Week 1

the Laplace Transform

Example of transformations:
Let's ponder for a moment some of the many fantastic transformations we've studied in the past:

1. $f(x) = x^2$
2. $\frac{d}{dx}x^2 = 2x$
3. $\int x^2 dx = \frac{1}{3}x^3 + C$
4. $\int_0^3 x^2 dx = g$

Definition of the Laplace Transform:
For a function $f(t)$,

$$L\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt, s > 0$$

where $L\{f(t)\}$ is a function of s.

Remark: L provides an efficient method to solve ODEs.

Theorem 1.1 Linearity of the Laplace transform
For a, b constants, and f, g functions, The Laplace transform satisfies

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

Definition of the kernel:
e^{-st} is called the **kernel** of the Laplace transform.

Definition of the domain:
The **domain** of $F(s) = L\{f(t)\}$ is the set of values of s such that $F(s)$ converges.

Example of computing $L\{t\}$:

Recall integration by parts: $\int udv = uv - \int vdu$

$$\begin{aligned}
L\{t\} &= \int_0^\infty te^{-st}dt \\
&= \lim_{\beta\to\infty} \int_0^\beta te^{-st}dt \\
&= \lim_{\beta\to\infty} t\left(-\frac{1}{s}\right)e^{st}|_0^\beta - \int_0^\beta \left(-\frac{1}{s}e^{-st}dt\right) \\
&= \lim_{\beta\to\infty} -\frac{\beta}{s}e^{s\beta} + \frac{1}{s}\int_0^\beta e^{-st}dt \\
&= \lim_{\beta\to\infty} -\frac{\beta}{s}e^{-s\beta} + \frac{1}{s}\left[-\frac{1}{s}e^{-st}\right]_0^\beta \\
&= \lim_{\beta\to\infty} -\frac{\beta}{s}e^{-s\beta} + \frac{1}{s^2}\left[1 - e^{s\beta}\right] \\
&= \frac{1}{s^2}
\end{aligned}$$

Wow, that computation sucked! Luckily, we will be given formula sheets to speed up these calculations. This is because the objective of this class is to solve ODEs using Laplace as a tool, not to practice computing integration by parts.

Theorem 1.2 Laplace transformations of power functions
We have, in general, $L\{t^p\} = \int_0^\infty t^p e^{-st}dt = \frac{1}{s^{p+1}}\int_0^\infty x^p e^- x dx$ where $x = st$. So

$$L\{t^p\} = \frac{1}{s^{p+1}}\Gamma(p+1)$$

Theorem 1.3 Properties of the Gamma function

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1. When $p > 0$, $\Gamma(p+1) = p\Gamma(p)$ (Recurrence Relation)
2. When $p \in \mathbb{N}$, $\Gamma(p) = (p-1)!$ (Generalization of Factorial)
3. When $p \leq 0$ and $p \in \mathbb{Z}$, $\Gamma(p)$ does not exist
4. When $p < 0$ and $p \notin \mathbb{Z}$, use successive applications of the Recurrence Relation $\Gamma(p) = \frac{1}{p}\Gamma(p+1)$