathbf{TS}_\omega$ seem to recover dual aspects of classicality.

The comparison between $\mathbf{ST}_\omega$ and $\mathbf{TS}_\omega$ is relevant for our purposes, for it extends to the systems we present in this paper. Logic **uLP** recovers every classical validity from the level 1 upwards, in the sense given by Fact 14. However, it does not recover antivalidities at any level, so it provides (at best) a positive characterization of classical logic at meta$_{n\geq 1}$inferential levels. Logic **uK3** is its dual: it does not recover classical validities, but it does recover the classical antivalidites from the level 1 upwards:

Fact 21. *For* $n \geq 0$, $\Gamma \Rightarrow^n \Delta$ *is antivalid in* $\widehat{\mathbf{CL}}$ *just in case* $\varnothing \Rightarrow^{n+1} \Gamma \Rightarrow^n \Delta$ *is antivalid in* **uK3**.

Hence, **uK3** provides a negative characterization of classical logic. There is a clear sense, then, in which systems **uLP** and **uK3** are both meta-classical: they recover dual aspects of classicality.

Now, what about the uniform variant of **S3**? One could perhaps conjecture that it fails to provide *either* a positive *or* a negative characterization of classical logic. But this is not so. Let us write $\mathbf{uLP}_k$ and $\mathbf{uK3}_k$ to denote the k-th elements of **uLP** and **uK3**, respectively. For each $n > 0$, $\mathbf{uS3}_n$ is the the $\textsc{vnm}_n$ defined as follows:

$$v \Vdash_{\mathbf{uS3}_n} \Gamma \Rightarrow^n \Delta \quad \text{iff} \quad (v \Vdash_{\mathbf{uLP}_n} \Gamma \Rightarrow^n \Delta \text{ and } v \Vdash_{\mathbf{uK3}_n} \Gamma \Rightarrow^n \Delta)$$

Them, the uniform variant of **S3** is straightforward:

Definition 22. **uS3** is the logic $\langle \mathbf{S3}, \mathbf{uS3}_1, ..., \mathbf{uS3}_n, ... \rangle$

Fact 23. **uS3** *is uniform under translation*

Proof. The result follows immediately from Facts 13 and 17. □

Obviously, logic **uS3** has less valid meta$_n$inferences than **uLP**. Hence, it does not give a positive characterization of classical logic in the way that this latter system does. However, it is easy to check that **uS3** has exactly the same antivalidities as **uK3**. Thus, it provides a negative characterization of classical logic.

So far we have explored two strategies to obtain non-classical logics that are to a greater or lesser extent meta-classical; one of the strategies delivers **mc**-logics, the other delivers **u**-logics. While different, these strategies share a feature that might result hard to swallow for some readers. In the logics they give rise to, there are some inferences that are valid (invalid) even though their pseudo-metavariants are invalid (valid). For instance, **uLP** invalidates MP, but it validates MP*. Dually, **uK3** invalidates MP* but validates MP. These facts might seem highly counterintuitive. Indeed, they violate a *prima facie* plausible principle that, for lack of a better name, we call the Equivalence Thesis:

(Equivalence Thesis) The meta$_n$inference $\Gamma \Rightarrow^n \Delta$ and the meta$_{n+1}$inference $\varnothing \Rightarrow^{n+1} \Gamma \Rightarrow^n \Delta$ are equivalent things, in the sense that any logic that validates the former also validates the latter, and *viceversa*.

We think that this principle is often implicitly assumed in the literature; for instance, when axioms of a sequent calculus are taken not just as metainferences without premises, but as inferences in their own law.[12] Moreover, Brian Porter [42] has explicitly suggested that for a sequence of mixed metainferential standards to constitute a logic, the standard for the level n (for any finite n) should be the same as the standard for the conclusions of metainferences of level $n+1$; no logic violating the Equivalence Thesis can accomplish this goal. Lastly, there is a longstanding tradition in the philosophy of logic, according to which a *logical truth* can be understood as the conclusion of a valid inference without any premises; the principle under scrutiny can be seen as a natural generalization of this standpoint. So, the third and last strategy we explore is meant to retain the Equivalence Thesis, that is, to deliver systems that respect it.

