Algebra (Winter) Notes

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Note: Theorem numbers come from the order they are presented in lecture, and do not correspond to any textbook or written course material.

Week 1

Introduction to Groups

1.1 What is a group?

Definition of a group:
A **group** G is a nonempty set together with a multiplication $G \times G \to G$ satisfying

1. $(ab)c = a(bc) \forall a, b, c, \in G$, (Associativity)
2. there exists $e \in G$ such that $ea = ae = a \forall a \in G$, (Identity)
3. and for every $a \in G$ there exists $b \in G$ such that $ab = ba = e$. (Inverse)

Example of a group:
Let $\mathbb{R}^{\times} = \mathbb{R}^{\dagger} = \{a \in \mathbb{R} : a \neq 0\}$ together with multiplication on $\mathbb{R}$.
Associativity is immediate.
The identity is $1 \in \mathbb{R}^{\times}$.
For every $a \in \mathbb{R}^{\times}$, $\frac{1}{a} \in \mathbb{R}$ and $a(\frac{1}{a}) = \frac{1}{a}(a) = 1$.
So $\mathbb{R}^{\times}$ is a group.

Remark: When we need to highlight the group multiplication we write a group as a pair of the set and the multiplication, e.g., $(\mathbb{R}, +), (\mathbb{R}, \cdot)$.
From now on, G is **always** a group.

Theorem 1.1
There is a unique identity element in G.

Theorem 1.2 Cancellation
Suppose $ba = ca$ for $a, b, c \in G$. Then $b = c$

Proof. Let $d \in G$ be an inverse for a, i.e. $da = ad = e$. Multiplying on the right by d, we obtain

$$\begin{aligned}(ba)d = (ca)d &\implies b(ad) = c(ad)\\ &\implies be = ce\\ &\implies b = c.\end{aligned}$$

□

Theorem 1.3 Uniqueness of Inverses
For every $a \in G$ there is a unique element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$.

Proof. Suppose $a \in G$ and $b, b' \in G$ are inverses of a, then

$$ba = e = b'a \implies b = b'$$

(by theorem 1.2) □

Example of inverses in different groups:

1. For $b \in \mathbb{R}^\times$, $b^{-1} = \frac{1}{b}$.
2. For $b \in \mathbb{R}$ under addition $b^{-1} = -b$.
3. For $b \in \mathbb{Z}_n$, $b^{-1} = n - b$.

Example of groups using a field F:

1. $(F, +)$ is a group (Imitate $(\mathbb{R}, +)$).
2. $(F^\times, \cdot)$ where $F^\times = F^\dagger = \{a \in F : a \neq 0\}$ is a group. In particular, if p is a prime number, then $\mathbb{Z}_p^\times = \{1, \ldots, p-1\}$ is a group.
3. The set of $m \times n$ matrices with entries in F, $M_{mn}(F)$ is a group under addition. When $n = 1$, $M_{m1}(F) = F^m$.
4. The set of invertible $m \times n$ matrices with entries in F, $GL(n, F) = \{A \in M_{mn}(F) : \det(A) \neq 0\}$ together with matrix multiplication is called (rank n) **general linear group** (over F). The identity matrix $I \in GL(n, F)$ is the identity. $\det(A) \neq 0 \implies \exists A^{-1} \in GL(n, F)$ such that $AA^{-1} = A^{-1}A = I$.

Example of the symmetries of the equilateral triangle:
Let σ = flip through the vertical axis. Let ρ = rotation by $\frac{2\pi}{3}$.
We can compose two symmetries, e.g., $\sigma\rho = \sigma \cdot \rho$.
We can show that the symmetries given by σ and ρ under composition are $\left\{e, \rho, \rho^2, \sigma, \sigma\rho, \sigma\rho^2\right\}$ where e = doing nothing.
We call this set D_3. It forms a group under composition. Clearly $\rho^3 = \rho\rho\rho = e$, $\sigma^2 = \sigma\sigma = e$, and $\sigma\rho\sigma = \rho^2 = \rho^{-1}$.

Definition of a dihedral group:
The **dihedral group** of order $2n$ is defined by

$$D_n = \left\{e, \rho, \ldots, \rho^{n-1}, \sigma, \sigma\rho, \ldots, \sigma\rho^{n-1}\right\}$$

where $p^n = e$, $\sigma^2 = e$, and $\sigma\rho\sigma = \rho^{-1}$. This is a group with the multiplication given by $\sigma\rho\sigma = \rho^{-1}$.

Remark: D_n is the group of symmetries of a regular n-gon.

Definition of an Abelian Group:
A group G is **abelian (commutative)** if $ab = ba$ for all $a, b \in G$

Example of classifying groups:
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1. $(F, +)$ where F is a field is Abelian.
2. $(F^\times, \cdot)$ where F is a field is Abelian.
3. $(M_{mn}(F), +)$ is Abelian.
4. $(GL(n, F), \cdot)$ is not Abelian.
5. D_n is not Abelian.

Definition of the group of units:
Let $n \geq 2$ and $U(n) = \{1 \leq k \leq n - 1 : \gcd(k, n) = 1\}$.
$U(n)$ is called the **group of units** of $\mathbb{Z}_n$

Recall Facts about $d = \gcd(a, b)$*:* .

1. $d \mid a$ and $d \mid b$, and d is the largest integer with this property
2. There exists $l, m \in \mathbb{Z}$ such that $\gcd(a, b) = la + mb$
3. $\gcd(a, b)$ is the smallest positive $\mathbb{Z}$-linear combination of a and b.
4. If $f \mid a$ and $f \mid b$ then f divides $\gcd(a, b) = la + mb \implies f \mid d$

Example of $U(n)$ together with multiplication $\mod n$ is a group:
Facts 2 and 3 tell us that $\gcd(k, n) = 1 \iff \exists l, m \in \mathbb{Z}$ such that $lk + mn = 1$.
So $U(2) = \{1\}, U(3) = \{1, 2\}, U(4) = \{1, 3\}, U(5) = \{1, 2, 3, 4\}$, etc.
So $U(p) = \{1, \ldots, p-1\} = \mathbb{Z}_p^{\times}$ where p is prime.

Definition of exponentiation:
Suppose $g \in G$.

1. $g^0 = e$
2. $g^n = g \cdot \dots \cdot g$ (n times)
3. $g^{-n} = (g^{-1})^n$

Theorem 1.4 Socks and Shoes
Suppose $a, b \in G$. Then $(ab)^{-1} = b^{-1}a^{-1}$ (only relevant for non-abelian groups)

Proof.

$$(ab)(b^{-1}a^{-1}) = aea^{-1} = aa^{-1} = e$$
$$(b^{-1}a^{-1})(ab) = b^{-1}eb = b^{-1}b = e$$

□

Definition of the order of a group and its elements:
The number of elements in G is called the **order** of G. Suppose $a \in G$. Then the **order of a** is the largest positive integer n such that $a^n = e$. If no such integer exists, we say a has **infinite order**. We denote the order of a by $|a|$.

