Topic 5: Continuity and Differentiability[1]

Keywords for Today

- Continuity
- Differentiability
- Derivatives
- Chain Rule
- Linearization
- Taylor's Theorem
- Mean Value Theorem
- L'Hôpital's Rule

Continuity

Definition 1 (Continuity at a Point)

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point $c \in \mathbb{R}$ if:

$$\lim_{x \to c} f(x) = f(c)$$

Formally, this means: $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $0 < |x - c| < \delta$.

[1] Instructors: Camilo Abbate and Sofia Olguin. This note was prepared for the 2025 UCSB Math Camp for Ph.D. students in economics. It incorporates materials from previous instructors, including Shu-Chen Tsao, ChienHsun Lin, and Sarah Robinson.

Let us compare this definition to the epsilon-delta definition of a limit. The difference in the epsilon-delta definition is that we changed $|f(x) - L| < \epsilon$ to $|f(x) - f(c)| < \epsilon$.

That is, we say that function $f(x)$ is continuous at point c if $f(c)$ is exactly the L on the epsilon-delta definition of a limit. This means there are no jumps, breaks, or holes at that point c.

Definition 2 (Continuity on an Interval)

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on an interval $I \subseteq \mathbb{R}$ if it is continuous at every point $c \in I$.

Definition 3 (Continuity of a Function)

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on its entire domain if it is continuous at every point in its domain.

Exercise 1

Determine whether the following functions f are continuous on $[-1, 1]$.

(1) $f(x) = |x|$

(2) $f(x) = x^2$

(3) $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$

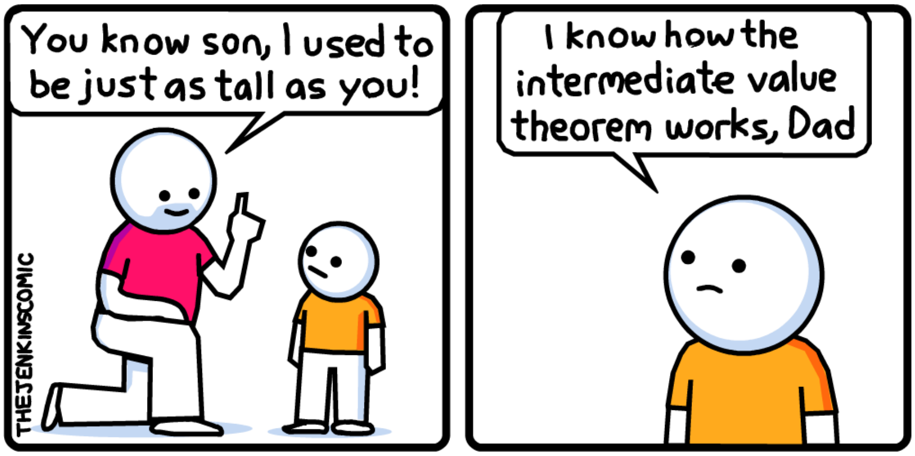
(4) $f(x) = \begin{cases} |x| & \text{if } x \text{ is rational} \\ -x^2 & \text{otherwise} \end{cases}$

There are three important properties of a continuous function.

Theorem 1: Intermediate Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous on the closed interval $[a, b]$, and L is any number between $f(a)$ and $f(b)$ ($\min\{f(a), f(b)\} < L \leq \min\{f(a), f(b)\}$), then there exists at least one $c \in [a, b]$ such that:

$$f(c) = L$$



(Figure source: The Jenkins Comic).

Exercise 2

In fact, the information in the comic above is not sufficient to ensure the dad's "argument" to hold. There are some implicit conditions assumed.

Let $f(t)$ be the mapping from time to the dad's height. Let $f(a)$ be the dad's height at birth, and let $f(b)$ be the dad's height now. Let L be the child's height now.

We observe in the comic that $L < f(b)$. Which other conditions do we need to guarantee the statement that "The dad used to be just as tall as the son"?

The intermediate value theorem may sound trivial, but note that we don't need f to be differentiable or monotone.