Intuitively, our strategy is to relativise the validity standards in play to whether or not a given meta$_n$inference has any premises. More precisely, we shall appeal to validity notions of the following kind:

Definition 24. For $n \geq 1$, let $\mathbf{V}_1$ be a $\textsc{vnm}_n$ and $\mathbf{V}_2$ a $\textsc{vnm}_{n-1}$, both on the space of interpretations val. Then, $\mathbf{V}_1\#\mathbf{V}_2$ is the $\textsc{vnm}_n$ defined by

$$v \Vdash_{\mathbf{V}_1\#\mathbf{V}_2} \Gamma \Rightarrow^n \Delta \qquad \text{iff} \qquad \begin{cases} v \Vdash_{\mathbf{V}_1} \Gamma \Rightarrow^n \Delta & \text{if } \Gamma \neq \varnothing \\ v \Vdash_{\langle \mathbf{V}_2, \mathbf{V}_2 \rangle} \Gamma \Rightarrow^n \Delta & \text{if } \Gamma = \varnothing \end{cases}$$

So, $\mathbf{V}_1\#\mathbf{V}_2$ evaluates a meta$_n$inference according to $\mathbf{V}_1$ if it has any premises, and according to the lifting of $\mathbf{V}_2$ if it has none. With this, we can easily modify the **mc**-logics in such a way that they respect the Equivalence Thesis. We first define the appropriate $\textsc{vnm}_n$s:

$$\begin{array}{ccc} \mathbf{eqLP}_0 = \mathbf{LP} & \mathbf{eqK3}_0 = \mathbf{K3} & \mathbf{eqS3}_0 = \mathbf{S3} \\ \mathbf{eqLP}_{n+1} = \mathbf{ST}_{n+1}\#\mathbf{eqLP}_n & \mathbf{eqK3}_{n+1} = \mathbf{ST}_{n+1}\#\mathbf{eqK3}_n & \mathbf{eqS3}_{n+1} = \mathbf{ST}_{n+1}\#\mathbf{eqS3}_n \end{array}$$

And then, the corresponding logics:

[12] Arguably, this is the reading of axioms that is in play whenever we focus on the internal (as opposed to the external) consequence relation of a sequent calculus (see [25] for the distinction).

Definition 25.

Logic **eqLP** is given by the sequence $\langle \mathbf{LP}, \mathbf{eqLP}_1, ..., \mathbf{eqLP}_n, ...\rangle$
Logic **eqK3** is given by the sequence $\langle \mathbf{K3}, \mathbf{eqK3}_1, ..., \mathbf{eqK3}_n, ...\rangle$
Logic **eqS3** is given by the sequence $\langle \mathbf{S3}, \mathbf{eqS3}_1, ..., \mathbf{eqS3}_n, ...\rangle$

Intuitively, one takes one's preferred validity notion for inferences, and then, at each level $n \geq 1$ one applies the operation # to the n-th validity notion of $\mathbf{ST}_\omega$ and the $n-1$-th validity notion obtained in the process.

Fact 26. *Logics* **eqLP***,* **eqK3** *and* **eqS3** *satisfy the Equivalence Thesis.*

(The proof is straightforward.) Of course, **eq**-logics do not satisfy a result analogous to Fact 9, that is, they are not coextensive with classical logic $\widehat{\mathbf{CL}}$ at every level. For instance, they invalidate the pseudo-metavariants of every inference that is valid in **CL** but not in the corresponding $\textsc{vnm}_0$ (thus, **eqLP** invalidates MP*, **eqK3** invalidates PR*, and so on); these meta$_1$inferences are all valid in $\widehat{\mathbf{CL}}$. However, there is a sense in which **eq**-logics are still meta-classical.

Fact 27. *A meta$_n$inference $\Gamma \Rightarrow^n \Delta$ is valid in $\widehat{\mathbf{CL}}$ just in case the meta$_{n+1}$inference*

$$\frac{\{\varnothing \Rightarrow^n \gamma \mid \gamma \in \Gamma\}}{\{\varnothing \Rightarrow^n \delta \mid \delta \in \Delta\}}$$

is valid in **eqLP***,* **eqK3** *and* **eqS3***.*

Thus, following the terminology of [6], we can say that the external logic of the **eq**-logics coincides with the internal logic of classical logic. Also, an inference $\Gamma \Rightarrow^0 \Delta$ is valid in $\widehat{\mathbf{CL}}$ just in case the meta$_1$inference

$$\frac{\bot \Rightarrow^0 \top}{\Gamma \Rightarrow^0 \Delta}$$

(where $\top$ is defined as $\neg\bot$) is valid in each one of the **eq**-logics. Thus, there are various ways in which **eq**-logics recover classical validities.

This last strategy has some limitations, however. It cannot be applied to the **u**-logics we have defined (that is, **uLP**, **uK3** and **uS3**). The resulting systems would respect the Equivalence Thesis, but they would not be equivalent under translation anymore (we leave the proof of this fact as an exercise to the reader), and equivalence under translation was the main motivation behind these systems.[13]