Example of the order of $\{e\}$:
We know $|\{e\}| = 1$, and $e^1 = e \implies |e| = 1$

Example of the order of $\mathbb{R}^{\times}$:
$\mathbb{R}^{\times}$ is an infinite group so it has infinite order.
Obviously, $|1| = 1$.
$|-1| = 2$ since $(-1)^2 = 1$ and $(-1)^1 \neq 1$.
All other real numbers in $\mathbb{R}^{\times}$ have infinite order.

Example of the order of D_3:
$|D_3| = 6$.
$|\sigma| = 2, |\rho| = 3, \left|\rho^2\right| = 3, |\sigma\rho| = 2, \left|\sigma\rho^2\right| = 2$.

1.2 Subgroups and subgroup tests

Definition of a subgroup:
A **subgroup** of G is a subset $H \subseteq G$ which is a group under the same group multiplication as G.

Example of subgroups:
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1. $\{\pm 1\} \subseteq \mathbb{R}^{\times}$ is a subgroup

2. $\mathbb{Z}_5 \subseteq \mathbb{Z}$ is not a subgroup of $\mathbb{Z}$ since they have different group multiplications

Theorem 1.5 2-step subgroup test
Suppose H is a non-empty subset of G. Then H is a subgroup of G if and only if:

1. $a, b \in H \implies ab \in H$ (closure under multiplication)
2. $a \in H \implies a^{-1} \in H$ (closure under inverse)

Theorem 1.6 1-test subgroup test
$\emptyset \neq H \subseteq G$ is a subgroup $\iff a, b \in H \implies ab^{-1} \in H$

Proof. The forward direction is immediate.
" $\impliedby$ " Suppose 1 and 2 hold. 1 tells us that the group multiplication on G restricts to a multiplicationon H. The associativity of this multiplication on H is inherited from the associativity of the group multiplication on G.
By 1 and 2, for any $a \in H$, $a^{-1} inH$ and $e = aa^{-1} \in H$. Therefore $e \in H$.
Finally, 2 is the inverse axiom for H. $\square$

Example of showing subgroup-ness:
Let $\mu_4 = \left\{a \in \mathbb{C}^{\times} : a^4 = 1\right\} = \{1, -1, i, -i\}$.
$\mu_4 \neq \emptyset$.
$a, b \in \mu_4 \implies (ab)^4 = a^4 b^4 = (1)(1) = 1 \implies ab \in \mu_4$
$a \in \mu_4 \implies (a^{-1})^4 = a^{-4} = (a^4)^{-1} = 1^{-1} = 1 \implies a^{-1} \in \mu_4$

Theorem 1.7 Finite subgroup test
Suppose $H \neq \emptyset$ is a finite subset $H \subseteq G$. Then H is a subgroup $\iff$ $a, b \in H \implies ab \in H$.

Proof. " $\implies$ " Follows from 2-step subgroup test.
" $\impliedby$ " By the 2-step subgroup test it is enough to show that if $a, b \in H \implies ab \in H$ then $b \in H \implies b^{-1} \in H$ also holds. Suppose $a, b \in H \implies ab \in H$ (*). Suppose $e \neq b \in H$. Let's prove $b^{-1} \in H$
By (*), $b^2 = bb \in H$, and by induction, $b^n \in H$ for all $n \geq 1$.
Since H is a finite set, $b^k = b^j$ for some $k > j \geq 1 \implies b^k b^{-j} = b^j b^{-k} = e \implies b^{k-j} = e$ for $k - j \geq 1$.
So $b^{-1} = b^{k-j-1}$. $k - j - 1$ cannot be zero, since then $b = e$. So $k - j - 1 \geq 1$ and so $b^{-1} = b^{k-j-1} \in H$. If $b = e \in H$, then its inverse (itself) is obviously also in H. □

Example of a finite subgroup:
Consider $\{1, i, -1, -i\} \subseteq \mathbb{C}^{\times}$. By the finite subgroup test, it suffices to show that $\{1, i, -1, -i\}$ is closed under multiplication to prove that it is a subgroup. This can be done by brute force.

Week 2

Cyclic Subgroups

Definition of a cyclic group:
A group G is called **cyclic** if there is an element $a \in G$ such that $G = \{a^j : j \in \mathbb{Z}\}$. a is called a **generator** of G. We indicate that G is a cyclic group generated by a with the notation $G = \langle a \rangle$.

Theorem 2.1
Suppose $a \in G$. Then $\langle a \rangle$ is a subgroup of G.

Proof. Suppose $a^m, a^n \in \langle a \rangle$ where $m, n \in \mathbb{Z}$. Then $a^m a^n = a^{m+n} \in \langle a \rangle$ since $m + n \in \mathbb{Z}$. Also $a^{-m} \in \langle a \rangle$ for all m since $-m \in \mathbb{Z}$, and $a^m a^{-m} = a^0 = e = a^0 = a^{-m} a^m$.
By the 2-step subgroup test $\langle a \rangle$ is a subgroup. $\square$

Definition of a cyclic subgroup:
The subgroup $\langle a \rangle \subseteq G$ is called the **cyclic subgroup** generated by $a \in G$.

Example of generators:
Take $G = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ together with addition mod 6.
$\mathbb{Z}_6 = \langle 1 \rangle$ since $n(1) = n \mod 6$. Note that we also have $\mathbb{Z}_6 = \langle 5 \rangle$.

Remark: In general, $\mathbb{Z}_n$ is cyclic and generated by $\langle -1 \rangle$. All finite cyclic are isomorphic to Z_n for some n.
Remark: For $a \in G$, $\langle a \rangle = \langle a^{-1} \rangle$.

Example of the integers:
Take $G = \mathbb{Z}$.

$\langle 1\rangle = \{j1 : j \in \mathbb{Z}\} = \mathbb{Z}$.
$\langle 2\rangle = \{j2 : j \in \mathbb{Z}\} =$ even numbers $\subset \mathbb{Z}$.
$\langle m\rangle = \{jm : j \in \mathbb{Z}\} =$ integers divisible by m for $m \neq 0$.
$\langle 0\rangle = \{0\}$.

Remark: Infinite cyclic groups are all isomorphic to $\mathbb{Z}$.

Definition of the centre of a group:
The **centre** of G is the subset

$$Z(G) = \{x \in G : xa = ax \forall a \in G\}$$

i.e., the elements that commute with everything in G.

Theorem 2.2
$Z(G)$ is a subgroup of G.

