

Multivariable Calculus Winter Notes

Camila Restrepo

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Note: Theorem numbers come from the order they are presented in lecture, and do not correspond to any textbook or written course material.

Week 1

Classifying Critical Points

Theorem 1.1 2nd Derivative Test

Let $f \in C^2(\Omega)$ and let $a \in \Omega (\Omega \subseteq \mathbb{R}^n)$ be a critical point of f .

1. If $H_f(a)$ is positive definite then f has a local minimum at a .
2. If $H_f(a)$ is negative definite then f has a local maximum at a .
3. If $H_f(a)$ is indefinite then f has a saddle point at a .

Recall: Any symmetric $n \times n$ matrix A can be diagonalized, i.e., \exists an orthonormal basis u_1, u_2, \dots, u_n in \mathbb{R}^n and real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $Au_i = \lambda_i u_i \forall i = 1, 2, \dots, n$.

Proposition 1.2

Let Q be the quadratic form associated with an $n \times n$ symmetric matrix A . Then:

1. Q is positive \iff all the eigenvalues of A are positive,
2. Q is negative \iff all the eigenvalues of A are negative,
3. Q is indefinite $\iff A$ has both positive and negative eigenvalues.

Corollary 1.3

Let a be a critical point of a C^2 function $f : \Omega \rightarrow \mathbb{R}$. If $\det H_f(a) \neq 0$, then f has either a local minimum or a local maximum or a saddle point at a .

Definition of degenerate critical points:

A critical point a of a C^2 function f is called non-degenerate if $\det H_f(a) \neq 0$ and degenerate otherwise.

Example of a degenerate critical point:

When $f(x, y) = x^3$ then $(0, 0)$ is a degenerate critical point of f , and f has neither a local extremum at $(0, 0)$ nor a saddle point.

Definition of the principal minors of a matrix:

Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ matrix. Given $k = 1, 2, \dots, n$, we will denote by A_k the $k \times k$ submatrix $A_k = (a_{ij})_{i,j=1}^k$.

The determinants $\det A_k$ are called the **principal minors of A**.

Proposition 1.4

Let A be a symmetric $n \times n$ matrix with $\det A \neq 0$. Then:

1. A is positive definite $\iff \det A_k > 0 \forall k = 1, 2, \dots, n$.
2. A is negative definite $\iff (-1)^k \det A_k > 0 \forall k = 1, 2, \dots, n$.
3. A is indefinite $\iff A$ is neither positive definite nor negative definite.

Corollary 1.5

Let $A = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$. Then:

1. A is positive definite $\iff \alpha > 0$ and $\alpha\gamma - \beta^2 > 0$
2. A is negative definite $\iff \alpha < 0$ and $\alpha\gamma - \beta^2 > 0$
3. A is indefinite $\iff \alpha\gamma - \beta^2 < 0$

Example of classifying a critical point:

We found that the function $f(x, y) = xye^{-x^2-y^2}$ has 5 critical points: $(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}})$, $(\pm \frac{1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}})$, and $(0, 0)$, with an absolute maximum at $(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}})$ and an absolute minimum at $(\pm \frac{1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}})$.

Investigate the nature of $(0, 0)$,

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} [y(1 - 2x^2)e^{-x^2-y^2}] = -4xye^{-x^2-y^2} - 2xy(1 - 2x^2)e^{-x^2-y^2} \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} [x(1 - 2y^2)e^{-x^2-y^2}] = -4xye^{-x^2-y^2} - 2xy(1 - 2y^2)e^{-x^2-y^2} \\ \frac{\partial^2 f}{\partial x \partial y} &= (1 - 2y^2)e^{-x^2-y^2} - 2x^2(1 - 2y^2)e^{-x^2-y^2}\end{aligned}$$

So $H_f(0, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is indefinite $\implies f$ has a saddle point at $(0, 0)$.

Example of non-degenerate critical points:

Find and classify the critical points of $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ where $f(x, y, z) = x^3 - y^3 + 3xy + z^2 - 2z$.

$$\begin{aligned}\nabla f &= (3x^2 + 3y, -3y^2 + 3x, 2z - 2) = 0 \implies x^2 = -y, y^2 = x, z = 1 \implies \\ x^2 &= -y, x^4 = x, z = 1 \implies x^2 = -y, x = 0 \text{ or } x = 1, z = 1 \implies (x, y, z) = \\ &= (0, 0, 1), (1, -1, 1).\end{aligned}$$

So $(0, 0, 1)$ and $(1, -1, 1)$ are the critical points. We have $H_f(x, y, z) = \begin{bmatrix} 6x & 3 & 0 \\ 3 & -6y & 0 \\ 0 & 0 & 2 \end{bmatrix}$,

so $H_f(0, 0, 1) = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is clearly indefinite since the first principal minor is

0 and $H_f(1, -1, 1) = \begin{bmatrix} 6 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is positive definite.

So we have non-degenerate critical points (as $\det H_f \neq 0$). Hence, $(0, 0, 1)$ is a saddle point; $(1, -1, 1)$ is a local minimum.

But f has no global extrema because $f(x, 0, 0) = x^3$ can take arbitrarily positive and negative values.

Example of a degenerate critical point:

Let $f(x, y) = x^4 + y^4$ (with $(x, y) \in \mathbb{R}^2$).

$$\nabla f = (4x^3, 4y^3) = 0 \iff (x, y) = (0, 0).$$

$$H_f(x, y) = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix}, H_f(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So $(0, 0)$ is a degenerate critical point and the 2nd derivative test does not apply. However, f has a global minimum at $(0, 0)$.

Week 2

Inverse Function Theorem and Implicit Function Theorem

Theorem 2.1

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$ is a continuous injective function. Then:

1. f is either strictly increasing or strictly decreasing.
2. $f(I)$ is an interval containing the same number of endpoints as I .
3. f is a homeomorphism of I onto $f(I)$.