[13] Throughout this Section, we have provided semantic presentations of **mc**-, **u**- and **eq**-logics. Nevertheless, we have said nothing about possible proof-systems for these logics. In [21], Da Ré and Pailos display a method to define sequent-calculi for mixed $\textsc{vnm}_n$s built with any finite combination of the notions of validity for inferences **LP**, **K3**, **ST** and **TS**. Thus, all **mc**- and **u**-logics except for the ones with **S3** as the inferential standard have a corresponding sound and complete proof-system of this kind. The method mentioned, however, cannot be applied in a straightforward manner to the **eq**-logics. In [19], Cobreros et. al. also present a labeled sequent-calculus for the validities of any mixed $\textsc{vnm}_n$ built with any finite combination of the four $\textsc{vnm}_0$s we have previously mentioned. Thus, once again, this system is also sound and complete with respect to the validities of the **mc**- and **u**-logic based on **LP** or **K3**. The sequent calculi displayed in Fjellstad [32] closely resembles the one introduced by Cobreros, so they can also be used for our logics. Finally, Golan [35] develops a sequent-calculus for $\mathbf{ST}_\omega$ that recovers every classical validity. Thus, it might be adapted for our **mc**-logics based on **LP** or **K3** without much trouble (though the rules for inferential validities should be the ones for **LP** or **K3** that Golan also introduces in his article).

5 Philosophical Discussion

In this section we address three issues related to the philosophical relevance of the meta-classical non-classical logics we have presented. The first one concerns what is the intuitive reading of validity in these systems; we will provide one plausible answer to this question. The second issue has to do with a potential concern that one may have, namely, that the systems under scrutiny are so non-standard that one might doubt whether they constitute logics in their own right; we give an argument to dispel such doubts. Lastly, the third issue concerns possible applications of our meta-classical non-classical logics; we suggest that our systems may be of substantial value for some non-classical logicians.

Let us begin with the issue of what is the intuitive meaning of validity in our meta-classical non-classical logics. Giving an answer to this question involves specifying, for each one of these logics and each metainferential level n, what is the intuitive reading of the claim that a meta$_n$inference is valid in that logic. Our approach to this issue builds upon the bilateralist reading of meta$_n$inferences recently advanced by Thomas Ferguson and Elisángela Ramírez-Cámara [27]. We should stress, however, that we do not think that our approach is the only way to go. We choose it for it strikes us as a particularly natural way to read multiple-conclusion logical consequenc at arbitrary metainferential levels. But other informal readings may be possible.[14]

Ferguson and Ramírez-Cámara evaluate two ways for interpreting metainferences, which they call the *operational* and the *bounds consequence* reading. Here, we will focus on the latter only. In few words, the bounds consequence reading extends, from inferences to metainferences, Ripley's bilateralist way of understanding validity in **ST** [e.g. 46]. According to Ripley, an inference is valid in **ST** if and only if it is "out-of-bounds" or "incoherent" to accept every premise while rejecting every conclusion. Analogously, in the bounds consequence reading a metainference of level n (for any n) is valid if and only if it is out of bounds to accept every premise while rejecting every conclusion.[15] The authors emphasize that, under this approach, a metainference $\varnothing \Rightarrow^n \phi$ and its pseudo-metavariant $\varnothing \Rightarrow^{n+1} \varnothing \Rightarrow^n \phi$ say different things:

> On this reading, the two [i.e. $\varnothing \Rightarrow^n \phi$ and $\varnothing \Rightarrow^{n+1} \varnothing \Rightarrow^n \phi$] seem to differ markedly in meaning; while $\varnothing \Rightarrow^n \phi$ sets a condition about how we should speak about ϕ, $\varnothing \Rightarrow^{n+1} \varnothing \Rightarrow^n \phi$ sets a condition about how we should speak about this *condition*.[16] To be more exact, the appearance of the sequent $\varnothing \Rightarrow^n \phi$ in the example constituted a *positive assertion* that denials of ϕ are out-of-bounds; the bounds consequence reading of $\varnothing \Rightarrow^{n+1} \varnothing \Rightarrow^n \phi$ makes only the claim that this positive assertion is not to be denied. ([27, p. 19])

[14]Indeed, we adhere to the position depicted in [3], according to which in general pure logics do not have something as an canonical informal interpretation.

[15]Remember that we are working with a local conception of metainferential validity. Accordingly, when we say that it is out of bounds to have certain attitudes (acceptance, rejection, etc.) towards certain metainferences, we always mean that it is out of bounds to have those attitutes towards the assertions that these metainferences hold (viz. are satisfied) at a particular valuation.

[16]We have changed the way the authors chose to represent metainferences to be consistent to how we have referred to them in the rest of the article. In fact, as will soon be apparent, here the authors are not talking about metainferences themselves, but about sequents that might be interpreted as representing metainferences. As we think that the way to understand sequents can be extended to interpret metainferences, we will not put too much weight on the differences between sequents and metainferences here.