Proof. Suppose $x, y \in Z(G) and a \in G$. Then $(xy)a = x(ya) = xay = axy = a(xy)$. Therefore $xy \in Z(G)$.
Moreover, $xa = ax \implies x^{-1}xa = x^{-1}ax \implies a = x^{-1}ax \implies ax^{-1} = x^{-1}axx^{-1} \implies ax^{-1} = x^{-1}a \implies x^{-1} \in Z(G)$.
By the 2-step subgroup test, $Z(G)$ is a subgroup of G. □

Remark:

1. G is abelian $\iff Z(G) = G$
2. $Z(G)$ is abelian (even when G is not)
3. $Z(D_3) = \{e\}$ (brute force)
4. $x \in Z(G) \iff xax^{-1} = a$ for all $a \in G \iff axa^{-1} = x$ for all $a \in G$

Example of a non-trivial center:
$Z(GL(2,\mathbb{R})) = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{R}^{\times} \right\}$

Definition of the centralizer:
Fix $b \in G$. The **centralizer** of b in G is

$$\begin{aligned} C_G(b) = C(b) &= \{a \in G : ab = ba\} \\ &= \left\{a \in G : aba^{-1} = b\right\} \end{aligned}$$

Theorem 2.3
For any $b \in G$, $C_G(b)$ is a subgroup.

Proof. Subgroup test. $\square$

Remark:

1. $C_G(e) = G$
2. $C_G(b) = G \iff b \in Z(G)$
3. $e \in C_G(b)$, $\langle b \rangle \subseteq C_G(b)$

Example of a centralizer:
$C_{GL(2,\mathbb{R})}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R}^\times\right\}$

Recall: G is cyclic if $G = \langle a \rangle = \left\{a^j : j \in \mathbb{Z}\right\}$ for some $a \in G$.

Theorem 2.4
Suppose $a \in G$. Then

1. If $|a| = \infty$, then $a^k = a^j \iff j = k$
2. If $|a| = n$, then $a^k = a^j \iff n$ divides $k - j$

Proof.

1. Suppose $|a| = \infty$. This means $a^n \neq e$ for any $n \geq 1$. Suppose now $a^k = a^j$ with $k \geq j$. Then $a^k a^{-j} = aja^{-j} = e \implies a^{k-j}$ for $k - j \geq 0$. Since $a^n \neq e \forall n \geq 1$, we have $k - j = 0 \implies k = j$.
2. Suppose $|a| = n$. This means $a^n = e$ and n is the least positive number satisfying this equation. Suppose $a^k = a^j$ with $k \geq j$. Then $a^{k-j} = e$ where $k - j \geq 0$. By definition of n, $n \leq k - j$. By the division algorithm, $k - j = qn + r$ where $q, r \in \mathbb{Z}$ are unique and $0 \leq r \leq n - 1$.
$e = a^{k-j} = a^{qn+r} = a^{qn}a^r = (a^n)^q a^r = e^q a^r = ea^r = a^r$, so $r = 0$ by the minimality of n, and so $k - j = qn \implies \frac{k-j}{n} = q \in Z \implies n$ divides $k - j$.
Conversely if $qn = k - j$, then $a^{k-j} = (a^n)^q = e^q = e \implies a^k = a^j$.

$\square$

Remark: In part 2., n divides $k - j \iff (k-j) \mod n = 0 \iff k \mod n = j \mod n$

Corollary 2.5
Suppose $|a| = n$. Then $a^k = e$ for some $k \in \mathbb{Z} \iff$ k is a multiple of $|a|$

Proof. Suppose $a^k = e$. Then $a^k = a^0$, so n divides $k - 0 = k$. ☐

Corollary 2.6
Suppose $a \in G$. Then

1. If $|a| = n$ then $\langle a \rangle = \left\{e, a^1, a^2, \ldots, a^{n-1}\right\}$ and $|\langle a \rangle| = |a|$.
2. If $|a| = \infty$, then $\langle a \rangle$ is infinite and $|\langle a \rangle| = |a| = \infty$

Proof. Didn't take notes for this one. ☐

Corollary 2.7
Suppose G is a finite group and $a, b \in G$. Then

1. $|a|, |b|$ are finite
2. If $ab = ba$ then $|ab|$ divides $|a|\,|b|$

Proof.

1. Suppose by way of contradiction that $|a|$ is infinite. Then $\langle a \rangle \subseteq G$ is infinite. But G is finite so $|\langle a \rangle| \leq |G|$ is a contradiction.
2. $(ab)^{|a||b|} = a^{|a||b|}b^{|a||b|} = (a^{|a|})^{|b|}(b^{|b|})^{|a|} = e^{|b|}e^{|a|} = e$

☐

2 examples omitted. Sorry, I'm prepping for my tutorial later!

Theorem 2.8
Suppose $a \in G$ and $|a| = n$. Then for any $k \geq 1$, $\langle a^k \rangle = \langle a^{\gcd(n,k)} \rangle$ and $\left|a^k\right| = \frac{n}{\gcd(n,k)}$

Theorem 2.9 Fundamental Theorem of Cyclic Groups
Suppose $G = \langle a \rangle$ is cyclic and $|G| = n$. Then

1. Every subgroup H is cyclic and $k = |H|$ divides $n = |G|$, i.e., k is a divisor of n

2. For every divisor k of n, there is a unique subgroup of G of order k and it is equal to $\langle a^{\frac{n}{k}} \rangle$

Proof. 1. Suppose H is a subgroup of G and $H \neq \langle e \rangle$. Let $m \geq 1$ be the least power of a such that $a^m \in H$. Since H is closed under multiplication and inversion, $\langle a^m \rangle \subseteq H$. Suppose $a^j \in H$. By the division algorithm, $j = qm + r$ with $0 \leq r \leq m \implies a^j = (a^m)^q a^r \implies a^j (a^m)^{-q} = a^r$, so since $a^j, (a^m)^{-q} \in H$, $a^r \in H \implies r = 0$ by the minimality of m.

2. Suppose k divides n, i.e. $\frac{n}{k}$ is an integer. Recall that $\left|\langle a^{\frac{n}{k}} \rangle\right| = \left|a^{\frac{n}{k}}\right| = k$. It follows that $\left|\langle a^{\frac{n}{k}} \rangle\right| = k$.

   Suppose $H \subseteq \langle a \rangle$ is a subgroup and $|H| = k$. By part 1, $H = a^m$ for some $m \geq 1$. By Theorem 2.8, $k = |H| = |\langle a^m \rangle| = |a^m| = \frac{n}{\gcd(m,n)} \implies \gcd(m, n) = \frac{n}{k}$.

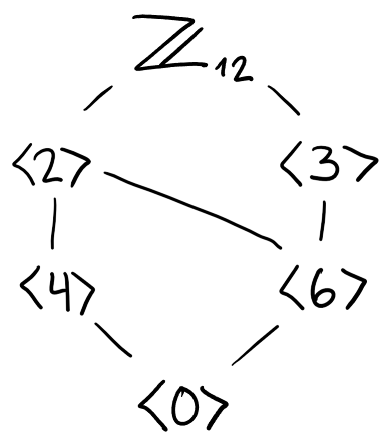
   By Theorem 2.8 again, $H = \langle a^m \rangle = \langle a^{\gcd(m,n)} \rangle = \langle a^{\frac{n}{k}} \rangle$.