Theorem 2: Extreme Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous on the closed interval $[a, b]$, then f attains both a maximum and a minimum value on $[a, b]$.

In other words, there exist points $c, d \in [a, b]$ such that:

$$f(c) \leq f(x) \leq f(d) \quad \text{for all } x \in [a, b]$$

The extreme value theorem is fundamental in optimization.

Exercise 3

Consider $f(x) = (x-1)^2$.

(1) Does f attain both maximum and minimum values on $[0, 2]$?
(2) Does f attain both maximum and minimum values on $(0, 2)$? Which part of Theorem 2 is violated?

Exercise 4

Consider

$$f(x) = \begin{cases} x & \text{if } x \neq 1 \\ -1 & \text{if } x = 1 \end{cases}$$

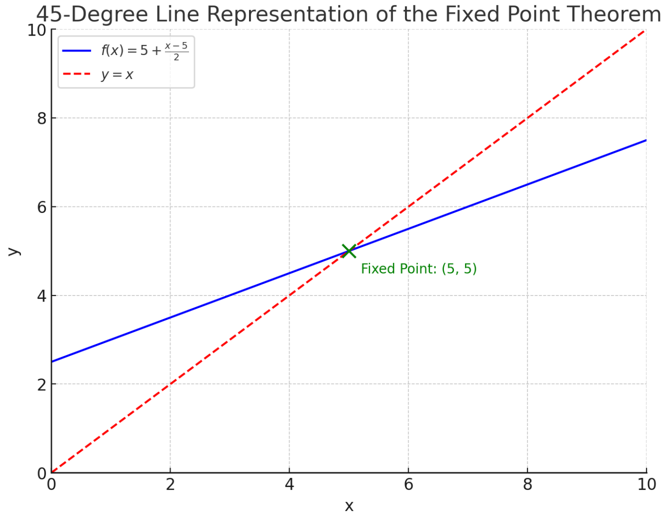
Does f attain both maximum and minimum values on $[0, 1]$? Which part of Theorem 2 is violated?

Theorem 3: Fixed Point Theorem

If $f : [a, b] \to [a, b]$ is continuous, then there exists at least one point $c \in [a, b]$ such that:

$$f(c) = c$$

The fixed-point theorem is fundamental in game theory. One of the most famous application of the fixed-point theorem is the Nash equilibrium. We also use the fixed-point theorem to derive the steady states in macroeconomics.



There is a corresponding fixed point theorem in $\mathbb{R}^N$, which is called the Brouwer's fixed point theorem. The intuition is as followed: Imagine compressing a sponge ball inward. Then, there will be at least one point inside the sponge ball that remains fixed.

The intermediate value theorem and extreme value theorem also hold in $\mathbb{R}^N$. Let's define the continuity in $\mathbb{R}^N$ formally.

Definition 4 (Continuity at a point in $\mathbb{R}^n$)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at a point $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ if:

$$\lim_{\mathbf{x} \to \mathbf{c}} f(\mathbf{x}) = f(\mathbf{c})$$

Formally, this means: $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\|f(\mathbf{x}) - f(\mathbf{c})\| < \epsilon$ whenever $\|\mathbf{x} - \mathbf{c}\| < \delta$.

Here, $\|\cdot\|$ denotes the Euclidean norm, which for a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ is given by:

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Definition 5 (Continuity on a Set in $\mathbb{R}^n$)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on a set $S \subseteq \mathbb{R}^n$ if it is continuous at every point $\mathbf{c} \in S$.

Differentiability

Definition 6 (Difference Quotient)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We define the difference quotient of f as follows:

$$\frac{f(x') - f(x)}{x' - x} \text{ for any } x, x' \in X.$$

Intuitively, this quotient is the **slope** of f across the points x and x'.

If we make the two points extremely close, then the quotient becomes the slope at the point, or the derivative.

Definition 7 (Derivatives)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on some open interval containing a point a.

The derivative of f at the point a, denoted by $f'(a)$, is defined as the limit (if it exists):

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Definition 8 (Differentiability)

A function f is said to be differentiable at a point a if the derivative $f'(a)$ exists.