Proof. 1. Let us first consider the case that $I = [a, b]$ ($a < b$). Since f is injective, either $f(a) < f(b)$ or $f(b) < f(a)$. Assume that $f(a) < f(b)$ (the other case can be done symmetrically). Let's show that f is strictly increasing on $[a, b]$, i.e., $f(x) < f(y)$ whenever $a \leq x < y \leq b$. We argue by contradiction, supposing that $f(x) > f(y)$ for some $a \leq x < y \leq b$.

Note that $f(y) > f(a)$, for otherwise $f(y) < f(a) < f(b)$ and by the Intermediate Value Theorem (IVT), $\exists \alpha \in (y, b)$ such that $f(\alpha) = f(a)$, contradicting the injectivity of f . Therefore $f(a) < f(y) < f(x)$ and so, again, by the IVT $\exists y' \in (a, x)$ such that $f(y') = f(y)$, again contradicting the injectivity of f .

Next, let I be any interval. Pick up any $a, b \in I$ with $a < b$. Suppose that $f(a) < f(b)$ (the case $f(a) > f(b)$ can be done symmetrically). By the previous paragraph, we know that f is strictly increasing on $[a, b]$. Now, if $x, y \in I$ and $x < y$, then with $\alpha = \min\{a, x\}$, $\beta = \max\{y, b\}$, we have $[a, b], [x, y] \subseteq [\alpha, \beta] \subseteq I$. Since f is strictly increasing on $[a, b]$, we must have (using the 1st paragraph again) $f(\alpha) < f(\beta)$ and f is strictly increasing on $[\alpha, \beta]$. Hence, we conclude that f is strictly increasing on I .

2. Since f is continuous, $J = f(I)$ is an interval. Suppose that f is strictly increasing. Note that the inverse function f^{-1} is then also strictly increasing.

Now, if I contains its left endpoint a , then $\forall x \in I$, $f(a) \leq f(x)$, so $f(a)$ is a left endpoint of J . Similarly, if I contains its right endpoint b , then $f(b)$ is the right endpoint of J . Applying the same argument with f^{-1} in place of f , we conclude if I contains its left (respectively, right) endpoint c , then $f^{-1}(c)$ is the left (respectively, right) endpoint of I . It follows that I and J contain the same number of endpoints.

3. If $I = [a, b]$, then f is a homeomorphism of I onto $f(I)$ because of our general result about continuous injective functions on compact sets.

Otherwise, it follows that $f|_{[a, b]}$ is a homeomorphism onto $f([a, b])$ for any $a, b \in I$ with $a \leq b$. This implies that $f^{-1} : f(I) \rightarrow I$ is continuous (at any $y \in f(I)$).

Indeed, let $y \in f(I)$ and consider any sequence (y_n) in $f(I)$ with $y_n \rightarrow y$. Then the set $S = \{y\} \cup \{y_n : n \in \mathbb{N}\}$ is compact, so it has both a smallest element $c = f(a)$ and a largest element $d = f(b)$. Assuming that f is strictly increasing we must have $a \leq b$, and $f([a, b]) = [c, d] \supseteq S$. Since $f|_{[a, b]}$ is a homeomorphism onto $[c, d]$ (i.e., $(f|_{[a, b]})^{-1} = f^{-1}|_{[c, d]}$ is continuous), we obtain $f^{-1}(y_n) = (f^{-1}|_{[c, d]})(y_n) \rightarrow (f^{-1}|_{[c, d]})(y) = f^{-1}(y)$. It follows that f^{-1} is continuous at any $y \in f(I)$.

□

Theorem 2.2

Let f be a bijection of a non-zero interval $I \subseteq \mathbb{R}$ onto an interval $J \subseteq \mathbb{R}$. If f is differentiable at $a \in I$, $f'(a) \neq 0$, and f^{-1} is continuous at $f(a)$ and $(f^{-1})'(f(a)) = \frac{1}{f'(a)}$

(Sketch).

□

Definition of a diffeomorphism:

Let f be a bijection of an open subset $U \subseteq \mathbb{R}^n$ onto an open subset $V \subseteq \mathbb{R}^n$. If both f and f^{-1} are differentiable (on U and V respectively), then f is called a **diffeomorphism** of U onto V . If both f and f^{-1} are C^k functions ($k = 1, 2, \dots, \infty$), then f is called a **diffeomorphism of class C^k** .

Corollary 2.3

Let f be a differentiable homeomorphism of an open subset $U \subseteq \mathbb{R}$ onto an open subset $V \subseteq \mathbb{R}$. If $f'(a) \neq 0$ for all $a \in U$, then f is a diffeomorphism of U onto V . Moreover, if $f \in C^k(U)$, then f is a C^k diffeomorphism.

Proof. If $b = f(a) \in V$ (where $a \in U$), then there exists an open interval $I \subseteq U$ such that $a \in I$. Then $f(I)$ is another open interval and $f|I$ is a homeomorphism onto $f(I)$ (by the Inverse Function Theorem), and $f|I$ satisfies the assumptions of the above theorem. Hence, $(f|I)^{-1} = f^{-1}|f(I)$ is differentiable at b . But this means that f^{-1} is differentiable at b . Since $b \in V$ is arbitrary, f^{-1} is differentiable on V and so f is a diffeomorphism.

We also have $(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$ for any $b = f(a) \in V$.

