

Linear Algebra & Applications:-

Matrix representation

$$A : \text{IN} \times \text{IN} \rightarrow \text{IF}$$

$\text{IF} = \text{Field } (\mathbb{R} \text{ or } \mathbb{C})$

→ G. Strang, Linear
Algebra

→ Hoffman &
Kunze, A,
Linear Algebra

Elementary row operations

- 1) Multiplication of any row by non-zero scalar c .
- 2) (Addition &) Replacement of s^{th} row by $s^{\text{th}} + (c * \text{row } s)$ where c is a non-zero scalar.
- 3) Interchange of any two rows.

First definition:-

→ It says to every elementary row operation there corresponds an elementary row operation which is inverse of the given elementary row operation.

In other words, if e is given elementary row operation & another elementary row operation e_1 such that

$$e(e_1(A)) = A = e_1(e(A))$$

Proof:

Given $A \in \mathbb{R}^{m \times n}$ & $B \in \mathbb{R}^{n \times p}$

Let us assume a matrix,

is invertible

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

invertible matrix

Let e be elementary operation i.e. multiplication with a scalar $c (\neq 0)$

$$\therefore \text{so, } e(cA) = \begin{bmatrix} c \cdot a_{11} & c \cdot a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Elementary row operation

Now, let e_1 be a elementary row operation i.e. multiplication with a scalar $\frac{1}{c}$.

So, Now,

$$e_1(e(cA)) = e_1 \begin{bmatrix} c \cdot a_{11} & c \cdot a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$e_1(e(cA)) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A$$

Definition:

Let A and B be given $m \times n$ matrices.

We say A & B are row equivalent if

one can be obtained from another by

using finitely many elementary row operations.

$$\text{Ex: } A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & c \\ 1 & 0 \end{bmatrix}$$

Here, A and B are row equivalent.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row-Reduced Form:-

Let R be $m \times n$ matrix over the field, F(Rank).
we say R is in row-reduced form if

1) the first non-zero entry in each non-zero rows of R is equal to 1.

2) Each column of R which contains the leading non-zero entry of some row has all its other entries '0'.

$$\text{Ex: } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$I = (I_{ij}) \quad I_{ij} = \begin{cases} 0; i \neq j \\ 1; i = j \end{cases}$$

$$\begin{bmatrix} 0 & 0 & 1 & 6 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \end{bmatrix} \xrightarrow{\text{row operations}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{row operations}}$$

→ Every $m \times n$ matrix can be reduced to its row reduced form by using finitely many row operations.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix} = A \quad \text{Ex: } E(\text{Row})$$

for A to be max matrix over the row & one row A must be same elimination

Row Reduced Form:

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

for A to be max matrix over the row & one row A must be same elimination

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

for each column to get zero entries

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

so we can divide by 2 to get rid of 0's

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = I$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \left\{ \begin{array}{l} \text{if } I \\ (\text{if } I) = I \end{array} \right.$$

Row reduced echelon form:

Let R be given $m \times n$ matrix, we say R is in row-reduced echelon form, if it satisfies the following

- 1) R is row-reduced
- 2) every row of R which has all its entry '0' occurs below every row which has non zero entry.

3) If rows $1, 2, \dots, r$ are the non zero rows of R
 and if the leading non zero entry of row i
 occurs in the column k_i where $k_1 < k_2 < k_3 < \dots < k_r$

then, $k_1 < k_2 < k_3 < \dots < k_r$

ex:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark \quad \begin{bmatrix} 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 4 & 6 & 8 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{LH}$$

$$\begin{bmatrix} 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{LH}$$

$$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -4 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\xrightarrow{H-N} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 8 & 1 & 0 \end{bmatrix} \xrightarrow{2) \quad \times 2 \rightarrow \dots \rightarrow \text{ex}} \begin{bmatrix} 1 & 0 & 2 & 7 \\ 3 & -1 & 7 & 0 \\ 6 & 2 & 1 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

$$\checkmark \begin{bmatrix} 1 & 0 & 2 & 7 \\ 3 & -1 & 7 & 0 \\ 6 & 2 & 1 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{ex}} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & -4 & 0 & 1 \\ 0 & -10 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -5 & -11 & -13 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{\text{ex}} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -10 & -11 & 0 \\ 0 & -4 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -5 & -11 & -13 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{\text{ex}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 19 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 6 & 3 \end{bmatrix} \xrightarrow{\text{ex}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{\text{ex}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{ex}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 0 & 11 \end{bmatrix} \xrightarrow{\text{ex}} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

* Any given $m \times n$ matrix is row equivalent to its row-reduced echelon form.

→ give examples of matrix A and B such that AB is defined, but BA is not defined.

AB is defined, but BA is not defined.

give ex. of AB such that AB, BA both defined.

but $AB \neq BA$.

$$\Rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, AB$$

$$\Rightarrow A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 6 \\ 8 & 9 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -9 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -6 \\ 0 & -7 & -21 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 7 & 21 \end{bmatrix}$$

$$B = xA$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix} = A$$

$$B = xU$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -9 \\ 0 & 1 & 6 \\ 0 & 0 & -3 \end{bmatrix}$$

Perform row operations to get echelon form.
 Row 2 - 7R1 \rightarrow $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -17 \\ 5 & 8 & 9 \end{bmatrix}$
 Row 3 - 5R1 \rightarrow $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -17 \\ 0 & 8 & 9 \end{bmatrix}$
 Same operation
 perform of identity matrix. that will be
 the elementary matrix E.

$$\text{So, } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2 = R_2 - 7R_1 \begin{bmatrix} 0 & 1 & 0 \\ 0 & -8 & -17 \\ 0 & 8 & 9 \end{bmatrix} = A$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 7 & 6 & 4 \\ 5 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -17 \\ 0 & 8 & 9 \end{bmatrix}$$

Rank:
 The no. of non zero rows in a row
 reduced echelon form.

Gauss elimination method to solve L.S.Eqn's

$$Ax = B$$



$$Ux = B'$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array} \right]$$

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array} \right]$$

Augmented Matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array} \right]$$

Now, we have to reduce to form,

$$\left[\begin{array}{ccc|c} u_{11} & u_{12} & u_{13} & b'_1 \\ 0 & u_{22} & u_{23} & b'_2 \\ 0 & 0 & u_{33} & b'_3 \end{array} \right]$$

$$u_{33} \cdot x_3 = b'_3$$

$$x_3 = \frac{b'_3}{u_{33}}$$

Now,

$$u_{22} x_2 + u_{23} x_3 = b'_2$$

$$\Rightarrow x_2 = \frac{b'_2 - u_{23} x_3}{u_{22}}$$

$$\text{Hence, } x_1 = \frac{b'_1 - u_{12} x_2 - u_{13} x_3}{u_{11}}$$

Ex

$$x + 2y - 4z = -4$$

$$5x + 11y - 21z = -22$$

$$3x - 2y + 3z = 11$$

Solve using Gauss elimination method.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 5 & 11 & -21 & -22 \\ 3 & -2 & 3 & 11 \end{array} \right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 0 & 1 & -1 & -2 \\ 0 & -8 & 15 & 23 \end{array} \right] \xrightarrow{\begin{matrix} T_A \\ T_B \\ T_B \end{matrix}} \left[\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 7 & 7 \end{array} \right]$$

$$x_3 = 1$$

$$\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right]$$

$$x_2 - x_3 = -2$$

$$\Rightarrow x_2 = -1$$

$$\Rightarrow x_1 = -4 - 2x_2 + 4x_3$$

$$x_1 = -4 + 2 + 4$$

$$x_1 = 2$$

$$\therefore x = 2$$

$$y = -1$$

$$z = 1$$

Trace of matrix = sum of principle diagonal elements

for a square matrix.

$$\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$$

$$\text{tr}(cA) = c\text{tr}(A)$$

$$(A+B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T$$

$$\left[\begin{array}{c|ccc} 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 \end{array} \right]$$

$AB \neq BA$ in general

$$\left[\begin{array}{c|cc} 1 & \text{when } AB = BA ? \\ 2 & \end{array} \right] \quad \left[\begin{array}{c|cc} 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 \end{array} \right]$$

① $AB = (AB)^T = B^T A^T = B A I$

② $(AB)^T = (BA)^T = A^T B^T$

If, $AB = BA$ then

$$A, B \text{ are symmetric. } \sum_{i=1}^n |A_i| = |A| = |A^T|$$

Determinants

* Any square matrix A can be expressed as a sum of a symmetric & skew symmetric matrix $|A^T A| = |A|$ ①

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = |A_3| \quad ②$$
$$= B + C \quad |B| + |A| \neq |B + A| \quad ③$$

$$B^T = B, C^T = -C \quad \text{Note: } A + A^T \rightarrow \text{symmetric} \quad ④$$
$$A - A^T \rightarrow \text{skew symmetric} \quad ⑤$$

Minor and cofactor of an element a_{ij} :

Suppose,

$$A = (a_{ij})_{n \times n} \quad |A| = |A_1| \quad ⑥$$

Minor of a_{ij} can be obtained by eliminating i^{th} row and j^{th} column from $A = (a_{ij})$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$\text{cofactor of } A_{ij} = (-1)^{i+j} M_{ij}$$

$M = \text{set of all square matrices.}$

Now, $f: M \rightarrow \mathbb{R}$ $= \begin{bmatrix} P+1 \\ S \\ E \end{bmatrix} = A$

$$A \in M \quad \begin{bmatrix} P+1 \\ S \\ E \end{bmatrix} = A$$

$E = \text{minor}$

$$f(A) = |A| = \sum_{i=1}^n (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^n (-1)^{i+j} a_{ij} M_{ij}$$

Determinant

Properties:-

$$\textcircled{1} \quad |A| = |A^T|$$

$$\textcircled{2} \quad |cA| = c^n |A| \quad \frac{1}{2} + (TA + A) \frac{1}{2} = A$$

$$\textcircled{3} \quad |A+B| \neq |A| + |B|$$

$$\textcircled{4} \quad |AB| = |A||B|$$

$\textcircled{5}$ Interchanging row will give same value of determinant except the sign.

$$j^{\text{th}} (-1)^j |A|$$

$\textcircled{6}$ if $|A| = 0$, then $|A| = 0$ not vice-versa.

Rank of a matrix :-

→ Rank (r) of a matrix A is a smallest positive integer r such that the matrix A has a submatrix of $r \times r$, whose determinant is non-zero.

→ Suppose $A_{m \times n}$, then $\text{rank } A \leq \min(m, n)$

→ $A = 0 \Leftrightarrow \text{rank } A = 0$

ex:

$$A = \begin{bmatrix} 1 & 4 & 9 \\ 3 & 6 & 2 \\ 4 & 1 & 3 \end{bmatrix} = 1(16 - 4(-12) + 9(-21)) = 16 - 4 - 189 = -177 \neq 0$$

$M_3 A$

rank = 3

② find the value of μ for which rank of matrix A is 3.

$$A = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & -11 & -6 & 1 \end{bmatrix} \sim \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & -11 & -6 & 1 \end{bmatrix} = A$$

$$\mu(\mu - 6) + 1(11) \neq 0 \quad \Rightarrow \mu(\mu - 6) = 1(0) \neq 0 \quad \Rightarrow \mu \neq 0$$

$$\Rightarrow \mu^2 - 6\mu + 11 \neq 0$$

$$\Rightarrow \mu(\mu^2 - 6\mu + 11) = 0$$

$$\mu[\mu(\mu - 6) + 11] = 0$$

$$\Rightarrow \mu[\mu(\mu - 6) + 11] + 6 = 0 \quad \text{(A) is } A$$

$$\Rightarrow \mu^3 - 6\mu^2 + 11\mu + 6 = 0 \quad |A|$$

$$\text{So, } \mu = 1 \text{ or } 2 \text{ or } 3 \quad r(A) = 3$$

otherwise, $r(A) = 4$.

Inverse:- (i) If $|A| = 0 \Rightarrow$ singular
 $|A| \neq 0 \Rightarrow$ non-singular

$$AB = BA = I$$

$$\text{then, } A^{-1} = B \quad (\text{or}) \quad B^{-1} = A$$

We get inverse for a non-singular matrix.

Adjoint of a matrix:

$\text{adj}(A) =$ transpose of the matrix whose entities are cofactors of each element.

Wichtige Formeln für Matrizenrechnung mit 3x3-Matrizen

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A^T$$

$$\text{adj}(A) = \frac{1}{\det(A)} \begin{bmatrix} a_{22} a_{33} - a_{23} a_{32} & a_{21} a_{33} - a_{23} a_{31} & a_{21} a_{32} - a_{22} a_{31} \\ a_{32} a_{33} - a_{33} a_{32} & a_{11} a_{33} - a_{13} a_{31} & a_{11} a_{32} - a_{12} a_{31} \\ a_{12} a_{33} - a_{13} a_{32} & a_{11} a_{23} - a_{13} a_{21} & a_{11} a_{22} - a_{12} a_{21} \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{adj}(A)}{\det(A)} = \frac{1}{2+4+(-4)} [1 + (2-4)] = \frac{1}{8} [1] = \frac{1}{8}$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \quad \det(A) = 1(1+3) + 1(5) + 1(1) = 10$$

$$\begin{aligned} \det(A) &= 1(1+3) + 1(5) + 1(1) \\ &= 4 + 5 + 1 \\ &= 10 \end{aligned}$$

$$\text{adj}(A) = \begin{bmatrix} 1 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

System mit 3 Gleichungen und 3 Unbekannten

$$A^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} / 10$$

Solution of system of $n \times n$ linear equations :-

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{array} \right.$$

(+) nibba
use of same place of each row
for solve (+) of each row

$$AX = b \quad \text{Eq. 1}$$

$$\text{if } |A| \neq 0 \quad \text{Eq. 2}$$

$$\Rightarrow A^{-1}AX = A^{-1}b \quad \text{Eq. 3}$$

$$\Rightarrow IX = A^{-1}b \quad \text{Eq. 4}$$

$$\Rightarrow X = A^{-1}b. \quad \text{Eq. 5}$$

Cramer's rule:

$$x_i = \frac{|A_i|}{|A|} \quad \text{Eq. 6}$$

$A_i = i^{\text{th}}$ column replaced by b .
when, $|A| = 0$, and $|A_i| \neq 0$ (no solution).

$|A| \neq 0$ (unique solution).

$|A| = 0$ and all $|A_i| = 0$ (infinitely many sol.).

consistent \rightarrow If a system has soln.

inconsistent \rightarrow system doesn't have soln.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_3$$

$$\begin{pmatrix} v \\ w \end{pmatrix} = v /$$

$$\begin{pmatrix} w \\ v+w \end{pmatrix} = w+v.$$

Vector spaces: - / Linear spaces

V = set of elements usually called vectors

F = Field

vector space has to satisfy some properties
adding (+) satisfies

① for any $v_1, v_2 \in V$, $v_1 + v_2 \in V$

② $v_1 + v_2 = v_2 + v_1 \quad \forall v_1, v_2 \in V$

③ $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$ ④ $0 \in V$ such that
 $v + 0 = v \quad \forall v \in V$

⑤ for any,

$v \in V \exists (-v)$ s.t. $v + (-v) = 0$

scalar multiplication (\cdot)

① for any scalar $c \in F$, $v \in V$, $c \cdot v \in V$

② $c(v_1 + v_2) = cv_1 + cv_2 \quad \forall v_1, v_2 \in V$

③ $(c_1 + c_2)v = c_1v + c_2v \quad \forall c_1, c_2 \in F$ and $v \in V$.

④ $\exists 1 \in F$ such that $1 \cdot v = v \quad \forall v \in V$.

⑤ $(c_1 c_2)v = c_1(c_2 v) \quad \forall c_1, c_2 \in F, v \in V$.

ex:

i) $V = \mathbb{R}^n = F$ and $m \times n$ matrix $\in F$

Then $V(\mathbb{R})$ is a vector space.

2) $V = \mathbb{R}^2$

$F = \mathbb{R}$

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$cv = \begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix}$$

$$v + w = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$$

Here it is not a vector space, because we don't have $-v$ as $v \in \mathbb{R}^2$, so it is (not) a vector space.

$$3) V = \mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R} \right\} \quad R = ?$$

$$\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right) + p \left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right) = r$$

$$4) V = \mathbb{R}^{m \times n} = \left\{ A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} : a_{ij} \in \mathbb{R} \right\}$$

This forms a vector space

$$5) V = \{0\} \quad 0 + 0 = 0 ; 0 \cdot 0 = 0$$

This is also vector space

$$6) V = \text{set of all polynomials of degree } < n$$

$$F = \mathbb{R} \quad p, q \in V$$

$$p(x) = p_0 + p_1 x + p_2 x^2 + \dots + p_n x^n$$

$$q(x) = q_0 + q_1 x + q_2 x^2 + \dots + q_n x^n$$

$$\textcircled{1} \quad (p+q)(x) = p(x) + q(x)$$

$$\textcircled{2} \quad c(p(x)) = c \cdot p(x)$$

This is a vector space.

If V = set of polynomial of degree n .

then not a vector space.

because, if $p(x) = p_0 + p_1 x + p_2 x^2 + \dots + p_n x^n$

then $(p+q)(x) \notin V$.

$$V + (V + V) = (V + V) + V \in$$

So, not a vector space.

$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_2 = x_1 + 1 \right\}$

$$F = \mathbb{R}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\left\{ \begin{array}{l} x+y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \\ x \in V, y \in V \end{array} \right\} = A$$

Here, $x_2 + y_2 = x_1 + y_1 + 2$

$$\text{so, } (x_2 + y_2) \neq (x_1 + y_1) + 1$$

so, not a vector space.

8) $V = \left\{ x : Ax = b \right\}$

$$x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

not a vector space

$A(x+y) \neq b$; if $Ax = 0$, then a vector space.

Subspace:

Suppose V is a vector space over the field \mathbb{R} .

Let $W \subset V$, we call W is a vector subspace of V if W is a vector space with respect to the same operation in V .

We need to check the distributive properties,

$$\Rightarrow v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$$

$$\Rightarrow c(v_1 + v_2) = cv_1 + cv_2.$$

$$\Rightarrow v_1 + v_2 \in V$$

- ① $v_1 + v_2 \in W$ & $v_1, v_2 \in W$
- ② $0 \in W$
- ③ $c \in F$ & $v \in W$, $c \in F$ $\Rightarrow cv \in W$

more simplified,

$$cv + w \in W \quad \text{or not.} \quad \Rightarrow cv + w \in W$$

Theorem:

A ~~different~~ non empty subset W of a vector space V is a subspace of V iff for any two vectors $v, w \in W$, $\exists c \in F$

$$cv + w \in W, c \in F$$

Ex: $V = \mathbb{R}^n = \{x = (x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}\}$

$$V = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_1 = 0 \right\} \quad \text{and } \subseteq \mathbb{R}^{n-1}$$

vector space / linear space:

Theorem: Let v_1 and v_2 are two subspaces of a vector space V . $v_1 \cap v_2$ is also a subspace.