If f is differentiable at every point in an interval, we say that f is differentiable on that interval.

Theorem 4: Differentiability implies Continuity

If a function f is differentiable at a point a, then f is continuous at a.

However, note that the continuity **does not** imply differentiability.

Exercise 5

Determine if $f(x) = |x|$ is continuous at $x = 0$. Determine if $f(x) = |x|$ is differentiable at $x = 0$.

Here are some derivatives for common functions.

(1) Constants: for any $r \in \mathbb{R}$,
$$f(x) = r \;\Rightarrow\; f'(x) = 0.$$

(2) Powers of x: for any $k \in \mathbb{R} \setminus \{0\}$,
$$f(x) = x^k \;\Rightarrow\; f'(x) = kx^{k-1}.$$

(3) Polynomial functions
$$f(x) = a_n x^n + \cdots + a_1 x + a_0 = \sum_{k=0}^{n} a_k x^k$$
$$\Rightarrow f'(x) = n a_n x^{n-1} + \cdots + a_1 = \sum_{k=1}^{n} k a_k x^{k-1}.$$

(4) The exponential function:
$$f(x) = \exp(x) \;\Rightarrow\; f'(x) = \exp(x).$$

(5) The natural logarithm function:
$$f(x) = \log(x) \;\Rightarrow\; f'(x) = \frac{1}{x}.$$

Theorem 5

Let f, g are defined on $[a, b]$ and are differentiable at $x \in [a, b]$. Then the functions $f + g$, fg, and f/g are differentiable, and

(1) $(f + g)'(x) = f'(x) + g'(x)$;
(2) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$;
(3) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$.

We sometimes use the differential operator $\frac{d}{dx}$ to express the operation "taking derivatives."

Hence, when f is differentiable,

$$\frac{d}{dx}f(x) \equiv \frac{df}{dx}(x) \equiv f'(x).$$

If we define Δ as:

$$\Delta x = x' - x, \ \Delta f(x) = f(x') - f(x)$$

then the difference quotient can then be expressed as $\frac{\Delta f(x)}{\Delta x}$. When we take the limit to the difference quotient, it becomes the notation that we are familiar with, $\frac{df(x)}{dx}$.

When you take derivatives of the composite functions, you will need to apply the chain rule.

Theorem 6

Suppose $f : X \to Y$ is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval $I \in Y$, and g is differentiable at $f(x)$.

Let $h(x) = g \circ f(x) = g(f(x))$, then

$$h'(x) = g'(f(x))f'(x)$$

.

Exercise 6

Find the derivative of the following functions.

(1) $f(x) = \sqrt{x-2}$

(2) $f(x) = \log(x^2)$

(3) $f(x) = (\log x)^2$

There is a strong connection between derivatives and monotonic behavior.

Theorem 7

Suppose f is differentiable in the open interval (a, b). Then

(1) f is monotonically increasing if $f'(x) \geq 0$;

(2) f is monotonically decreasing if $f'(x) \leq 0$;

(3) f is a constant if $f'(x) = 0$.

Linearization

An application of derivatives is to approximate the function values around a specific point (say, a). Let's review the definition of derivatives of f at a. Linearization is used in macroeconomics where the functions of interest might not have analytical representation.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

By rearranging the terms,

$$\lim_{h \to 0} \{ \frac{f(a+h) - f(a)}{h} - f'(a) \} = 0$$

Define $\varepsilon(h) = \frac{f(a+h)-f(a)}{h} - f'(a)$, and note that $\lim_{h \to 0} \varepsilon(h) = 0$. We can rearrange the terms in this definition of $\varepsilon(h)$ to have:

