Multivariable Calculus Winter Notes

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Note: Theorem numbers come from the order they are presented in lecture, and do not correspond to any textbook or written course material.

Week 1

Classifying Critical Points

Theorem 1.1 2nd Derivative Test
Let $f \in C^2(\Omega)$ and let $a \in \Omega(\Omega \subseteq \mathbb{R}^n)$ be a critical point of f.

1. If $H_f(a)$ is positive definite then f has a local minimum at a.
2. If $H_f(a)$ is negative definite then f has a local maxiumum at a.
3. If $H_f(a)$ is indefinite then f has a saddle point at a.

Recall: Any symmetric $n \times n$ matrix A can be diagonalized, i.e., $\exists$ an orthonormal basis $u_1, u_2, \ldots, u_n$ in $\mathbb{R}^n$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $Au_i = \lambda_i u_i \forall i = 1, 2, \ldots, n$.

Proposition 1.2
Let Q be the quadratic form associated with an $n \times n$ symmetric matrix A. Then:

1. Q is positive $\iff$ all the eigenvalues of A are positive,
2. Q is negative $\iff$ all the eigenvalues of A are negative,
3. Q is indefinite $\iff$ A has both positive and negative eigenvalues.

Corollary 1.3
Let a be a critical point of a C^2 function $f : \Omega \to \mathbb{R}$. If $\det H_f(a) \neq 0$, then f has either a local minimum or a local mximum or a saddle point at a.

Definition of degenerate critical points:
A critical point a of a C^2 function f is called non-degenerate if $\det H_f(a) \neq 0$ and degenerate otherwise.

Example of a degenerate critical point:
When $f(x, y) = x^3$ then $(0, 0)$ is a degenerate critical point of f, and f has neither a local extremum at $(0, 0)$ nor a saddle point.

Definition of the principal minors of a matrix:
Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ matrix. Given $k = 1, 2, \ldots, n$, we will denote by A_k the $k \times k$ submatrix $A_k = (a_{ij})_{i,j=1}^k$.
The determinants $\det A_k$ are called the **principal minors of A**.

Proposition 1.4
Let A be a symmetric $n \times n$ matrix with $\det A \neq 0$. Then:

1. A is positive definite $\iff$ $\det A_k > 0 \forall k = 1, 2, \ldots, n$.
2. A is negative definite $\iff$ $(-1)^k \det A_k > 0 \forall k = 1, 2, \ldots, n$.
3. A is indefinite $\iff$ A is neither positive definite nor negative definite.

Corollary 1.5
Let $A = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$. Then:

1. A is positive definite $\iff$ $\alpha > 0$ and $\alpha\gamma - \beta^2 > 0$
2. A is negative definite $\iff$ $\alpha < 0$ and $\alpha\gamma - \beta^2 > 0$
3. A is indefinite $\iff$ $\alpha\gamma - \beta^2 < 0$

Example of classifying a critical point:
We found that the function $f(x, y) = xye^{-x^2-y^2}$ has 5 critical points: $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$, $\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right)$, and $(0, 0)$, with an absolute maximum at $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$ and an absolute minimum at $\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right)$.

Investigate the nature of $(0,0)$,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[y(1-2x^2)e^{-x^2-y^2}\right] = -4xye^{-x^2-y^2} - 2xy(1-2x^2)e^{-x^2-y^2}$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[x(1-2y^2)e^{-x^2-y^2}\right] = -4xye^{-x^2-y^2} - 2xy(1-2y^2)e^{-x^2-y^2}$$
$$\frac{\partial^2 f}{\partial x \partial y} = (1-2y^2)e^{-x^2-y^2} - 2x^2(1-2y^2)e^{-x^2-y^2}$$

So $H_f(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is indefinite $\implies$ f has a saddle point at (0,0).

Example of non-degenerate critical points:
Find and classify the critical points of $f : \mathbb{R}^3 \to \mathbb{R}$ where $f(x,y,z) = x^3 - y^3 + 3xy + z^2 - 2z$.
$\nabla f = (3x^2 + 3y, -3y^2 + 3x, 2z - 2) = 0 \implies x^2 = -y, y^2 = x, z = 1 \implies x^2 = -y, x^4 = x, z = 1 \implies x^2 = -y, x = 0$ or $x = 1, z = 1 \implies (x,y,z) = (0,0,1), (1,-1,1)$.

