Recursive

A question

- People often give out their telephone number as a word representing the seven-digit number. For example, if a number is 866-2665, I could tell people my number was “TOOCOOL” instead of the hard-to-remember seven-digit number.
- Not that many other possibilities (most of which are nonsensical) can represent 866-2665. You can see how letters correspond to numbers on a telephone keypad.
- Write a function that takes a seven-digit telephone number and prints out all of the possible “words” or combinations of letters that can represent the given number. Because the 0 and 1 keys have no letters on them you should change only the digits 2-9 to letters.

Objectives

- To understand how to think recursively
- To learn how to trace a recursive method
- To learn how to write recursive algorithms and methods for using arrays and linked lists
- To understand how to use recursion to calculate permutation
- To understand how to use recursion to solve the Towers of Hanoi problem

Iteration

- Iteration
  - Loops: for, while

Recursion

- Recursion is a problem-solving approach in which a problem is solved using repeatedly applying the same solution to smaller instances.
  - Each instance to the problem has size.
  - An instance of size *n* can be solved by putting together solutions of instances of size at most *n*-1.
  - An instance of size 1 or 0 can be solved very easily.

Introduction

- Iteration
  - Loops: for, while
- Recursion
  - A function refers to itself in its own definition
  - Function call: f1 calls f2
  - Function call: f1 calls f1?
  - Provides an elegant and powerful alternative for performing repetitive tasks

Outline

- Factorial
- Recursion trace
- Linear recursion, tail recursion, binary recursion, multiple recursion
- Fractal
- Towers of Hanoi, Complexity
- Reasoning

Several questions

- Factorial

$$n! = \begin{cases} 1 \quad if \quad n = 0 \\ n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 \quad if \quad n \geq 1 \end{cases}$$

$$factorial(n) = \begin{cases} 1 \quad if \quad n = 0 \\ n \cdot factorial(n-1) \quad if \quad n \geq 1 \end{cases}$$

One or more base cases

One or more recursive cases

Implementation

```
public int recursiveFactorial(int n){
        if (n==0) return 1;
        else return (n*recursiveFactorial(n-1));
}
```

Greatest Common Divisor

- The greatest common divisor, sometimes also called the highest common divisor, of two positive integers *a* and *b*, denoted as gcd(a,b), is the largest divisor common to *a* and *b*.
- The greatest common divisor g is the largest natural number that divides both a and b without leaving a remainder.
  - E.g., gcd(3,8)=1, gcd(4,16)=4
- Brute forth algorithm

```
public static int gcdI (int a, int b){
    int n = (a<b)?a:b;
    int gcd = 1, i = 1;
    while (i <= n) {
        if (a % i == 0 && b % i == 0) gcd = I;
        i++;
    }
    return gcd;
}
```

Euclidean algorithm (Euclid’s algorithm) gcd(a,b)

- It is named after the ancient Greek mathematician Euclid.
- If a= 0, gcd(a,b)=b
- If b=0, gcd(a,b) = a
- If a<>0 and b<>0, gcd(a,b)=gcd(b,a%b)

Lemma

- **Lemma.** Suppose $d,u,v \in Z$. Then $d \mid u$ and $d \mid v$ if and only if $d \mid v$ and $d \mid u-qv$ where $q \in Z$.
- Proof:
- ($\Rightarrow$) If $d|u$ and $d|v$, then clearly $d|v$. Since $d|u$, there exists $n \in Z$ such that $u=nd$. Similarly, there is $m \in Z$ such that $v = md$. Then $u-qv = nd-qmd = (n-qm)d$, so $d \mid u-qv$ as well.
- ($\Leftarrow$) Again, it is trivial to see that if $d|v$ and $d|u-qv$, then $d|v$. We can write $v=md$ and $u-qv=nd$ for some $m,n \in Z$. Then $u=nd+qv=nd+qmd=(n+qm)d$, so $d|u$.

- Notation $d \mid u$: u is divisible by d, e.g., 15 is divisible by 3.