Proof: $v_1, v_2 \in V, v_1 \cap v_2 = W$

$c_1v_1 + c_2v_2 \in W$

\Rightarrow 2 to prove

$W = V_1 \cup V_2$ may not be holding the above theorem.

$$W = V_1 \cup V_2 = \left\{ \begin{pmatrix} n \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

To W $\forall v_1 \in V_1 : v_1 = \begin{pmatrix} n \\ 0 \\ 0 \\ 0 \end{pmatrix}, n \in \mathbb{R}$ is a multiple of A .

For $v_2 \in V_2 : v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$V_2 = \left\{ \begin{pmatrix} 0 \\ 1 \\ y \\ 0 \end{pmatrix} : y \in \mathbb{R} \right\}$$

$$\text{Now, } V_1 + V_2 = \{n+0y : n, y \in \mathbb{R}\} \neq W.$$

∴ union of two subspaces is not a subspace.

Linear combination of vectors:

Suppose, $v_1, v_2, v_3, \dots, v_n$ are a set of vectors, then the linear combination of these would be

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

where,

$$c_i \in F = \mathbb{R}$$

Spanning Set:- Let V be a vector space and $S \subset V$. Then set S is called spanning set of V if S spans V if every element of V can be expressed as a linear combination of elements of S .

Ex: Let V be the vector space of 2×2 matrices.

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So, S spans V .

Ex: Let V be the vector space of all polynomial of degree ≤ 3 .

$$S = \{ t^3, t^2 + t, t^3 + t + 1 \}$$

Check whether S spans V or not.

Sol:-

$$c_1 t^3 + c_2 (t^2 + t) + c_3 (t^3 + t + 1) \quad \text{①}$$

$$= P(t) = P_0 + P_1 t + P_2 t^2 + P_3 t^3 \quad \text{②}$$

Here, it can't be expressed as linear comb.

so, S doesn't spans V . degree 0

Linearly dependent set of vectors :

Suppose $S = \{v_1, v_2, v_3, \dots, v_n\}$ be the set of vectors from a vector space V . We call the set S is linearly dependent if there exists scalars c_1, c_2, \dots, c_n (not all zero) such that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n = 0$$

(At least there should be one non zero c_i).

if all c_i 's are zero, "Linearly independent".

$$\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} = 2$$

$$v_i = \frac{c_1}{c_i} v_1 + \frac{c_2}{c_i} v_2 - \frac{c_3}{c_i} v_3 - \dots - \frac{c_n}{c_i} v_n = \begin{pmatrix} d & 0 \\ b & 0 \end{pmatrix}$$

$$\dots - \left(\frac{c_{i-1}}{c_i} v_{i-1} - \frac{c_{i+1}}{c_i} v_{i+1} \right) \dots$$

No 2 zero vectors reduce & left

If $\{v_1, v_2\}$ given,

if, $v_2 = c_1 v_1$, then linearly dependent

Properties :-

- ① Any set which contains linearly dependent set is linearly dependent.
- ② Any subset of a linearly independent set is linearly independent.
- ③ Any set containing zero vector is always linearly dependent.

Ex:-

Let $V = \mathbb{R}^2$

$v_1 = (1, 1)$ $v_2 = (-3, 2)$ $v_3 = (2, 4)$

$c_1 v_1 + c_2 v_2 + c_3 v_3$

Dependent

(1) If $c_1 = 0$ then $c_2 v_2 + c_3 v_3 = 0$

$$c_1 - 3c_2 + 2c_3 = 0$$

$$c_1 + 2c_2 + 4c_3 = 0$$

$$\underline{5c_2 + 2c_3 = 0}$$

$$\text{solution: } c_2 = -\frac{2}{5}c_3, \quad c_3 \neq 0$$

so, c_1 can be arbitrary.

②

$$V = \mathbb{R}^3$$

$$v_1 = (1, -1, 0)$$

$$v_2 = (0, 1, 1)$$

$$v_3 = (0, 0, 1)$$

$$c_1 - c_2 + 0 = 0$$

$$c_2 - c_3 \neq 0$$

$$c_3 = 0$$

$$\Rightarrow c_1 = c_2 = c_3 = 0$$

"Independent."

$$c_1 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$$

$$-c_1 + c_2 = 0$$

$$-c_2 + c_3 = 0$$

zero vector $\Rightarrow 0$

zero scalar $\Rightarrow 0$

Theorem:-

Any set containing zero vectors is linearly dependent set.

Pf

Let,

$v_1, v_2, v_3, \dots, v_n, 0$ are the set of vectors

from a vector space V over \mathbb{R}

$$cv_1 + cv_2 + \dots + cv_n + e\theta = 0 + 0 + 0 + \dots + 0 = 0,$$

$e \neq 0$

So, there exists at least one non zero scalar, $e \neq 0$. So, the set containing zero vector is linearly dependent.

Theorem:-

A collection of vectors which contains a collection of linearly dependent vectors is linearly dependent i.e any super set of a linearly dependent set of vectors is linearly dependent.

Proof:-

Let $v_1, v_2, \dots, v_{p-1}, v_p, v_{p+1}, v_{p+2}, \dots, v_n$ are set of vectors from a vector space $V(\mathbb{R})$ in which $\{v_1, v_2, \dots, v_p\}$ is linearly dependent.

$$c_1v_1 + c_2v_2 + \dots + c_pv_p = 0$$

$$c_1v_1 + c_2v_2 + \dots + c_pv_p + 0v_{p+1} + 0v_{p+2} + \dots + 0v_n = 0$$

So, the set containing $v_1, v_2, v_3, \dots, v_p, v_{p+1}, \dots, v_n$ is linearly dependent.

\therefore at least one $c_i \neq 0$ v_i .

Theorem: Any ~~set~~ subset of linearly independent set is linearly independent.

Proof:-

Same as previous theorem

$$AX = 0 \text{ or } AR = b.$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

$$v_1 = (a_{11}, a_{12}, \dots, a_{1n})$$

$$v_2 = (a_{21}, a_{22}, \dots, a_{2n})$$

$$\vdots$$

$$v_m = (a_{m1}, a_{m2}, \dots, a_{mn})$$

Here $v_i \in \mathbb{R}^m$

$$\text{rank } \leq \min\{\text{min}\}$$

If $\text{rank}(A) = m$, then
the set is linearly independent.

If $\text{rank}(A) < m$,
then linearly dependent.

Ex:

$$A = \begin{bmatrix} 2 & 4 & 8 & 12 & 8 \\ 1 & 2 & 4 & 6 & 4 \\ 1 & 2 & 2 & 2 & 2 \\ -1 & 0 & 2 & 4 & 2 \end{bmatrix}_{4 \times 5}$$

$$m=4 \\ n=5$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 & 4 \\ 1 & 2 & 4 & 6 & 4 \\ 1 & 2 & 2 & 2 & 2 \\ -1 & 0 & 2 & 4 & 2 \end{bmatrix}$$

$$r \leq 4$$

$$r = \min(4, 5)$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 2 \\ -1 & 0 & 2 & 4 & 2 \end{bmatrix}$$

Transformed into row echelon form

Linear algebra is linear algebra of linear algebra

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 & 4 \\ -1 & 0 & 2 & 4 & 2 \\ 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 & 4 \\ 0 & 2 & 6 & 10 & 6 \\ 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of matrix is 3

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 & 4 \\ 0 & 1 & 3 & 5 & 3 \\ 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 & 4 \\ 0 & 1 & 3 & 5 & 3 \\ 0 & 0 & -2 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 & 4 \\ 0 & 1 & 3 & 5 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 & -4 & -2 \\ 0 & 1 & 3 & 5 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\delta(A) = 3 \cdot \langle (m=4) \rangle$$

So, A is linearly Dependent.

Replacement theorem:-

If the set $\{v_1, v_2, \dots, v_n\}$ to spans

Problem :-

Let $v_1, v_2, v_3, \dots, v_n$ are subsets of a vector space V .

$$W = v_1 + v_2 + \dots + v_n = \left\{ v : v = v_1 + v_2 + v_3 + \dots + v_n, \text{ where } v_i \in V \right\}$$

where, W is a vector subspace of V .

$$V = \{1, 2, 3, 4\}$$

$$V_1 = \{1, 2\}$$

$$V_2 = \{3, 4\}$$

$$V_1 + V_2 = \{1+3, 1+4, 2+3, 2+4\}$$

Basis and dimensions of vector space:-

Suppose V is a vector space over \mathbb{R} . A basis for V is a linearly independent set of vectors in V which spans the space V .

Suppose B is basis then

- ① B is linearly independent
- ② B spans V .

→ Dimension is the total number of vectors in a basis.

→ Basis is not unique for a vector space.

Ex:

$$V = \mathbb{R}^3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$B = \{(1, -1, 0), (0, 1, -1), (0, 0, 1)\}$$

Ex. Find the dimension and of the vector space of \mathbb{R}^4 spanned by the set

$$\{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,1,1)\}.$$

Hence, find its basis.

Sol:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$c_1(1,0,0,0) + c_2(0,1,0,0) + c_3(1,2,0,1) + c_4(0,0,1,1) = 0$$

$$\Rightarrow (c_1+c_3, c_2+2c_3, c_4, c_3+c_4) = (0, 0, 0, 0)$$

$$\Rightarrow c_4 = 0$$

$$c_3+c_4=0$$

$$\Rightarrow c_3 = 0$$

$$\Rightarrow c_2 = c_1 = 0$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

So, linearly independent.

$$\text{Dimension} = 4, \{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,1,1)\}$$

Replacement theorem :- If the set $\{v_1, v_2, \dots, v_n\}$ be a basis of a vector space V over R . Suppose, $v \neq 0 \in V$ such that v can be expressed as a linear combination $\{av_1, \dots, av_n, w\} = V$

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n \text{ where } c_j \neq 0 \text{ for some } j.$$

Then if v_j is replaced by v then

$\{v_1, v_2, v_3, \dots, v_{j-1}, v, v_{j+1}, \dots, v_n\}$ is also a basis.

Theorem :- Suppose a basis of a vector space V contains n number of elements and if another linearly independent set contains m elements then $m \leq n$.

Proof:

$A = \{v_1, v_2, \dots, v_n\}$ is a basis of a vector space V .

$B = \{w_1, w_2, \dots, w_m\}$ is a linearly independent set. Here, $m \leq n$

$$w_1 = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

claim $c_i \neq 0$ for atleast one i .

Assume, $c_1 \neq 0$

[at ... in $\{v_1, v_2, \dots, v_n\}$ to get w_1 -> linearly independent transformation
 $v = \{v_1, v_2, \dots, v_n\}$ so w_1, w_2, \dots, w_m is linearly independent in V
so by replacement theorem, w_1, w_2, \dots, w_m is basis of V .

$$V = \{w_1, w_2, \dots, w_m\}$$

Now, tip shows $w_1 + \dots + c_m w_m = v$.

$$w_2 = c_1 w_1 + c_2 w_2 + c_3 w_3 + \dots + c_m w_m$$

say, $c_2 \neq 0$

By replacement theorem,

$$w_3 = c_1 w_1 + c_2 w_2 + c_3 w_3 + \dots + c_m w_m.$$

After m steps, $\{w_1, w_2, w_3, \dots, w_m, v_{m+1},$

$v_{m+2}, \dots, v_n\}$

Here, we can conclude that $m \leq n$.

Extension theorem :- A linearly independent set of vectors can be extended to a basis

In the last theorem if $m = n$, then it will be basis or else we have to extend w .

Theorem :-

Any two bases of a vector space have same number of elements.

Proof:

$$A = \{v_1, v_2, \dots, v_n\}$$

$$B = \{w_1, w_2, \dots, w_m\}$$

are two bases.

Here, A is linearly independent and B is a basis. $n \leq m$. $(A) \cap (B) = \emptyset$ so $A \cup B$ is linearly independent and A is a basis, $m \leq n$.

$$\begin{aligned} & n \leq m \text{ and } m \leq n \\ \Rightarrow & n = m. \end{aligned}$$

$$A \leftarrow M : T$$

$$A \text{ assert} = (A)T$$

Theorem:- If $\{v_1, v_2, \dots, v_n\}$ be a basis of a vector space V then any vector V can be expressed uniquely as a linear combination of v_1, v_2, \dots, v_n .

$$\text{If } v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

$$v = d_1 v_1 + d_2 v_2 + \dots + d_n v_n$$

$$0 = (c_1 - d_1)v_1 + (c_2 - d_2)v_2 + \dots + (c_n - d_n)v_n$$

$$\Rightarrow c_i = d_i \forall i$$

Linear Transformation:-

$$\begin{aligned} T: V &\rightarrow W \\ \text{any two } v_1, v_2 \in V & \quad \left| \begin{array}{l} T(c_1 v_1 + c_2 v_2) \\ = c_1 T(v_1) + c_2 T(v_2). \end{array} \right. \\ T(v_1 + v_2) &= T(v_1) + T(v_2) \end{aligned}$$

$T(\alpha v) = \alpha T(v)$, for any scalar α and $v \in V$.

$$\text{Trace}(A+B) = \text{Tr}(A) + \text{Tr}(B)$$

$$\text{tr}(CA) = C \cdot \text{tr}(A)$$

$M_{n \times n}$

$$T: M \rightarrow \mathbb{R}$$

$$T(A) = \text{trace } A.$$

Sum and direct sum of vector spaces:

Let W_1, W_2, \dots, W_n are subspaces of a vector space V . Then

$W = W_1 + W_2 + W_3 + \dots + W_n$ is a vector subspace of V .

Proof: let $u, v \in W$, $c_1 u + c_2 v \in W$

$$cu + cv \in W$$

$$u = w_1 + w_2 + \dots + w_n \quad w_i \in W$$

$$v = w'_1 + w'_2 + \dots + w'_n \quad w'_i \in W$$

$$c(w_1 + w_2 + \dots + w_n) + (w'_1 + w'_2 + \dots + w'_n)$$

$$= (cw_1 + w'_1) + (cw_2 + w'_2) + \dots + (cw_n + w'_n)$$

$$\text{Here, } cw_1 + w'_1 \in W_1$$

$$cw_2 + w'_2 \in W_2$$

$\therefore W$ is a subspace of V .

V is a vector space $\dim(V)=n$

W is a vector subspace $\dim(W)=m$

So, always $\dim(W) \leq \dim(V)$.

$$W = W_1 + W_2$$

$W_1 \cap W_2 \rightarrow$ vector space

$$W_1 \cap W_2 \subset W_1 + W_2$$

Theorem:

If W_1 and W_2 are finite dimensional vector subspaces of a vector space V ,

Then $W_1 + W_2$ is finite dimensional and

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Proof: $W_1 \cap W_2 \subset W_1$ | $\dim(W_1 + W_2) \leq \dim(W_1)$
 $W_1 \cap W_2 \subset W_2$ | $\dim(W_1 + W_2) \leq \dim(W_2)$

Since, $W_1 \cap W_2$ is finite dimensional, it has a finite basis.

$$\text{Say, } \beta_1 = \{v_1, v_2, \dots, v_k\}$$

β_1 is a part of $\{v_1, v_2, v_3, \dots, v_k, u_1, u_2, u_3, \dots, u_n\}$

of some basis of W_1 . $\dim(W_1) = k+n$

By, β_2 will be a part of $\{v_1, v_2, \dots, v_k, w_1, w_2, \dots, w_m\}$ of some basis of W_2 .

$$\dim(W_2) = k+m$$

$\dim(W)$ units \Rightarrow no. of vectors in W is n
 & $\dim(V)$ units \Rightarrow no. of vectors in V is m
 $\dim(W_1 + W_2) = k+n+m$
 (V) units $\geq (W)$ units \Rightarrow $n \leq m$
 $\therefore W_1 + W_2 = \{v_1, v_2, v_3, \dots, v_k, u_1, u_2, \dots, u_n, w_1, w_2, \dots, w_m\}$
 $\dim(W_1 + W_2) = k+n+m$

Now,

$$\dim(W_1) = k+n \quad SW + W \supset SW \cup W$$

$$\dim(W_2) = k+m$$

$$\dim(W_1 \cap W_2) = k$$

Linearly independent vectors in $W_1 \cap W_2$

Linearly independent vectors in $W_1 \cap W_2$

$$\therefore \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

$$(m+n) = m+k+n+k+m-k = (m+n+k)$$

$$\begin{aligned} m &= k+n+m \\ (m) &= \dim((W_1 \cup W_2) \text{ units}) \\ (m) &= \dim(W_1 + W_2). \end{aligned}$$

$$\therefore \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

above we have to prove that $(W_1 + W_2)$ is a basis.

$$\{v_1, v_2, \dots, v_k, u_1, u_2, \dots, u_n, w_1, w_2, \dots, w_m\}$$

Span $(W_1 + W_2)$.

$$\sum_{i=1}^k x_i v_i + \sum_{j=1}^n y_j u_j + \sum_{l=1}^m z_l w_l = 0$$

Let it to be L.I., then $x_i, y_j, z_l \neq 0$. v_i, u_j, w_l

$$m+n+k = (W_1 + W_2) \text{ units}$$

$$-\sum z_k w_k = \sum x_i v_i + \sum y_j u_j \in W_1$$

vector space \$W_1\$

$$\in W_2 = \{0\}$$

$$\sum z_k w_k = \sum c_i v_i \Rightarrow z_k = 0 \forall k, w$$

vector space \$W_2\$

$$\Rightarrow \theta = \sum x_i v_i + \sum y_j u_j \in W_1$$

$$\Rightarrow x_i = y_j = 0 \quad \forall i, j$$

vector space \$W_1 \cap W_2 = \{0\}\$

Here,

$$\{v_1, v_2, \dots, v_k, u_1, u_2, \dots, u_l, w_1, w_2, \dots, w_m\} =$$

spans \$W_1 + W_2\$ and is linearly independent.

So, basis

$$\dim(W_1 + W_2) = k + n + m$$

$$\dim(W_1) = k + n$$

$$\dim(W_2) = k + m$$

$$\dim(W_1 \cap W_2) = k$$

$$\Rightarrow \dim(W_1 + W_2) = (k+n)+(m+k)-(k)$$

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

direct sum:

$$W = W_1 \oplus W_2 ; \quad W_1 + W_2 = \{\theta\}$$

notation of direct sum

$$w_1, w_2, w_3, \dots, w_k$$

$$W = W_1 \oplus W_2 \oplus \dots \oplus W_k, \quad w_j \cap (W_1 + W_2 + \dots + W_{j-1}) = \{\theta\}$$

for, \$2 \leq j \leq k\$

$$\rightarrow w_1 + w_2 + \dots + w_k = \theta \Rightarrow w_i = 0 \quad \forall i.$$

$\text{defn: } M_{n \times n} = \text{set of all } n \times n \text{ matrices}$

$W_1 = \text{set of all symmetric matrix}$

$W_2 = \text{set of skew symmetric matrix}$

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$
$$= B + C$$

$\Rightarrow B \in W_1$

$C \in W_2$

$$V = W_1 + W_2$$

$$\text{Here, } W_1 \cap W_2 = \{0\}$$

so, the above example is of direct sum of sub

②

$M_{n \times n} = \text{set of all } n \times n \text{ matrix}$

$(W_1 = \text{set of all upper triangular matrix})$

$W_2 = \text{set of all lower triangular matrix}$

$$V = W_1 + W_2$$

$$\text{But, } W_1 \cap W_2 = ?$$

$$\{0\} = (\text{set of all upper triangular matrix}) \cap (\text{set of all lower triangular matrix})$$

$$\{0\} = \{0\}$$

$$V = W_1 + W_2 \Leftrightarrow 0 = \text{sum of elements of } W_1 + \text{sum of elements of } W_2$$

Linear transformation:-

$T(v) = 0$ trivial linear transformation

$T(v) = v$ non trivial. $v \neq 0$ $\Rightarrow T(v) = v$, word

ex:- i) Let A be a fixed $m \times n$ matrix, Define

$$T: F^{n \times 1} \rightarrow F^{m \times 1}$$

$T(x) = Ax$, then T is a linear transformation.

$$T(cx_1 + x_2) = cT(x_1) + T(x_2)$$

Take, $x_1, x_2 \in F^{n \times 1}$, $c \in F$

then,

$$T(cx_1 + x_2) = cT(x_1) + T(x_2)$$

$$= A(cx_1 + x_2)$$

$$= A(cx_1) + A(x_2)$$

$$= cA(x_1) + A(x_2)$$

$$= cT(x_1) + T(x_2)$$

2) Let P be a fixed $m \times m$ matrix over the field

F and Q be a fixed $n \times n$ matrix.