$\square$

Example of the subgroups of $\mathbb{Z}_{12}$:
The divisors of $n = 12$ are $1, 2, 3, 4, 6, 12$

- $k = 1$: $\langle 0 \rangle$
- $k = 2$: $\langle 6 \rangle = \{0, 6\}$
- $k = 3$: $\langle 4 \rangle = \{0, 4, 8\}$
- $k = 4$: $\langle 3 \rangle = \{0, 3, 6, 9\}$
- $k = 6$: $\langle 2 \rangle = \{0, 2, 4, 6, 8, 10\}$
- $k = 12$: $\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

Note: The lattice of subgroups of $\mathbb{Z}_{12}$ illustrates the containment relationships.



Remark: In $\mathbb{Z}_n$, clearly $\langle m \rangle \subseteq \langle k \rangle \iff m \in \langle k \rangle \iff ka = m \iff$ k divides m.

Example of subgroups of $\mathbb{Z}_p$:
Consider $\mathbb{Z}_p$ where p is prime. The only subgroup of $\mathbb{Z}_p$ is $\langle 0 \rangle$.

Week 3

Permutation Groups (Symmetric Groups)

Definition of the Euler ϕ-function:
The Euler ϕ-function is defined for every positive integer $d \geq 1$ by

$$\phi(d) = \begin{cases} 1 \text{ if } d = 1 \\ |\{1 \leq j \leq d-1 : \gcd(j,d) = 1\}| \end{cases}$$

Definition of:
Suppose $A \neq \phi$ is a set. A **permutation** of A is a bijection $\beta : A \to A$ (1-1, onto). The **permutation group** (**symmetric group**) of A is the set of permutations of A under composition.

Recall some facts about functions: Let S_A be the symmetric group of $A \neq \phi$.

If $\alpha, \beta \in S_A$ then $\alpha \circ \beta(a) = \alpha(\beta(a))$ for all $a \in A$.

From MATH1800 composition of 1-1 and onto functions is again 1-1 and onto, i.e., $\alpha \circ \beta \in S_A$.

From MATH1800 $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$ for all $\alpha, \beta, \gamma \in S_A$. α permutation $\iff$ α is invertible under composition.s

Remark: Define $e \in S_A$ by $e(a) = a$ for all $a \in A$. Clearly $e \circ \alpha(a) = e(\alpha(a)) = \alpha(a)$ for all $a \in A \implies e \circ a = a$. We see that S_A truly is a group.

Example of:
Take $A = \{1, 2, 3\}$. What are the permutations in $S_3 = S_A$?

- $e \in S_3 : e(1) = 1, e(2) = 2, e(3) = 3$.
- $\beta \in S_3$ where $\beta(1) = 2, \beta(2) = 3, \beta(3) = 1$.

Let's rewrite β as follows: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \mathbb{R}$.

In general for any $\alpha \in S_3$, we may rewrite it as $\begin{bmatrix} 1 & 2 & 3 \\ \alpha(1) & \alpha(2) & \alpha(3) \end{bmatrix} = \mathbb{R}$.

The number of permutations is given by the number of choices. This is $3! = 3 \cdot 2 \cdot 1$. We just proved that $|S_3| = 3! = 6$.

Similar reasoning tells us that $|S_n| = n!$ for every $n \geq 1$.

Question
Paul Mezo said we "know everything" about linear algebra. What does that mean?

Answer
There are no unsolved problems in finite linear algebra.

3.1 Cycle Notation

Consider S_3 and $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \in S_3$. We rewrite this permutation as follows: $(1\ 2\ 3)$.

Notice that $\alpha = (1\ 2\ 3) \neq (1\ 3\ 2) = \beta$, but they are both 3-cycles.

Also, $\gamma = (1\ 2)$ is the permutation such that $\gamma(1) = 2, \gamma(2) = 1, \gamma(3) = 3$. It's a 2-cycle.

We omit 1-cycles.

The six permutations in S_3 in cycle notation are $e, (12), (13), (23), (123), (132)$.

Example of cycles of S_4:
Consider S_4. $|S_4| = 24 = 4!$.

- e,
- $(1\ 2), (1\ 3), (1\ 4), (2\ 3), (2\ 4), (3\ 4)$
- $(1\ 2\ 3), (1\ 3\ 4), \ldots$
- $(1\ 2\ 3\ 4), (1\ 2\ 4\ 3), \ldots$
- $(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)$

Definition of disjoint cycles:
Two cycles $(a_1\ a_2\ \ldots\ a_m), (b_1\ b_2\ \ldots\ b_k) \in S_n$ are **disjoint** if $a_j \neq b_l$ for any j, l. Their product can be written equally in either order.

Composition of permutations is interpreted as products of cycles as follows:

Example of compositions in S_7:

- $(6\ 2\ 3)(1\ 2) = (1\ 3\ 6\ 2)$
- $(1\ 2)(3\ 4\ 7)(2\ 3) = (2\ 4\ 7\ 3\ 1)$
- $(1\ 3)(2\ 4\ 5\ 6\ 7)(3\ 2)(1\ 2\ 5) = (2\ 6\ 7)(5\ 3\ 4)$

Remark:

1. Some authors move from left to right, one cycle to the next. We move from right to left.
2. Cycles don't tell us which S_n they live in.

Example of powers of a k-cycle:
Consider $(a_1\ \dots\ a_k) \in S_n$.

1. $(a_1\ \dots\ a_k)^2 = (a_1\ a_2\ a_3\ \dots\ a_k)(a_1\ a_2\ a_3\ \dots\ a_k)$ sends a_1 to a_3, and a_l to a_{l+2} if $l \leq k-2$. Sends $a_{k-1} \to a_1$, $a_k \to a_2$.
2. $(a_1\ \dots\ a_k)^j$ sends a_l to $a_{(l+j) \mod k}$.

In particular $(a_1\ \dots\ a_k)^k$ sends a_l to $a_{(l+j) \mod k} = a_{l \mod k} = a_l$ for $1 \leq l \leq k$, so $(a_1\ \dots\ a_k)^k = e \implies |(a_1\ \dots\ a_k)| = k$.

Theorem 3.1
Every permutation in S_n is a product of disjoint cycles. The products of disjoint cycles $\alpha, \beta \in S_n$ commute, i.e., $\alpha\beta = \beta\alpha$.

Proof. Proof omitted. □

Remark: Products of disjoint cycles can be written in more than one way to represent a single permutation in S_n.

$$(1\ 2\ 3)(5\ 6) = (5\ 6)(1\ 2\ 3) = (5\ 6)(2\ 3\ 1)$$

Question
Dr. Mezo said they were unique "modulo" changing the order. Why use this language? What's the connection to modulo here?

Definition of the least common multiple:
The **least common multiple** of $m, n \geq 1$ is the smallest positive integer k such that m divides k and n divides k. We write $k = \operatorname{lcm}(m, n)$.

Example of finding LCM:

.