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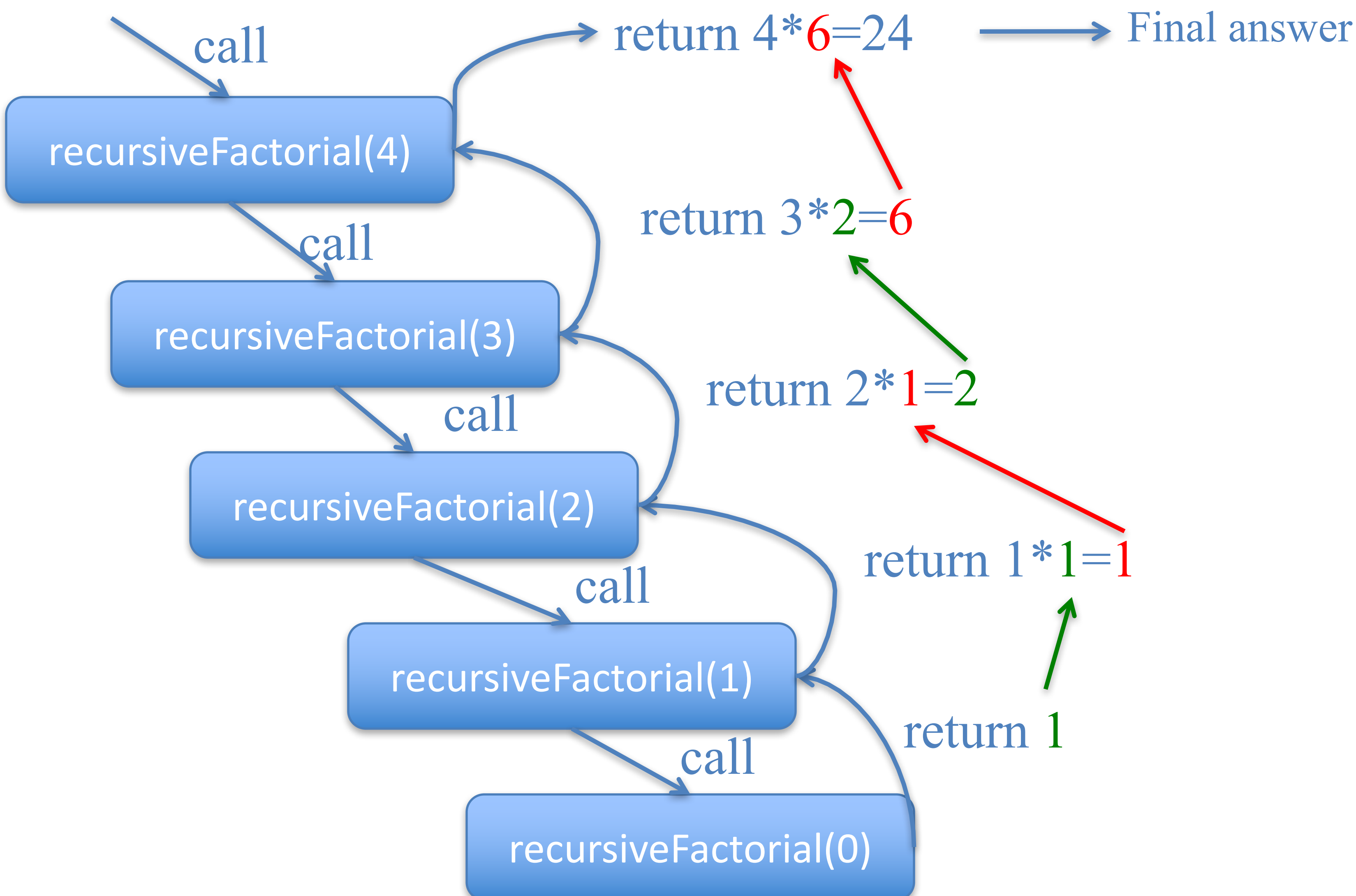
Recursion trace

- Illustrate the execution of a recursive function definition by means of recursion trace.
  - Each entry of the trace corresponds to a recursive call.
  - Each new recursive function call is indicated by an arrow to the newly called function.
  - When the function returns, an arrow showing this return is drawn. The return value may be indicated with this arrow.

A recursive trace for the call recursiveFactorial (4)



System Processing of a Recursion

- Push onto a stack the information of the current execution
- Execute the recursive call
- Retrieve the information from the stack by pop
- Too many recursive calls without pop will result in stack overflow.

Recursion versus Iteration

- Recursive methods are often more expensive than iterative methods because the stack overhead is larger than the loop overhead
- Recursive methods are easier to write and conceptualize.