$$f(a+h) = f(a) + f'(a)h + \varepsilon(h)h$$

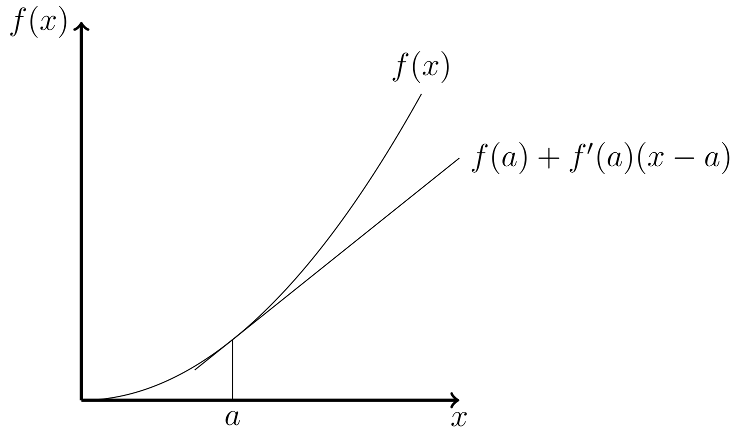
Define $x = a + h$, then we have:

$$f(x) = f(a) + f'(a)(x-a) + \varepsilon(x-a)(x-a)$$

Hence we can loosely write

$$f(x) \approx f(a) + f'(a)(x-a).$$

With this process, we approximate $f(x)$ with a linear function of the intercept of $f(a)$ and a slope of $f'(a)$. This is called a **linear approximation** of f at a.



Higher Order Derivatives

Let $g = \frac{df}{dx}$. If g is also differentiable, we can denote

$$g' \equiv \frac{d}{dx}\frac{df}{dx} \equiv \frac{d^2 f}{(dx)^2} \equiv f''.$$

This function f'' is called the **second order derivative** of f. The interpretation of f'' is *the change in slopes*. We can further define the higher order derivatives with the similar manner, and we denote the k-th derivative of f with $f^{(k)}$.

We say f is in the family of functions $\mathcal{C}^k$ if the derivatives f', f'',$\ldots$,$f^{(k)}$ exists and are continuous. The function f is **smooth** or **infinitely differentiable** if for any $k \in \mathbb{N}$, $f^{(k)}$ exists, and we denote $f \in \mathcal{C}^\infty$.

Taylor's theorem

Taylor's theorem provides a way to approximate the function values around a specific point.

Theorem 8

Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is $(n+1)$-times continuously differentiable on an open interval containing the point a.

Then, $\forall x$ in this interval, the function $f(x)$ can be expressed as the **Taylor polynomial**:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

where the remainder term $R_n(x)$ is given by the **Lagrange form of the remainder**:

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Exercise 7

Write out the Taylor polynomial of $f(x)$ around point a, with $n = 0$.
Write out the Taylor polynomial of $f(x)$ around point a, with $n = 1$. Compare it with the linearization.

Suppose $f(x) = x^2$ and $a = 1$. Draw the above two Taylor polynomials.

When $n = 0$, the Taylor theorem is called the mean value theorem.

Theorem 9: The mean value theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).

Then, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Here is an exercise for writing down a model and derive results using math tools.

Exercise 8

The distance between downtown SB and UCSB is 10 miles. Suppose your friend tells you it took them 7.9 minutes to drive from downtown SB to UCSB.

Define speeding as there exists at least one point in time such that the speed, which exists in $\mathbb{R}$, is greater than 75 miles per hour. Use the mean value theorem to prove that your friend was speeding according to this definition.[a]

[a] Extension: What if your friend's vehicle can teleport? Can we still prove that they are speeding according to this definition? If not, which condition for the mean value theorem is violated?

L'Hôpital's rule

L'Hôpital's rule is used to find the limit of a quotient of two functions when the limit results in an indeterminate form, such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Theorem 10: L'Hôpital's rule

Suppose functions f and g are both differentiable on an open interval I containing c, except possibly at c itself.

If:

- $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, or
- $\lim_{x \to c} f(x) = \pm\infty$ and $\lim_{x \to c} g(x) = \pm\infty$,

and the limit of their derivatives exists:

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L,$$

Then,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} = L.$$

Exercise 9

1. Evaluate the following limit: $\lim_{x \to 0} \frac{e^x - 1}{x}$. Find the limit if it exists.

2. Evaluate the following limit: $\lim_{x \to 0} \frac{|x|}{x}$. Find the limit if it exists.