Thus, $(f^{-1})' = \frac{1}{f'} \circ f^{-1}$. That f^{-1} is C^k when f is C^k follows by induction on $k = 1, 2, \dots$: When $k = 1$, then $\frac{1}{f'}$ is continuous (as $f \in C^1(U)$), and f^{-1} is continuous, so $(f^{-1})' = \frac{1}{f'} \circ f^{-1}$ is continuous. Assuming that our claim is true for C^k functions, consider $f \in C^{k+1}(U)$. Then $f' \in C^k(U)$, and as $f \in C^k(U)$, $f^{-1} \in C^k(V)$ by induction. Hence, $(f^{-1})' = \frac{1}{f'} \circ f^{-1}$ is a C^k function as the composition of two C^k functions. Therefore $f^{-1} \in C^k(V)$ □

Corollary 2.4 Inverse Function Theorem in 1 variable

Let $I \subset \mathbb{R}$ be an open interval and $f : I \rightarrow \mathbb{R}$ a C^k function such that $f'(x) \neq 0$ for all $x \in I$. Then f is a C^k diffeomorphism of I onto $f(I)$.

Proof. By the IVT either $f'(x) > 0$ for all $x \in I$ (i.e., f is strictly increasing) or $f'(x) < 0$ for all $x \in I$ (i.e., f is strictly decreasing). Hence, f is injective and is a homeomorphism of I onto an open interval J . The assumption of the previous corollary are satisfied, hence the conclusion. \square

Corollary 2.5 Inverse Function Theorem in 1 variable, local version

Let $U \subset \mathbb{R}$ be open and $f : U \rightarrow \mathbb{R}$ be a C^k function. If $f'(a) \neq 0$ at some $a \in U$, then there exists an open interval I such that $a \in I \subseteq U$ and $f|_I$ is a C^k diffeomorphism of I onto $f(I)$.

How do these results generalize to functions of n variables?

Theorem 2.6

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $f : \Omega \rightarrow \mathbb{R}^n$ be injective. Then $f(\Omega)$ is open and f is a homeomorphism of Ω onto $f(\Omega)$.

Proof. Omitted due to high difficulty. \square

Lemma 2.7

If $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an invertible linear transformation then there exists a $c > 0$ such that for all $x \in \mathbb{R}^n$, $\|T(x)\| \geq c\|x\|$.

Proof. Recall that T^{-1} is a Lipschitz function, i.e., there exists $M > 0$ such that $\|T^{-1}(x)\| \leq M\|x\|$ for all $x \in \mathbb{R}^n$. Hence, for all $x \in \mathbb{R}^n$, $\|x\| = \|T^{-1}(T(x))\| \leq M\|T(x)\|$, so $\|T(x)\| \geq \frac{1}{M}\|x\|$. \square

Theorem 2.8

Let f be a bijection of an open subset $U \subseteq \mathbb{R}^n$ onto an open subset $V \subseteq \mathbb{R}^n$. If f is differentiable at $a \in U$, $\det(Df(a)) \neq 0$, and f^{-1} is continuous at $b = f(a)$, then f^{-1} is differentiable at b and $D_{f^{-1}}(b) = (Df(a))^{-1}$.

Proof. Let $T = D_f(a)$, $b = f(a)$. It suffices to show that

$$\lim_{y \rightarrow b} \frac{f^{-1}(y) - f^{-1}(b) - T^{-1}(y - b)}{\|y - b\|} = 0$$

But,

$$\frac{f^{-1}(y) - f^{-1}(b) - T^{-1}(y - b)}{\|y - b\|} = -T^{-1} \left(\frac{y - b - T(f^{-1}(y) - f^{-1}(b))}{\|y - b\|} \right)$$

So it suffices to show that

$$\lim_{y \rightarrow b} \frac{y - b - T(f^{-1}(y) - f^{-1}(b))}{\|y - b\|} = 0$$

and this will be done if we show that

$$\lim_{k \rightarrow \infty} \frac{y_k - b - T(f^{-1}(y_k) - f^{-1}(b))}{\|y_k - b\|} = 0$$

For every sequence $(y_k) \in V \setminus \{b\}$ with $y_k \rightarrow b$. Let $x_k = f^{-1}(y_k) \in U \setminus \{a\}$ (i.e., $y_k = f(x_k)$). Then $x_k \rightarrow f^{-1}(b) = a$ because f^{-1} is continuous at b . Thus we need to show that

$$\lim_{k \rightarrow \infty} \frac{f(x_k) - f(a) - T(x_k - a)}{\|f(x_k) - f(a)\|} =$$

$$\lim_{k \rightarrow \infty} \left[\frac{\|x_k - a\|}{\|f(x_k) - f(a)\|} \frac{f(x_k) - f(a) - T(x_k - a)}{\|x_k - a\|} \right] = \lim_{k \rightarrow \infty} A_k B_k = 0$$

Now, as $T = D_f(a)$, $\lim_{k \rightarrow \infty} B_k = 0$ (by the definition of the derivative). So to complete the proof it is enough to show that the sequence (A_k) is bounded. But

$$\frac{1}{A_k} = \left\| \frac{f(x_k) - f(a) - T(x_k - a)}{\|x_k - a\|} + T \left(\frac{x_k - a}{\|x_k - a\|} \right) \right\| =$$

$$\left\| B_k + T \left(\frac{x_k - a}{\|x_k - a\|} \right) \right\| \geq \left\| T \left(\frac{x_k - a}{\|x_k - a\|} \right) \right\| - \|B_k\|$$

and by the lemma, there exists a $c > 0$ such that $\left\| T \left(\frac{x_k - a}{\|x_k - a\|} \right) \right\| \geq c$ for all k . As $B_k \rightarrow 0$, there exists a k_0 such that for all $k > k_0$ $\frac{1}{A_k} \geq \frac{c}{2}$ and so for all $k \in \mathbb{N}$ $\frac{1}{A_k} \geq \min \left\{ \frac{c}{2}, \frac{1}{A_1}, \frac{1}{A_2}, \dots, \frac{1}{A_{k_0}} \right\} > 0$. Hence, (A_k) is bounded. \square

Corollary 2.9

Let f be a differentiable homeomorphism of an open subset $U \subseteq \mathbb{R}^n$ onto an open subset $V \subseteq \mathbb{R}^n$. If $\det(D_f(x)) \neq 0$ for all $x \in U$, then f is a diffeomorphism of U onto V . Moreover, if $f \in C^k(U)$ then f is a C^k diffeomorphism.