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Linear recursion

- A function is defined so that it makes at most one recursive call each time it is invoked. (Simplest form of recursion)
- SumLinearRecursive(A,n)

```
public static int SumLinearRecursive(int[] A,int n)
{
    if(n==1) return A[0];
    else return (SumLinearRecursive(A,n-1)+A[n-1]);
}
```

- Recursion trace for SumLinearRecursive(A,5)
- *n* calls for an array of size *n*, time proportional to *n*, space proportional to *n*

Tail recursion

- An algorithm uses tail recursion if
  - it uses linear recursion and
  - it makes a recursive call as its very last operation
- SumLinearRecursive does not use tail recursion

```
public static int SumLinearRecursive(int[] A,int n)
{
    if(n==1) return A[0];
    else return (SumLinearRecursive(A,n-1)+A[n-1]);
}
```

Tail recursion

- ReverseArrayRecursive uses tail recursion

  call ReverseArrayRecursive(A,0,A.length-1);
- Method implementation

```
public static void ReverseArrayRecursive(int[] A, int i,int j){
        if(i<j){
                int tmp = A[i];
                A[i]=A[j];
                A[j]=tmp;
                ReverseArrayRecursive(A,i+1,j-1);
        }
}
```

Tail recursion

- An algorithm uses tail recursion if
  - it uses linear recursion and
  - it makes a recursive call as its very last operation
- ReverseArrayRecursive uses tail recursion
- When an algorithm uses tail recursion, we convert the recursive algorithm into a non-recursive one, by iterating through the recursive calls rather can calling them explicitly.

```
public static void ReverseArray (int[] A, int i,int j){
        while i<j do{
                swap A[i] and A[j]
                i=i+1
                j=j-1
        }
}
```

Binary Recursion

- An algorithm makes two recursive calls
- Solve two similar halves of some problem

- SumBinaryRecursive(A,i,n)

- Recursion trace
- The maximum number of function instances that are active at the same time: $1+\log_2 n$
  - Space (improves SumLinearRecursive)
  - Time: proportional to n (2n-1 total function calls)

Multiple Recursion

- A function can make *m* recursive calls where m>2
- Very commonly used when we want to enumerate various configurations in order to solve a combinatorial puzzle
- Permutation

Permutation

{1,2,3}, permutations: n!

- [1]{2,3}
  - [1,2]{3}→[1,2,3]
  - [1,3]{2}→[1,3,2]
- [2]{1,3}
  - [2,1]{3} → [2,1,3]
  - [2,3]{1} →[2,3,1]
- [3]{1,2}
  - [3,1]{2}→[3,1,2]
  - [3,2]{1}→[3,2,1]

permutation

public static <E> void PermuteArray (E[] array, int prefixLen)

- Base case: prefixLen =array.length
- Recursive case: prefixlen <array.lenth
- For (each position from [prefixlen] to array.length)
  - swap array[i] and array[prefixLen]
  - permutate all the elements in array[prefixLen+1] ... array[arrayl.length-1] by recursive call, PermuteArray (array,prefixLen+1);
  - swap back array[prefixLen] and array[i]
- Main function call

  Integer[] A= new Integer []{1,2,3}

  PermuteArray(A,0)

  A={1, 2, 3 }

  PermuteArray({1,2,3},0)

permutation

PermuteArray({1,2,3},0)

--swap A[0] and A[0], PermuteArray({1,2,3},1), swap back A[0] and A[0]

--swap A[0] and A[1], PermuteArray({2,1,3},1), swap back A[1] and A[0]

--swap A[0] and A[2], PermuteArray({3,2,1},1), swap back A[2] and A[0]

PermuteArray({1,2,3},1)

--swap A[1] and A[1], PermuteArray({1,2,3},2), swap back A[1] and A[1]

--swap A[1] and A[2], PermuteArray({1,3,2},1), swap back A[1] and A[2]

PermuteArray({1,2,3},2)

--swap A[2] and A[2], PermuteArray({1,2,3},3), swap back A[2] and A[2]

PermuteArray({1,2,3},2) print 1,2,3

Why using recursion?