Define $T: F^{n \times n} \rightarrow F^{m \times n}$ by $T(A) = PAQ$ $A \in F^{n \times n}$

Take, $A, B \in F^{n \times n}$, $c \in F$

then,

$$T(cA + B) = cT(A) + T(B)$$

$$= P(CA + B)Q$$

$$= P(CA)Q + PBQ$$

$$= cPAQ + PBQ = cT(A) + T(B)$$

Let $T: V \rightarrow W$ be a linear transformation
 Show, $T(\theta_v) = \theta_w$
 $T(\theta + \theta) = T(\theta) + T(\theta)$
 $\Rightarrow T(\theta) = \theta \in W$

→ Image of a zero vector is a zero vector.

Theorem:

Let V be a finite dimensional vector space over the field F and let $\{v_1, v_2, \dots, v_n\}$ be an ordered basis of V .

Suppose W is another vector space over the same field F and

$\{w_1, w_2, \dots, w_n\}$ are any vectors in W .

Then there is precisely one linear transformation from V to W such that

$$T(v_j) = w_j ; j = 1, 2, 3, \dots, n$$

Proof:

We prove that there is a linear transformation $T: V \rightarrow W$ such that

$$T(v_j) = w_j ; j = 1, 2, 3, \dots, n$$

$$(A) T + (B) T = (A + B) T$$

$$(A) T \cdot (B) T = A \cdot B T$$

$\{v_1, v_2, v_3, \dots, v_n\}$ is a ordered basis of V , so \exists scalars x_1, x_2, \dots, x_n such that any vector

$v \in V$ can be written as $v = v$

$$v = x_1 v_1 + x_2 v_2 + x_3 v_3 + \dots + x_n v_n \quad (1)$$

Let us define $T: V \rightarrow W$ by

$$T(v) = x_1 w_1 + x_2 w_2 + \dots + x_n w_n$$

$$v = x_1 v_1 + x_2 v_2 + \dots + x_n v_n$$

$$v' = x'_1 v'_1 + x'_2 v'_2 + \dots + x'_n v'_n$$

$$T(cv + v') = \sum_{i=1}^n (cx_i + x'_i) w_i$$

$$= cx_1 w_1 + x'_1 w_1 + cx_2 w_2 + x'_2 w_2 + \dots + cx_n w_n + x'_n w_n$$

$$= c(x_1 w_1 + x_2 w_2 + \dots + x_n w_n) + (x'_1 w_1 + x'_2 w_2 + \dots + x'_n w_n)$$

$$= c T(v) + T(v')$$

Let us define $T: V \rightarrow W$ by

$$T_i: V \rightarrow W \quad T_i(v_j) = w_j$$

$$\text{Suppose, } v = \sum x_i v_i$$

$$T_i(v) = T_i(\sum x_i v_i) = (\sum x_i) T$$

$$(\sum x_i) T = (\sum x_i) T_i(v_i) = (\sum x_i v_i) T$$

$$= \sum x_i w_j$$

$$= T(v) \Rightarrow T = T^{-1}$$

Theorem: If $T: V \rightarrow W$ is a linear transformation, then $T(v_1 + v_2) = T(v_1) + T(v_2)$ and $T(cv) = cT(v)$.

Def'n: $V = \mathbb{R}^2$, $W = \mathbb{R}^3$

$$(1) \quad v_1 = (1, 2), v_2 = (3, 4)$$

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$T(v_1) = (3, 2, 1) \quad \text{and } W \rightarrow V: T \text{ sends } v_1 \mapsto v_1$$

$$T(v_2) = (6, 5, 4) \quad \text{and } v_2 \mapsto v_2$$

$$T(1, 0) = ?$$

Def'n: (Range of T): Suppose $T: V \rightarrow W$ then

$$R(T) = \{T(v) : v \in V\}$$

Theorem: $R(T)$ forms a vector space and subspace of W (Range space of W).

Proof:

$$\begin{aligned} R(T) &= \{T(v) : v \in V\} \\ &= \{w : w = T(v) \text{ for some } v\} \end{aligned}$$

$w_1, w_2 \in R(T)$, $c \in \mathbb{F}$ then

\exists some $v_1, v_2 \in V$ such that

$$T(v_1) = w_1, T(v_2) = w_2$$

$$\begin{aligned} T(cv_1 + v_2) &= cT(v_1) + T(v_2) \\ &= cw_1 + w_2 \end{aligned}$$

So, $cw_1 + w_2 \in R(T)$.

Hence, $R(T)$ is a vector space.

Defⁿ Null Space: $N(T) = \{v : T(v) = 0\}$ is a subspace of V .

Proof: $v_1, v_2 \in N(T)$, $c \in F$, $T(v_1) = 0, T(v_2) = 0$
 $T(cv_1 + v_2) = cT(v_1) + T(v_2) = 0$
 $\Rightarrow cv_1 + v_2 \in N(T)$.

Defⁿ Suppose V is a finite dimensional vector space, dimension of $R(T)$ is known as the rank of T .

dimension of $N(T)$ is called nullity of T .

Theorem:-

Let V and W be two vector spaces over some field F

$T: V \rightarrow W$ be a linear transformation. Then

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

Proof:

Suppose $\{v_1, v_2, \dots, v_k\}$ is a basis of $N(T)$

Then there exists vectors $v_{k+1}, v_{k+2}, \dots, v_n$ such that $\{v_1, v_2, \dots, v_n\}$ is a basis of V ($\dim V = n$)

Claim: $\{T(v_{k+1}), T(v_{k+2}), \dots, T(v_n)\}$ forms a basis of $R(T)$

Clearly, $\{T(v_1), T(v_2), \dots, T(v_n)\}$ spans $R(T)$
 $\Rightarrow \{v_i = (v)T : v\} = \{v\}N$
 To show,

$T(v_{k+1}), T(v_{k+2}), \dots, T(v_n)$ spans $R(T)$

$$0 = (av)T + (bv)T \dots = (av + bv)T$$

Since, $T(v_1) = T(v_2) = \dots = T(v_k) = 0$

Now, we have to show that the set, $\{T(v_{k+1}), T(v_{k+2}), \dots, T(v_n)\}$ is L-independent.

$$\text{Let, } \sum_{i=k+1}^n c_i T(v_i) = 0$$

$$\Rightarrow \sum T(c_i v_i) = 0$$

$$\Rightarrow T \left(\sum c_i v_i \right) = 0$$

$$\Rightarrow \sum c_i v_i \in N(T)$$

Since, $\{v_1, v_2, \dots, v_k\}$ is a basis of $N(T)$

$$\text{and } \sum_{i=k+1}^n c_i v_i \in N(T)$$

$$\sum_{i=k+1}^n c_i v_i = \sum_{i=1}^k d_i v_i \quad \{ \text{since } T(v_i) \in N(T) \text{ for all } i \}$$

$$\Rightarrow \sum_{i=k+1}^n c_i v_i - \sum_{i=1}^k d_i v_i = 0$$

$$\Rightarrow c_i = d_i = 0 \quad \forall i$$

$\{T(v_{k+1}), T(v_{k+2}), \dots, T(v_n)\}$ is a basis

Hence $\{T(v_{k+1}), T(v_{k+2}), \dots, T(v_n)\}$ is a basis
of $R(T)$.

$$\text{Therefore, } \dim(R(T)) = n - k$$

$$\Rightarrow \text{Rank}(T) = n - k$$

$$\text{and Rank of } N(T)$$

$$\Rightarrow \text{nullity}(T) = k$$

$$\text{Rank}(T) + \text{nullity}(T)$$

$$= n - k + k$$

$$= n$$

$$= \dim(V)$$

$$\therefore \text{Rank}(T) + \text{nullity}(T) = \dim(V)$$

Defn:- Let A be a $m \times n$ matrix over the field F

The row vectors of A are v_1, v_2, \dots, v_m in F^n

defined by

$$v_i = (a_{i1}, a_{i2}, \dots, a_{in}) \quad i = 1, 2, \dots, m$$

Row space is a vector space spanned by
the row vectors.

By, column space is defined.

Theorems:-

If A is a $(m \times n)$ matrix over the Field F , then
 $\text{rank}(A) = \text{column rank}(A)$

Defines $T: F^{n \times 1} \rightarrow F^{m \times 1}$ by $T(x) = Ax$

$$N(T) = \{x: Tx = 0 \Rightarrow Ax = 0\}$$

$$R(T) = \{Y: Ax = Y \text{ for some } x \in F^{n \times 1}\}$$

Suppose, A_1, A_2, \dots, A_n be the columns of A

$$T(x) = Ax = x_1 A_1 + x_2 A_2 + \dots + x_n A_n$$

$R(T)$ is spanned by the columns of A .

$\Rightarrow R(T)$ is a column space.

$\text{rank}(T) = \text{column rank of } A$

$$\dim(R(T)) + \dim N(T) = \dim V$$

$$\dim N(T) + \text{column rank of } A = \dim V = n$$

Suppose r is the dimension of row space of A

then, $N(T)$ has a basis with $(n-r)$ elements

$$\Rightarrow n-r + \text{column rank} = n$$

$\Rightarrow \text{column rank} = r = \text{row rank of } A$

Algebra of Linear transformation :-

Theorem :- Let V and W are two vector spaces over the field F . Suppose T_1 and T_2 are two linear transformation from V to W . Then the function $(T_1 + T_2)$ is defined by

$$(T_1 + T_2)(v) = T_1(v) + T_2(v) \quad \forall v \in V,$$

linear transformation.

If $c \in F$, the function $(cT_1)v$ is defined by

$$(cT_1)(v) = cT_1(v) \quad \forall v \in V$$

→ The set of all linear transformation from V to W , together with the above two operation forms a vector space over the field.

Proof :-

Let A be the set of all linear transformation from V to W .

$$\text{Let } T_1, T_2 \in A$$

$$\Rightarrow T_1 + T_2 \in A$$

Take $v_1, v_2 \in V$ and $c \in F$.

To show,

$$(T_1 + T_2)(cv_1 + v_2) = c(T_1 + T_2)v_1 + (T_1 + T_2)v_2$$

$$= T_1(cv_1 + v_2) + T_2(cv_1 + v_2)$$

$$= cT_1v_1 + T_1v_2 + T_2cv_1 + T_2v_2$$

$$= c((T_1 + T_2)v_1) + (T_1 + T_2)v_2$$

$A = L(V, W) = \text{set of all L.T from } V \rightarrow W$

Note: If V and W are n -dimensional vector spaces, then $L(V, W)$ is n^m -dimensional vector space.

① Suppose V is an n -dimensional vector space and W is an m -dimensional vector space. Then $L(V, W)$ is finite dimensional vector space and $\dim(L(V, W)) = mn$.

② Suppose U, V, W are three vector spaces over some field F .

$T_1: U \rightarrow V$ be a L.T.

$T_2: V \rightarrow W$ be a L.T.

Define $(T_2 T_1)v = T_2(T_1(v))$

Show that $T_2 T_1: U \rightarrow W$ is L.T.

$$(T_2 T_1)(cv_1 + v_2) = c(T_2 T_1)v_1 + (T_2 T_1)(v_2)$$

$$\Rightarrow T_2 T_1(cv_1) + T_2 T_1(v_2)$$

$$= c T_2 T_1(v_1) + T_2 T_1(v_2)$$

$$T_2 T_1(cv_1 + v_2)$$

$$= T_2(T_1(cv_1 + v_2)) \Rightarrow T_2 T_1: U \rightarrow W \text{ is}$$

$$= T_2(c T_1(v_1) + T_1(v_2)) \text{ a linear transformation}$$

$$= c T_2 T_1(v_1) + T_2 T_1(v_2)$$

$$= cv(T_2 T_1(v_1)) + T_2 T_1(v_2)$$

② There can be a T such that $T: V \rightarrow V$.

$T: V \rightarrow V$ is called linear transformation on V .

Example :-

$$A \in F^{m \times n}$$

$$T_1: F^{n \times 1} \rightarrow F^{m \times 1} \text{ by } T(X) = AX \quad A \in F^{m \times n}$$

$$T_2: F^{m \times 1} \rightarrow F^{p \times 1} \text{ by } T(Y) = BY \quad B \in F^{p \times m}$$

$$T_2 T_1: F^{n \times 1} \rightarrow F^{p \times 1} \quad (T_2 T_1)(X) = (B(A(X)))$$

$$T_2 T_1(X)$$

$$= T_2(T_1(X))$$

$$= T_2(A(X))$$

$$= AT_1(X) = B(A(X))$$

$$= AB(X) = (BA)X$$

$$\cancel{= (AB)X} \quad \therefore T_2 T_1(X) = (BA)X$$

Def

Proof:

Let $T: V \rightarrow W$ be a function. T is called invertible if there exists a function

$T_1: W \rightarrow V$ such that $T_1 T$ is a identity

$(T_1 T = I)$ on V .

$$T T_1 = I \text{ on } W$$

T invertible we use $(T^{-1} = T_1)$

iff, ① T is 1-1

② T is onto $R(T) = W$

$\forall v \in V : T$ don't have $T(v) = 0$ and non singular \Leftrightarrow
 $\forall v$ suppose T is a L.T. from $V \rightarrow W$ then $T^{-1} : W \rightarrow V$
 is also a L.T? (yes)

Proof of above prev page def:-

$$T(x) = T(y) \Rightarrow x = y \text{ for one-one func.}$$

Suppose, T is a L.T.

$$T(x-y) = T(x) - T(y)$$

$$T(x) = T(y) \quad (\text{if } T \text{ is L.T.})$$

$$\Rightarrow T(x-y) = 0$$

$$\text{As per nullity, } N(T) = \{v : T(v) = 0\}$$

$$= \{0\}$$

$$\Rightarrow x-y = 0$$

$$x(Ag) = (x)gA =$$

$$\Rightarrow x = y. \quad x(Ag) = (x)gA = x(gA) =$$

A linear transformation is 1-1 iff $N(T) = \{0\}$.

Example: $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$T(x_1, x_2) = (x_1 + x_2, x_1)$$

Show that T is non singular L.T. and

compute T^{-1} .

Q8:-

$$T(x_1, x_2) = 0$$

$$\Rightarrow T(x_1, x_2) = 0 \quad \text{if } T(0) = 0 \\ w = (0, 0) \text{ onto at } T \quad \text{if } T \neq 0$$

$$x_1 + x_2 = 0; \quad x_1 \neq 0$$

$$\{0\} = T(\{0\})$$

$$\Rightarrow x_2 = 0$$

$$\Rightarrow x_1 = x_2 = 0.$$

$\Rightarrow N(T) = \{0\}$ so, T is 1-1.

$$z_1, z_2 \in \mathbb{R}^2$$

$$z_1 = x_1 + x_2$$

$$z_2 = x_1$$

$$\text{So, } T(z_1, z_2) = T(z_2, z_2 - z_1)$$

Theorem:

Let T be a linear transformation from $V \rightarrow W$. Then T is non-singular iff T carries each linearly independent subset of V onto a linearly independent subset of W .

Proof: Suppose T is non-singular

To prove T carries each of linearly independent set of V to a L.I. set of W .

Let $S = \{v_1, v_2, \dots, v_k\}$ is a linearly independent set.

$\{T(v_1), T(v_2), \dots, T(v_k)\}$ is linearly independent.

Let us consider,

$$c_1 T(v_1) + c_2 T(v_2) + \dots + c_k T(v_k) = 0$$

$$\forall c_i \in \mathbb{R}$$

$$T(c_1v_1 + c_2v_2 + \dots + c_kv_k) = 0$$

$$c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$$

$$\Rightarrow c_1 = c_2 = \dots = c_k = 0$$

Now, the converse,

Let us assume T carries each of the linearly independent subset of V onto a linearly independent subset of W .

To prove T is non singular

Suppose, $v \neq 0$ is a vector in V

$$S = \{T(v)\} \Rightarrow T(v) \neq 0 \quad [\because T(v) \text{ is L.I.}]$$

$v \neq 0$

$$\Rightarrow T(v) \neq 0$$

$$\Rightarrow N(T) = 0$$

Hence, T is non-singular.

Now, $\{((x_1)v_1), ((x_2)v_2), \dots, ((x_n)v_n)\}$

$$0 = (x_1)v_1 + \dots + (x_n)v_n$$

Matrix representation of linear transformation:

Let V be a n -dimensional vector space over the field F . Let W be an m -dimensional vector space over the same field F .

Let $B = \{v_1, v_2, \dots, v_n\}$ is a basis of V

$B' = \{w_1, w_2, \dots, w_m\}$ is a basis of W

$T: V \rightarrow W$ is a linear transformation.

$$T(v_j) = \sum_{i=1}^m a_{ij} w_i, \quad j = 1, 2, \dots, n$$

$$T(v_j) \in W$$

$$T(v_1) = a_{11} w_1 + a_{21} w_2 + a_{31} w_3 + \dots + a_{m1} w_m$$

The scalars $a_{1j}, a_{2j}, \dots, a_{mj}$ are the coordinates of $T(v_j)$ with respect to B' .

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

This $m \times n$ matrix $A = (a_{ij})$ is called matrix of T relative to the pair of B, B' .

How the matrix A determines Linear transformation?

