Problem Set 3 - Analysis [1]

Due date: 09/09

Question 1 Let X be a random variable with $\mathbb{E}[X] = 0$. Show that $\mathbb{E}[X^2] > 0$ if X takes more than one value with positive probability.
(Hint: Use Jensen's inequality.)

Question 2 Consider the following expression for the reservation wage w_R:

$$w_R = b + \frac{\beta}{1-\beta} \int_{w_R}^{\overline{w}} (w - w_R)\, dF(w)$$

Use integration by parts to show that this expression can be rewritten as:

$$w_R = b + \frac{\beta}{1-\beta} \int_{w_R}^{\overline{w}} [1 - F(w)]\, dw$$

where $F(w)$ is the (cumulative) distribution of wages, with support $(\underline{w}, \bar{w})$.
(Hint: consider how the CDF behaves at the boundaries of its support.)

Question 3 Consider the following constrained utility maximization problem:

$$\max_{x_1, x_2} \alpha \ln(x_1) + \beta \ln(x_2) \qquad m \geq p_1 x_1 + p_2 x_2, \quad x_1 \geq 0, \quad x_2 \geq 0$$

where $\alpha > 0$, $\beta > 0$, $m > 0$, $p_1 > 0$ and $p_2 > 0$.

(a) Find the demand functions (or correspondences) $x_1^*(p_1, p_2, w)$ and $x_2^*(p_1, p_2, w)$, assuming the budget constraint binds.

(b) Find the demand functions (or correspondences) using the Karush-Kuhn-Tucker conditions.

(c) Verify that the demand functions solve the maximization problem. (You may refer to the theorems discussed in class.)

(d) Find the indirect utility function $v(p_1, p_2, m) = \alpha \ln(x_1^*) + \beta \ln(x_2^*)$.

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Question 4 Consider the following constrained expenditure minimization problem:

$$\min_{h_1,h_2} p_1h_1 + p_2h_2 \qquad \alpha \ln(h_1) + \beta \ln(h_2) \geq u, \quad h_1 \geq 0, \quad h_2 \geq 0$$

where $\alpha > 0$, $\beta > 0$, $u > 0$, $p_1 > 0$ and $p_2 > 0$.

(a) Find the demand functions (or correspondences) $h_1^*(p_1, p_2, u)$ and $h_2^*(p_1, p_2, u)$.

(b) Find the expenditure function $e(p_1, p_2, u) = p_1h_1^* + p_2h_2^*$.

(c) Verify the duality property:

$$x^*(p_1, p_2, e(p_1, p_2, u)) = h^*(p_1, p_2, u) \qquad \text{and} \qquad h^*(p_1, p_2, v(p_1, p_2, m)) = x^*(p_1, p_2, m).$$

using the demands x_1^*, x_2^*, and indirect utility function v derived in Question 3.

Question 5 Suppose the motion of capital, $\dot{k}$, satisfies the differential equation:

$$\dot{k} = 0.03k + 0.01$$

(a) Find the general solution ($k(t)$) to this ODE.

(b) Let $k(0) = 100$. Find the particular solution ($k(t)$) to this ODE.

Question 6 Assume the capital-output ratio $x(t) = k(t)/y(t)$ evolves according to:

$$\dot{x}(t) = s_1(1 - \alpha) - (\delta + n)(1 - \alpha)x(t)$$

(a) Find the general solution ($x(t)$) to this ODE.

(b) Let $x(0) = \frac{s_0}{\delta+n}$. Find the particular solution ($x(t)$) to this ODE.