1. $\operatorname{lcm}(2, 3) = 6$

2. $\text{lcm}(6, 12) = 12$

3. $\text{lcm}(12, 8) = \text{lcm}\left(2^3 \cdot 3^1, 2^3 \cdot 3^0\right) = 2^3 \cdot 3^1 = 24$

Theorem 3.2
Let $\alpha_1, \ldots, \alpha_k \in S_n$ be disjoint cycles. Then $|\alpha_1 \ldots \alpha_k| = \text{lcm}(|\alpha_1|, \ldots, |\alpha_k|)$

Proof. Proof omitted. □

Example of the theorem:
$|(15)(37124)(986)| = 12 = \text{lcm}(2, 4, 3)$

Definition of a transposition:
A **transposition** in S_n is a 2-cycle.

Example of transpositions:
Note

- $(1\ 2) = (2\ 1) = (1\ 2)^{-1}$

- $(1\ 2)(1\ 2) = (1)(2) = e$

- Similarly, $(a\ b) = (b\ a) = (ab)^{-1}$

- $(1\ 2\ 3) = (1\ 3)(1\ 2)$

- $(1\ 2\ 3) = (a\ c)(a\ b)$

Theorem 3.3
Every permutation in S_n is a product of transpositions.

Proof. $e = (1\ 2)(2\ 1) = (1\ 2)(1\ 2)$.
Suppose $\sigma \in S_n, \sigma \neq e$. Then by a previous theorem, $\sigma = \beta_1 \dots \beta_k$ for disjoint cycles $\beta_1, \dots, \beta_k \in S_n$. If each B_j is a product of transpositions then so is σ.
Let $\beta = (a_1\ \dots\ a_k)$ be a k-cycle in S_n. Let's prove by induction on $k \geq 2$ that $(a_1\ \dots\ a_k) = (a_1 a_k)(a_1 a_{k-1}) \dots (a_1 a_2)$.
Base case is obvious. Assume it's true for k.
Let $\beta = (a-1\ \dots\ a_{k+1}), \alpha = (a_1\ a_{k+1}), \gamma = (a_1\ \dots a_k)$.
By induction $\gamma = (a_1\ \dots\ a_k) = (a_1\ a_k)(a_1\ a_{k-1}) \dots (a_1\ a_2)$.
It suffices to show that $\beta = \alpha\gamma$.
Let $1 \leq l \leq k-1$. Then $\beta(a_l) = a_{l+1}$, and $\alpha\gamma(a_l) = \alpha(a_{l+1}) = a_{l+1} \implies \beta(a_l) = \alpha\gamma(a_l)$.
So $\beta(a_k) = a_{k+1}$ and $\alpha\gamma(a_k) = \alpha(a_1) = a_{k+1} \implies \beta(a_k) = \alpha\gamma(a_k)$.
So $\beta(a_{k+1}) = a_1$ and $\alpha\gamma(a_{k+1}) = \alpha(a_{k+1}) = a_1 \implies \beta(a_{k+1}) = \alpha\gamma(a_{k+1})$
Rest of the proof was erased before I could get to it :($\square$

Lemma 3.4
If $e = \alpha_1 \dots \alpha_k$ is a product of transpositions $\alpha_1, \dots, \alpha_k \in S_n$, then k is even.

Proof. Proof omitted. $\square$

Theorem 3.5
Suppose $\alpha \in S_n$ and $\beta_1 \dots \beta_r = \alpha = \gamma \dots \gamma_s$ where $\beta_1, \dots, \beta_r, \gamma_1, \dots, \gamma_s \in S_n$ are transpositions.
Then either r and s are both even, or they are both odd (i.e., $r \mod 2 = s \mod 2$).

Proof. $\gamma_1 \dots \gamma_s = \beta_1 \dots \beta_r \implies \gamma_1^{-1}\gamma_1 \dots \gamma_s = \gamma_1^{-1}\beta_1 \dots \beta_r \implies e = \gamma_s \dots \gamma_1\beta_1 \dots \beta_r$, so the identity is a product of transpositions.
By the lemma, $r + s$ is even. $\square$

Definition of parity:
We say that $\alpha \in S_n$ is **even** if it is a product of even number of transpositions, we say α is odd if it is a product of an odd number of transpositions.

Example of parity of cycles:

1. $(a\ b)$ odd
2. $(a_1\ a_2\ a_3) = (a_1\ a_3)(a_1\ a_2)$ even
3. $(a_1\ a_2\ a_3\ a_4) = (a_1\ a_4)(a_1\ a_3)(a_1\ a_2) = (a_1\ a_4)(a_1\ a_2\ a_3)$

Remark: A k-cycle is even for odd k and is odd for even k.

Theorem 3.6
Let $A_n \subseteq S_n, n \geq 2$ be the subset of even elements in S_n. Then A_n is a subgroup (called the **alternating group**).

Proof. $e \in A_n$ so $A_n \neq \emptyset$. Suppose $\alpha = \beta_1 \dots \beta_s$ and $\sigma = \gamma_1 \dots \gamma_r$ for transpositions $\beta_1, \dots, \beta_s, \gamma_1, \dots, \gamma_r \in S_n$, i.e., s and r are even. Then $\alpha\sigma = \beta_1, \dots, \beta_s, \gamma_1, \dots, \gamma_r$ is a product of $r + s$ transpositions.
Since $r + s$ is even, $\alpha\sigma \in A_n$.
$\alpha^{-1} = (\beta_1 \dots \beta_s)^{-1} = \beta_s^{-1} \dots \beta_1^{-1} = \beta_s \dots \beta_1$ a product of s transpositions. Since s is even, $\alpha^{-1} \in A_n$. □

Example of alternating groups:

- $S_2 = \{e, (1\ 2)\} \supseteq \{e\} = A_2$
- $S_3 = \{e\} \cup$ 2-cycles $\cup$ 3-cycles $\supseteq \{e\} \cup$ 3-cycles $= \{e, (1\ 2\ 3), (1\ 3\ 2)\} = A_3 = \langle (1\ 2\ 3) \rangle$
- $S_4 = \{e\} \cup$ 2-cycles $\cup$ 3-cycles $\cup$ 4-cycles $\cup$ products of disjoint 2-cycles $\supseteq \{e\} \cup$ 3-cycles $\cup$ products of disjoint 2-cycles $= A_4$

Theorem 3.7
$|A_n| = \frac{n!}{2}$ for all $n \geq 2$.