Proof. Clearly, the assumptions of the previous theorem are satisfied for each $a \in U$, so f^{-1} is differentiable at each $b = f(a)$, and f is thus a diffeomorphism of U onto V . \square

Remark: The following example shows that the 1-dimensional Inverse Function Theorem cannot be generalized to n -dimensions.

Example of Polar Coordinate Mapping:
Let $f : (0, \infty) \times \mathbb{R}$ be given by $f(s, t)$

Theorem 2.10 Inverse Function Theorem (IFT)

Let $f : \Omega \rightarrow \mathbb{R}^n$ be a C^k function where $\Omega \subseteq \mathbb{R}^n$ is open (and $k = 1, 2, \dots, \infty$). If $\det(D_f(a)) \neq 0$ for some $a \in \Omega$, then there exists an open set $U \subseteq \Omega$ with $a \in U$ and an open set $V \subseteq \mathbb{R}^n$ with $f(a) \in V$ such that $f|_U$ is a C^k diffeomorphism of U onto V .

Corollary 2.11 Open Mapping Theorem

Let $F : \Omega \rightarrow \mathbb{R}^n$ be C^1 function where $\Omega \subseteq \mathbb{R}^n$ is open. If $\det(D_f(x)) \neq 0$ for all $x \in \Omega$, then f is an open mapping, i.e., for every open subset $W \subseteq \Omega$, $f(W)$ is open in \mathbb{R}^n .

Proof. Let $W \subseteq \Omega$ be open. To conclude that $f(W)$ is open, it suffices to show that for all $b \in f(W)$ there exists an open V such that $b \in V \subseteq f(W)$. But $b = f(a)$ for some $a \in W$ and $f|_W$ and $a \in W$ satisfy the assumption of the IFT. Thus, there exists open $U \subseteq W$ and open $V \subseteq \mathbb{R}^n$ such that $a \in U$, $b \in V$ and $f(U) = (f|_W)(U) = V$. Clearly, $b \in V \subseteq f(W)$. \square

Corollary 2.12

Let $f : \Omega \rightarrow \mathbb{R}^n$ be a C^k function where $\Omega \subseteq \mathbb{R}^n$ is open. If f is injective and $\det(D_f(x)) \neq 0$ for all $x \in \Omega$, then $f(\Omega)$ is open and f is a C^k diffeomorphism of Ω onto $f(\Omega)$.

Proof. By a previous corollary, it suffices to show that $f(\Omega)$ is open and f is a homeomorphism of Ω onto $f(\Omega)$. But by the previous corollary, f is an open mapping, so, in particular, $f(\Omega)$ is open. Thus, it remains to prove that $f^{-1} : f(\Omega) \rightarrow \Omega$ is continuous. Recall that this will be true if for each open $U \subseteq \mathbb{R}^n$, $(f^{-1})^{-1}(U)$ is open relative to $f(\Omega)$, i.e., is open in \mathbb{R}^n because $f(\Omega)$ is open. But $(f^{-1})^{-1}(U) = (f^{-1})^{-1}(U \cap \Omega) = f(U \cap \Omega)$ is indeed open in \mathbb{R}^n by the Open Mapping Theorem. \square

Example of determining a diffeomorphism:

The polar coordinate mapping $f(r, \theta) = (r \cos \theta, r \sin \theta)$ (considered on $(0, \infty) \times \mathbb{R}$), is an open mapping of $(0, \infty) \times \mathbb{R}$ onto $\mathbb{R}^2 \setminus \{(0, 0)\}$ because $\det(D_f(r, \theta)) = r > 0$ for all $(r, \theta) \in (0, \infty) \times \mathbb{R}$.

Note that $\varphi = f|((0, \infty) \times (-\pi, \pi))$ is injective. Hence, by the last corollary φ is a C^∞ diffeomorphism on $(0, \infty) \times (-\pi, \pi)$ onto $\varphi((0, \infty) \times (-\pi, \pi)) = \mathbb{R}^2 \setminus ((-\infty, 0] \times \mathbb{R})$.

$$D_{\varphi^{-1}}(r \cos \theta, r \sin \theta) = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}^{-1} = \frac{1}{r} \begin{bmatrix} r \cos \theta & r \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

Similarly $\varphi|((0, \infty) \times (a, b))$, where $b - a = 2\pi$ is a C^∞ diffeomorphism on $(0, \infty) \times (a, b)$ onto $\mathbb{R}^2 \setminus \{r(\cos \theta, \sin \theta) : r \geq 0\}$.

Definition of an implicit function:

Let $\Omega_n \subseteq \mathbb{R}^n$, $\Omega_m \subseteq \mathbb{R}^m$, $F : \Omega_n \times \Omega_m \rightarrow \mathbb{R}^m$, and $c \in \mathbb{R}^m$.

Consider the equation

$$F(x, y) = c \quad (x \in \Omega_n, y \in \Omega_m)(*)$$

which we suppose needs to be solved for y . If for every $x \in \Omega_n$ this equation has a solution, then by choosing for each $x \in \Omega_n$ a solution $y \in \Omega_m$ and calling it $f(x)$, we obtain a function $f : \Omega_n \rightarrow \Omega_m$ such that $F(x, f(x)) = c$ for all $x \in \Omega_n$. Any such function is called an **implicit function** defined by Eq. (*).

Note: If for all $x \in \Omega_n$ there exists a unique $y \in \Omega_m$ such that $F(x, y) = c$, then Eq. (*) defines a unique implicit function, but in general, implicit functions are not unique.