- The subtask is simpler

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Fractals

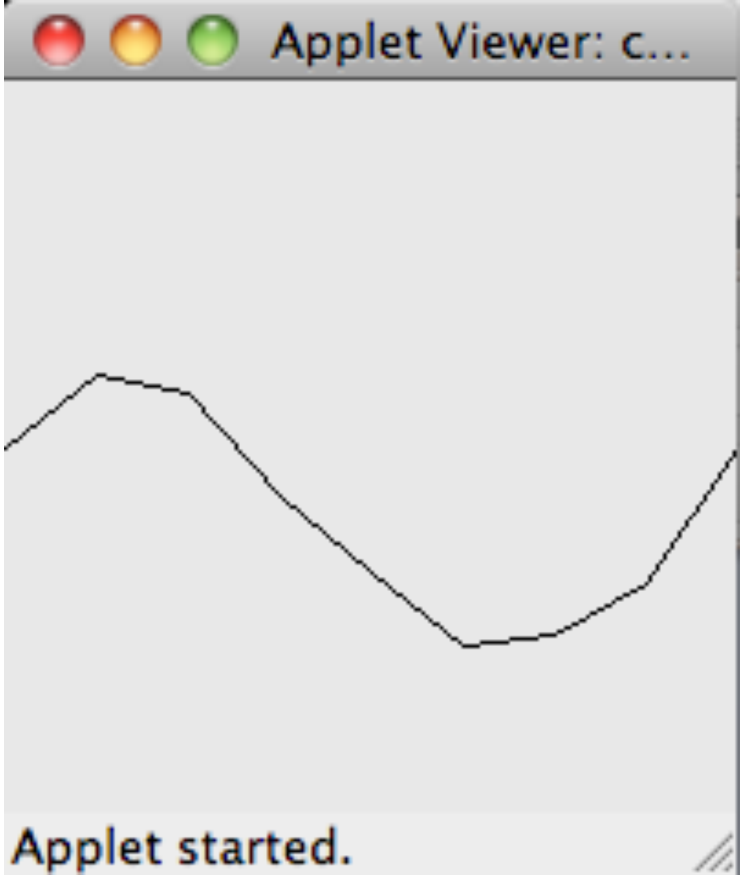
- Term coined by the mathematician Benoit Mandelbrot to describe objects
  - Exhibit some kind of similarity under magnification
- Random fractal

Java classes

- java.applet.Applet
  - A small program that is intended not be to run on its own, for being embedded in other applications (e.g., web page)
  - init(): called by the browser or applet viewer to inform this applet that it has been loaded into the system.
- java.awt.Graphics
  - Abstract base class for all graphics contexts
- java.awt.Image

```
public void Generate(int leftX,int leftY,int rightX, int rightY,int indentation,
Graphics drawingArea) {
        final int STOP = 25;

        if((rightX-leftX)<=STOP){
                drawingArea.drawLine(leftX, leftY, rightX, rightY);
                return;
        }

        //1. Calculate the middle point
        int midX = (leftX+rightX)/2;
        int midY = (leftY+rightY)/2;

        // 2. Calculate the shift on the Y coordinate
        //    Make sure that the shift is not too abrupt, limit it to be <= half of the x
span
        //    random --> [0,1), random-0.5 --> [-0.5,0.5)
        int shift = (int)((Math.random()-0.5) * (rightX - leftX));

        //3. Add the shift value to Y
        midY+=shift;

        // 4. Recursion
        Generate(leftX, leftY, midX, midY, indentation+1, drawingArea);
        Generate(midX, midY, rightX, rightY, indentation+1, drawingArea);
```

```
recursion[u]: (0,100) to (100,134)
  recursion[u]: (0,100) to (50,85)
    recursion[u]: (0,100) to (25,80)
      base case: draw line from (0,100) to (25,80)
    recursion[d]: (25,80) to (50,85)
      base case: draw line from (25,80) to (50,85)
  recursion[d]: (50,85) to (100,134)
    recursion[u]: (50,85) to (75,113)
      base case: draw line from (50,85) to (75,113)
    recursion[d]: (75,113) to (100,134)
      base case: draw line from (75,113) to (100,134)
recursion[d]: (100,134) to (200,100)
  recursion[u]: (100,134) to (150,151)
    recursion[u]: (100,134) to (125,154)
      base case: draw line from (100,134) to (125,154)
    recursion[d]: (125,154) to (150,151)
      base case: draw line from (125,154) to (150,151)
  recursion[d]: (150,151) to (200,100)
    recursion[u]: (150,151) to (175,137)
      base case: draw line from (150,151) to (175,137)
    recursion[d]: (175,137) to (200,100)
      base case: draw line from (175,137) to (200,100)
```



