Mathematical Methods Notes

by Camila Restrepo

Last updated January 16, 2024

Note: Theorem numbers come from the order they are presented in lecture, and do not correspond to any textbook or written course material.

Week 1

the Laplace Transform

Example of transformations:
Let's ponder for a moment some of the many fantastic transformations we've studied in the past:

1. $f(x) = x^2$
2. $\frac{d}{dx}x^2 = 2x$
3. $\int x^2 dx = \frac{1}{3}x^3 + C$
4. $\int_0^3 x^2 dx = g$

Definition of the Laplace Transform:
For a function $f(t)$,

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt, s > 0$$

where $\mathcal{L}\{f(t)\}$ is a function of s.

Remark: $\mathcal{L}$ provides an efficient method to solve ODEs.

Theorem 1.1 Linearity of the Laplace transform
For a, b constants, and f, g functions, The Laplace transform satisfies

$$\mathcal{L}\{af(t) + bg(t)\} = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}$$

Definition of the kernel:
e^{-st} is called the **kernel** of the Laplace transform.

Definition of the domain:
The **domain** of $F(s) = \mathcal{L}\{f(t)\}$ is the set of values of s such that $F(s)$ converges.

Example of computing $\mathcal{L}\{t\}$:

Recall integration by parts: $\int udv = uv - \int vdu$

$$\begin{aligned}
\mathcal{L}\{t\} &= \int_0^\infty te^{-st}dt \\
&= \lim_{\beta\to\infty}\left(\int_0^\beta te^{-st}dt\right) \\
&= \lim_{\beta\to\infty}\left(t\left(-\frac{1}{s}\right)e^{st}\Big|_0^\beta - \int_0^\beta\left(-\frac{1}{s}e^{-st}dt\right)\right) \\
&= \lim_{\beta\to\infty}\left(-\frac{\beta}{s}e^{s\beta} + \frac{1}{s}\int_0^\beta e^{-st}dt\right) \\
&= \lim_{\beta\to\infty}\left(-\frac{\beta}{s}e^{-s\beta} + \frac{1}{s}\left[-\frac{1}{s}e^{-st}\right]_0^\beta\right) \\
&= \lim_{\beta\to\infty}\left(-\frac{\beta}{s}e^{-s\beta} + \frac{1}{s^2}\left[1 - e^{s\beta}\right]\right) \\
&= \frac{1}{s^2}
\end{aligned}$$

Wow, that computation sucked! Luckily, we will be given formula sheets to speed up these calculations. This is because the objective of this class is to solve ODEs using Laplace as a tool, not to practice computing integration by parts.

Recall: $\Gamma(n) = (n-1)!$

Theorem 1.2 Laplace transformations of power functions
We have, in general, $\mathcal{L}\{t^p\} = \int_0^\infty t^p e^{-st}dt = \frac{1}{s^{p+1}}\int_0^\infty x^p e^- x dx$ where $x = st$. So

$$\mathcal{L}\{t^p\} = \frac{1}{s^{p+1}}\Gamma(p+1)$$

When $p = n \in \mathbb{N}$, we have

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Theorem 1.3 Properties of the Gamma function
.

1. When $p > 0$, $\Gamma(p+1) = p\Gamma(p)$ (Recurrence Relation)
2. When $p \in \mathbb{N}$, $\Gamma(p) = (p-1)!$ (Generalization of Factorial)
3. When $p \leq 0$ and $p \in \mathbb{Z}$, $\Gamma(p)$ does not exist
4. When $p < 0$ and $p \notin \mathbb{Z}$, use successive applications of the Recurrence Relation $\Gamma(p) = \frac{1}{p}\Gamma(p+1)$

Remark: From the definition we can find $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Example of $\Gamma\left(\frac{3}{2}\right)$:

$$\begin{aligned}\Gamma\left(\frac{3}{2}\right) &= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\\ &= \frac{1}{2}\sqrt{\pi}\end{aligned}$$

Example of $\mathcal{L}\left\{t^{\frac{1}{2}}\right\}$:

$$\mathcal{L}\left\{t^{\frac{1}{2}}\right\} = \Gamma\left(\frac{3}{2}\right)$$