Example of:

Let $n = m = 1$, $\Omega_n = \Omega_m = [-1, 1]$, $F(x, y) = x^2 + y^2$, $c = 1$. Then the functions $f_{\pm}(x) = \pm \sqrt{1 - x^2}$ are implicit functions defined by (*) (i.e., eg. $x^2 + y^2 = 1$) and there are many other implicit functions.

If we replace Ω_m by $[0, 1]$, then f_+ will be the unique implicit function defined by (*) ($f_+(x) = \sqrt{1 - x^2}$).

Question

Under what conditions does an implicit function exist; is unique; is it differentiable? If it is differentiable how can we obtain its derivative?

Note: Let $F : \Omega \rightarrow \mathbb{R}^m$ be a C^k function where $\Omega \subseteq \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ is open. We will write the elements of $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ as (x, y) where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. Then

$$D_f(x, y) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(x, y) & \dots & \frac{\partial F_1}{\partial x_n}(x, y) & \frac{\partial F_1}{\partial y_1}(x, y) & \dots & \frac{\partial F_1}{\partial y_m}(x, y) \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1}(x, y) & \dots & \frac{\partial F_m}{\partial x_n}(x, y) & \frac{\partial F_m}{\partial y_1}(x, y) & \dots & \frac{\partial F_m}{\partial y_m}(x, y) \end{bmatrix}$$

with the first $m \times n$ block will be named $\frac{\partial F}{\partial x}(x, y)$ and the second $m \times m$ block will be named $\frac{\partial F}{\partial y}(x, y)$.

Thus, we can write $D_F(x, y) = \begin{bmatrix} \frac{\partial F}{\partial x}(x, y) & \frac{\partial F}{\partial y}(x, y) \end{bmatrix}$

Theorem 2.13 Implicit Function Theorem (IPFT)

Let $F : \Omega \rightarrow \mathbb{R}^m$ be a C^k function where $\Omega \subseteq \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ is open. Suppose that for $(a, b) \in \Omega$ and $c \in \mathbb{R}^m$, $F(a, b) = c$ and $\det \left(\frac{\partial F}{\partial y}(a, b) \right) \neq 0$. Then there exist open sets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ that satisfy:

1. $(a, b) \in U \times V$,
2. for all $x \in U$, there exists a unique $y \in V$ such that $F(x, y) = c$.

Moreover, the unique implicit function $f : U \rightarrow V$ defined by the equation $F(x, y) = c$ ($x \in U$, $y \in V$) is a C^k function.

Proof. Define $G : \Omega \rightarrow \mathbb{R}^{n+m}$ by $G(x, y) = (x, F(x, y))$. This is a C^k function, $G(a, b) = (a, c)$ and

$$D_G(x, y) = \begin{bmatrix} I_n & 0 \\ \frac{\partial F}{\partial x}(x, y) & \frac{\partial F}{\partial y}(x, y) \end{bmatrix}$$

Thus $\det(D_G(a, b)) = (\det I_n) \left(\det \left(\frac{\partial F}{\partial y}(a, b) \right) \right) \neq 0$.

Thus by the IFT, there exists an open subset $\Omega_1 \subseteq \Omega$ with $(a, b) \in \Omega_1$ and an open subset $W \subseteq \mathbb{R}^{n+m}$ with $(a, c) = G(a, b) \in W$ such that $G|_{\Omega_1}$ is a C^k diffeomorphism of Ω_1 onto W . Let $H = (G|_{\Omega_1})^{-1} : W \rightarrow \Omega_1$. Then $H(x, y) = (j(x, y), k(x, y))$ where $j : W \rightarrow \mathbb{R}^n$ and $k : W \rightarrow \mathbb{R}^m$ are C^k functions. Note that $(x, y) = G(H(x, y)) = (j(x, y), F(k(x, y)))$ for all $(x, y) \in W$. Hence, $j(x, y) = x$ and $F(k(x, y)) = y$ for all $(x, y) \in W$. Thus $H(x, y) = (x, k(x, y))$ and so for all $(x, y) \in W$,

$$(x, k(x, y)) \in \Omega_1 \text{ and } F(x, k(x, y)) = y$$

Note that we may assume that $\Omega_1 = U' \times V$ where $U' \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open. [Indeed, $(a, b) \in \Omega_1$ and Ω_1 is open, so there exists an $r > 0$ such that $B_r^{n+m}(a, b) \in \Omega_1$. But $B_r^{n+m}(a, b) \supseteq B_{\frac{r}{2}}^n(a) \times B_{\frac{r}{2}}^m(b)$. So we can take $U' = B_{\frac{r}{2}}^n(a)$, $V = B_{\frac{r}{2}}^m(b)$ and replace Ω_1 with $U' \times V$ and W with $G(U' \times V)$].

Moreover, since $(a, c) \in W$, we can find an open set U such that $a \in U \subseteq U'$ and $U \times \{c\} \subseteq W$. Then for all $x \in U$, $(x, c) \in W$ and so $F(x, k(x, c)) = c$. Thus when $f : U \rightarrow V$ is given by $f(x) = k(x, c)$, then f is an implicit function defined by the equation $F(x, y) = c$ (for $x \in U, y \in V$). It is clear that f is a C^k function.

It remains to confirm that for all $x \in U$ there exists a unique $y \in V$ such that $F(x, y) = c$. But if $y_1, y_2 \in V$ and $F(x, y_1) = c = F(x, y_2)$, then $G(x, y_1) = (x, c) = G(x, y_2)$, and so $y_1 = y_2$ as $G|_{U \times V}$ is injective. \square

Week 3

IPFT Practice and Constraints

Corollary 3.1

With the assumptions and notation of the IPFT, let $S = \{(x, y) \in \Omega : F(x, y) = c\}$. Then $S \cap (U \times V) = \{(x, y) \in \mathbb{R}^{n+m} : x \in U \text{ and } y = f(x)\}$.