It should be noted that this divergence in meaning between a metainference and its pseudo-metavariant harmonizes well with the abandonment of the Equivalence Principle, which is essential for the plausibility of various of our meta-classical non-classical logics (more precisely, the **mc**- and **u**-systems). The bounds consequence reading advanced by the authors is based on a Strict-Tolerant reading of inferential validity, which is not the only mixed reading available. For example, on a Tolerant-Strict reading (i.e., the one adopted by a supporter of **TS**), the assertion of an inference's validity should be understood as taking as out-of-bounds to non-reject (which may involve accepting, but not necessarily) every premise while at the same time non-accepting (but not necessarily rejecting) every conclusion. On a Tolerant-Tolerant reading of inferential validity (i.e., the one adopted by a supporter of **LP**), the assertion of an inference's validity should be understood as taking as out-of-bounds to non-reject every premise while at the same time rejecting every conclusion. Finally, on a Strict-Strict reading of inferential validity (i.e., the one adopted by a supporter of **K3**), the assertion of an inference's validity should be understood as taking as out-of-bounds to accept every premise while at the same time non-accepting every conclusion. In general, a Strict standard for premises means acceptance and a Tolerant standard for premises means non-rejection, while a Tolerant standard for conclusions means rejection, and a Strict standard for conclusions means non-acceptance. What do these different versions of bounds consequence for inferential validity have in common? That validity is understood as the incoherence of having one attitude towards a set of sentences (i.e., the premises) while at the same time having another attitude with respect to another set of sentences (i.e., the conclusions). In the case of the bounds consequence reading of metainferential validity, both sets of sentences have as members only sentences that express that a given (meta)inference holds. And these sentences should be understood in terms of the bounds consequence relation, which makes it fully recursive.

As we have just seen, there are different options regarding how to understand the variety of metainferential validity standards. We will explain in each case which one we take as the most relevant, and why. Also, we will give examples of the informal readings of particular metainferences—we choose cases that are particularly challenging from an intuitive standpoint, for they involve failures of the Equivalence Principle.

In the case of **mc**-logics, we will adopt for the metainferential levels, in each case, the bounds consequence reading associated with the basic inferential standard of the given logic. Thus, in the case of **mcLP**, we will adopt a Tolerant-Tolerant reading of metainferential validity, while in the case of **mcK3** we will adopt a Strict-Strict reading.[17] Let us start with **mcLP**. MP does not hold in this system. Nevertheless, MP* holds. On a bounds consequence reading, though, this is neither unpleasant nor strange. The **mcLP**-theorist foregoes MP because according to her logic it is in bounds to non-reject every premise while rejecting every conclusion—i.e., she looks at the validity of $A, A \to B \Rightarrow B$ through the lenses of the Tolerant-Tolerant approach. But she embraces MP* because she *does not reject* (here is the Tolerant-Tolerant reading of metainferences) that it is out of bounds to accept each

[17]This is not the only available option, though. As each of these logics adopts as metainferential standards of level n the ones that correspond, via suitable translations, with valid inferences of **ST**, then it seems also reasonable to adopt for them the kind of bounds consequence reading favored by **ST**'s supporters like Dave Ripley—i.e., that it is incoherent to accept each premise while denying each conclusion. Notice, though, that for a supporter of **LP** (**K3**), probably a Tolerant-Tolerant (Strict-Strict) reading for metainferential validities might sound more plausible, at least if they take metainferences to be just another type of inferences, as [8] and [40] take them to be.

premise of $A, A \to B \Rightarrow^0 B$ while rejecting every conclusion—notice that now she looks at the validity of this inference through the lenses of the Strict-Tolerant approach, and this is correct, given how **mcLP** is defined. Consider, now, the case of **mcK3**. We know that the Law of Excluded Middle

$$\varnothing \Rightarrow^0 A \lor \neg A \tag{LEM}$$

does not hold in the system. Nevertheless, its pseudo-metavariant, namely

$$\frac{\varnothing}{\varnothing \Rightarrow^0 A \lor \neg A} \tag{LEM*}$$

holds. Again, on a bounds consequence reading this goes as expected. The **mcK3**-theorist foregoes LEM because according to her logic it is in bounds to non-accept $A \lor \neg A$—i.e. she looks at the validity of $\varnothing \Rightarrow A \lor \neg A$ through the lens of the Strict-Strict approach. But she embraces LEM* because she *accepts* (here is the Strict-Strict reading of metainferences) that it is out-of-bounds to reject $A \lor \neg A$—now, she looks at the validity of $\varnothing \Rightarrow A \lor \neg A$ through the lenses of the Strict-Tolerant approach, and this is again correct, given the way in which **mcK3** is defined. Finally, regarding **mcS3**, this bounds consequence reading can be applied in a straightforward way, demanding that both the conditions for **mcLP** and **mcK3** obtain. Again, in the bounds-consequence reading of metainferential validity, the equivalence between meta$_n$inferences and their pseudo-metavariants breaks apart.

