Problem Set 2 - Analysis

Question 1 Find the Taylor polynomial of degree 5 ($n = 5$) for the function $f(x) = e^x$ around the point $x = 0$. Then, evaluate the polynomial at $x = 1$.

Question 2 Let $x(t)$ be differentiable. Show that $\frac{\frac{dx(t)}{dt}}{x} = \frac{d\log(x(t))}{dt}$.

Question 3 Determine whether the following functions are convex/concave/quasi-convex/quasi-concave.
(1) $f(x, y) = \sqrt{x} + \sqrt{y}$
(2) $g(x, y) = \sqrt{xy}$

Question 4 Let (X, d) be a metric space, where $X = \mathbb{R}$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}$, define $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$.
(1) Consider $I_n = (-\frac{1}{n}, \frac{1}{n}), n = 1, 2, ...$ For any n, is I_n an open set or a closed set (or both)? Find $\cap_{n=1}^{\infty} I_n$. Is it an open set or a closed set (or both)?
(2) Consider $J_n = [\frac{1}{n+1}, 1 - \frac{1}{n+1}], n = 1, 2, ...$ For any n, is I_n an open set or a closed set (or both)? Find $\cup_{n=1}^{\infty} J_n$. Is it an open set or a closed set (or both)?

Question 5 The distance between downtown SB and UCSB is 10 miles. Your friend claims it took them 7.9 minutes to drive this distance. Define speeding as the existence of at least one point in time during the trip where the instantaneous speed exceeds 75 miles per hour.
(1) Use the Mean Value Theorem to prove that your friend was speeding according to this definition.
(2) Suppose your friend's vehicle can teleport. Can you still prove that they were speeding according to this definition? If not, identify which condition of the Mean Value Theorem is violated. Clearly state what is your definition of "teleport" mathematically in your argument.

Question 6 Prove that $\mathbb{E}[X^2] - \mathbb{E}[X]^2 \geq 0$ using one of the inequalities that we studied (i.e., not using the property that $Var(X) \geq 0$ directly). For simplicity, consider that X is a discrete random variable with some p.d.f. $P(X)$.

Question 7
(1) Let $Q = f(K, L)$ be a differentiable homogeneous function of degree one. Prove that

$$f(K, L) = K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L}$$

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(2) Let $Q = f(K, L)$ be a differentiable homogeneous function of degree r. We can also have a similar formula:

$$f(K, L) = g(r)(K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L})$$

where $g(r)$ is some function of r. Find $g(r)$.

Question 8 and 9 are Optional.

Question 8

Prove that $f(K, L) = AK^{\alpha}L^{(1-\alpha)}$ is quasiconcave.

Question 9

Consider the following sets. Determine if they are bounded. If the set is bounded, provide an M and a $\mathbf{x}$ such that $B_M(\mathbf{x})$ contains the set.

1. $A = \{x | x \in \mathbb{R} \wedge x^2 \leq 10\}$
2. $B = \left\{x | x \in \mathbb{R} \wedge x + \frac{1}{x} < 5\right\}$
3. $C = \{(x, y) | (x, y) \in \mathbb{R}^2_+ \wedge xy < 1\}$
4. $D = \{(x, y) | (x, y) \in \mathbb{R} \wedge |x| + |y| \leq 10\}$