Remark: Note that when $m = 1$, then $\det \left(\frac{\partial F}{\partial y} \right) = \frac{\partial F}{\partial y}$. So if $\frac{\partial F}{\partial y}(a, b) \neq 0$ then the level set $S = \{(x, y) \in \mathbb{R}^{n+1} : F(x, y) = c\}$ in a neighbourhood of (a, b) is the graph of the implicit function.

Example of:

(IPFT, level set, and graph) Consider the level set $S = \{(x, y) \in \mathbb{R}^2 : x^3 y^2 + y^3(x - 1)^2 = 1\}$ of $F(x, y) = x^3 y^2 + y^3(x - 1)^2$.

1. Show that S is not the graph of any function $y = f(x)$, i.e., $S \neq \{(x, y) \in \mathbb{R}^2 : y = f(x)\}$.
2. Show that in a neighbourhood of $(1, 1)$, S is the graph of a smooth function f and find the slope of the tangent line to the graph of f at $(1, 1)$.

Solutions:

1. $(1, -1), (1, 1) \in S$, so no such function exists.
2. $\frac{\partial F}{\partial y}(1, 1) = 2x^3 y + 3^2(x - 1)^2 \Big|_{x=1, y=1} = 2 \neq 0$. So by the IPFT (with $a = b = c = 1$) and the corollary there exist open sets $U, V \subseteq \mathbb{R}$ with $(1, 1) \in U \times V$ and a smooth function $f : U \rightarrow V$ such that $f(1) = 1$, $F(x, f(x)) = 1 = x^3 f(x)^2 + f(x)^3(x - 1)^2 = 1$ for all $x \in U$, and $S \cap (U \times V) = \{(x, y) : x \in U \text{ and } y = f(x)\}$.

The slope is $f^{-1}(1)$: Since $x^3 f(x)^2 + f(x)^3 (x-1)^2 = 1$ for all $x \in U$, so $0 = \frac{d}{dx} [x^3 f(x)^2 + f(x)^3 (x-1)^2] = 3x^2 f(x)^2 + 2x^3 f(x) f'(x) + 3f(x)^2 f'(x) (x-1)^2 + 2f(x)^3 (x-1)$. When $x = 1$, $f(1) = 1$, and so $0 = 3 + 2f'(1)$. Thus $f'(1) = -\frac{3}{2}$.

Example of:

(Finding the derivative without the function) Consider the problem of solving the system of equations: $\begin{cases} xy^2 + xzu + yv^2 = 3 \\ u^3 yz + 2xv - u^2 v^2 = 2 \end{cases}$ (*) for u and v in terms of x, y, z near $x = y = z = u, v = 1$ and computing the partial $\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}$.

Let $a = (1, 1, 1), b = (1, 1), c = (3, 2)$, and $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by

$$F(x, y, z, u, v) = (xy^2 + xzu + yv^2, u^3 yz + 2xv - u^2 v^2).$$

$$\text{Then } F(a, b) = c, \frac{\partial F}{\partial (u, v)} = \begin{bmatrix} xz & 2yv \\ 3u^2 yz - 2uv^2 & 2x - 2u^2 v \end{bmatrix}.$$

$$\det \left(\frac{\partial F}{\partial (u, v)} (a, b) \right) = \det \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = -2 \neq 0.$$

Hence, by the IPFT there exists a smooth function $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z))$ defined on a neighbourhood U of $u = (1, 1, 1)$ such that $F(x, y, z, f(x, y, z)) = (3, 2) = c$ for all $(x, y, z) \in U$ and $f(1, 1, 1) = (1, 1)$: $u = f_1(x, y, z), v = f_2(x, y, z)$ are the expressions of u and v in terms of x, y, z . To find $\frac{\partial u}{\partial z}$ and $\frac{\partial v}{\partial z}$ we differentiate Eqs(*) with respect to z , treating u and v as functions of x, y, z :

$$\begin{aligned} 0 &= \frac{\partial}{\partial z} (xy^2 + xzu + yv^2) = xu + xz \frac{\partial u}{\partial z} + 2yv \frac{\partial v}{\partial z} \\ 0 &= \frac{\partial}{\partial z} (u^3 yz + 2xv - u^2 v^2) = 3u^2 \frac{\partial u}{\partial z} yz + u^3 y + 2x \frac{\partial v}{\partial z} - 2u \frac{\partial u}{\partial z} v^2 - u^2 2v \frac{\partial v}{\partial z} \end{aligned}$$

With $(x, y, z) = (1, 1, 1), (u, v) = (1, 1)$ we get

$$1 + \frac{\partial u}{\partial z} + 2 \frac{\partial v}{\partial z} = 0, \frac{\partial u}{\partial z} + 1 = 0.$$

$$\text{Hence, } \frac{\partial f_1}{\partial z} = \frac{\partial u}{\partial z} (1, 1, 1) = -1, \frac{\partial f_2}{\partial z} = \frac{\partial v}{\partial z} (1, 1, 1) = 0.$$

Proposition 3.2 Implicit Differentiation

Let $F : \Omega_n \times \Omega_m \rightarrow \mathbb{R}^m$ be a C^1 function where $\Omega_n \subset \mathbb{R}^n$ and $\Omega_m \subset \mathbb{R}^m$ are open and let $c \in \mathbb{R}^m$. If $f : \Omega_n \rightarrow \Omega_m$ is a differentiable function such that $F(x, f(x)) = c$ for all $x \in \Omega_n$, then

$$\frac{\partial F}{\partial y}(x, f(x)) D_f(x) = -\frac{\partial F}{\partial x}(x, f(x))$$

and

$$D_f(x) = -\left[\frac{\partial F}{\partial y}(x, f(x))\right]^{-1} \frac{\partial F}{\partial x}(x, f(x))$$

provided $\det\left(\frac{\partial F}{\partial y}(x, f(x))\right) \neq 0$.