The strategy that we have sketched can be more or less straightforwardly extended to the 'uniform' logics we have previously introduced. In fact, as every meta$_n$inferential validity of **uLP** can be translated into a valid inference of **LP**, a proper reading of metainferential validity of any level might be in terms of the Tolerant-Tolerant approach, that is, in terms of it being out-of-bounds to *non-reject* (which again, does not necessarily involve accepting) every premise while at the same time *rejecting* every conclusion. Thus, the following seems a reasonable way to interpret **uLP**. As its standard for meta$_1$inferences is $\langle \mathbf{ST}_0, \mathbf{ST}_0 \rangle$, Cut is not valid. To illustrate this, let's expand the language by adding a new 1/2-constant λ. Now take an instance of Cut without context and with λ as the cut-formula:

$$\frac{\varnothing \Rightarrow^0 \lambda \qquad \lambda \Rightarrow^0 \varnothing}{\varnothing \Rightarrow^0 \varnothing}$$

The **uLP** theorist foregoes this instance of Cut because according to her logic it is in bounds to non-reject $\varnothing \Rightarrow^0 \lambda$ and $\lambda \Rightarrow^0 \varnothing$ while at the same time rejecting $\varnothing \Rightarrow^0 \varnothing$, all of these inferences being read according to the Strict-Tolerant approach.[18] Nevertheless, as **uLP**'s standard for meta$_2$inferences is $\langle \mathbf{ST}_1, \mathbf{ST}_1 \rangle$, that is $\langle \langle \mathbf{TSST} \rangle, \langle \mathbf{TSST} \rangle \rangle$, the system validates the pseudo-metavariant of Cut, and this includes the instance

$$\begin{array}{l} 2 \\ 1 \end{array} \frac{\varnothing}{\dfrac{\varnothing \Rightarrow^0 \lambda \qquad \lambda \Rightarrow^0 \varnothing}{\varnothing \Rightarrow^0 \varnothing}} \tag{Cut*}$$

(where labels next to horizontal lines indicate the level of the corresponding metainference). This means that a supporter of this logic is willing to *non-reject* (here is the Tolerant-Tolerant

[18]Note that, according to this approach, $\varnothing \Rightarrow^0 \lambda$ and $\lambda \Rightarrow^0 \varnothing$ say that λ should be non-rejected and non-accepted, respectively.

reading in action) the instance of Cut under consideration when its premises are read according to **TS** and its conclusion according to **ST**. What we have said about the uniform **uLP** applies, *mutatis mutandis*, to **uK3**: since every meta$_n$inferential validity of this logic can be translated into a valid inference of **K3**, it seems reasonable to adopt a reading of metainferential validity in terms of the Strict-Strict approach; we leave the detailed examples to the reader. Once again, regarding **uS3**, this bounds consequence reading can be applied in a straightforward way, demanding that both the conditions for **uLP** and **uK3** obtain.

Finally, and though it might seem initially more complicated to give a bounds consequence account of validity for **eq**-logics, this is not the case at all. In fact, in all of these logics the metainferential validity should be understood in the same way as we have interpreted **mc**-logics, but with one important distinction: if the metainference at case has an empty set of premises, it should be understood as a metainference (or inference) of the immediate lower level, and interpreted accordingly. And this is exactly what should be expected of supporters of the Equivalence Thesis as the **eq**-logicians are.

Let us now move to the second issue to be addressed in this section. Admittedly, our meta-classical non-classical logics are highly non-standard. Even if one admits that a logic is an infinite collection containing one validity notion for each metainferential level n, one might feel dubious about them. They all differ from the default logics of their respective validity notions for inferences. Moreover, **mc**-logics and **u**-logics violate the Equivalence Thesis, which, we argued, seems to be a quite reasonable condition. In contrast, **eq**-logics respect the Equivalence Thesis, but at the cost of relativizing the notion of validity in play at any given level to whether or not a metainference has any premises. The question arising, then, is whether our systems constitute genuine logics. We argue for a positive answer.

Over the last years, the literature on philosophical logic has slowly welcomed the idea that validities of level n do not determine validities of level $n+1$. Thus, for instance, we have David Ripley claiming

> **ST** and $\widehat{\mathbf{ST}}$ are distinct: the first says only when a model is a counterexample to a meta$_0$inference, while the second says when a model is a counterexample to a meta$_n$inference for any level n. As far as I know, nobody has so far put forward any endorsement of $\widehat{\mathbf{ST}}$, only of **ST**. And (...) an advocate of **ST** (...) has taken on no commitments at all regarding meta$_n$counterexample relations for $n \geq 1$. [47]

We next explain how it is that this idea began to spread, and why the supporters of the strong Kleene logics $\widehat{\mathbf{LP}}$, $\widehat{\mathbf{K3}}$ and $\widehat{\mathbf{S3}}$ have good reasons to accept it.