Matrix representation of linear transformation

$T(v) = x_1 v_1 + x_2 v_2 + \dots + x_n v_n$

$T(v) = x_1 T(v_1) + x_2 T(v_2) + \dots + x_n T(v_n)$

$$v \in \sum_{i=1}^n x_i T(v_i)$$

Here, $T(v_i) \in W$

So,

$$\begin{aligned} &= \sum_{i=1}^n x_i \sum_{j=1}^m a_{ij} w_j \\ &= \sum_{i=1}^n \sum_{j=1}^m (x_i a_{ij}) w_j \end{aligned}$$

Coordinates of $v = \{x_1, x_2, \dots, x_n\} = X$
 Coordinates of $T(v) = AX$
 → representation $[v]_B = \{x_1, x_2, \dots, x_n\}$

$V \rightarrow n$ dimensional vector space

$W \rightarrow m$ dimensional vector space

$[T(v)]_B = A[v]_B$

$T: V \rightarrow W$

Ansatz, but not Ansatz

$$\text{Ex: } T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

maps to linear mapping to standard basis = V
 $T(x_1, x_2) = (x_1, 0)$. & other basis are correct
check it is L.T. Then find the matrix of T
relative to standard basis.

$$B = \{(1, 0), (0, 1)\}$$

$$V \leftarrow V \circ T$$

$$B' = \{(1, 0), (0, 1)\} \rightarrow (1, 0, 0, 1)$$

$$T(1, 0) = (1, 0) = 1(1, 0) + 0(0, 1)$$

$$T(0, 1) = (0, 1) = 0(1, 0) + 1(0, 1)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$\text{Ex: } T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$T(x, y, z) = (y - z, y + z)$$

w.r.t S-Basis:

$$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

$$B' = \{(1, 0), (0, 1)\}$$

$$T(1, 0, 0) = (0, 0)$$

$$= 0(1, 0) + 0(0, 1)$$

$$T(0, 1, 0) = (1, 1)$$

$$= 1(1, 0) + 1(0, 1) \rightarrow A$$

$$T(0, 0, 1) = (-1, -1)$$

$$= -1(1, 0) + -1(0, 1)$$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

③

$V = \text{space of all polynomial of degree less than or equal to } 3$. $(0,1,2) = (x, x^2)$

T to xist from set basis itself. This is a standard basis.

$$T(f) = f'(x)$$

$$T: V \rightarrow V$$

$$\{f : f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3\}$$

$$B = \{(1, 0), (0, 1), (0, 0, 1)\} \quad B' = \{1, x, x^2\}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A$$

$$T(1) = 0$$

$$= 0(1) + 0(x) + 0(x^2) + 0(x^3)$$

$$T(x) = 1 \quad (f, g, h, k) = (f, g, h, k)T$$

$$= 1(1) + 0(x) + 0(x^2) + 0(x^3)$$

$$T(x^2) = 2x$$

$$= 0(1) + 2(x) + 0(x^2) + 0(x^3)$$

$$T(x^3) = 3x^2$$

$$= 0(1) + 0(x) + 3(x^2) + 0(x^3)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = (1, 0, 0)T$$

$$(1, 0)T + (0, 1)T =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A$$

Suppose U, V , and W are vector spaces over the field F with respective dimensions n, m , and p .

$$B = \{u_1, u_2, \dots, u_n\}$$

$$B' = \{v_1, v_2, \dots, v_m\}$$

$$B'' = \{w_1, w_2, \dots, w_p\}$$

$T_1: U \rightarrow V$ be a LT

$T_2: V \rightarrow W$ be a LT

Let A be the matrix of T_1 relative to the basis B, B' .

Let B be the matrix of T_2 relative to the basis B', B'' .

$T_2 T_1: U \rightarrow W$ is a linear transformation

C is the matrix of $T_2 T_1$ relative to the basis B, B''

$$C = BA$$

$$[T_1(u)]_{B'} = A[u]_B$$

$$[T_2(v)]_{B''} = B[v]_{B'}$$

$$[T_2 T_1(u)]_{B''} = T_2[T_1(u)] = B[T_1(u)] = BA[u]_B$$

set of two edges between sets W_{B1}, V_1, W_{B2} , V_2 respectively
 $A = (a_{ij})_{m \times n}$

$B = (b_{ij})_{n \times p}$

$$AB = \sum_{k=1}^n (a_{ik} b_{kj})_{m \times p}$$

Matrix multiplication using LT

$$(T_2 T_1)(v_j) = T_2(T_1(v_j)) \quad j=1, 2, \dots, n$$

$$= T_2 \left(\sum_{k=1}^m a_{kj} v_k \right)$$

$$= \sum_{k=1}^m a_{kj} T_2(v_k)$$

$$= \sum_{k=1}^m a_{kj} \sum_{i=1}^p b_{ik} w_i$$

$$= \sum_{k=1}^m \sum_{i=1}^p a_{kj} b_{ik} w_i$$

Their may be

Theorem:

Let U, V, W be the finite dimensional spaces over the field F . Let T_1 be a linear transformation from U to V and T_2 be a linear transformation from V to W .

$$g(U) A g =$$

If B, B', B'' are bases of U, V and W , respectively and A is the matrix of T_1 , relative to the bases B, B' and B is the matrix of T_2 relative to B', B'' then $C = BA$ is the matrix of $T_1 T_2$ relative to basis B, B'' .

Change of ordered Basis:-

Suppose $T: V \rightarrow W$, V is finite dimensional

$$B = \{v_1, v_2, \dots, v_n\} \quad B' = \{v'_1, v'_2, \dots, v'_n\}$$

A is the matrix of T relative to basis B

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{\text{basis } B} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = q$$

$$T[v]_B = A[v]_B \quad q A^T q = 8$$

$$T[v']_{B'} = B[v]_B \quad q A^T q = 8$$

Def (similar matrix)

A matrix A is said to be similar to B if there exists a non singular matrix P such that $B = P^{-1}AP$

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{basis } B} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = q$$

$$[T]_{B'} = P^{-1} [T]_B P$$

$$P = [P_1, P_2, \dots, P_n]_{n \times n}$$

$$P_j = [v_j]_B$$

IT is $R^2 \rightarrow R$ with A basis $\{x_1, x_2\}$
 and B has basis $\{B_1, B_2\}$ and set of vectors
 $T(x_1, x_2) = (x_1, 0)$
 $AB = I$ invertible set of vectors let T be system
 second of vectors T is system with ai
 $[T]_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$B' = \{(1,2), (2,1)\}$$

longer unique solution in V, $V \leftarrow T'$ orthogonal.

$$(1,2) = 1(1,0) + 2(0,1)$$

$$\{av_1, av_2\} = \{av_1, v_2\} = \{av_1, v_1, v_2\} = \{v_1, v_2\}$$

$$(2,1) = 2(1,0) + 1(0,1)$$

$$P = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}^T$$

$$B = P^{-1} A P$$

$$B = -\frac{1}{3} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^T$$

$$B = -\frac{1}{3} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$B = -\frac{1}{3} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -1/3 & -2/3 \\ 2/3 & 4/3 \end{bmatrix}$$

$$g(r) = g(T)$$

26/08

A similar to B

$\text{rank} = (\text{W}, \text{V}) \perp$

B similar to C

\rightarrow ~~similar to SV to get b/w row~~

$$C = Q^T B Q$$

$$= Q^{-1} P^T A P Q$$

$$= (PQ)^{-1} A (PQ)$$

$\text{rank} = (\text{F}, \text{V}) \perp$

Hence, C similar to A.

Defn

(Linear function)

If V is a vector space over the field F, a linear transformation $T: V \rightarrow F$ is called a linear function.

Ex:- $T: F^{n \times n} \rightarrow \mathbb{R}$

$T(A) = \text{Trace of } A$

2) $T(f) = \int_a^b f \quad f \in V = C[a, b]$

3) $T: F^n \rightarrow F$

$$T(x_1, x_2, \dots, x_n) = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

$$[T]_B = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & a_3 & 0 & 0 & \dots & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & a_n \end{bmatrix}$$

$$[T]_B = A = \text{Span}[a_1, a_2, \dots, a_n]$$

$L(V, W) = mn$
 $L(V, W) \rightarrow$ set of all L.T from $V \rightarrow W$.
 which would be a vector space.
 $L(V, F) \rightarrow$ collection of all L.F from $V \rightarrow W$.
 $L(V, F) = mx1$

$$V^* = L(V, F) = \{ f : f: V \rightarrow F \text{ is a L.T} \}$$

= Dual Space of V.

$$\dim(V^*) = \dim(V)$$

$$B = \{v_1, v_2, \dots, v_n\}$$

we need to find the basis of V^* ,

so, take

$$f_i(v_j) = \delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

$$\therefore \{f_1, f_2, \dots, f_n\}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = g[t]$$

Eigen vector:

$$AX = \lambda X$$

↓

Eigen vector.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = A$$

Defn

Suppose $T: V \rightarrow V$ be a linear transformation operator and V is finite dimensional. A non-zero vector ($v \neq 0$) and a scalar ($\lambda \in F$) are called eigenvectors and eigenvalues of the linear operator T if $T(v) = \lambda v$.

λ = eigenvalue

v = eigenvector

$$0 = |IV - A|$$

λ -eigen space :- Collection of all eigenvectors corresponding to the eigenvalue λ .

spectrum :- Collection of all eigen values of a linear operator is called spectrum of the linear operator.

Algebraic multiplicity :- No. of repetition of eigen value

Geometric multiplicity : Number of linearly independent eigenvectors corresponding to eigen value λ is called geometric multiplicity.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 2 & 4 & 6 \end{bmatrix}$$

$$A \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 4 & 6-\lambda & 7 \\ 2 & 4 & 6-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$(1-\lambda) \times$$

$$(1-\lambda)[(6-\lambda)(6-\lambda) - 28]$$

$$- 2[4(6-\lambda) - 14]$$

$$+ 3[16 - 2(6-\lambda)] = 0$$

$$\Rightarrow (1-\lambda)[36 + \lambda^2 - 12\lambda - 28] - 2[16 - 4\lambda]$$

$$+ 3[4 + 2\lambda] = 0$$

phosphorus

$$\Rightarrow \lambda^2 - 12\lambda + 8 - \lambda^3 + 12\lambda^2 - 8\lambda - 20 + 8\lambda \\ + 12 + 6\lambda = 0$$

$$\Rightarrow -\lambda^3 + 13\lambda^2 - 6\lambda - 14 = 0$$

$$\Rightarrow \lambda^3 - 13\lambda^2 + 6\lambda + 14 = 0$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & 6 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 7 \\ 0 & 0 & 6-\lambda \end{bmatrix}$$

$$\Rightarrow (1-\lambda)[(1-\lambda)(6-\lambda)] = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda)(6-\lambda) = 0$$

$$\lambda = 1, 1, 6$$

$$\lambda \neq 1$$

$$T(v) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$(1, 0, 0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$T(v_1) = (1, 0, 0)$$

$$T(v_2) = (2, 0, 0)$$

$$T(v_3) = (3, 7, 6)$$

$$\text{for } \lambda = 1,$$

algebraic multiplicity = 2

geometric multiplicity = 1

$$Kx + 0y - Kz = Kx - Kz + K - 8 + Kz = 0$$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 7 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5z = 0$$

$$\Rightarrow z = 0$$

$$\Rightarrow y = 0$$

$$v = K \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} -5 & 2 & 3 \\ 0 & -5 & 7 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$z = K$$

$$-5y + 7z = 0$$

$$-5y + 7z = 0$$

$$5y = 7K$$

$$y = 7K/5$$

$$-5x + 2y + 3z = 0$$

$$\begin{array}{l|l} z = K & y = l \\ y = 7K/5 & z = 5l/7 \\ x = \frac{29}{25}K & x = \frac{19l}{75} \end{array}$$

$$5x = \frac{14K}{5} + 3K (y)$$

$$5x = \frac{29K}{5} \quad \underline{\text{wrong}}$$

$$x = \frac{29K}{25}$$

$$\text{so, } v = K \begin{bmatrix} 29/25 \\ 7/5 \\ 1 \end{bmatrix}$$

$$K \begin{bmatrix} 29/25 \\ 7/5 \\ 1 \end{bmatrix} \sim P \begin{bmatrix} 1 \\ 35/29 \\ 25/29 \end{bmatrix}$$

$$L \begin{bmatrix} 29/35 \\ 1 \\ 5/7 \end{bmatrix}$$

All these are same

$$x = P$$

$$2x(7z = 5y) = 10y - 14z = 0$$

$$5(x - 2y + 3z = 5P) = 10y + 15z = 25P$$

$$y = 25/29 P \quad z = 25P/29 \quad 29z = 25P$$

Theorem: Each λ -eigen space is a subspace of V .

Proof:

Let $v_1, v_2 \in V$ be two eigenvectors corresponding to λ

$$cv_1 + v_2 \in \lambda\text{-eigen space}$$

$$\begin{aligned} T(cv_1 + v_2) &= T(cv_1) + T(v_2) \\ &= cT(v_1) + T(v_2) \\ &= c\lambda v_1 + \lambda v_2 \\ &= \lambda(cv_1 + v_2) \end{aligned}$$

so, λ -eigen space is subspace of V .

Hence proved

09/09/19
Q:- whether eigen value and eigen vector exists for every L.T

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$T(x_1, x_2) = (-x_2, x_1)$$

$$T(x_1, x_2) = \lambda(x_1, x_2), \lambda \neq 0$$

$$(-x_2, x_1) = (\lambda x_1, \lambda x_2)$$

$$x_1 = \lambda x_2$$

$$x_2 = -\lambda x_1$$

$$\Rightarrow x_1 = -\lambda^2 x_1 \Rightarrow x_1(1 + \lambda^2) = 0$$

~~V to associate or assign angles & does conversion~~

and since $x \neq 0$ and $V \neq 0$, we have
 $\lambda^2 + 1 = 0$
 $\lambda = i$, implies $\lambda^2 + 1 = 0$

Here, $(1+i)$ is a positive quantity. So,

$$x_1 = 0 \Rightarrow x_2 = 0$$

So, there exists no non zero x .

$$T(1,0) = (0,1) \quad \text{So, this is rotational}$$

$$T(0,1) = (-1,0) \quad \text{operator } (90^\circ)$$

But, $T = \lambda x$ is translational operator.

So, there are no non zero "x".

$$A = [T]_B$$

$$B = [T]_{B'}$$

Whether A, B have same eigenvalues or not?

$$AX = \lambda X$$

$$BX = \lambda_1 X$$

$$|A - \lambda_1 I| = 0 \quad |B - \lambda_1 I| = 0$$

$$A - \lambda_1 I = B - \lambda_1 I = 0$$

Annn,

then $|A - \lambda I| = 0$ would give us a n-degree polynomial called the characteristic polynomial.

we know,

$$B = P^{-1}AP$$

$$\Rightarrow |B| = |P^{-1}P||A|$$

$$\Rightarrow |B| = |\mathbb{I}| |A|$$

$$\Rightarrow |B| = |A|$$

$$\underbrace{B - \lambda I} = P^{-1}AP - \lambda I$$

$$= P^{-1}AP - \lambda P^{-1}P$$

$$= P^{-1}(\underbrace{A - \lambda I})P$$

so, $B - \lambda I$ is equivalent to $A - \lambda I$.

$$\text{so, } |B - \lambda I| = |A - \lambda I|$$

So, since the characteristic polynomials are same for A and B, so the eigenvalues are also the same.

Ex:

Defn

$$f: L(V) \rightarrow F \text{ by}$$

$$f(T) = \det(T) = |[T]_B|$$

for some basis B.

② $A \in \mathbb{R}^{3 \times 3}$

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 6 \end{bmatrix}$$

eigenvalue, vectors, G.M, A.M

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} (3-\lambda) & 1 & -1 \\ 2 & (2-\lambda) & -1 \\ 2 & 2 & -\lambda \end{vmatrix} = 0$$

$$(3-\lambda)[-2\lambda + \lambda^2 + 2] - 1[-2\lambda + 2] - 1[4 - 4 + 2\lambda] = 0$$

$$\Rightarrow (3-\lambda)[\lambda^2 - 2\lambda + 2] + 2\lambda - 2 - 2\lambda = 0$$

$$\Rightarrow 3\lambda^2 - 6\lambda + 6 - \lambda^3 + 2\lambda^2 - 2\lambda - 2 = 0$$

$$\Rightarrow -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda = 1$$

$$1 \left[\begin{array}{cccc} 1 & -5 & 8 & -4 \\ 0 & 1 & -4 & 4 \\ \hline 1 & -4 & 4 & 0 \end{array} \right]$$

$$\lambda = \left| \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -2 & 2 \end{array} \right|$$

$$(\lambda-1)(\lambda^2 - 4\lambda + 4) = 0$$

$$(\lambda-1)(\lambda-2)^2 = 0$$

$$\lambda = 1, 2, 2$$

Eigen vectors:

$$\lambda = 1$$

$$\left[\begin{array}{ccc} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{array} \right] \left[\begin{array}{c} x \\ y \\ z \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] = B - A$$

$$\left[\begin{array}{ccc} 2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right] \left[\begin{array}{c} x \\ y \\ z \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$$

$$y = 0$$

$$2x + y - z = 0$$

$$2x = z$$

$$\text{if } z = k$$

$$x = k/2$$

$$\text{So, } \lambda = 1$$

$$A - M = I$$

$$G \cdot M = I$$

$$\text{So, } k \left[\begin{array}{c} 4/2 \\ 0 \\ 1 \end{array} \right]$$

$$\lambda = 2$$

$$\begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{vmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y - z = 0$$

$$x - y = 0$$

$$if y = k$$

$$x = k$$

$$z = 2k$$

$$\Rightarrow k \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

for a linear transformation,

$$\text{then } [T]_B = D$$

\downarrow condition is
Diagonal matrix $\rightarrow A \cdot M = G \cdot M$

LT is Diagonalisable.

Symmetric and Skew symmetric-

$$A = A^T \quad (\text{field is real}) \quad A = -A^T \quad (\text{skew symmetric})$$

(real)

$$A = (\bar{A})^T \quad (\text{Hermitian})$$

$$A = -(\bar{A})^T \quad (\text{complex})$$

(S.S.)
(S. Hermitian)

→ Symmetric matrices have real eigen values.

$$\bar{a}_{ij} = a_{ji}$$

→ Diagonal elements of a hermitian matrix are real.

for skew hermitian,

$$(\bar{A})^T = -A$$

for diagonal elements,

$$+\bar{a}_{ii} + a_{ii} = 0$$

$$\bar{a}_{ij} = -a_{ji}$$

⇒ diagonal elements

$$\Rightarrow \bar{a}_{ij} + a_{ji} = 0 \quad \text{are either purely imaginary or zero.}$$

Theorem :-

If $H = P+iQ$ be a hermitian matrix then the diagonal elements of H are all real numbers and P is symmetric, Q is skew-symmetric.

Proof:

$$(\bar{H})^T = H$$

$$(\bar{P+iQ})^T = H$$

$$\Rightarrow (\bar{P+iQ})^T = P+iQ$$

$$\Rightarrow (\bar{P})^T + (\bar{iQ})^T = P+iQ$$

$$\Rightarrow (\bar{P})^T - i(\bar{Q})^T = P+iQ$$

$$\Rightarrow \boxed{\begin{aligned} (\bar{P})^T &= P \\ +(\bar{Q})^T &= -Q \end{aligned}}$$

Suppose, $H = P + iQ$

If diagonal elements of H are purely imaginary or zero, then P is skew-symmetric and Q is symmetric.