Proof. Recall that every element in S_n is either even or odd. So $S_n = A_n \cup B$ is a disjoint union where B is the set of odd permutations.
So $|S_n| = |A_n| + |B| \implies n! = |A_n| + |B|$.
So if we prove $|A_n| = |B|$ then $n! = 2\,|A_n| \implies |A_n| = \frac{n!}{2}$.
To prove $|A_n| = |B|$, we define $f : A_n \to B$ by $f(\alpha) = (1\ 2)\alpha$ for all $\alpha \in A_n$.
Clearly, $(1\ 2)\alpha$ is odd since α is even. To show injectivity, suppose $\alpha, \beta \in A_n$ with $f(\alpha) = f(\beta) \implies (1\ 2)\alpha = (1\ 2)\beta \implies (1\ 2)(1\ 2)\alpha = (1\ 2)(1\ 2)\beta \implies \alpha = \beta$, so f is injective.
Suppose $\sigma \in B$. Then $f((1\ 2)\sigma) = (1\ 2)(1\ 2)\sigma = \sigma$. This proves that f is a bijection and $|A_n| = |B|$. □

Week 4

Isomorphisms

Remark:

- Cayley's Theorem says that every finite group is isomorphic to a subgroup of some S_n.
- Historically, the idea of a group comes from work with S_n.

Example of:
Recall $D_3 = \left\{e, \rho, \rho^2, \sigma, \sigma\rho, \sigma\rho^2\right\}$

Note that S_3 also has order 6.

We may identify the elements in D_3 by how they permute the vertices of a triangle:

- $e \leftrightarrow e$
- $\rho \leftrightarrow (1\ 2\ 3)$
- $\rho^2 \leftrightarrow (1\ 3\ 2)$
- $\sigma \leftrightarrow (1\ 3)$
- $\sigma\rho \leftrightarrow (1\ 2)$
- $\sigma\rho \leftrightarrow (2\ 3)$

If we define $\phi : D_3 \to S_3$ by $\phi(e) = e, \phi(\rho) = (1\ 2\ 3)$ etc. as above, then it remains to show that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in D_3$.

After a brute force check, we can verify that the above holds.

Definition of an isomorphism:
Suppose G and $\bar{G}$ are groups. An **isomorphism** is a map $\phi : G \to \bar{G}$ which is bijective and $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$.

Example of:
Let $G = \langle a \rangle = \left\{a^j : j \in \mathbb{Z}\right\}$ be an infinite cyclic group. ($|a| = \infty$).

Define $\phi : \mathbb{Z} \to G$ by $\phi(j) = a^j$.

This function is bijective. Proof omitted because I'm sleepy! It's not too hard, do it as an exercise.

Finally, $\phi(j+k) = a^{j+k} = a^j a^k = \phi(j)\phi(k)$. This proves that $\phi : \mathbb{Z} \to \langle a \rangle$ is an isomorphism.

Definition of isomorphic groups:
We say groups G and $\bar{G}$ are **isomorphic** if there is an isomorphism $\phi : G \to \bar{G}$. In this case we write $G \cong \bar{G}$.

Theorem 4.1
Suppose $\phi : G \to \bar{G}$ is a group isomorphism. Then

1. $\phi(e) = \bar{e}$ is the identity in $\bar{G}$
2. $\phi(b^n) = (\phi(b))^n$ for all $b \in G$
3. $ab = ba$ in $G \implies \phi(a)\phi(b) = \phi(b)\phi(a)$ in $\bar{G}$
4. $G = \langle b \rangle \implies \bar{G} = \langle \phi(b) \rangle$
5. $|b| = |\phi(b)|$ for all $b \in G$
6. Omitted.
7. $|G| = \left|\bar{G}\right|$ (In particular G finite $\iff$ $\bar{G}$ finite)

Proof. Sketch of proof

1. $\phi(e) = \phi(ee) = \phi(e)\phi(e) \implies \bar{e} = \bar{e}\phi(e) = \phi(e)$.
2. Prove by induction on $n \geq 1$. For $n \leq -1$, replace b by $b^{-1} \in G$.
3. $ab = ba \in G \implies \phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a)$
4. Suppose $G = \langle b \rangle$. Let $a \in \mathbb{Z}$. Since ϕ is a bijection, there exists a unique $a \in G$ such that $\phi(a) = \bar{a}$. Since $G = \langle b \rangle$, $a = b^j$ for some $j \in \mathbb{Z}$. By 2, $\bar{a} = \phi(a) = \phi\left(b^j\right) = \phi(b)^j \implies \bar{a} \in \langle \phi(b) \rangle \implies \bar{G} \subseteq \langle \phi(b) \rangle \implies \bar{G} = \langle \phi(b) \rangle$.
5. Suppose $|b| = n < \infty$. Then $\phi(b)^n = \phi(b^n) = \phi(e) = \bar{e}$. n must be the lowest of these, otherwise we arrive at $b^m = e$ for $m < n$ is a contradiction. A similar proof by contradiction is reached if $|b| = \infty$.

$\square$

Example of:
Define $\mu_n = \left\{e^{2\pi i \frac{k}{n}} : 0 \leq k \leq n-1\right\} = \{z \in \mathbb{C}^\times : z^n = 1\} \subseteq \mathbb{C}^\times$.

μ_n is a subgroup of $\mathbb{C}^\times$.

Define $\phi : \mathbb{Z}_n \to \mu_n$ by $\phi(k) = e^{2\pi i \frac{k}{n}}$ for all $0 \leq k \leq n-1$.

Exercise: ϕ is a bijection.

Note: Examples omitted from lecture today; sorry I'm sleepy and need to do tutorial prep.

Theorem 4.2
Suppose $\phi : G \to \bar{G}$ is a group isomorphism. Then

1. $\phi^{-1} : \bar{G} \to G$ is an isomorphism
2. G abelian $\iff$ $\bar{G}$ is abelian
3. G cyclic $\iff$ $\bar{G}$ is cyclic
4. K subgroup of $G \implies \phi(K) = \{\phi(k) : k \in K\}$ subgroup of $\bar{G}$
5. $\bar{K}$ subgroup of $\bar{G} \implies \phi^{-1}\left(\bar{K}\right)$ subgroup of G.
6. $\phi(Z(G)) = Z\left(\bar{G}\right)$

Proof. 1. $\phi^{-1} : \bar{G} \to G$ exists and is a bijection since ϕ is a bijection. Must prove $\phi^{-1}\left(\bar{a}\bar{b}\right) = \phi^{-1}(\bar{a})\,\phi^{-1}\left(\bar{b}\right)$ for $\bar{a}, \bar{b}, \in \bar{G}$. Since ϕ is a bijection, there exist unique $a, b \in G$ such that $\phi(a) = \bar{a}$ and $\phi(b) = \bar{b}$. So

$$\phi^{-1}\left(\bar{a}\bar{b}\right) = \phi^{-1}(\phi(a)\,\phi(b)) = \phi^{-1}(\phi(ab)) = ab = \phi^{-1}(\bar{a}) = \phi^{-1}\left(\bar{b}\right)$$

2. Let's prove G abelian $\iff$ $\bar{G}$ abelian. Suppose G is abelian, and $\bar{a}, \bar{b} \in \bar{G}$. Let $a, b, \in G$ such that $\phi(a) = \bar{a}, \phi(b) = \bar{b}$, then

$$\bar{a}\bar{b} = \phi(a)\,\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\,\phi(a) = \bar{b}\bar{a}.$$

