Week 1

Classifying Critical Points

Theorem 1.1 2nd Derivative Test
Let $f \in C^2(\Omega)$ and let $a \in \Omega (\Omega \subseteq \mathbb{R}^n)$ be a critical point of f.

1. If $H_f(a)$ is positive definite then f has a local minimum at a.
2. If $H_f(a)$ is negative definite then f has a local maxiumum at a.
3. If $H_f(a)$ is indefinite then f has a saddle point at a.

Recall: Any symmetric $n \times n$ matrix A can be diagonalized, i.e., $\exists$ an orthonormal basis $u_1, u_2, \ldots, u_n$ in $\mathbb{R}^n$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $Au_i = \lambda_i u_i \forall i = 1, 2, \ldots, n$.

Proposition 1.2
Let Q be the quadratic form associated with an $n \times n$ symmetric matrix A. Then:

1. Q is positive $\iff$ all the eigenvalues of A are positive,
2. Q is negative $\iff$ all the eigenvalues of A are negative,
3. Q is indefinite $\iff$ A has both positive and negative eigenvalues.

Corollary 1.3
Let a be a critical point of a C^2 function $f : \Omega \to \mathbb{R}$. If $\det H_f(a) \neq 0$, then f has either a local minimum or a local mximum or a saddle point at a.

Definition of degenerate critical points:
A critical point a of a C^2 function f is called non-degenerate if $\det H_f(a) \neq 0$ and degenerate otherwise.

Example of a degenerate critical point:
When $f(x, y) = x^3$ then $(0, 0)$ is a degenerate critical point of f, and f has neither a local extremum at $(0, 0)$ nor a saddle point.

Definition of the principal minors of a matrix:
Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ matrix. Given $k = 1, 2, \ldots, n$, we will denote by A_k the $k \times k$ submatrix $A_k = (a_{ij})_{i,j=1}^k$.
The determinants $\det A_k$ are called the **principal minors of A**.

Proposition 1.4
Let A be a symmetric $n \times n$ matrix with $\det A \neq 0$. Then:

1. A is positive definite $\iff \det A_k > 0 \forall k = 1, 2, \ldots, n$.
2. A is negative definite $\iff (-1)^k \det A_k > 0 \forall k = 1, 2, \ldots, n$.
3. A is indefinite $\iff$ A is neither positive definite nor negative definite.