Proof:

$$(\bar{H})^T = -H$$

$$\Rightarrow (\overline{P+iQ})^T = -P-iQ$$

$$\Rightarrow (\bar{P})^T + (\bar{iQ})^T = -P-iQ$$

$$\Rightarrow (\bar{P})^T + i(\bar{Q})^T = -P-iQ$$

$$\Rightarrow (\bar{P})^T = -P \text{ and } (\bar{Q})^T = Q$$

Note: If A is any complex matrix then

$A + (\bar{A})^T$ is Hermitian and

$A - (\bar{A})^T$ is Skew-Hermitian

$$\text{and } A = \frac{1}{2} [(A + (\bar{A})^T) + (A - (\bar{A})^T)]$$

Defn: A matrix A is called orthogonal if $AA^T = I$

A complex matrix A is called unitary

matrix if $A(\bar{A})^T = I$ (or) $A^*A = I$, where $A^* = (\bar{A})^T$

Theorem: If A is an unitary matrix, then A is non-singular and $|A| = 1$.

Proof: Given A is unitary matrix,

$$\text{So, } AA^* = I$$

$$\Rightarrow A(\bar{A})^T = I$$

Taking determinant on both sides,

$$|A| \cdot |(\bar{A})^T| = |I|$$

$$\Rightarrow |A| |\bar{A}| = |I|$$

$$\Rightarrow |A|^2 = |I|$$

$$\Rightarrow |A|^2 = 1$$

$$\Rightarrow |A| = \pm 1$$

$$\text{So, } \text{mod}(|A|) = 1.$$

Inverse of a unitary matrix, A^{-1} is ?

$$A^{-1} = (\bar{A})^T$$

$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix} \quad \text{verify unitary or not?}$$

$$(\bar{A})^T = \begin{bmatrix} +\frac{1-i}{2} & \frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}^T = \begin{bmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

Theorem: Eigenvalues of a Hermitian matrix are all real.

Proof: Suppose λ be a eigenvalue of Hermitian matrix A .

$$\Rightarrow |A - \lambda I| = 0$$

$$\Rightarrow (A - \lambda I)x = 0 \text{ for some } x \neq 0$$

Suppose $x_1 \neq 0$ is the eigen vector to λ .

$$\Rightarrow (A - \lambda I)x_1 = 0$$

$$\Rightarrow (A - \lambda I)x_1 = 0 \text{ & } x_1 \neq 0$$

$$\Rightarrow Ax_1 = \lambda x_1$$

$$\Rightarrow (\bar{A})(\bar{x}_1) = (\bar{\lambda})(\bar{x}_1)$$

$$\Rightarrow (\bar{A})(\bar{x}_1)^T = (\bar{x}_1)^T(\bar{x}_1)^T \quad (\bar{A})^T = (\bar{x}_1)^T$$

$$\Rightarrow A(\bar{x}_1)^T = (\bar{x}_1)^T(\bar{x}_1)^T \quad (\bar{x}_1)^T(\bar{A})^T = (\bar{x}_1)^T(\bar{\lambda})^T$$

$$\Rightarrow (\bar{x}_1)^T A = (\bar{x}_1)^T(\bar{\lambda})^T$$

λ is some scalar, so $(\bar{\lambda})^T = (\bar{\lambda})$

$$\Rightarrow (\bar{x}_1)^T A = (\bar{x}_1)^T(\bar{\lambda})$$

$$\Rightarrow (\bar{x}_1)(\bar{x}_1)^T A \cdot (x_1) = (\bar{x}_1)^T(\bar{\lambda})x_1$$

$$\Rightarrow AX_1 = \bar{\lambda} X_1$$

$$\text{and } A\bar{X}_1 = \lambda \bar{X}_1$$

So,

$$\boxed{\lambda = \bar{\lambda}}$$

Hence, the eigen values are real for Hermitian matrix.

For skew-Hamiltonian,

$$AX_1 = \lambda X_1$$

$$\Rightarrow (\bar{AX}_1)^T = (\bar{\lambda} \bar{X}_1)^T$$

$$\Rightarrow (\bar{X}_1)^T (\bar{A})^T = (\bar{X}_1)^T (\bar{\lambda})^T$$

$$\Rightarrow (\bar{X}_1)^T (\bar{A})^T = (\bar{X}_1)^T (\bar{\lambda})$$

$$\Rightarrow -(\bar{X}_1)^T A = (\bar{X}_1)^T \bar{\lambda}$$

$$\Rightarrow (\bar{X}_1)^T A X_1 + (\bar{X}_1)^T \bar{\lambda} X_1 = 0$$

$$\Rightarrow (\bar{X}_1)^T \lambda X_1 + (\bar{X}_1)^T \bar{\lambda} X_1 = 0$$

$$\Rightarrow (\bar{X}_1)^T (\lambda + \bar{\lambda}) X_1 = 0$$

$$\Rightarrow \boxed{\lambda + \bar{\lambda} = 0}$$

Hence, the eigen values are purely imaginary for skew-Hamiltonian matrix.

Theorem:- The eigenvector corresponding to two distinct eigenvalues of symmetric matrix are orthogonal.

Proof: Let λ_1, λ_2 are two distinct eigen-values

$$AX_1 = \lambda_1 X_1$$

$$AX_2 = \lambda_2 X_2$$

$$(\bar{A}X_1)^T = (\bar{\lambda}_1 \bar{X}_1)^T$$

$$\Rightarrow (\bar{X}_1)^T (\bar{A})^T = (\bar{X}_1)^T (\bar{\lambda}_1)$$

$$\Rightarrow (\bar{X}_1)^T A = (\bar{X}_1)^T (\bar{\lambda}_1)$$

$$\Rightarrow (\bar{X}_1)^T A X_2 = (\bar{X}_1)^T (\bar{\lambda}_1) X_2$$

$$\Rightarrow (\bar{X}_1)^T \lambda_2 X_2 = (\bar{X}_1)^T \lambda_1 X_2 \quad (\text{Since for Hermitian, } \lambda \text{ are real,})$$

$$\Rightarrow (\bar{X}_1)^T (\lambda_2 - \lambda_1) X_2 = 0$$

$$\Rightarrow (\lambda_2 - \lambda_1) (\bar{X}_1)^T X_2 = 0$$

$$\lambda_2 \neq \lambda_1$$

$$\text{So, } (\bar{X}_1)^T X_2 = 0$$

Hence, X_1, X_2 are orthogonal.

Eigen value and eigen vector of linear transformation:

$T: V \rightarrow V$ $V = \text{vector space over field } F (\mathbb{R} \text{ or } \mathbb{C})$

we say $\lambda \in F$ is a eigen value of T if $\exists v \neq 0 \in V$ such that $Tv = \lambda v$, then v is called the eigen vector corresponding to eigen value λ .

$$E_\lambda = N(T - \lambda I) = \{v \in V \mid (T - \lambda I)v = 0\}$$

$$= \{v \in V \mid T\vec{v} = \lambda v\}$$

E_λ is called as eigen space of T .

$\rightarrow \lambda$ is eigenvalue of $E_\lambda \neq \{0\}$

$\rightarrow E_\lambda$ is subspace of V .

Proof:

Let $v_1, v_2 \in E_\lambda$

$$Tv_1 = \lambda v_1, \quad Tv_2 = \lambda v_2$$

$$\begin{aligned} T(c_1 v_1 + c_2 v_2) &= c_1 T(v_1) + c_2 T(v_2) \\ &= c_1 \lambda v_1 + c_2 \lambda v_2 \\ &= \lambda(c_1 v_1 + c_2 v_2) \end{aligned}$$

$\Rightarrow c_1 v_1 + c_2 v_2$ also belongs to E_λ .

$$c_1 v_1 + c_2 v_2 \in E_\lambda.$$

So, E_λ is a subspace of V .

① Show that if T is linear transformation on a finite dimensional space V , then show the following are equivalent.

- (i) λ is eigen value of T
- (ii) The operator $T - \lambda I$ is singular
- (iii) $\det([T]_B - \lambda I) = 0$

where $[T]_B$ is the matrix representation of T w.r.t basis B .

$$T: V \rightarrow V$$

$$Tv = 0 \Rightarrow v = 0 \text{ (non singular)}$$

$$Tv = 0, v \neq 0 \text{ then (singular)}$$

① $\rightarrow \lambda$ is eigen value of T

$$\Rightarrow \exists v \neq 0 \in V \text{ such that } Tv = \lambda v.$$

$$\Rightarrow (T - \lambda I)v = 0$$

$$\Rightarrow T - \lambda I = 0 \text{ ie i.g. singular} (\because v \neq 0)$$

②

③

equation ③ is called as "characteristic equation of a linear operator."

Find the eigenvalue and corresponding eigen vector
of a L.T. $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$. $T(x, y, z) = (3x+2y+2z, x+2y+2z, -x-2y)$.

$$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

$$[T]_B = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$$

$$[T]_B - \lambda I = \begin{bmatrix} 3-\lambda & 2 & 2 \\ 1 & 2-\lambda & 2 \\ -1 & -1 & -\lambda \end{bmatrix}$$

$$| [T]_B - \lambda I | = 0$$

$$\Rightarrow (3-\lambda)(-2\lambda + \lambda^2 + 2) - 2(-\lambda + 2) + 2[-1 + 2 - \lambda] = 0$$

$$3\lambda^2 - 6\lambda + 6 - \lambda^3 + 2\lambda^2 - 2\lambda + 2\lambda - 4 + 2 - 2\lambda = 0$$

$$\Rightarrow -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda = 1$$

$$\begin{array}{r} 1 \\ | \begin{array}{cccc} 1 & -5 & 2 & -4 \\ 0 & 1 & -4 & 4 \\ \hline 1 & -4 & 4 & 0 \end{array} \end{array}$$

$$(\lambda-1)(\lambda^2 - 4\lambda + 4) = 0$$

$$\Rightarrow \lambda = 1, 2, 2$$

for $\lambda = 1$,

$$v = (x \ y \ z)^T$$

so t

$$[T]_{\beta} v = I v$$

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \text{Rank } ([T]_{\beta} - I) = 2$$

$$N([T]_{\beta} - I) = 3 - 2 = 1 = \text{eigen space } (\lambda = 1)$$

$$x + y + z = 0$$

$$z = 0$$

$$\Rightarrow (-1, 1, 0)$$

So, null space of $([T]_{\beta} - I)$ has dimension 1
i.e the eigen space corresponding to eigen value 1 has dimension 1.

$$\lambda = 2;$$

$$\begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & 2 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x + 2z = 0; y = 0$$

$$x + 2z = 0; y = 0$$

(-2, 0, 1)

$$N(T - \lambda I) = 3 - 2 = 1$$

→ Whether every linear operator on a finite dimensional space has a eigen value or not.

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$T(\alpha, y) = (-y, \alpha)$$

$$T(1, 0) = (0, 1)$$

$$T(0, 1) = (-1, 0)$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(T - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

It does not have root in \mathbb{R} .

But over the complex field it has an eigen value.

- ② Let λ is the eigen value of a linear operator T and the corresponding eigen vector is v , then show that λv is also a eigen vector corresponding to the same eigen value.

$\exists v \neq 0$, s.t

$$T v = \lambda v$$

$$\begin{aligned} T(Kv) &= K T(v) \\ &= K \lambda v \\ &= \lambda(Kv) \end{aligned}$$

So, Kv is also an eigen-vector.

→ Find the characteristic polynomial of the following L-operator.

$$(a) T: \mathbb{R}^2 \rightarrow \mathbb{R}^2; T(x, y) = (3x+4y, 2x-7y)$$

$$(b) D: V \rightarrow V$$

V = space of all func whose basis is

$$\beta = \{\sin t, \cos t\}$$

$$D(f) = f' = \frac{df}{dt}$$

$$D(\sin t) = \cos t$$

$$= 0(\sin t) + 1(\cos t)$$

$$D(\cos t) = -\sin t$$

$$= -1(\sin t) + 0(\cos t).$$

$$[D]_{\beta} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$T[D]_{\beta} - \lambda I = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$$

$\therefore \lambda^2 + 1$ is the characteristic polynomial.

$$(a) \begin{bmatrix} 3 & 4 \\ 2 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 3-\lambda & 4 \\ 2 & -7-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)(\lambda+7) + 8 = 0$$

$$= -\lambda^2 - 4\lambda + 29$$

Show that if T is invertible, and λ is the eigenvalue of T , then λ^{-1} is a eigenvalue of T^{-1} .

Sol: T is invertible and $\lambda = \text{eigenvalue}$.

\Rightarrow if $\lambda = 0$, then

$$T(v) = \lambda v = 0$$

$$\Rightarrow T v = 0$$

$\Rightarrow v = 0$ which is a contradiction.

$\therefore \lambda \neq 0$ and

$$T v = \lambda v$$

$$T^{-1}(T v) = T^{-1}(\lambda v)$$

$$I v = \lambda T^{-1}(v)$$

$$T^{-1}(v) = \frac{1}{\lambda} (v)$$

$\Rightarrow \frac{1}{\lambda}$ is the eigen value corresponding to T^{-1} .

Similar matrices:

A & B are two matrices we say that A is similar to B if \exists invertible matrix P such that $B = P^{-1}AP$.

→ Similar matrices have same eigen values.

$$\begin{aligned}\text{Note that } \det(B - \lambda I) &= \det(P^{-1}AP - \lambda I) \\ &= \det(P^{-1}AP - \lambda P^{-1}P) \\ &= \det(P^{-1}(A - \lambda I)P) \\ &= \det P^{-1} \det(A - \lambda I) \det P \\ &= \det(A - \lambda I)\end{aligned}$$

$$\therefore \det(B - \lambda I) = \det(A - \lambda I)$$

⇒ characteristic equation of matrix B is same as characteristic equation of A.

→ Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigen values of operator T and v_1, v_2, \dots, v_k be corresponding vectors then $\{v_1, v_2, \dots, v_k\}$ are L.I.

Proof: Suppose the results is not true.

Let $\{v_1, v_2, \dots, v_s\}$ be a minimal set of vectors for which theorem is not true.

We have $s > 1$, $v_i \neq 0$

Then by assumption $\{v_2, v_3, \dots, v_s\}$ will be linearly independent.

So, $\exists c_1, c_2, c_3, \dots, c_s$ not all zero such that

$$v_1 = c_2 v_2 + \dots + c_s v_s$$

$$v_1 = c_2 v_2 + \dots + c_s v_s - \textcircled{1}$$

$$T(v_1) = T(c_2 v_2 + c_3 v_3 + \dots + c_s v_s)$$

$$\lambda_i T v_i = \lambda_i v_i ; i=1, 2, \dots, s$$

$$\lambda_1 v_1 = c_2 \lambda_2 v_2 + \lambda_3 v_3 + \dots + \lambda_s v_s - \textcircled{2}$$

$$\lambda_1 \textcircled{1} - \textcircled{2} \Rightarrow$$

$$0 = c_2 (\lambda_1 - \lambda_2) v_2 + c_3 (\lambda_1 - \lambda_3) v_3 + \dots + c_s (\lambda_1 - \lambda_s) v_s$$

if $\{v_2, v_3, \dots, v_s\}$ is L.I, then

$$c_2 (\lambda_1 - \lambda_2) = 0, c_3 (\lambda_1 - \lambda_3) = 0, \dots, c_s (\lambda_1 - \lambda_s) = 0$$

$$\Rightarrow c_2 = c_3 = \dots = c_s = 0.$$

which is a contradiction.

→ Show that if λ is eigen value of operator T , then λ^k is the eigen value of T^k ; $k > 0$

$$\text{Sol: } T v = \lambda v$$

$$T(Tv) = T(\lambda v)$$

$$T^2(v) = \lambda T(v) = \lambda(\lambda v)$$

$$T^2(v) = \lambda^2 v$$

$$\text{Say, } T^{n-1} v = \lambda^{n-1} v$$

$$T(T^{n-1} v) = T(\lambda^{n-1} v)$$

By induction,

$$T^n(v) = \lambda^n v$$

$$\boxed{T^n(v) = \lambda^n v}$$

Let λ be eigenvalue of T , then λ is polynomial,

$$P(T) = a_0 + a_1 T + \dots + a_n T^n$$
 is poly. of degree n .

Show that $P(\lambda)$ is eigenvalue of $P(T)$.

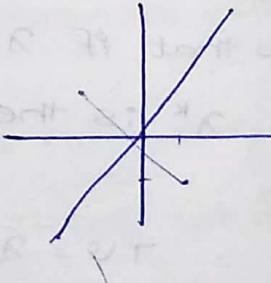
$$\text{Sd: } P(T) = a_0 + a_1 T + \dots + a_n T^n$$

→ We have to prove the previous theorem and then we should conclude this one for exam.

Let T be the linear transformation on \mathbb{R}^2 that reflects $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, which reflects each points across the line, $y = kx$ where k is considered (> 0). Show that $(1, k)$ is a eigen vector of T .

$$T(1, k) = (1, k)$$

Here, $(1, k)$ lies on $y = kx$,



$$\text{Hence, } T\mathbf{v} = \lambda \mathbf{v}$$

$$\text{So, } \lambda = 1 \text{ and } \mathbf{v} = (1, k).$$

$$T(1, -k) = -1 (1, -k)$$

$$\lambda = -1, \mathbf{v} = (1, -k)$$

Def:
Let T be linear
vector space V
if a basis of
an eigen vector
diagonizable
spans V .

Proof: Let $\dim V = n$
Suppose \exists

$$B = \{v_1, v_2, \dots, v_n\}$$

is an eigen

$$TV_i = \lambda_i v_i$$

Then,

$$[T]_B = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Ex Show that
operator T

$$T: \mathbb{C}^n \rightarrow \mathbb{C}^n$$

is diagonalizable

Def:
 Let T be linear operator on finite dimensional vector space V , we say T is diagonalisable if \exists a basis of V , in which each vector is an eigen vector of T . In other words, T is diagonalisable if characteristic vectors of T spans V .