This proves $\bar{G}$ is abelian/

3. Was done previously

4. Suppose $K \subseteq G$ is a subgroup. Suppose $\phi(K), \phi(k') \in \phi(K)$ where $k, k' \in K$. Then $(\phi(k))^{-1}\,\phi(k') = \phi\left(k^{-1}\right)\phi(k') = \phi\left(k^{-1}k'\right)$. Since $k^{-1}k' \in K$, $\phi\left(k^{-1}k'\right) = (\phi(k))^{-1}\,\phi(k') \in \phi(K)$

5. Follows from 4

6. First we prove $\phi(Z(G)) \subseteq Z\left(\bar{G}\right)$. Suppose $z \in Z(G)$ and $\bar{a} \in \bar{G}$. Let $a \in G$ such that $\phi(a) = \bar{a}$. Then $\phi(z)\,\bar{a} = \phi(z)\,\phi z = \phi(za) = \phi(az) = \phi(a)\,\phi(z) = \bar{a}\phi(z) \implies \phi(z) \in Z\left(\bar{G}\right) \implies \phi(Z(G)) \subseteq Z\left(\bar{G}\right)$. By symmetry, $Z\left(\bar{G}\right) \subseteq \phi(Z(G))$ so they are equal.

$\square$

Definition of an automorphism:
An **automorphism** of G is an isomorphism $\phi : G \to G$. The set of automorphisms of G is denoted by $\mathrm{Aut}(G)$.

Theorem 4.3
$\mathrm{Aut}(G)$ is a group under composition of functions.

Week 5

Cosets and Lagrange's Theorem

Definition of a coset:
A (left) **coset of H** is of the form $xH = \{xh : h \in H\}$ where $x \in G$. A (right) **coset of H** is of the form $Hx = \{hx : h \in H\}$ where $x \in G$.
In both cases, x is called a (coset) **representative for** xH (or Hx).
$|xH|$ is the number of elements in xH.

Lemma 5.1
Let H be a subgroup of G and $x, y \in G$. Then

1. $x \in xH$
2. $xH = H \iff x \in H$
3. $x(yH) = (xy)H$
4. $xH = yH \iff x \in yH$
5. Either $xH = yH$ or $xH \cap yH = \emptyset$
6. $xH = yH \iff y^{-1}x \in H \iff x^{-1}y \in H$
7. $|xH| = |H|$
8. $xH = Hx \iff xHx^{-1} = H$
9. xH is a subgroup $\iff x \in H \iff xH = H$

Theorem 5.2 Lagrange's Theorem
Suppose H is a subgroup of a finite group G. Then $|H|$ divides $|G|$ and the number of cosets is $\frac{|G|}{|H|}$.

Proof. $G = x_1 H \bigcup \cdots \bigcup x_m H$ a disjoint union $\implies |G| = \sum_{j=1}^{m} |x_j H| = m\,|H|$. ☐

Definition of coset spaces and index:
Suppose H is a subgroup of G, then the number of (left) cosets is called the **index of H in G** and is denoted by $|G : H|$. The set of (left) cosets is denoted by $\frac{G}{H} = \{gH : g \in G\}$ and is called the **coset space**. So

$$\left|\frac{G}{H}\right| = |G : H|$$

Example of:
If G is finite then $\left|\frac{G}{H}\right| = |G : H| = \frac{|G|}{|H|}$.

- Take $G = \mathbb{Z}$ and $H = \langle 2 \rangle$. Then $\frac{\mathbb{Z}}{\langle 2 \rangle} = \{0 + \langle 2 \rangle, 1 + \langle 2 \rangle\} \implies |\mathbb{Z} : \langle 2 \rangle| = 2$.
- $|\mathbb{Z} : \langle 3 \rangle| = 3$
- $|\mathbb{Z} : \langle 0 \rangle| = \infty$
- $|D_3 : \langle \rho \rangle| = \frac{6}{3} = 2$ by Lagrange's theorem.

Corollary 5.3
Suppose G is finite and $x \in G$. Then $|x|$ divides $|G|$.

Proof. $|x| = |\langle x \rangle|$ divides $|G|$ by Lagrange. ☐

Corollary 5.4
Suppose $|G| = p$ is a prime number. Then G is cyclic and $G \cong \mathbb{Z}_p$.

Proof. Suppose $x \in G, x \neq e$. Then $1 \neq |\langle x \rangle|$ divides p by Lagrange's theorem. So $|\langle x \rangle| = p$. However $\langle x \rangle \subseteq G$ so $\langle x \rangle = G$.
For the desired isomorphism let $\phi(k) = x^k$, $0 \leq k \leq p - 1$. ☐

Corollary 5.5
Suppose G is finite and $x \in G$. Then $x^{|G|} = e$.

Corollary 5.6 Fermat's Little Theorem
Suppose $m \in \mathbb{Z}$ and p is prime. Then $m^p \mod p = m \mod p$.

5.1 External Direct Products

Definition of a direct product:
Suppose $G_1, \ldots, G_n$ are groups. Then $G_1 \oplus \cdots \oplus G_n = G_1 \times \cdots \times G_n = \{(g_1, \ldots, g_n) : g_i \in G_i\}$ together with multiplication defined by $(g_1, \ldots, g_n)(g_1', \ldots, g_n')$. $|G_1 \oplus \cdots \oplus G_n| = \Pi_{i=1}^{n} |G_i|$

Theorem 5.7
Suppose $G_1 \oplus \cdots \oplus G_n$ is a direct product of groups and $(g_1, \ldots, g_n) \in G_1 \oplus \cdots \oplus G_n$. Then $|(g_1, \ldots, g_n)| = \mathrm{lcm}(|g_1|, \ldots, |g_n|)$.

Week 6

Theorem 6.1
Suppose $G_1 \oplus \cdots \oplus G_n$ is a direct product of finite groups and $(g_1, \ldots, g_n) \in G_1 \oplus \cdots \oplus G_n$. Then $|(g_1, \ldots, g_n)| = \operatorname{lcm}(|g_1|, \ldots, |g_n|)$.