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Solve a problem recursively

- Trick: assume that you know how to solve the problem on a smaller input
- Solution: break a problem to smaller problems
  - Figure out what is/are the sub-problems that come up in solving your problem
  - Figure out how to compose the solution to your original problem from the solution to the sub-problems
  - Provide a base case
- Correctness (termination, correctness)
- Efficiency: time/space complexity

Other examples

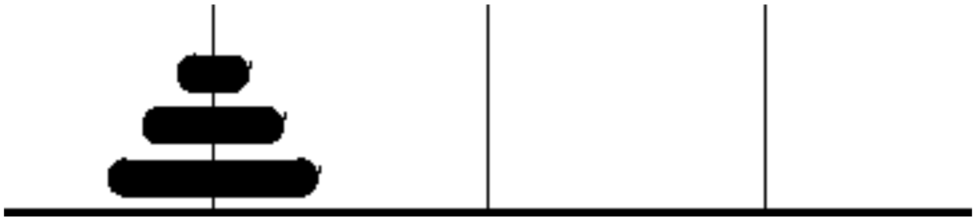
- Search a given value *e* from an array *A* whose values are sorted ascending
- Calculate the number of digits of a given number *x*
  - E.g., when x=10, the number of digits is 2
  - When x=134, the number of digits is 3
- Towers of Hanoi

Number of digits

```
int digits(int n)
{
    if(n<10)&&(n>-10) return 1;
    else return (1+digits(n/10));
}
```