Proof: Let $\dim V = n$

Suppose \exists basis of V

$B = \{v_1, v_2, \dots, v_n\}$ in which each vector is an eigen vector

$$Tv_i = \lambda_i v_i, i=1, 2, 3, \dots, n$$

Then,

$$[T]_B = \begin{bmatrix} c_1 & 0 & 0 & \dots & 0 \\ 0 & c_2 & 0 & \dots & 0 \\ 0 & 0 & c_3 & \dots & 0 \\ \vdots & 0 & 0 & c_4 & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots & c_n \end{bmatrix} = \text{diagonal matrix.}$$

Ex Shbut that check whether the linear operator T , from $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$T(x, y, z) = (2x+y, -2z, 2x+3y-4z)$$

$$x+y-z$$

is diagonalisable or not?

$$T(1,0,0) = (2, 2, 1)$$

$$T(0,1,0) = (1, 3, 1)$$

$$T(0,0,1) = (-2, -4, -1)$$

$$[T]_B = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{bmatrix}$$

Q

$$|[T]_B - \lambda I| = 0$$

$$\Rightarrow \begin{bmatrix} (2-\lambda) & 1 & -2 \\ 2 & (3-\lambda) & -4 \\ 1 & 1 & (-1-\lambda) \end{bmatrix} = 0$$

$$(2-\lambda) \left[(3-\lambda)(-1-\lambda) + 4 \right]$$

$$-1 \left[2(-1-\lambda) + 4 \right]$$

$$-2 \left[2 - 3 + \lambda \right] = 0$$

$$(2-\lambda) \left[\lambda^2 - 2\lambda + 1 \right] - 1 \left[-2\lambda + 2 \right]$$

$$-2 \left[\lambda - 1 \right] = 0$$

$$(2-\lambda)(\lambda-1)^2 + 2(\lambda-1) - 2(\lambda-1) = 0$$

$$\Rightarrow \lambda = 1, 1, 2$$

$\lambda = 1$

$$\begin{bmatrix} 1 & 1 & -2 \\ 2 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Rank}(A) = 1$$

$$\text{so, } N(C) = 3 - 1 = 2$$

$$x + y - 2z = 0$$

In general, for

$$x = 0, y = 1 \Rightarrow z = 1/2$$

L-I, we take

$$x = 1, y = 0 \Rightarrow z = 1/2$$

like this.

$$(1, 0, 1/2)$$

$$(0, 1, 1/2)$$

for $\lambda = 2$,

$$\begin{bmatrix} 0 & 1 & -2 \\ 2 & 1 & -4 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & -4 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{R}(A) = 2 \Rightarrow N(C) = 1.$$

$$x - z = 0$$

$$y - 2z = 0$$

$$z = k,$$

$$x = k$$

$$y = 2k$$

$$\Rightarrow v = k(1, 2, 1)$$

Here, we got in total 3 L.I vectors which can span \mathbb{R}^3 . So, diagonisable.

$$\beta' = \{(1, 0, 1), (0, 1, 1), (1, 2, 1)\} \text{ spans } \mathbb{R}^3$$

$$\begin{aligned} T(1, 0, 1) &= (1, 0, 1) \\ &= 1v_1 + 0v_2 + 0v_3 \end{aligned}$$

$$T(0, 1, 1) = 0v_1 + 1v_2 + 0v_3$$

$$T(1, 2, 1) = 0v_1 + 0v_2 + 2v_3$$

$$[T]_{\beta'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

T is diagonisable, then $\exists P$ (invertible) such that

$$D = P^{-1}[T]_{\beta'}P$$

where D is diagonisable matrix

$$D = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}.$$

→ Here, we
keep that
in mind
that given

Q- Let T be, $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$T(x, y) = (7x + 3y, 3x - y)$$

$$T(1, 0) = (7, -3)$$

$$T(0, 1) = (3, 1)$$

$$\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$$

$$(7-\lambda)(-1-\lambda) - 9 = 0$$

$$(7-\lambda)(1+\lambda) + 9 = 0$$

$$-\lambda^2 + 6\lambda + 16 = 0$$

$$\lambda^2 - 6\lambda - 16 = 0$$

$$\lambda^2 - 8\lambda + 2\lambda - 16 = 0$$

$$\lambda(\lambda - 8) + 2(\lambda - 8) = 0$$

$$(\lambda + 2)(\lambda - 8) = 0$$

$$\lambda = -2, 8$$

$$\lambda = -2,$$

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$NC(T) = 1$$

$$9x + 3y = 0 \Rightarrow K(1, -3)$$

$$\lambda = 3$$

$$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x + 3y = 0$$

$$3y = x$$

$$y = \frac{1}{3}, \Rightarrow x = \frac{1}{3}$$

$$\in \left\{ 1, \frac{1}{3} \right\}$$

$$P = \begin{bmatrix} 1 & 1 \\ \frac{1}{3} & -3 \end{bmatrix}$$

So, $\left\{ (1, -3), (1, \frac{1}{3}) \right\}$ spans \mathbb{R}^2 .

Hence, T is diagonalizable.

$$D = P^{-1} [+]_B P$$

$$\Rightarrow \begin{bmatrix} 3 & -1 \\ -\frac{1}{3} & -1 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{3} & -3 \end{bmatrix} / \det(P)$$

$$\Rightarrow \begin{bmatrix} 18 & 10 \\ -16/3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{3} & -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 18 & 10 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Q1) Show that the L.T., $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, that reflects each point across the line $y = kx$, where $k > 0$, is diagonalisable, find its diagonal representation.

Q2) Let T be the L.T., $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $T(x,y,z)$

$$= (4x+y-z, 2x+5y-2z, x+y+2z)$$

check whether the L.T. is diagonalisable or not
and if diagonalisable find the matrix P , s.t

$$D = P^{-1}[T]_B P$$

Remarks: Let T be a linear operator on finite dimensional vector space V . Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of T and let $W_i = N(T - \lambda_i I)$ (eigen space corresponding to eigen value λ_i)

Then the following are equivalent,

(i) T is diagonalizable

(ii) The characteristic polynomial of T is,

$$f = (x - \lambda_1)^{d_1} * (x - \lambda_2)^{d_2} \dots (x - \lambda_k)^{d_k}, \text{ where}$$

(d_i = multiplicity of eigen value λ_i)

$$\text{(iii)} \dim W_1 + \dim W_2 + \dots + \dim W_k = \dim V.$$

① \Rightarrow ②

T is diagonalizable \Rightarrow \exists basis in each element is characteristic vector

Let $\dim V = n$

$$B = \{v_1, v_2, \dots, v_n\}$$

$$Tv_i = \lambda_i v_i$$

Suppose λ_i is occurring d_i times.

$$[T]_{\beta} = \begin{bmatrix} \lambda_1 I_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 I_2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_k I_k & \end{bmatrix}$$

I_i = identity matrix of order $d_i \times d_i$.

$$f = (x - \lambda_1)^{d_1} (x - \lambda_2)^{d_2} \cdots (x - \lambda_k)^{d_k}$$

Defn: A polynomial $p(t)$ is said to be annihilating polynomial of the operator $T: V \rightarrow V$, if

$$p(T) = 0 \Rightarrow p(T)v = 0 \quad \forall v \in V.$$

Minimal polynomial: A polynomial, $p(t)$ is said to be minimal polynomial, if it satisfies the following three conditions.

(i) $p(T) = 0$

(ii) $p(t)$ is a monic polynomial (monic polynomial means the highest power term's coefficient should be 1).

(iii) no polynomial whose degree is less than the degree of $p(t)$ can annihilate T .

If $\deg. q(t) \leq \deg. p(t)$

then, $q(T) \neq 0$

→ Q. Show that if V is finite dimensional vector space and $T: V \rightarrow V$, then there exist annihilating polynomial for T .

Sol: Say, $\dim V = n$

Take space $L(V, V)$

We know that $\dim(L(V, V)) = n^2$.

Now, we take set $\{I, T, T^2, \dots, T^{n^2}, T^{n^2+1}\} = S$

As, S has n^2+1 elements, $\Rightarrow S$ is L.D

$\exists c_0, c_1, c_2, \dots, c_{n^2}$ not all 0, such that

$$c_0 I + c_1 T + c_2 T^2 + \dots + c_{n^2} T^{n^2} = 0$$

$$p(T) = c_0 I + c_1 T + c_2 T^2 + \dots + c_{n^2} T^{n^2}$$

then, $p(T)$ is a annihilating polynomial of T .

Q- Let T' be a linear operator on finite dimensional space V , $T: V \rightarrow V$ and $\dim(V) = n$. Then show that the characteristic and minimal polynomial of T will have same roots, except for multiplicity.

Sol: Let λ be root of characteristic eqn of T . i.e λ is an eigen value.

Let $p(T)$ be minimal polynomial of T

We show $p(\lambda) = 0$

As λ is eigen value, \exists a non zero vector v , such that $Tv = \lambda v \Rightarrow p(T)v = p(\lambda)v$.

\Rightarrow As $p(T)$ is a minimal polynomial,

$$p(T) = 0 \Rightarrow p(\lambda)v = 0$$

$$\Rightarrow p(\lambda) = 0 \quad (\because v \neq 0)$$

Hence, $\boxed{p(\lambda) = 0}$

Converse, suppose C is a root of $p(t)$ (minimal polynomial)

$$\text{i.e. } p(c) = 0$$

we can write,

$$p(t) = (t - c) q(t)$$

$$\text{As, deg. } q(t) \leq \text{deg. } p(t)$$

$$\text{So, } q(t) \neq 0$$

$$\text{i.e. } \exists \beta \in V$$

$$\text{s.t. } q(T)\beta \neq 0$$

$$\text{Take, } v = q(T)\beta$$

we have,

$$P(T)\beta = 0 \quad (\because P(t) \text{ is minimal polynomial})$$

$$\Rightarrow (T - CI) \underbrace{q(t)}_{v \neq 0} \beta = 0$$

$$\text{So, } (T - CI) = 0 \rightarrow \exists v \neq 0 \text{ s.t. } T^v = C^v \\ \Rightarrow C \text{ is a eigen value.}$$

Remark:

The Cayley-Hamilton theorem, says

Let T be a linear operator on a finite dimensional vector space V , $T: V \rightarrow V$. and if $f(t)$ is a characteristic polynomial of T , then $f(T) = 0$. Also, minimal polynomial divides characteristic polynomial.

We say that, poly. g divides poly. f, if
 $f = hg$, where poly. h has degree < degree(f).

Q. Find characteristic and minimal polynomial
 of a linear operator $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and
 $T(x, y, z) = (2x+3y+z, 2x+7y+2z, -5x-15y-4z)$

$$T(1, 0, 0) = (2, 2, -5)$$

$$T(0, 1, 0) = (3, 7, -15)$$

$$T(0, 0, 1) = (1, 2, -4)$$

$$\begin{bmatrix} 2-\lambda & 2 & -5 \\ 3 & 7-\lambda & -15 \\ 1 & 2 & -4-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(7-\lambda)(-4-\lambda) + 30] - 2[3(-4-\lambda) + 15] - 5[6 - 7 + \lambda] = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 2 & 7 & 2 \\ -5 & -15 & -4 \end{bmatrix} = 0$$

$$\begin{vmatrix} 2-\lambda & 3 & 1 \\ 2 & 7-\lambda & 2 \\ -5 & -15 & -4-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(7-\lambda)(-4-\lambda) + 30] - 3[-8 - 2\lambda + 10] + 1[-30 + 35 - 5\lambda] = 0$$

$$(2-\lambda)[\lambda^2 - 3\lambda + 2] + 6[\lambda - 1] - 5[\lambda - 1] = 0$$

$$(2-\lambda)(\lambda-2)(\lambda-1) + (\lambda-1) = 0$$

$$\Rightarrow (\lambda-1)[1 + (\lambda-2)^2] = 0$$

$$(\lambda-2)^2 = 1$$

$$\begin{array}{c|c|c} \lambda-2 = 1 & \lambda-2 = -1 & \lambda-1 = 0 \\ \lambda = 3 & \lambda = 1 & \lambda = 1 \end{array}$$

$$\Rightarrow (\lambda-1)^2(\lambda-3) = 0$$

$$f(t) = (t-1)^2(t-3)$$

So, $P(t)$ can be,

$$\left. \begin{array}{l} P_1(t) = (t-1)^2(t-3) \\ \text{(or)} \\ P_2(t) = (t-1)(t-3) \end{array} \right\} \text{because roots should be same}$$

Now,

$$(T-I)(T-3I) = 0$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 2 \\ -5 & -15 & -5 \end{bmatrix} \begin{bmatrix} -1 & 3 & 1 \\ 2 & 4 & 2 \\ -5 & -15 & -7 \end{bmatrix}$$

$$= 0$$

Here, $P_2(t)$ chosen, because degree ≤ 3 .

Q - Find the minimal polynomial of a matrix

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} (4-\lambda) & 1 & -1 \\ 2 & (5-\lambda)-2 & \\ 1 & 1 & (2-\lambda) \end{bmatrix} = 0$$

$$(4-\lambda)[(5-\lambda)(2-\lambda)+2] - 1[2(2-\lambda)+2] - 1[2-5+\lambda] = 0$$

$$\Rightarrow (4-\lambda)[\lambda^2 - 7\lambda + 12] - 1[-2\lambda + 6] - 1[\lambda - 3] = 0$$

$$\Rightarrow (4-\lambda)(\lambda-4)(\lambda-3) + 2(\lambda-3) - 1(\lambda-3) = 0$$

$$\Rightarrow -(\lambda-3)(\lambda-4)^2 + (\lambda-3) = 0$$

$$\Rightarrow (\lambda-3)[1 - (\lambda-4)^2] = 0$$

$$\Rightarrow (\lambda-3)[\lambda^2 + 16 - 8\lambda - 1] = 0$$

$$\Rightarrow (\lambda-3)[\lambda^2 - 8\lambda + 15] = 0$$

$$\Rightarrow (\lambda-3)^2(\lambda-5) = 0$$

→ minimal polynomial can be, $(t-3)^2(t-5)$.

$$(t-3)^2(t-5) \text{ or } (t-3)(t-5).$$

Q6 Q9 19

Suppose matrix is block triangular matrix.

$$M = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = M$$

where A_1 and A_2 are square matrix.

Then characteristic polynomial of M is

product of characteristic polynomials of the diagonal block.

$$(e+e^2+e^3+e^4)(e+e^2+e^3) =$$

Ex: Find the characteristic polynomial of

$$\begin{aligned}
 M &= \left[\begin{array}{cc} \begin{bmatrix} 9 & -1 \\ 8 & 3 \end{bmatrix} & \begin{bmatrix} 5 & 7 \\ 2 & -4 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 3 & 6 \\ 1 & 8 \end{bmatrix} \end{array} \right] \\
 &= [(9-\lambda)(3-\lambda)+8] [(3-\lambda)(8-\lambda)-6] \\
 &= [(\lambda^2-12\lambda+35)(\lambda^2-11\lambda+18)]
 \end{aligned}$$

Here, M is a block triangular form

$$\begin{aligned}
 \det(M-\lambda I) &= \det(A_1-\lambda I) \cdot \det(A_2-\lambda I) \\
 &= (\lambda^2-12\lambda+35)(\lambda^2-11\lambda+18) \\
 &= (\lambda-5)(\lambda-7)(\lambda-9)(\lambda-2)
 \end{aligned}$$

② Find, for

$$M = \begin{bmatrix} 2 & 5 & 11 \\ 1 & 4 & 2 \\ 0 & 0 & 6-5 \\ 0 & 0 & 2 \\ 3 \end{bmatrix} = M$$

$$\begin{aligned}
 \text{ch. poly} &= [(2-\lambda)(4-\lambda)-5][(6-\lambda)(3-\lambda)+10] \\
 &= [\lambda^2-6\lambda+3][\lambda^2-9\lambda+28]
 \end{aligned}$$

Remark:

Suppose the matrix M is a block diagonal matrix, with diagonal blocks A_1, A_2, \dots, A_r , then minimal polynomial of M is equal to least common multiples (LCM) of the minimal polynomials of the diagonal blocks.

Ex Find characteristic polynomial and minimal polynomial of

$$\begin{bmatrix} \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix} & 0 & 0 & 0 \\ 0 & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & 0 & 0 \\ 0 & 0 & \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix} & 0 \\ 0 & 0 & 0 & \begin{bmatrix} 7 \end{bmatrix} \end{bmatrix}$$

C-Poly,

$$= ((2-\lambda)^2)((4-\lambda)(5-\lambda)-6)(7-\lambda)$$

$$= (\lambda-2)^2 [\lambda^2 - 9\lambda + 14] [(\lambda-7)]$$

$$= (\lambda-2)^2 [(\lambda-7)] [\lambda-2] [\lambda-7]$$

$$= -(\lambda-2)^3 (\lambda-7)^2$$

Minimal polynomial,

$$A_1 = (2-\lambda)^2$$

$$A_2 = (\lambda-2)(\lambda-7)$$

$$A_3 = (7-\lambda)$$

$$\text{So, L.C.M} = \underline{(\lambda-2)(\lambda-7)}^2 \text{ is the minimal polynomial}$$

of M .

Q-Find characteristic and minimal polynomial of

$$\left\{ \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

Q>Show that if T is a linear transformation from $V \rightarrow V$, V is finite dimensional, Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigen values of T and T is diagonalizable show that the minimal polynomial of T will be of the form

$$P = (t - \lambda_1)(t - \lambda_2) \dots (t - \lambda_k)$$

Inner Product on vector space :-

Let V be a vector space over the field \mathbb{F} (\mathbb{R} or \mathbb{C}).

Inner product on V is function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$ satisfying following conditions

① $\langle v, v \rangle \geq 0 \quad \forall v \in V$

& $\langle v, v \rangle = 0 \text{ iff } v = 0$

② $\langle v, w \rangle = \overline{\langle w, v \rangle} \quad \forall v, w \in V$

③ $\langle cv, w \rangle = c \langle v, w \rangle$

$\forall v, w \in V$

and $c \in \mathbb{F}$

$$④ \langle v+u, w \rangle = \langle u, w \rangle + \langle v, w \rangle \quad \forall v, u, w \in V$$

$\rightarrow V$ together with inner product i.e $(V \langle \cdot, \cdot \rangle)$ is called as inner product space.