One of the most characteristic features of classical logic is that the Deduction Theorem holds in both directions. That is, we have

$$\Gamma \Rightarrow A \text{ is valid} \quad \text{iff} \quad \Rightarrow \bigwedge \Gamma \to A \text{ is valid} \tag{DT}$$

where $\to$ is the material conditional and $\bigwedge \Gamma$ the conjunction of the sentences in Γ. One popular way of explaining DT consists in saying that the material conditional *internalizes* the notion of logical consequence in the object language. It is thought by some that any decent conditional should fulfill this internalizing function.

Supporters of logics $\widehat{\mathbf{LP}}$, $\widehat{\mathbf{K3}}$ and $\widehat{\mathbf{S3}}$ give up DT. They embrace a notion of validity for inferences that does not play nice with the material conditional: $\widehat{\mathbf{LP}}$ violates the right-to-left direction of DT (e.g. MP* is valid but MP is not), $\widehat{\mathbf{K3}}$ the left-to-right direction (R is valid but PR is not),[19] and $\widehat{\mathbf{S3}}$ invalidates both directions. Arguably, then, these logics do not have a material conditional that counts as a decent conditional. In exchange, they can handle paradoxes of various kinds without triviality.

The mentioned systems and classical logic have an important feature in common, though. They all validate the following result, which we might call *Meta Deduction Theorem*:

$$\frac{\Rightarrow^0 A_1 \ \ldots \ \Rightarrow^0 A_n \ \ldots}{\Rightarrow^0 B} \text{ is valid} \quad \text{iff} \quad \{A_1, ..., A_n, ...\} \Rightarrow^0 B \text{ is valid} \qquad (\text{MDT})$$

Thus, for instance, MP and MMP are both valid in $\widehat{\mathbf{CL}}$, and both invalid in $\widehat{\mathbf{LP}}$. A plausible way of explaining MDT consists in saying that the validity of inferences *internalizes* the validity of metainferences. If one understands a validity notion for inferences $\Rightarrow^0$ as a kind of strict conditional, then it is reasonable to expect that any decent validity notion for inference will fulfill this internalizing condition.

Supporters of $\widehat{\mathbf{ST}}$ and $\widehat{\mathbf{TS}}$ go one step further, and give up MDT. They embrace notions of validity for inferences and for metainferences that do not play nice together: $\widehat{\mathbf{ST}}$ violates the right-to-left direction of MDT (e.g. MP is valid but MMP is not), and $\widehat{\mathbf{TS}}$ the left-to-right direction (MEx is valid but Ex is not). On this base, one could think that these systems do not have a decent notion of validity for inferences. In exchange, they can also handle various paradoxical phenomena without triviality, and moreover, they regain DT, so their conditional could be regarded as better, if that matters.

Lastly, supporters of $\mathbf{ST}_\omega$ and $\mathbf{TS}_\omega$ go not one, but infinite steps further. They embrace logics which, following Scambler's [50] terminology, are not closed under their own rules. This means, roughly, that if the base propositional language is sufficiently expressive, then for each level n there are meta$_n$inferences $\Gamma \Rightarrow^n \phi$ that are valid even though each $\gamma \in \Gamma$ is valid and ϕ is invalid. For instance, suppose again that we extend $\mathcal{L}$ with the constant λ; then, $\mathbf{ST}_\omega$ validates the metainference

$$\frac{\Rightarrow^0 \lambda \qquad \Rightarrow^0 \lambda \to \bot}{\Rightarrow^0 \bot}$$

as well as the inferences $\Rightarrow^0 \lambda$ and $\Rightarrow^0 \lambda \to \bot$; however, it does not validate inference $\Rightarrow^0 \bot$. In exchange, these logics can also handle paradoxical phenomena; moreover, they regain the deduction theorem at every metainferential level, that is, they satisfy DT as well as, for every $n \geq 0$, the principle

$$\frac{\Rightarrow^n \gamma_1 \ \ldots \ \Rightarrow^n \gamma_n \ \ldots}{\Rightarrow^n \delta} \text{ is valid} \quad \text{iff} \quad \{\gamma_1, ..., \gamma_n, ...\} \Rightarrow^n \delta \text{ is valid} \qquad (\text{M}_n\text{DT})$$

What should we make of all this? Clearly, it is not our aim to compare the relative merits of all the logics mentioned. We just want to point out that all these systems have something in common, namely, the idea that entailments of some level (and this includes material entailments, viz. sentences of the form $A \to B$) do not determine entailments of higher levels. Indeed, we think that from the literature we can extract an argument that goes more or less like this:

[19]Notice that HP, which as mentioned in Sect. 4 is invalid in $\widehat{\mathbf{K3}}$, is an instance of the left-to-right direction of DT.