So $(0,0,1)$ and $(1,-1,1)$ are the critical points. We have $H_f(x,y,z) = \begin{bmatrix} 6x & 3 & 0 \\ 3 & -6y & 0 \\ 0 & 0 & 2 \end{bmatrix}$,
so $H_f(0,0,1) = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is clearly indefinite since the first principal minor is 0 and $H_f(1,-1,1) = \begin{bmatrix} 6 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is positive definite.
So we have non-degenerate critical points (as $\det H_f \neq 0$). Hence, $(0,0,1)$ is a saddle point; $(1,-1,1)$ is a local minimum.
But f has no global extrema because $f(x,0,0) = x^3$ can take arbitrarily positive and negative values.

Example of a degenerate critical point:
Let $f(x,y) = x^4 + y^4$ (with $(x,y) \in \mathbb{R}^2$).
$\nabla f = (4x^3, 4y^3) = 0 \iff (x,y) = (0,0)$.
$H_f(x,y) = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix}, H_f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
So $(0,0)$ is a degenerate critical point and the 2nd derivative test does not apply.
However, f has a global minimum at $(0,0)$.

Theorem 1.6 The inverse function theorem
Th. A. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ is a continuous injective function. Then:

1. f is either strictly increasing or strictly decreasing.
2. $f(I)$ is an interval containing the same number of endpoints as I.
3. f is a homeomorphism of I onto $f(I)$.

Proof. 1. Let us first consider the case that $I = [a, b] (a < b)$. Since f is injective, either $f(a) < f(b)$ or $f(b) < f(a)$. Assume that $f(a) < f(b)$ (the other case can be done symmetrically). Let's show that f is strictly increasing on $[a, b]$, i.e., $f(x) < f(y)$ whenever $a \leq x < y \leq b$. We argue by contradiction, supposing that $f(x) > f(y)$ for some $a \leq x < y \leq b$.
Note that $f(y) > f(a)$, for otherwise $f(y) < f(a) < f(b)$ and by the Intermediate Value Theorem (IVT), $\exists \alpha \in (y, b)$ such that $f(\alpha) = f(a)$, contradicting the injectivity of f. Therefore $f(a) < f(y) < f(x)$ and so, again, by the IVT $\exists y' \in (a, x)$ such that $f(y') = f(y)$, again contradicting the injectivity of f.
Next, let I be any interval. Pick up any $a, b \in I$ with $a < b$. Suppose that $f(a) < f(b)$ (the case $f(a) > f(b)$ can be done symmetrically). By the previous paragraph, we know that f is strictly increasing on $[a, b]$. Now, if $x, y \in I$ and $x < y$, then with $\alpha = \min\{a, x\}, \beta = \max\{y, b\}$, we have $[a, b], [x, y] \subseteq [\alpha, \beta] \subseteq I$. Since f is strictly increasing on $[a, b]$, we must have (using the 1st paragraph again) $f(\alpha) < f(\beta)$ and f is strictly increasing on $[\alpha, \beta]$. Hence, we conclude that f is strictly increasing on I.

2. Since f is continuous, $J = f(I)$ is an interval. Suppose that f is strictly increasing. Note that the inverse function f^{-1} is then also strictly increasing.
Now, if I contains its left endpoint a, then $\forall x \in I$, $f(a) \leq f(x)$, so $f(a)$ is a left endpoint of J. Similarly, if I contains its right endpoint b, then $f(b)$ is the right endpoint of J. Applying the same argument with f^{-1} in place of f, we conclude if I contains its left (respectively, right) endpoint c, then $f^{-1}(c)$ is the left (respectively, right) endpoint of I. It follows that I and J contain the same number of endpoints.

3. If $I = [a, b]$, then f is a homeomorphism of I onto $f(I)$ because of our general result about continuous injective functions on compact sets.
Otherwise, it follows that $f|[a, b]$ is a homeomorphism onto $f([a, b])$ for any $a, b \in I$ with $a \leq b$. This implies that $f^{-1} : f(I) \to I$ is continuous (at any $y \in f(I)$).
Indeed, let $y \in f(I)$ and consider any sequence (y_n) in $f(I)$ with $y_n \to y$. Then the set $S = \{y\} \cup \{y_n : n \in N\}$ is compact, so it has both a smallest element $c = f(a)$ and a largest element $d = f(b)$. Assuming that f is strictly increasing we must have $a \leq b$, and $f([a, b]) = [c, d] \supseteq S$. Since $f|[a, b]$ is a homeomorphism onto $[c, d]$ (i.e., $(f|[a, b])^{-1} = f^{-1}|[c, d]$ is continuous), we obtain $f^{-1}(y_n) = (f^{-1}|[c, d])(y_n) \to (f^{-1}|[c, d])(y) = f^{-1}(y)$. It follows that f^{-1} is continuous at any $y \in f(I)$.

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