Ex: 1) Show that inner product of $\langle v, cw \rangle$

$$= \bar{c} \langle v, w \rangle \quad \forall v, w \in V \text{ and } c \in F$$

$$\begin{aligned} & \langle v, cw \rangle - V \text{ form } \Rightarrow \text{left side} \\ & = \langle \overline{cw}, v \rangle \quad \text{by } \langle \overline{vw}, u \rangle = \overline{\langle v, w \rangle} \\ & = \langle \overline{c} \bar{w}, \bar{v} \rangle \quad (\text{if } c \neq 0, (c, u, v) = u \text{ for } (u, v)) \\ & = \bar{c} \langle \bar{w}, \bar{v} \rangle \quad \langle v, w \rangle \text{ brif. } \bar{v} = v \\ & = \bar{c} \langle w, v \rangle \quad (\bar{c} + \bar{d} - \bar{d} + \bar{c} = c) = \langle v, w \rangle \\ & = \bar{c} \langle v, w \rangle \end{aligned}$$

2) $V = \mathbb{R}^n$ and $F = \mathbb{R}$

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow F$$

$$x = (x_1, x_2, \dots, x_n)$$

$$y = (y_1, y_2, \dots, y_n)$$

we defined it n

$$\langle x, y \rangle := \sum_{i=1}^n x_i y_i = x^T y$$

Show that the $\langle \cdot, \cdot \rangle$ defines inner product on \mathbb{R}^n .

$$\text{Sol: 1)} \langle x, x \rangle = \sum_{i=1}^n x_i^2 \geq 0 \Rightarrow x_i = 0 \quad \forall i \Rightarrow \bar{x} = \bar{0}$$

$$\begin{aligned} 2) \quad \langle x, y \rangle &= \sum_{i=1}^n x_i y_i \quad \Rightarrow \langle x, y \rangle \\ \langle y, x \rangle &= \sum_{i=1}^n y_i x_i \quad = \langle y, x \rangle \\ &\quad (\text{real field}) \end{aligned}$$

$$3) \langle c\alpha, y \rangle = \sum_{i=1}^n c\alpha_i y_i \\ = c \langle \alpha, y \rangle$$

$$4) \langle x+y, y \rangle = \sum x_i y_i + \sum y_i^2 \\ = \langle x, y \rangle + \langle y, y \rangle$$

3) Try above for $F = \mathbb{C}$ and $V = \mathbb{R}^n$

$$\langle \alpha, y \rangle := \sum_{i=1}^n \alpha_i \bar{y}_i$$

Ex 4) Let $u = (1, 3, -4, 2)$, $v = (4, -2, 2, 1)$

$$v = \mathbb{R}^4, \text{ find } \langle u, v \rangle$$

$$\langle u, v \rangle = (4 + -6 + -8 + 2)$$

$$\therefore \langle u, v \rangle = -8$$

$$w = (5, -1, -2, 6)$$

$$\langle 3u - 2v, w \rangle$$

$$(-5, 13, -16, 4)$$

$$= -25 - 13 + 32 + 24$$

$$= 18$$

30/11/19

Inner product space:-

Ex: $C[a, b]$ $F = \mathbb{R}$

$$(f+g)x = f(x) + g(x)$$

$$(cf)(x) = c \cdot f(x)$$

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx$$

if $F = \int_a^b f(x) dx$

$\langle f, g \rangle = \int_a^b f(x)g(x) dx$

$$\textcircled{1} \quad \langle f \cdot f \rangle = \int_a^b f(x)^2 dx \geq 0$$

$$\langle f \cdot f \rangle = 0 \Leftrightarrow \int_a^b f(x)^2 dx = 0$$

$$\Leftrightarrow f(x) = 0 \forall x$$

$$\Leftrightarrow f = 0$$

$$\textcircled{2} \quad \langle f \cdot g \rangle = \int_a^b f(x)g(x) dx$$

$$= \int_a^b g(x)f(x) dx$$

$$= \langle g, f \rangle$$

$$\textcircled{3} \quad \langle cf \cdot g \rangle = \int_a^b c \cdot f(x)g(x) dx$$

$$= c \langle f \cdot g \rangle$$

$$\textcircled{4} \quad \langle f+g, h \rangle = \int_a^b (f(x)+g(x))h(x) dx$$

$$= \int_a^b f(x)h(x) dx + \int_a^b g(x)h(x) dx$$

$$= \langle f, h \rangle + \langle g, h \rangle$$

Let, $f(t) = 3t-5$ and $g(t) = t^2$ be the elements of polynomial space $P(t)$.

Find the following,

$P[0, 1]$

$$\textcircled{1} \quad \langle f \cdot g \rangle$$

$$\textcircled{2} \quad \langle f \cdot f \rangle$$

$$\textcircled{3} \quad \langle g \cdot g \rangle$$

$$\textcircled{1} \quad \int_0^1 (3t-5)t^2 dt = \int_0^1 (3t^3 - 5t^2) dt = \frac{3}{4} - \frac{5}{3} = \frac{-11}{12}$$

$$\textcircled{2} \quad \int_0^1 (3t-5)^2 dt = \frac{(3t-5)^3}{(3)3} = \frac{-27+125}{9} = \frac{98}{9} = +\frac{117}{9}$$

$$\textcircled{3} \quad \int_0^1 t^4 dt = \frac{1}{5}$$

$$\text{Ex} \quad x = M_{m \times n}$$

$$\langle A, B \rangle = \text{trace}(B^T A)$$

Verify defines inner product on $M_{m \times n}$.

V -vector space over field \mathbb{F} , Norm of V is function

$\|\cdot\| : V \rightarrow \mathbb{R}$ satisfying following properties

$$1) \quad \|v\| \geq 0 \quad \forall v \in V \quad \text{if} \quad \|v\| = 0 \text{ iff } v = 0$$

$$2) \quad \|cv\| = |c| \|v\| \quad \forall v \in V \text{ and } c \in \mathbb{F}$$

$$3) \quad \|v + w\| \leq \|v\| + \|w\|$$

$$\text{Ex: } 1) \quad V = \mathbb{R} ; \mathbb{F} = \mathbb{R}$$

$$\|\cdot\| : V \rightarrow \mathbb{R}$$

$$\|v\| = |v|$$

$\Rightarrow V = \mathbb{R}^n, F = \mathbb{R}$

i. $\| \cdot \| : V \rightarrow \mathbb{R}$

$$\| v \|_1 = \underbrace{|x_1| + |x_2| + |x_3| + \dots + |x_n|}_{1\text{-norm}}$$

Verify is this a norm?

i) $\| v \| \geq 0$ (True)

ii) $\| cv \| = c|x_1| + c|x_2| + \dots + c|x_n|$

$$= c(|x_1| + |x_2| + \dots + |x_n|) \quad (\text{True})$$

iii) $\| x+y \| = \cancel{|x_1+y_1|}$

$$|x_1+y_1| + |x_2+y_2| + \dots + |x_n+y_n|$$

$$\leq |x_1| + |y_1| + |x_2| + |y_2| + \dots + |x_n| + |y_n|$$

$$\leq (|x_1| + |x_2| + \dots + |x_n|) + (|y_1| + |y_2| + \dots + |y_n|)$$

$$\leq \| x \| + \| y \| \quad (\text{True})$$

Hence, it is a Norm.

iv) $V: v$

$$\| \cdot \|_\infty = \max \{ |x_1|, |x_2|, \dots, |x_n| \}$$

∞ norm

Verify $\| \cdot \|_\infty$ defines norm on \mathbb{R}^n .

Sol:- i) $\| \cdot \|_\infty \geq 0$ (v)

ii) $\| cv \|_\infty = |c| \| v \|_\infty$ (v)

iii) $|x_i+y_i| \leq |x_i| + |y_i| \quad \forall i$

$$\max(|x_i| + |y_i|) \leq \max(|x_i| + |y_i|)$$

$$\Rightarrow \| x+y \|_\infty \leq \| x \|_\infty + \| y \|_\infty.$$

3) $X : \mathbb{R}^n$

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \dots + |x_n|^p \right)^{\frac{1}{p}} \quad p=1, 2, \dots$$

Ex: $x \in \mathbb{R}^5$ $\rightarrow x = (1, 1, -2, 3, -1)$

$$\|x\|_1 = 1 + 1 + 2 + 3 + 1 = 8$$

$$\|x\|_\infty = 3$$

$$\|x\|_2 = \sqrt{1+1+4+9+1} = 4$$

Note: $y = \frac{x}{\|x\|}$ \rightarrow can be any norm.

then, $\|y\|_1 = 1$
same norm as x.

V - inner product space

$$\|v\|_1 = \langle v, v \rangle^{\frac{1}{2}}$$

Schwartz inequality :- Let V be inner product space and $v, w \in V$ then

$$|\langle v, w \rangle| \leq \|v\| \cdot \|w\|$$

$$\text{where, } \|v\| = \langle v, v \rangle^{\frac{1}{2}}$$

$$\|w\| = \langle w, w \rangle^{\frac{1}{2}}$$

$v, w \in V$ and $c \in \mathbb{F}$

Proof $\|v - cw\|^2 = \langle v - cw, v - cw \rangle$

$$= \langle v, v \rangle - \langle v, cw \rangle - \langle cw, v \rangle + \langle cw, cw \rangle$$
$$= \|v\|^2 - \langle v, cw \rangle - \overline{\langle v, cw \rangle} + c\bar{c} \|w\|^2$$
$$\leq \|v - cw\|^2 = \|v\|^2 - 2 \operatorname{Re} \langle v, cw \rangle + |c|^2 \|w\|^2 \quad \textcircled{1}$$

Take, $c = \frac{\langle v, w \rangle}{\|w\|^2}$

$$\langle v, cw \rangle = \langle v, \frac{\langle v, w \rangle}{\|w\|^2} w \rangle$$

$$= \frac{\langle v, w \rangle}{\|w\|^2} \langle v, w \rangle$$

$$= \frac{|\langle v, w \rangle|^2}{\|w\|^2} \quad \left| \begin{array}{l} \operatorname{Re}(\langle v, w \rangle) \geq 0 \\ \Rightarrow |\langle v, w \rangle|^2 = \frac{|\langle v, w \rangle|^2}{\|w\|^2} \end{array} \right.$$

Now, $c^2 = \frac{|\langle v, w \rangle|^2}{\|w\|^4}$

Now in $\textcircled{1}$,

$$0 \leq \|v\|^2 - 2 \frac{|\langle v, w \rangle|^2}{\|w\|^2} + \frac{|\langle v, w \rangle|^2}{\|w\|^4} \|w\|^2$$

$$0 \leq \|v\|^2 - \frac{|\langle v, w \rangle|^2}{\|w\|^2}$$

$$\Rightarrow |\langle v, w \rangle| \leq \|v\| \cdot \|w\|.$$

Remark: If v and w are linearly dependent, then equality occurs in a Schwartz inequality.

Q- $\|v\| = \sqrt{\langle v, v \rangle}$ defines norm on inner product space V . This norm is called as "Norm induced by inner product."

Properties

$$1) \langle v, v \rangle \geq 0 \Rightarrow \sqrt{\langle v, v \rangle} \geq 0 \Rightarrow \|v\| \geq 0$$

Now if,

$$\|v\| = 0 \rightarrow \langle v, v \rangle = 0 \Leftrightarrow v = 0$$

$$2) \|cv\|^2 = \langle cv, cv \rangle = c\bar{c} \langle v, v \rangle$$

$$= c^2 \langle v, v \rangle$$

$$= c^2 \|v\|^2$$

$$\Rightarrow \|cv\| = c \|v\|$$

$$3) \|v+w\|^2 = \langle v+w, v+w \rangle$$

$$= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle$$

$$= \|v\|^2 + 2 \operatorname{Re} \langle v, w \rangle + \|w\|^2$$

$$(\text{By Schwartz Ineq}) \leq \|v\|^2 + 2 \|v\| \|w\| + \|w\|^2$$

$$\Rightarrow \|v+w\| \leq \|v\| + \|w\|$$

$$\|x\| = \sqrt{\langle x, x \rangle} = \left(\sum_{i=0}^n x_i^2 \right)^{1/2}$$

↑
Norm induced by inner product (ie 2 norm)

→ Suppose v, w are two vectors in a inner product space V and θ is the angle b/w v and w . S.t. by Cauchy-Schwarz inequality

$$\cos \theta = \frac{|\langle v, w \rangle|}{\|v\| \cdot \|w\|}$$

Ex find the angle b/w the vectors, $v = (2, 3, 5)$ and $w = (1, -4, 3)$ in \mathbb{R}^3 .

$$\cos \theta = \frac{|\langle v, w \rangle|}{\|v\| \cdot \|w\|}$$

$$|\langle v, w \rangle| = |2 + (-12) + (15)| = |3 + 2| = 5$$

$$\|v\|^2 = \langle v, v \rangle = 4 + 9 + 25 = 38$$

$$\|w\|^2 = \langle w, w \rangle = 1 + 16 + 9 = 26$$

$$\Rightarrow \cos \theta = \frac{5}{\sqrt{38} \cdot \sqrt{26}}$$

$$\Rightarrow \theta = \cos^{-1} \left[\frac{5}{\sqrt{38} \cdot \sqrt{26}} \right]$$

Q Let $f(t)$ be $3t - 5$ and $g(t) = t^2$ and on $P(t)$ where $\langle f, g \rangle = \int_0^1 f(t)g(t) dt$.

find a b/w f and g.

$$\cos \theta = \frac{|\langle f, g \rangle|}{\|f\| \|g\|} = \frac{\frac{11}{12} \sqrt{5}}{\sqrt{13}} = \frac{\frac{11}{12} \sqrt{5}}{\sqrt{13}}$$

$$\text{So, } \theta = \cos^{-1} \left[\frac{\frac{11}{12} \sqrt{5}}{\sqrt{13}} \right]$$

3) Find the angle θ b/w the matrix

$$A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$B^T A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}_{2 \times 3}$$

$$= \begin{bmatrix} 33 & 28 & 23 \\ 48 & 41 & 34 \\ 63 & 54 & 45 \end{bmatrix}$$

$$\langle A, B \rangle = 119$$

$$\langle A, A \rangle =$$

$$\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}^T \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix} = 119$$

→ Let V be inner product space we say $v, w \in V$ are orthogonal vectors if $\langle v, w \rangle = 0$.

$$\langle 0, v \rangle = 0 \quad \forall v \in V$$

so 0 is orthogonal to each element of V .

→ Let $S \subseteq V$ (inner product space = ips)

$$S = \{v_1, v_2, \dots, v_n\}$$

we say S is orthogonal subset of V if

$$\langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ c \neq 0 & \text{if } i=j \end{cases}$$

→ A subset $S = \{v_1, v_2, \dots, v_n\}$ is said to be orthonormal if

$$\langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i=j \end{cases}$$

→ i) Show that the vector $u = (1, 1, 1)$ and $v = (1, 2, -3)$ are orthogonal to each other.

$$\text{Sol: } \langle u, v \rangle = (1)(1) + (1)(2) + (1)(-3) \\ = 0$$

Hence, u, v are orthogonal to each other.

2) $C[-\pi, \pi]$, Show that the $f(t) = \sin t$ and $g(t) = \cos t$ are orthogonal.

$$\text{Sol: } \langle f, g \rangle = \int_{-\pi}^{\pi} \sin t \cos t dt \\ = \frac{1}{2} \left[\frac{-\cos 2t}{2} \right]_{-\pi}^{\pi} = -\frac{1}{4} [1 - 1] = 0$$

3. Find a non-zero vector which is orthogonal to $u(1,2,1)$ and $w(2,5,4)$ in \mathbb{R}^3 .

Sol:- Let nonzero vector be, $v = (x, y, z)$

Now,

$$\langle u \cdot v \rangle = 0 \Rightarrow x + 2y + z = 0$$

$$\langle w \cdot v \rangle = 0 \Rightarrow 2x + 5y + 4z = 0$$

$$\Rightarrow y + 2z = 0$$

$$\text{take, } z = 1$$

$$\Rightarrow y = -2$$

$$\Rightarrow x = 3$$

so, $v = (3, -2, 1)$ is orthogonal to u, w .

4. Find the value of k such that the vector

$(1, 2, k, 3)u$ and $v(3, k, 7, -5)$ are orthogonal

Sol:- $\langle u \cdot v \rangle = 0$

$$\Rightarrow 3 + 2k + 7k - 15 = 0$$

$$\Rightarrow 9k = 12$$

$$\Rightarrow k = 4/3$$

5. Show that orthogonal subset of an inner product space containing non zero element is linearly independent.

Sol:- Let V be the inner product space.

$S = \{v_1, v_2, \dots, v_n\}$ be orthogonal subset of

and $v_i \neq 0$ $i = 1, 2, \dots, n$

By taking linear combination,

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

$$\Rightarrow \langle c_1 v_1 + c_2 v_2 + \dots + c_n v_n, v_i \rangle = 0$$

$$\Rightarrow c_1 \langle v_1, v_i \rangle + c_2 \langle v_2, v_i \rangle + \dots + c_n \langle v_n, v_i \rangle = 0$$

$$\Rightarrow c_1 \langle v_1, v_i \rangle + c_2 \langle v_2, v_i \rangle + \dots + c_i \langle v_i, v_i \rangle + \dots + c_n \langle v_n, v_i \rangle = 0$$

$$\Rightarrow c_i \|v_i\|^2 = 0$$

$$\Rightarrow c_i = 0, \forall i$$

so, they are linearly independent.

Similar proof, would work for the orthonormal subset.

Remark: Orthonormal subset of inner product space containing non zero elements is L.I.

Definition:

Orthogonal subset of inner product space V , which spans V is called as the "orthogonal basis of V ".

Similarly, orthonormal set which spans inner product space V is called as the orthonormal basis of V .

$$\text{Ex: } \{(1,0,0), (0,1,0), (0,0,1)\}$$

Here, this is orthogonal and orthonormal basis of \mathbb{R}^3 .

$\{(1,1,0), (0,1,1), (1,1,1)\}$ is neither orthogonal nor orthonormal basis.

→

$$V = \mathbb{R}^n$$

$$S \subseteq V$$

$$S^\perp = \{ v \in V \mid \langle v, u \rangle = 0 \ \forall u \in S \}$$

Now,

$$V = \mathbb{R}^2$$

$$S = \{ (x, 0) \mid x \in \mathbb{R} \}$$

$$S^\perp = \{ (0, y) \mid y \in \mathbb{R} \}$$

>Show that S^\perp is a subspace of V .

Sol:

$$\langle 0, u \rangle = 0 \ \forall u \in S$$

$$\Rightarrow 0 \in S^\perp$$

Let $v, w \in S^\perp$

$$\langle c_1 v + c_2 w, u \rangle \quad u \in S$$

$$= \langle c_1 v, u \rangle + \langle c_2 w, u \rangle$$

$$= c_1 \langle v, u \rangle + c_2 \langle w, u \rangle$$

$$= c_1 (0) + c_2 (0)$$

$$= 0$$

$$\Rightarrow c_1 v + c_2 w \in S^\perp$$

Hence, S^\perp is a subspace of V .

Note: S^\perp is also called as orthogonal complement of S .

Q-Let W be a subspace of \mathbb{R}^5 spanned by
 $u = (1, 2, 3, 1, -2)$ and $v = (2, 4, 7, 2, -1)$. Find
orthogonal complement of W .

Sol: $W^\perp = \{ w \in \mathbb{R}^5 \mid \langle w, u \rangle = 0 \text{ and } \langle w, v \rangle = 0 \}$

We find $w(x, y, z, s, t) \in \mathbb{R}^5$ such that

$$\langle w, u \rangle = 0$$

$$\langle w, v \rangle = 0$$

$$\Rightarrow x + 2y + 3z + s - 2t = 0$$

$$\Rightarrow 2x + 4y + 7z + 2s - t = 0$$

~~$$z + 1/4(4s - 5t) = 0$$~~

~~$$z + 3t = 0$$~~

Take y, s, t , $y = 1, s = 0, t = 0$

$$\Rightarrow z = 0$$

$$\Rightarrow x = -2$$

$$\text{So, } w = (-2, 1, 0, 0, 0)$$

If $y = 0, s = 1, t = 0$

$$\Rightarrow z = 0$$

$$\Rightarrow x = -1$$

$$w = (-1, 0, 0, 1, 0)$$

If $y = 0, s = 0, t = 1$

$$\Rightarrow z = -3$$

$$\Rightarrow x = 11$$

$$w = (11, 0, -3, 0, 1)$$

So,

$$W^\perp = \{ (-2, 1, 0, 0, 0), (-1, 0, 0, 1, 0), (11, 0, -3, 0, 1) \}$$

Forms basis of W^\perp .

$$\dim(\mathbb{R}^5) = \dim(W) + \dim(W^\perp)$$

Let $v = (1, 2, 3, 1) \in \mathbb{R}^4$ find orthogonal basis of V^\perp .