(1) It is acceptable to espouse a material conditional that does not internalize meta$_0$validity (initial $\widehat{\mathbf{LP}}$, $\widehat{\mathbf{K3}}$ and $\widehat{\mathbf{S3}}$'s predicament)
(2) If the above is the case, then it is also acceptable to espouse a notion of meta$_0$validity that does not internalize meta$_1$validity ($\widehat{\mathbf{ST}}$ and $\widehat{\mathbf{TS}}$'s predicament)
(3) If the above is the case, then for each $n \geq 1$ it is also acceptable to espouse a notion of meta$_n$validity that does not internalize meta$_{n+1}$validity ($\mathbf{ST}_\omega$ and $\mathbf{TS}_\omega$'s predicament).

The upshot of this line of reasoning would be a liberal approach to the link between meta$_n$inferences and meta$_{n+1}$inferences. According to this approach, if one endorses a certain validity notion for inferences (or a certain sequence containing one VNM_n for each n up to some k), one need not endorse the default logic of this validity notion (or sequence). More formally,

(Weak Metafreedom) By endorsing a validity notion $\mathbf{L}$ for meta$_n$inferences one has not thereby endorsed $\uparrow\mathbf{L}$. *A fortiori*, by endorsing a sequence $\mathbf{L}$ of validity notions, one for each level up to some n, one has not thereby endorsed $\widehat{\mathbf{L}}$.

We submit that the supporters of **LP**, **K3**, and **S3** have good reasons to accept the above argument and thus endorse Weak Metafreedom. To begin with, they have already committed to the first premise of the argument, and the others seem to be plausible statements by analogy.[20] If for whatever reason one has already accepted a material conditional that does not match one's notion of validity for inferences, what prevents one from accepting a notion of validity for inferences that does not match one's notion of validity for metainferences? And, if one has already done the latter, what prevents one from going even further, climbing the meta$_n$inferential hierarchy? The burden of the proof seems to lie on those who reject the legitimacy of these moves.

Admittedly, Weak Metafreedom is not enough to justify the idea that our meta-classical non-classical systems are logics in their own right. That is, one may endorse Weak Metafreedom and still deny that these systems constitute genuine logics. The idea would be that these systems are too liberal in the way that metainferential levels relate to each other. For instance, one may endorse Weak Metafreedom but insist on the Equivalence Thesis at the same time. This would amount to the claim that, given a certain VNM_n $\mathbf{V}$, the only admissible VNM_{n+1}s are those whose standard for conclusions is identical to $\mathbf{V}$; in other words, if one endorses $\mathbf{V}$, then in selecting a VNM_{n+1} one has freedom to choose among various different standard for premises, but one has to choose $\mathbf{V}$ as the standard for conclusions. This position certainly undermines the legitimacy of our **mc**- and **u**-logics. But we do not find it ultimately convincing. Once we adopt a liberal stance towards the standard for premises of meta$_{n+1}$inferences, it seems kind of arbitrary not to allow the same freedom for choosing the standard for conclusions. What would be the reasons for such an asymmetry? This is why we suggest the following strengthened version of the liberal stance towards the link between metainferential validity of different levels:

(Strong Metafreedom) By endorsing a validity notion $\mathbf{L}$ for meta$_n$inferences one has not thereby endorsed $\uparrow\mathbf{L}$, *or any other particular notion of validity for meta$_{n+1}$inferences. A*

[20] Moreover, this slippery-slope argument resembles another famous slippery-slope argument that Priest himself put forward in [44] (distinguishing three levels of paraconsistency; Beall and Restall in [12] also mentioned a fourth level) and that leads from the rejection of Explosion to embracing Dialetheism, i.e., the thesis that there are some inconsistent but non-trivial true theories.

fortiori, by endorsing a sequence **L** of validity notions, one for each level up to some n, one has not thereby endorsed $\widehat{\mathbf{L}}$, *or any other particular logic.*

Indeed, Strong Metafreedom is the position implicit in Ripley's quote above,[21] and we think that, more in general, it underlies the kind of liberal spirit that the study of metainferences prompted in the literature on philosophical logic. As is easy to see, Strong Metafreedom vindicates our meta-classical non-classical systems. Thus, insofar as the thesis is reasonable, our systems can be regarded as genuine logics.

We end this section by arguing that several of our systems are of substantive value for the supporter of **LP**, **K3** or **S3**. The reason is that they are useful in overcoming a difficult challenge that this non-classical logician faces. The challenge stems from the fact that non-classical logicians often use classical logic to prove important metalogical results that, for all we know, would otherwise be unavailable to them. But this is regarded by many as an unacceptable double-standard in the choice o valid patterns of inference. For instance, we have Burgess complaining

> How far can a logician who professes to hold that [her favored logic provides] the correct criterion of a valid argument, but who freely accepts and offers standard mathematical proofs, in particular for theorems about [this] logic itself, be regarded as sincere or serious in objecting to classical logic? [14]

The idea is that, form an epistemic standpoint, the non-classical logician *is not as she ought to be* when she disapproves a logical principle but uses it to reason. Let us call this the *hypocrisy objection* to non-classical logics. Following Rosenblatt [48], we can say that the non-classical logician is in a dilemma: either she uses classical logic in her metatheory or she does not; if she does, then the hypocrisy objection seems to apply; but if she does not, then it seems that she must give up many important metalogical results.