Proof. Let $t = |(g_1, \ldots, g_n)|$ and $m = \operatorname{lcm}(|g_1|, \ldots, |g_n|)$.
Then $m = q_j |g_j|$ for some $q_j \in \mathbb{Z}$ and any $1 \leq j \leq n$. So

$$(g_1, \ldots, g_n)^m = (g_1^m, \ldots, g_n^m) = \left(g_1^{q_1|g_1|}, \ldots, g_n^{q_n|g_n|}\right)$$

$$= \left(\left(g_1^{|g_1|}\right)^{q_1}, \ldots, \left(g_n^{|g_n|}\right)^{q_n}\right) = (e^{q_1}, \ldots, e^{q_n}) = (e, \ldots, e),$$

so m is divisible by $|(g_1, \ldots, g_n)| = t \implies m \geq t$.
In addition,

$$\begin{aligned}
(e, \ldots, e) &= (g_1, \ldots g_n)^t \\
&= (g_1^t, \ldots, g_n^t) \\
&\implies g_j^t = e \text{ for all } 1 \leq j \leq n \\
&\implies |g_j| \text{ divides } t \text{ for all } 1 \leq j \leq n \\
&\implies t \text{ is a common multiple of } |g_1|, \ldots, |g_n| \\
&\implies t \geq m = \text{ the least common mutliple.}
\end{aligned}$$

□

Remark: For $n = p_1^{a_1} \ldots p_k^{a_k}$, $m = p_1^{b_1} \ldots p_k^{b_k}$, we have

$$\begin{aligned}
\operatorname{lcm}(n, m) &= p_1^{\max\{a_1, b_1\}} \ldots p_k^{\max\{a_k, b_k\}} \\
\gcd(n, m) &= p_1^{\min\{a_1, b_1\}} \ldots p_k^{\min\{a_k, b_k\}},
\end{aligned}$$

so

$$\operatorname{lcm}(n, m) \gcd(n, m) = p_1^{a_1 + b_1} \ldots p_k^{a^k + b^k}$$

Example of the direct product of prime groups:
Suppose $\gcd n, m = 1$ are prime.

$(1,1) \in \mathbb{Z}_n \oplus \mathbb{Z}_m$, $|(1,1)| = \operatorname{lcm}(|1|, |1|) = \operatorname{lcm}(n, m) = (n)(m) = nm$.

Also $\langle(1,1)\rangle \subseteq \mathbb{Z}_n \oplus \mathbb{Z}_m$ and $|\langle(1,1)\rangle| = |(1,1)| = nm \implies \langle(1,1)\rangle = \mathbb{Z}_n \oplus \mathbb{Z}_m$.

Example of non-cyclic direct products:
Take $n \geq 2$. Consider $\mathbb{Z}_n \oplus \mathbb{Z}_n$. Then $\operatorname{lcm}(n,n) = \gcd(n,n) = n$.

Let $(a,b) \in \mathbb{Z}_n \oplus \mathbb{Z}_n$. By Lagrange's Theorem, $|a|$ divides $n = |\mathbb{Z}_n|$, so $|b|$ divides n. Thus n is a common multiple of $|a|, |b|$.

By the theorem, $|a,b| = \operatorname{lcm}(|a|, |b|) \leq n$. More directly $(a,b)^n = (a^n, b^n) = (e,e)$ by Lagrange's theorem since $n = \mathbb{Z}_n$. So $|(a,b)|$ divides n.

It follows $|\langle(a,b)\rangle| = |(a,b)| \leq n < n^2 = |\mathbb{Z}_n \oplus \mathbb{Z}_n|$.

Therefore $\langle(a,b)\rangle \neq \mathbb{Z}_n \oplus \mathbb{Z}_n$ and $\mathbb{Z}_n \oplus \mathbb{Z}_n$ is not cyclic.

Theorem 6.2
Suppose G and H are finite cyclic groups ($G \cong \mathbb{Z}_n, H \cong Z_m$). Then $G \oplus H$ is cyclic if and only if $1 = \gcd(|G|, |H|)$.

Proof. Let $n = |G|$ and $m = |H|$. Suppose $G = \langle g \rangle$ and $H = \langle h \rangle$ so that $|g| = n$ and $|h| = m$.
" $\Longleftarrow$ " Suppose $\gcd(n,m) = 1$. Then $nm = \operatorname{lcm}(n,m)\gcd(n,m) = \operatorname{lcm}(n,m)$ by theorem 8.1 $|(g,h)| = \operatorname{lcm}(n,m) = nm$.
Moreover $\langle(g,h)\rangle \subseteq G \oplus H$ and $|\langle(g,h)\rangle| = nm = |G| \oplus |H|$. So $\langle(g,h)\rangle = G \oplus H$ and $G \oplus H$ is cyclic.
" $\implies$ " Suppose $G \oplus H = \langle(a,b)\rangle$ so that $|(a,b)| = |\langle(a,b)\rangle| = |G \oplus H| = nm$.
Let $d = \gcd(n,m)$. Then $(a,b)^{\frac{nm}{d}} = \left(a^{\frac{nm}{d}}, b^{\frac{nm}{d}}\right) = \left((a^n)^{\frac{m}{d}}, (b^m)^{\frac{n}{d}}\right) = \left(e^{\frac{m}{d}}, e^{\frac{n}{d}}\right) = (e,e)$.
Thus $|(a,b)|$ divides $\frac{nm}{d}$ implies nm divides $\frac{nm}{d} \implies d = 1$ ☐

Example of a non-cyclic group:
$\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is not cyclic since $\gcd(2,4) = 2 \neq 1$.

Example of $\mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{13}$:
$|(1,1,1)| = \operatorname{lcm}(|1|, |1|, |1|) = \operatorname{lcm}(2,10,13) = (2)(5)(13) = 130 \neq 260 = |\mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{13}|$.

$\langle(1,1,1) \subseteq \mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_13\rangle$ and $|\langle(1,1,1)\rangle| = |(1,1,1)| \neq |\mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{13}|$. That is, the element $(1,1,1)$, made up of the generators of each group, does not generate the direct product of the groups. Observe that $\gcd(2,10,13) = 1$.

Corollary 6.3
Suppose $G_1, \ldots, G_n$ are finite cyclic groups. Then $G_1 \oplus \cdots \oplus G_n$ is cyclic if and only if $\gcd(|G_j|, |G_k|) = 1$ for all $j \neq k$.

Proof. Proof omitted. □

Corollary 6.4
Let $m = n_1 \ldots n_k$. Then

$$\mathbb{Z}_m \cong \mathbb{Z}_{n_1} \oplus \cdots \mathbb{Z}_{n_k} \iff \gcd(n_j, n_l) = 1 \forall j \neq l.$$

Proof. (Sketch). Take $G_j = \mathbb{Z}_{n_j}$. Then $|G_1 \oplus \cdots \oplus G_k| = n_1 n_2 \ldots n_k = m$. By corollary 6.3 $G_1 \oplus \cdots \oplus G_k \cong Z_m \iff \gcd(n_j, n_l) = 1 \forall j \neq l$. □

Example of using the theorem:
Proofs left as an exercise, but notice $12 = 2^2 \cdot 3 = 6 \cdot 2 = 4 \cdot 3$, but by corollary 6.4,

- $\mathbb{Z}_{12} \ncong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$
- $\mathbb{Z}_{12} \ncong \mathbb{Z}_6 \oplus \mathbb{Z}_3$
- $\mathbb{Z}_{12} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_3$

It is not hard to see that if m has prime factorization $m = p_1^{a_1} \ldots p_k^{a_k}$ $(a_j \geq 0)$, then $\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{a_1}} \oplus \cdots \oplus \mathbb{Z}_{p_p^{a_k}}$ by corollary 6.4 (since their gcd is 1).