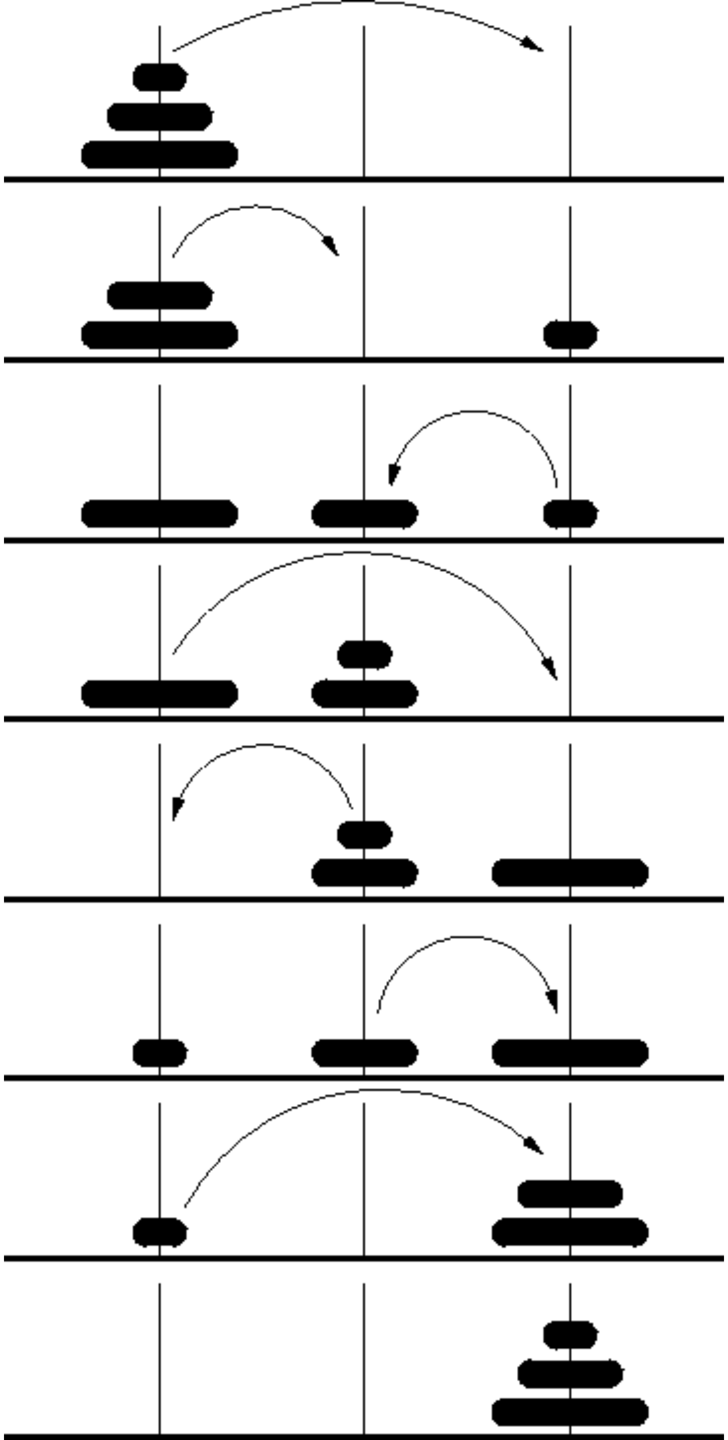
Towers of Hanoi

- The Towers of Hanoi puzzle was invented in the 1880s by Douard Lucas, a French mathematician.
- The puzzle consists of three pegs on which a set of disks, each with a different diameter, may be placed.
- Initially the disks are stacked on the leftmost peg, in order of size, with the largest disk on the bottom.

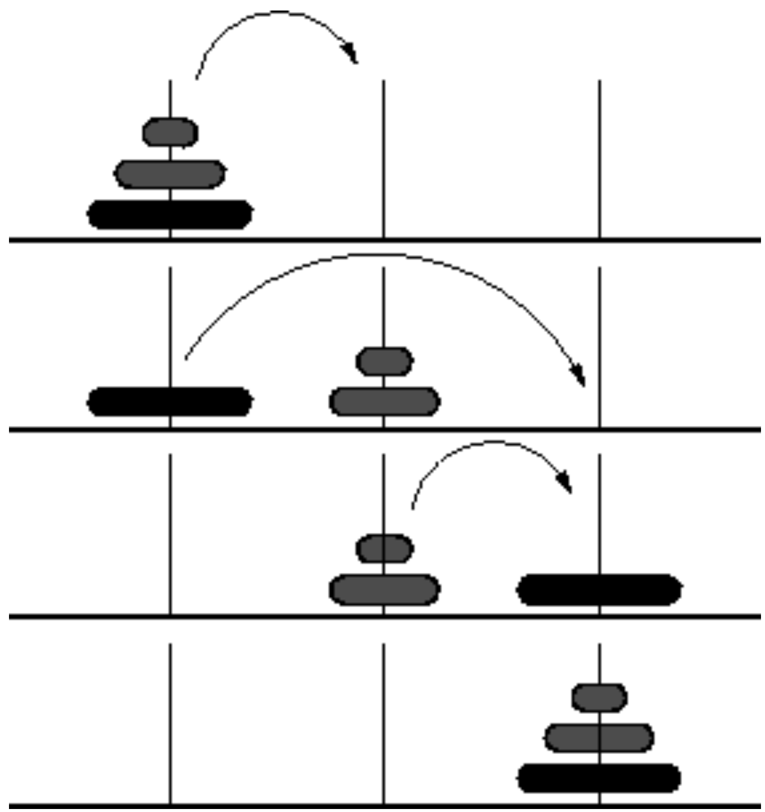


Goal

- The goal of the puzzle is to move all the disks from the leftmost peg to the rightmost peg, adhering to the following rules
  - Move only one disk at a time.
  - A larger disk may not be placed on top of a smaller disk.
  - All disks, except the one being moved, must be on a peg.



Strategy



Strategy

- The rules imply that smaller disks must be ``out of the way'' to move larger disks from one peg to another.
- General strategy for moving disks from the original peg to the destination peg:
  - Move the N-1 topmost disks from the original peg to the extra peg.
  - Move the largest disk from original peg to destination peg.
  - Move N-1 disks from the extra peg to the destination peg.

Recursive solution

- This strategy lends itself to a recursive solution.
- Step 1 and 3 are the same problems over and over again: move a stack of disks.
- The base case for this problem occurs when we want to move a ``stack'' that consists of only one disk. This step can be accomplished withouth recursion.

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Complexity

- Let the time required for moving n disks be T(n) .
- There are 2 recursive calls for n-1 disks and one constant time operation to move a disk from ‘from’ peg to ‘to’ peg . Let it be k1.Therefore,
  - T(n) = 2 T(n-1) + k1
- Analysis
  - T(1) = k1
  - T(2) = 2 k1 + k1
  - T(3) = 4 k1 + 2k1 + k1
  - T(4) = 8 k1 + 4k1 + 2k1 + k1
  - $T(n) = (2^{n-1}+2^{n-2}+\ldots+2^{1}+2^{0})k1=(2^{n}-1)\ k1$