Various responses have been given to address this objection. Some authors stick to the first horn of the dilemma, and justify themselves by assuming some sort of instrumentalist attitude towards metatheory.[22] Others, stick to the second horn, and wholeheartedly embrace the project of developing a non-classical metatheory for their favorite non-classical logic.[23] Lastly, several authors adhere to what has come to be known as the 'recapture strategy'; they claim that, first appearances notwithstanding, the dilemma is false. To develop her metatheory, the non-classical logician does not need to assume classical logic in general; on the contrary, it suffices if she accepts certain *instances* of principles that are valid in classical logic but not in the relevant non-classical system. Rejecting classical logic and accepting those instances—the argument goes—is a coherent and justifiable move.[24] Typically, the recapture strategy proceeds by taking the relevant non-classical theory and strengthening it with the appropriate instances of classical principles; this can be done either by extending the language [e.g. 28, 31], or by just adding axioms and/or principles [e.g. 11, 30]. Then, a 'recapture result' is provided, which shows that the strengthened theory has the desired deductive power—that is, it can prove whatever metalogical results were at stake.

[21] Note that endorsing the Equivalence Thesis is incompatible with Ripley's claim that the supporter of **ST** has taken "no commitments at all regarding meta$_n$counterexample relations for $n \geq 1$".

[22] See e.g. Beall [10]

[23] See Dummett [26] and Badia et. al. [2] for the cases of intuitionism and dialetheism, respectively.

[24] See [53, 36] and [48, 49] for arguments against and in favor of this strategy, respectively.

Many of our meta-classical non-classical logics can be viewed as providing a novel and elegant kind of recapture result. Consider those of our systems that recover positive aspects of classical logic (viz. validities) as opposed to negative aspects (antivalidities). These are all the **mc**-logics, the **eq**-logics and **uLP**. All these systems allow the non-classical logician to stick with her preferred non-classical notion of validity for inferences, while at the same time recovering, by means of the appropriate metainferences, any piece of classical reasoning she wants to perform. To see this, we refresh some of the results from Section 4. In the case of the **mc**-logics and **uLP**, for any classically valid inference $\Gamma \Rightarrow^0 \Delta$ the systems validate the meta$_1$inference

$$\frac{}{\Gamma \Rightarrow^0 \Delta}$$

(Facts 10 and 14.) In the case of the **eq**-logics, for any classically valid inference $\Gamma \Rightarrow^0 \Delta$ the systems validate the meta$_1$inferences

$$\frac{\bot \Rightarrow^0 \top}{\Gamma \Rightarrow^0 \Delta} \qquad\qquad \frac{\{\varnothing \Rightarrow^0 \gamma : \gamma \in \Gamma\}}{\{\varnothing \Rightarrow^0 \delta : \delta \in \Delta\}}$$

Besides, both **mc**-logics and **eq**-logics coincide with $\widehat{\mathbf{CL}}$ in every meta$_n$inference with non-empty premises. Thus, for instance, if a supporter of **LP** wants to perform Modus Ponens (viz. MP), and she endorses **mcLP**, she can apply the pseudo-metavariant MP*; for another case, if a supporter of **K3** want to perform Contraposition (C), and she endorses **eqK3**, she can apply the principle as it stands. The distinctive feature of the recapture result provided by our meta-classical non-classical logics is that, unlike other results present in the literature, it does not require either extending the language of the object theory or beefing the theory up with additional principles. At the level of inferences, the theory stays as it stands; the additional strength comes form the metalevels. This is, we take it, a useful innovation.

6 Conclusions

The understanding of a *logic* as containing validity notions for metainferences of each finite level opens a wide (and in our opinion quite fascinating) range of new possibilities. This paper explored one of these possibilities, namely, that of defining systems that differ from classical logic at the level of inferences but, nonetheless, recover some aspects of classical logics at the metainferential level. We presented three families of such systems: the **mc**-logics, the **u**-logics and the **eq**-logics. We gave informal readings for them, we argued that they deserve to be regarded as *logics* in their own right, and we suggested that they may enjoy important applications. Much work remains to be done in the future. For instance, we would like to develop in full the proof-theory of all these systems. Also, to study how they can be adapted to admit so-called mixed metainferences—studied in [27]. Lastly, some of us think that our meta-classical non-classical systems may be helpful in the context of the philosophical conundrum known as the Adoption Problem, put forward by Kripke [39]; we think that this issue deserves closer inspection.

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