Proof. Define $g : \Omega_n \rightarrow \Omega_n \times \Omega_m$ by $g(x, f(x))$. Then g is differentiable and

$$D_g(x) = \begin{bmatrix} I_n \\ D_f(x) \end{bmatrix}.$$

Since $(F \circ g)(x) = c$, the chain rule yields $0 = D_{F \circ g}(x) = D_F(g(x)) D_g(x) = \begin{bmatrix} \frac{\partial F}{\partial x}(g(x)) & \frac{\partial F}{\partial y}(g(x)) \end{bmatrix} \begin{bmatrix} I_n \\ D_f(x) \end{bmatrix} = \frac{\partial F}{\partial x}(x, f(x)) + \frac{\partial F}{\partial y}(x, f(x)) D_f(x)$. Hence, the result. \square

3.1 Constrained Extrema and Lagrange Multipliers

Let $\Omega \subseteq \mathbb{R}^n$ be open and $f, g_1, g_1, \dots, g_m : \Omega \rightarrow \mathbb{R}$ be C^1 functions. Suppose that for some $c_1, c_2, \dots, c_m \in \mathbb{R}$, $S = \{x \in \Omega : g_1(x) = c_1, g_2(x) = c_2, \dots, g_m(x) = c_m\} \neq \emptyset$. The problem of finding the extreme values of f on the set S (i.e., the extrema of $f|_S$) is referred to as the problem of finding the extreme values of f subject to (or with) the constraints $g_1(x) = c_1, \dots, g_m(x) = c_m$.

E.g., finding the extreme values of $f(x, y, z) = \sin(x + y) \cos(y + z)$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 1$ means finding the extreme values of f on the sphere $S_1(0, 0, 0) = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$.

Theorem 3.3

Let $f, g : \Omega \rightarrow \mathbb{R}$ be C^1 functions where $\Omega \subseteq \mathbb{R}^{n+1}$ and let $S = \{x \in \Omega : g(x) = c\}$ (where $c \in \mathbb{R}$). If $f|_S$ attains an extreme value at some $s \in S$ where $\nabla g(s) \neq 0$, then there exists an $\lambda \in \mathbb{R}$ (called a Lagrange multiplier) such that $\nabla f(s) = \lambda \nabla g(s)$.

Proof. Since $\nabla g(s) \neq 0$, $\frac{\partial g}{\partial x_i}(s) \neq 0$ for some $i = 1, 2, \dots, n+1$. Let us first consider the case that $\frac{\partial g}{\partial x_{n+1}}(s) \neq 0$. Let $a = (s_1, \dots, s_n)$, $b = s_{n+1}$ (so $s = (a, b)$). Then $g(a, b) = c$ and $\frac{\partial g}{\partial x_{n+1}}(a, b) \neq 0$. Hence by the IPFT there exist open sets $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}$ such that $s = (a, b) \in U \times V$ and a C^1 function $\varphi : U \rightarrow V$ such that $\varphi(a) = b$ and $g(x, \varphi(x)) = c$ (i.e., $(x, \varphi(x)) \in S$) for all $x \in U$. Define $\tilde{f} : U \rightarrow \mathbb{R}$ by $\tilde{f}(x) = f(x, \varphi(x))$. Clearly, \tilde{f} is a C^1 function and \tilde{f} has an extremum at $x = a$, so $\nabla \tilde{f}(a) = 0 = D_{\tilde{f}}(a)$. Note that $\tilde{f} = f \circ h$ where $h : U \rightarrow S \subseteq \mathbb{R}^{n+1}$ is given by $h(x) = (x, \varphi(x))$. Hence, by the Chain Rule

$$0 = D_{\tilde{f}}(a) = D_f(h(a)) D_h(a) = D_f(s) \begin{bmatrix} I_n \\ D_{\varphi}(a) \end{bmatrix}$$

or

$$0 = \frac{\partial f}{\partial x_i}(s) + \frac{\partial f}{\partial x_{n+1}} + \frac{\partial \varphi}{\partial x_i}(a) \forall i = 1, 2, \dots, n.$$

But by the Implicit Differentiation Formula,

$$\begin{aligned} D_{\varphi}(a) &= - \left[\frac{\partial g}{\partial x_{n+1}}(a, \varphi(a)) \right]^{-1} \left[\frac{\partial g}{\partial x_1}(a, \varphi(a)), \dots, \frac{\partial g}{\partial x_n}(a, \varphi(a)) \right] \\ &= - \left[\frac{\partial g}{\partial x_{n+1}}(s) \right]^{-1} \left[\frac{\partial g}{\partial x_1}(a, \varphi(a)), \dots, \frac{\partial g}{\partial x_n}(s) \right] \end{aligned}$$

Therefore,

$$0 = \frac{\partial f}{\partial x_1}(s) - \frac{\partial f}{\partial x_{n+1}}(s) \left(\frac{\partial g}{\partial x_{n+1}}(s) \right)^{-1} \frac{\partial g}{\partial x_i}(s) \forall i = 1, 2, \dots, n$$

Note that this equality also trivially holds when $i = n+1$. Thus, with

$\lambda = \frac{\partial f}{\partial x_{n+1}}(s) \left(\frac{\partial g}{\partial x_{n+1}}(s) \right)^{-1}$ we obtain $\nabla f(s) = \lambda \nabla g(s)$.

If $\frac{\partial g}{\partial x_{n+1}}(s) = 0$, we can choose $p = 1, 2, \dots, n$ such that $\frac{\partial g}{\partial x_p}(s) \neq 0$.