Space complexity

- $T(n)=T(n-1)+k$
- $T(1) = k$
- $T(2) = 2k$
- $T(n) = nk$
- $O(n)$

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Reasoning about recursion

- Prove two facts
  - Termination
    - There is no infinite recursion
  - Correctness
    - Recursive method’s results are correct

Termination

- Find a variant expression and a threshold
  1. Between recursive calls, the value of the variant expression decrease at least some fixed amount
  2. If the value of the variant expression <= threshold, the method terminates (stopping condition)

Termination

- One-Level recursion
  - A stopping case
  - Or makes a recursive call that is a stopping case
- Multiple-level (deeper) recursion calls
  - Variant expression

Proof of correctness

- Recursive programming is related to mathematical induction
  - Induction: a technique of proving that some statement is true for *n*
- Proof of correctness is similar to Proof-by-induction
  - Proof by induction
    - Prove the statement is true for the base case (size 0, 1, or whatever)
    - Show that if the statement is assumed true for n, then it must be true for n+1
  - E.g., prove by induction $1 + 2 + 3 + \ldots + n = n(n + 1)/2$

Proof of Correctness

- Recursive proof is similar to induction. Verify that:
  - The base case is recognized and solved correctly
  - Each recursive case makes progress towards the base case
  - If all smaller problems are solved correctly, then the original problem is also solved correctly.

Correctness - details

- Induction, in each call
  - Meet precondition and postcondition
  - Correct logic
- Base step: Whenever the method makes no recursive calls → it meets its precondition/postcondition
- Induction step: Whenever the method is activated and all the recursive calls meet their precondition/postcondition → the original call also meets its precondition and postcondition

Example

- Design a recursive method to compute the number of digits in an integer *n*. And prove (1) it can terminate, (2) it is correct.

```
int digits(int n)
{
    if(n<10)&&(n>-10) return 1;
    else return (1+digits(n/10));
}
```

Termination Proof - # of digits

- Proof of termination
  - Let us first define a variant expression. In this problem, we define a variant expression to be “the number of digits in n”
  - Next we check how this variant expression changes. This variant expression decreases 1 in every recursive call.
  - The stopping case is the base case when the variant expression is one.

Proof of Correctness- # of digits

- Base case:
  - If n is in (-10, 10), the number of digits is 1, which is correct.
- Induction step:
  - Induction hypothesis: suppose that n/10 has the correct number of digits, which is x-1.
  - Now we show that we an calculate the number of digits of n correctly.
    - Since n/10 reduces the number of digits by 1, # of digits of n is # of digits of n/10 +1, which is the recursive code.

Towers of Hanoi

- Proof of Correctness
  - Base case: works correctly for moving one disk
  - Induction step:
    - Assume that it works correctly for moving (n-1) disks
    - It follows that it works correctly for moving n disks.

gcd

```
int gcd(int a, int b) {
    /*Pre: a>b i b≥0*/
    /* Post: gcd(a, b) = GCD(a, b) */
    if(b==0) return a;
    else return gcd(b, a%b);
}
```

Proof of correctness

- We prove its correctness by induction over N, the number of recursive calls.
- Base case: b=0, N=0, there for gcd(a,b)=gcd(a,0)=a. CORRECT.
- Inductive case:
  - Let a_n and b_n be the a and b in gcd(a,b) after N recursive calls where N>0.
  - Let q and r be the quotient and the remainder of dividing a_n into b_n, so that $a_n = qb_n + r$ and $0 <= r < b_n$ since $r = a_n \% b_n$.
    - $gcd(a_n, b_n)=gcd(b_n, a_n)=gcd(b_n, qb_n + r) = gcd(b_n, r)$
  - The next recursive call
    - $gcd(a_{n-1}, b_{n-1})= gcd(b_n, r)$

Example

- Power(base,exp) = $base^{exp}$
- Rules:
  - If exp>0, $base^{exp}$ = base* base * base * …
  - If exp=0, $base^{0}$=1
  - If exp<0, base!=0, $base^{exp}$ = 1/$base^{-exp}$
  - If exp<0, base=0, illegal

Summary

- Why do we need to have recursion?
- Recursive algorithm
  - Base cases
  - Recursive cases
- Recursion: linear, binary, multiple, tail recursive
- Algorithms
  - Factorial, randomFactal, Sum, Fibonacci, Tower of Hanoi, Permutation
  - Recursive algorithm design!!!!!
- Reasoning
  - No infinite recursion
  - Correctness