

Mathematical Methods Notes

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Last updated January 12, 2024

1 the Laplace Transform

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Note: Theorem numbers come from the order they are presented in lecture, and do not correspond to any textbook or written course material.

Week 1

the Laplace Transform

Example of transformations:

Let's ponder for a moment some of the many fantastic transformations we've studied in the past:

1. $f(x) = x^2$
2. $\frac{d}{dx}x^2 = 2x$
3. $\int x^2 dx = \frac{1}{3}x^3 + C$
4. $\int_0^3 x^2 dx = g$

Definition of the Laplace Transform:

For a function $f(t)$,

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt, s > 0$$

where $\mathcal{L}\{f(t)\}$ is a function of s .

Remark: \mathcal{L} provides an efficient method to solve ODEs.

Theorem 1.1 Linearity of the Laplace transform

For a, b constants, and f, g functions, The Laplace transform satisfies

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Definition of the kernel:

e^{-st} is called the **kernel** of the Laplace transform.

Definition of the domain:

The **domain** of $F(s) = \mathcal{L}\{f(t)\}$ is the set of values of s such that $F(s)$ converges.

Example of computing $\mathcal{L}\{t\}$:

Recall integration by parts: $\int u dv = uv - \int v du$

$$\begin{aligned}
 \mathcal{L}\{t\} &= \int_0^{\infty} t e^{-st} dt \\
 &= \lim_{\beta \rightarrow \infty} \left(\int_0^{\beta} t e^{-st} dt \right) \\
 &= \lim_{\beta \rightarrow \infty} \left(t \left(-\frac{1}{s} \right) e^{st} \Big|_0^{\beta} - \int_0^{\beta} \left(-\frac{1}{s} e^{-st} dt \right) \right) \\
 &= \lim_{\beta \rightarrow \infty} \left(-\frac{\beta}{s} e^{s\beta} + \frac{1}{s} \int_0^{\beta} e^{-st} dt \right) \\
 &= \lim_{\beta \rightarrow \infty} \left(-\frac{\beta}{s} e^{-s\beta} + \frac{1}{s} \left[-\frac{1}{s} e^{-st} \right]_0^{\beta} \right) \\
 &= \lim_{\beta \rightarrow \infty} \left(-\frac{\beta}{s} e^{-s\beta} + \frac{1}{s^2} [1 - e^{s\beta}] \right) \\
 &= \frac{1}{s^2}
 \end{aligned}$$

Wow, that computation sucked! Luckily, we will be given formula sheets to speed up these calculations. This is because the objective of this class is to solve ODEs using Laplace as a tool, not to practice computing integration by parts.

Theorem 1.2 Laplace transformations of power functions

We have, in general, $\mathcal{L}\{t^p\} = \int_0^{\infty} t^p e^{-st} dt = \frac{1}{s^{p+1}} \int_0^{\infty} x^p e^{-x} dx$ where $x = st$. So

$$\mathcal{L}\{t^p\} = \frac{1}{s^{p+1}} \Gamma(p+1)$$

Theorem 1.3 Properties of the Gamma function

1. When $p > 0$, $\Gamma(p+1) = p\Gamma(p)$ (Recurrence Relation)
2. When $p \in \mathbb{N}$, $\Gamma(p) = (p-1)!$ (Generalization of Factorial)
3. When $p \leq 0$ and $p \in \mathbb{Z}$, $\Gamma(p)$ does not exist
4. When $p < 0$ and $p \notin \mathbb{Z}$, use successive applications of the Recurrence Relation $\Gamma(p) = \frac{1}{p}\Gamma(p+1)$