Define a linear isomorphism $T : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ by $T(x_1, x_2, \dots, x_{n+1}) = (x_1, x_2, \dots, x_{p-1}, x_{n+1}, x_p, x_{p+1}, \dots, x_n)$, and let $\Omega_* = T^{-1}(\Omega)$, $S_* = T^{-1}(S)$, $s_* = T^{-1}(s)$, $f_* = f \circ T : \Omega_* \rightarrow \mathbb{R}$, $g_* = g \circ T : \Omega_* \rightarrow \mathbb{R}$. Then $S_* = \{x \in \Omega_* : g_*(x) = c\}$ and $f_*|_{S_*}$ has an extremum at s_* . Moreover, $\frac{\partial g_*}{\partial x_{n+1}}(s_*) = \frac{\partial g}{\partial x_p}(s) \neq 0$. So by the 1st part of the proof, there exists a $\lambda \in \mathbb{R}$ such that $\nabla f_*(s_*) = \lambda \nabla g_*(s_*)$. But

$$\frac{\partial f_*}{\partial x_i}(s_*) = \begin{cases} \frac{\partial f}{\partial x_i}(s) & \text{for } i = 1, 2, \dots, p-1 \\ \frac{\partial f}{\partial x_{i+1}}(s) & \text{for } i = p, p+1, \dots, n \\ \frac{\partial f}{\partial x_p}(s) & \text{for } i = n+1 \end{cases}$$

and similarly for g_* . Hence, $\nabla f(s) = \lambda \nabla g(s)$. \square

Example of Minimum distance with the Lagrange multiplier:

Find the minimum distance from the point $(1, 2, 0)$ to the surface $z^2 = x^2 + y^2, z \geq 0$, using the Lagrange multiplier.

The distance from $(1, 2, 0)$ to a point (x, y, z) is $d = \sqrt{(x-1)^2 + (y-2)^2 + z^2}$ and it suffices to minimize d^2 , i.e., the function $f(x, y, z) = (x-1)^2 + (y-2)^2 + z^2$ on the set $\tilde{S} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0, z \geq 0\}$. Recall that in Lecture 25 we solved this problem by eliminating z .

In particular, we found that f attains a global min value of \tilde{S} but there does not exist a global max. Note also that $z = 0 \implies x^2 + y^2 = 0$, and $f(0, 0, 0) = 5$ while $f(0, 1, 1) = 3 < 5$. So f attains a global min on $S = \{(x, y, z) : x^2 + y^2 - z^2 = 0 \text{ and } z > 0\}$ and does not have a global max on S . We can apply our theorem to:

$\Omega = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$. $f : \Omega \rightarrow \mathbb{R}$, $f(x, y, z) = (x-1)^2 + (y-2)^2 + z^2$, $g : \Omega \rightarrow \mathbb{R}$, $g(x, y, z) = x^2 + y^2 - z^2$, and $S = \{(x, y, z) \in \Omega : g(x, y, z) = 0\} = \{(x, y, z) \in \Omega : x^2 + y^2 - z^2 = 0\}$ ($c = 0$).

Note that $\nabla g(x, y, z) = (2x, 2y, -2z) \neq 0$ for all $(x, y, z) \in \Omega$, so by the theorem if a minimum occurs at $(x, y, z) \in S$ then $\nabla f(x, y, z) = (2(x-1), 2(y-2), 2z) = \lambda(2x, 2y, -2z)$ for some $\lambda \in \mathbb{R}$. So we need to solve the system:

$$\begin{cases} 2(x-1) = 2\lambda x \\ 2(y-2) = 2\lambda y \\ 2z = -2\lambda z \\ x^2 + y^2 - z^2 = 0 \end{cases} \implies \lambda = -1 \implies x = \frac{1}{2}, y = 1 \implies z = \sqrt{\frac{5}{4}}$$

So a minimum occurs at $(\frac{1}{2}, 1, \sqrt{\frac{5}{4}})$ and the min distance is $d_{\min} = \sqrt{f(\frac{1}{2}, 1, \sqrt{\frac{5}{4}})} = \sqrt{\frac{5}{2}}$.

Example of Maximum volume with the Lagrange multiplier:

Consider rectangular boxes $[-x, x] \times [-y, y] \times [-z, z]$ (x, y, z) incubed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (i.e., with vertices on the ellipsoid). Find the values of x, y, z which maximize the volume of such a box and the maximum volume.

Intuitively, it seems clear that the maximum exists. Can we confirm this mathematically?

Note that $\tilde{S} = \{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$ is compact, so by the EVT $f(x, y, z) = 8xyz$ attains its absolute maximum on \tilde{S} . It is clear that the maximum value is strictly positive, so (among other possibilities), it is attained at a point where $x, y, z > 0$. Hence, our problem has a solution.

Formally we work with the open set $\Omega = \{(x, y, z) : x, y, z > 0\}$ with the constraint function $g : \Omega \rightarrow \mathbb{R}$ given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and the function to maximize is $f(x, y, z) = 8xyz$. Note that $\nabla g(x, y, z) = (\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}) \neq 0$ for all $(x, y, z) \in \Omega$. By the theorem, the max occurs at a point $(x, y, z) \in S$ where $\nabla f(x, y, z) = (8yz, 8xz, 8xy) = \lambda(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2})$ for some $\lambda \in \mathbb{R}$. So we need to solve the system:

$$\left\{ \begin{array}{l} 8yz = \lambda \frac{2x}{a^2} \\ 8xz = \lambda \frac{2y}{b^2} \\ 8xy = \lambda \frac{2z}{c^2} \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \end{array} \right. \implies \begin{array}{l} 4xyz = \lambda \frac{x^2}{a^2} \\ 4xyz = \lambda \frac{y^2}{b^2} \\ 4xyz = \lambda \frac{z^2}{c^2} \end{array} \implies 2xyz = \lambda \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) = \lambda$$

Given that $x, y, z > 0$, $\frac{1}{b} = \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} \implies x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$.
 The max volume is then $f\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right) = \frac{8abc}{3\sqrt{3}}$.