Sol:- Let $w = (x, y, z, t)$

$$\Rightarrow x + 2y + 3z + t = 0$$

$$x=1, y=0, z=0, t=-1$$

$$x=0, y=1, z=0, t=-2$$

$$x=0, y=0, z=1, t=-3$$

$$\text{So, } V^\perp = \{(1, 0, 0, -1), (0, 1, 0, -2), (0, 0, 1, -3)\}$$

is the subspace of V .

$$\begin{array}{l} \text{Now, } (1, 0, 0, -1) \\ \quad (x, y, z, t) \end{array} \left[\begin{array}{l} \text{Orthogonal} \end{array} \right]$$

$$x-t=0$$

$$\therefore \text{and } x+2y+3z+t=0$$

$$\Rightarrow y=1$$

$$z=0$$

$$\Rightarrow x=-1$$

$$t=-1$$

$$(-1, 1, 0, -1)$$

$$y=\emptyset$$

$$z=1$$

$$x=-\frac{1}{2}$$

$$t=-\frac{3}{2}$$

$$(-\frac{1}{2}, 0, 1, -\frac{3}{2})$$

(x, y, z, t) such that

$$x-t=0$$

$$x+y+t=0$$

$$\text{and } x+2y+3z+t=0$$

$$\Rightarrow x = t$$

$$y = 2t$$

$$\Rightarrow t + 4t + 3z + t = 0$$

$$\Rightarrow z = -2t$$

$$t = 1$$

$$\Rightarrow x = 1$$

$$y = 2$$

$$z = -2$$

$$\text{so, } (1, 2, -2, 1)$$

so, the elements of orthogonal basis are

$$\{(-1, 0, 0, -1), (-1, 1, 0, -1), \underline{(1, 2, -2, 1)}\}$$

soln

Let x, y be orthogonal vectors in inner product space V , then show that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

(Pythagoras Theorem)

$$\text{Sol:- } \|x+y\|^2 = \langle x+y, x+y \rangle$$

$$\begin{aligned} &= \langle x \cdot x \rangle + \underbrace{\langle x \cdot y \rangle}_{=0} + \underbrace{\langle y \cdot x \rangle}_{=0} \\ &\quad + \langle y \cdot y \rangle \end{aligned}$$

since x, y are orthogonal.

$$\therefore \|x+y\|^2 = \|x\|^2 + \|y\|^2$$

Hence, proved

$\rightarrow X$ ips

$$x, y \in X$$

Then verify that,

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

$$\underline{\text{Sol:}} \quad \|x+y\|^2 = \|x\|^2 + \|y\|^2$$

$$\begin{aligned} \|x-y\|^2 &= \langle x-y, x-y \rangle \\ &= \langle x-x \rangle - \underbrace{\langle x, y \rangle}_{=0} - \underbrace{\langle y, x \rangle}_{=0} + \langle y, y \rangle \\ &= \|x\|^2 + \|y\|^2 \end{aligned}$$

$$\Rightarrow \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Hence, proved.

Let V be a subspace and the set consists of v_1, v_2, \dots, v_n

$S = \{v_1, v_2, \dots, v_n\}$ be orthogonal basis of V . Then show that any $x \in V$ can be written

$$\text{as } x = \sum_{i=1}^n \frac{\langle x, v_i \rangle v_i}{\langle v_i, v_i \rangle}$$

$$= \frac{\langle x, v_1 \rangle v_1}{\langle v_1, v_1 \rangle} + \frac{\langle x, v_2 \rangle v_2}{\langle v_2, v_2 \rangle} + \dots + \frac{\langle x, v_n \rangle v_n}{\langle v_n, v_n \rangle}$$

Here, $\frac{\langle x, v_i \rangle}{\langle v_i, v_i \rangle}, i=1, 2, \dots, n$ are called as

Fourier coefficients.

Proof: As S is a basis of V

$x \in V$ can be written as

$$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

$$x = \sum_{i=1}^n c_i v_i \quad \text{--- (1)}$$

$$\langle x, v_j \rangle = \sum_{i=1}^n \langle c_i v_i, v_j \rangle$$

$$\langle x \cdot v_j \rangle = c_j \langle v_j \cdot v_j \rangle$$

$$c_j = \frac{\langle x \cdot v_j \rangle}{\langle v_j \cdot v_j \rangle}$$

so, ① \Rightarrow

$$x = \sum_{i=1}^n \frac{\langle x \cdot v_i \rangle}{\langle v_i \cdot v_i \rangle} v_i \quad \text{hence, proved.}$$

* Suppose, S is orthonormal basis, then just the denominator i.e $\langle v_i \cdot v_i \rangle$ would be 1.

These two expressions i.e for the above two expressions are called Fourier Expansions.

Let $S \{v_1, v_2, \dots, v_n\}$ be orthonormal basis of V

$$\text{Then } \|x\|^2 = \sum_{i=1}^n |\langle x \cdot v_i \rangle|^2$$

$$\text{Proof: } \|x\|^2 = \langle x \cdot x \rangle$$

$$= \left\langle \sum_{i=1}^n \frac{\langle x \cdot v_i \rangle}{\langle v_i \cdot v_i \rangle} v_i \cdot x \right\rangle$$

$$= \left\langle \sum_{i=1}^n \langle x \cdot v_i \rangle v_i \cdot x \right\rangle$$

$$= \sum_{i=1}^n \langle x \cdot v_i \rangle \langle v_i \cdot x \rangle$$

$$= \sum_{i=1}^n \langle x \cdot v_i \rangle \overline{\langle x \cdot v_i \rangle}$$

$$= \sum_{i=1}^n |\langle x \cdot v_i \rangle|^2$$

Let $\omega \in \mathbb{R}^4$ and $S = \left\{ \frac{1}{2}(1,1,1,1), \frac{1}{\sqrt{6}}(-1,-1,0,2), \right.$
 $\left. \frac{1}{5\sqrt{2}}(1,3,-6,2) \right\}$ which is a orthonormal basis
 & ω . Find Fourier coefficient of the element
 $(1,1,2,4)$.

Sol:-

$$\langle (1,1,2,4) \mid \frac{1}{2}(1,1,1,1) \rangle$$

$$c_1 = \frac{\langle (1,1,2,4) \mid \frac{1}{2}(1,1,1,1) \rangle}{\langle \frac{1}{2}(1,1,1,1) \mid \frac{1}{2}(1,1,1,1) \rangle}$$

$$= \frac{\frac{1}{2} + \frac{1}{2} + \frac{5}{2} + \frac{9}{2}}{\langle \frac{1}{2}(1,1,1,1) \mid \frac{1}{2}(1,1,1,1) \rangle}$$

$$\bullet \frac{1}{2}[4]$$

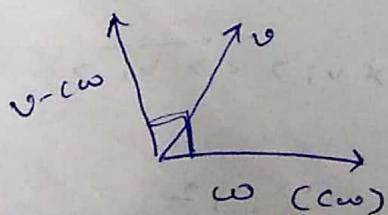
$$= \frac{8}{2} = 4$$

$$c_2 = \sqrt{6}$$

$$c_3 = 0$$

Remark: Let V be ips and w belongs to V then
 projection of v (v is another vector in V)
 along vector w

$$\text{proj}(v, w) = \frac{\langle v, w \rangle}{\langle w, w \rangle} w$$



$$v' = v - cw$$

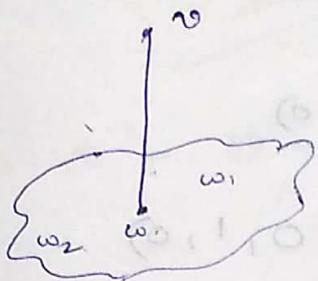
$$\langle v - cw, w \rangle \Rightarrow c = \frac{\langle v, w \rangle}{\langle w, w \rangle}$$

Let v be a vector given as $v = (1, 3, 5, 7)$ and W
 $= \text{span} \{(1, 1, 1, 1), (1, -3, 4, -2)\}$. Find proj. of v
onto W .

$$\text{Sol: } \parallel \text{proj}(v, w_1) = \frac{\langle v, w_1 \rangle}{\langle w, w_1 \rangle} w_1$$

$$= \left(\frac{16}{4} \right) (1, 1, 1, 1)$$

$$= (4, 4, 4, 4) \parallel$$



$\text{Proj}(v, W)$

$$= \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

Gram-Schmidt orthogonalisation :-

Let V be inner product space and u_1, u_2, \dots, u_n are linearly independent vectors in V . Then, there exists vectors v_1, v_2, \dots, v_n in V which are orthogonal s.t,

$$\text{span}\{u_1, u_2, \dots, u_k\} = \text{span}\{v_1, v_2, \dots, v_k\}$$

$$\text{for } k = 1, 2, \dots, n$$

In fact the vectors are defined as

$$v_1 = u_1$$

$$v_{k+1} = u_{k+1} - \sum_{i=1}^k \frac{\langle u_{k+1}, v_i \rangle}{\langle v_i, v_i \rangle} v_i$$

$$k = 1, 2, \dots, n-1$$

Let $V = \mathbb{R}^3$ and $\beta = \{(1,0,0), (1,1,0), (1,1,1)\}$
 using Gram-Schmidt orthogonalisation process
 find orthogonal basis of \mathbb{R}^3 .

Sol:- $v_1 = u_1 = (1,0,0)$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= (1,1,0) - \frac{1}{1} (1,0,0)$$

$$= (1,1,0) - (1,0,0) = (0,1,0)$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$= (1,1,1) - (1,0,0) - (0,1,0)$$

$$= (0,0,1)$$

So, orthogonal basis = $\{(1,0,0), (0,1,0), (0,0,1)\}$

Let V be a vector space of polynomials with inner product $\langle f, g \rangle = \int f(t)g(t) dt$. Apply

Gram-Schmidt orth... process to set $\{1, t, t^2\}$

to get orthogonal basis for space $P_2(t)$.

$$v_1 = u_1 = 1$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= t - \frac{\int_1^t dt}{\int_1^t dt} \quad (1)$$

$$= t - 0 \\ = t$$

$$v_3 = u_3 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$= t^2 - 0 - \frac{\int t^2 \cdot t dt}{\int t^2 dt} (t)$$

$$= t^2 - 0 - 0 = t^2$$

Gram Schmidt Orthogonalisation:-

we construct v_1, v_2, \dots, v_n in V s.t.

$$\text{span}\{u_1, u_2, \dots, u_n\} = \text{span}\{v_1, v_2, \dots, v_n\}$$

$$\text{Let } v_1 = u_1$$

$$\text{we define } u_2 = \alpha v_1 + v_2$$

$$\text{where } \alpha \text{ is such that } \langle v_2, v_1 \rangle = 0$$

$$v_2 = u_2 - \alpha v_1$$

$$v_2 = u_2 - \alpha u_1$$

$$\text{Now, } \langle v_2, v_1 \rangle = 0$$

$$\Rightarrow \langle u_2 - \alpha u_1, v_1 \rangle = 0$$

$$\Rightarrow \langle u_2, v_1 \rangle - \alpha \langle u_1, v_1 \rangle = 0$$

$$\Rightarrow \alpha = \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle}$$

So,

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

Now, write

$$u_3 = (\alpha_1 v_1 + \alpha_2 v_2) + v_3$$

$$v_3 = u_3 - (\alpha_1 v_1 + \alpha_2 v_2)$$

α_1, α_2 are s.t.

$$\langle v_1, v_3 \rangle = 0 \text{ and } \langle v_2, v_3 \rangle = 0$$

$$\Rightarrow \langle v_3, v_1 \rangle = 0$$

$$\langle u_3 - (\alpha_1 v_1 + \alpha_2 v_2), v_1 \rangle = 0$$

$$\langle u_3, v_1 \rangle - \alpha_1 \langle v_1, v_1 \rangle - \alpha_2 \underbrace{\langle v_2, v_1 \rangle}_{=0} = 0$$

$$\alpha_1 = \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle}$$

$$\Rightarrow \langle v_3, v_2 \rangle = 0$$

$$\langle u_3 - (\alpha_1 v_1 + \alpha_2 v_2), v_2 \rangle = 0$$

$$\langle u_3, v_2 \rangle - \alpha_1 \underbrace{\langle v_1, v_2 \rangle}_{=0} - \alpha_2 \langle v_2, v_2 \rangle = 0$$

$$\Rightarrow \langle u_3, v_2 \rangle = \alpha_2 \langle v_2, v_2 \rangle$$

$$\alpha_2 = \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle}$$

so,

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle v_1}{\langle v_1, v_1 \rangle} - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

similarly,

$$\text{Span}\{u_1, v_2, v_3\} = \text{Span}\{u_1, u_2, u_3\}$$

$$v_{k+1} = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k + v_{k+1}$$

where,

$$v_{k+1} = u_{k+1} - (\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k)$$

where, $\alpha_1, \alpha_2, \dots, \alpha_k$ are such that,

$$\langle v_{k+1}, v_i \rangle = 0, \text{ for } i = 1, 2, \dots, k$$

$$\Rightarrow \alpha_i = \frac{\langle u_{k+1}, v_i \rangle}{\langle v_i, v_i \rangle}$$

$$\Rightarrow v_{k+1} = u_{k+1} - \sum_{i=1}^k \frac{\langle u_{k+1}, v_i \rangle}{\langle v_i, v_i \rangle} v_i$$

$$\text{Span}\{u_1, u_2, \dots, u_{k+1}\} = \text{Span}\{v_1, v_2, \dots, v_{k+1}\}$$

Q- Consider the subspace U of \mathbb{R}^4 spanned by the vectors $u_1 = (1, 1, 1, 1)$, $u_2 = (1, 1, 2, 4)$

$$u_3 = (1, 2, -4, -3)$$

Find @ an orthogonal basis of U

② an orthonormal basis of U

Sol:-

$$v_1 = u_1 = (1, 1, 1, 1)$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= (1, 1, 2, 4) - \frac{8}{4} (1, 1, 1, 1)$$

$$= (1, 1, 2, 4) - (2, 2, 2, 2)$$

$$v_2 = (-1, -1, 0, 2)$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$= (1, 2, -4, -3) + \frac{4}{4} (1, 1, 1, 1) + \frac{9}{8} (-1, 0, 1)$$

$$= (1, 2, -4, -3) + (1, 1, 1, 1) + \left(-\frac{5}{3}, \frac{5}{3}, 0\right)$$

$$= \left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)$$

$$S = \{(1, 1, 1, 1), (-1, -1, 0, 2), \left(\frac{1}{2}, \frac{3}{2}, -3, 1\right)\}$$

$$S' = \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$$

$$\|v_1\| = (\langle v_1, v_1 \rangle)^{1/2}$$

$$\|v_1\| = 2$$

$$\|v_2\| = \sqrt{6}$$

$$\|v_3\| = \frac{5}{\sqrt{2}}$$

Q- show that every finite dimensional inner product space has an orthonormal basis.

→ we taken $\{u_1, u_2, \dots, u_n\}$ be finite dim. ips and by Gram... we would produce the orthogonal and orthonormal basis.

Q- Suppose V is a finite dimensional ips and V_0 is subspace of V , then show that \exists a subspace W of V such that $V = V_0 + W$ and $V_0 \perp W$.

Sol:- Case, $V_0 = V$

Take, $W = \{0\}$

then $V = V_0 + W$ & $V_0 \perp W$.

Case, $V_0 \neq V$ i.e. $V_0 \subset V$

Let $\dim V_0 = k < \dim V$

Let $\beta \{v_1, v_2, \dots, v_k\}$ basis of V_0 .

So, β can be extended that as $\{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n\}$ to form basis of V .

We use G-S process to get $\beta = \{w_1, w_2, \dots, w_n\}$ as orthogonal basis of V .

Take, $W = \{w_{k+1}, \dots, w_n\}$

then, $V = V_0 + W$

and $V_0 \perp W$

Ex: $V = \mathbb{R}^3$

$V_0 = \{(x, 0, 0) | x \in \mathbb{R}\}$

$$\beta = \{(1,0,0)\}$$

$$\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$W = \text{span}\{(0,1,0), (0,0,1)\}$$

$$\mathbb{R}^3 = V_0 \oplus W$$

Def: A real matrix P is said to be orthogonal if $P^{-1} = P^T$.

Remark: Let P be a real matrix then the following are equivalent.

① P is orthogonal

② The rows of P forms an orthonormal set

③ The columns of P forms an orthogonal set

Ex:- $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} \end{bmatrix}$

Find the orthogonal matrix P whose first row is $(1/\sqrt{3}, 2/\sqrt{3}, 2/\sqrt{3})$.

Sol:

$$P = \begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{3} & 2/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -4/\sqrt{18} & 1/\sqrt{18} & 1/\sqrt{18} \end{bmatrix}$$

$$\langle (x, y, z) | (1/\sqrt{3}, 2/\sqrt{3}, 2/\sqrt{3}) \rangle = 0$$

$$\frac{x}{3} + \frac{2y}{3} + \frac{2z}{3} = 0$$

$$x + 2y + 2z = 0$$

$$(0, 1, -1)$$

$$(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$$

$$x + 2y + 2z = 0$$

$$y - z = 0$$

$$y = z$$

$$\begin{cases} z = 1 \\ y = 1 \end{cases} \quad (-4, 1, 1)$$

$$x = -4 \quad \left(-\frac{4}{\sqrt{18}}, \frac{1}{\sqrt{18}}, \frac{1}{\sqrt{18}}\right)$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 1 & -5 & 2 \end{bmatrix}$$

We thought here rows are orthogonal. A is not orthogonal.
So, to make it orthogonal divide each row with its norm.

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ \frac{1}{\sqrt{78}} & \frac{-5}{\sqrt{78}} & \frac{2}{\sqrt{78}} \end{bmatrix}$$

Q. Show the following

- (1) P is orthogonal iff P^T is orthogonal.
- (2) P is orthogonal iff P^{-1} is orthogonal.
- (3) If P and Q are orthogonal, if product exists,
then PQ is also orthogonal.

Sol: (1) $PP^T = I = P^T P$

so, P^T is also orthogonal.

(2) $PP^{-1} = I = P^{-1}P$

(3) $(PQ)^{-1} = Q^{-1}P^{-1} = Q^T P^T = (PQ)^T$

$$\Rightarrow \lambda = 2\lambda + 1 = 0$$

$$\lambda_1 = 1$$

$$f(1) = 1$$

$$1 + 2\lambda + 2\lambda^2 + 2\lambda^3 - 3\lambda^4 + \lambda^5$$

$$(x-1)(x-1) = 0$$

$$x^2 - 2x + 1 = 0$$

$$x^2 - 2x + 1 - 3 = 0$$

$$x^2 - 2x + 1 - 3 = 0$$

$$I = P^T P$$

$$[0] = d \cdot n$$

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Orthogonal matrices:-

$$P^{-1} = P^T$$

Show that P is orthogonal matrix, then

$$\textcircled{1} \quad \langle P\mathbf{u}, P\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle \quad \forall \mathbf{u}, \mathbf{v} \in V$$

$$\textcircled{2} \quad \|P\mathbf{u}\| = \|\mathbf{u}\| \quad \forall \mathbf{u} \in V$$

$$\textcircled{1} \quad \langle P\mathbf{u}, P\mathbf{v} \rangle = (P\mathbf{u})^T (P\mathbf{v})$$

$$= \mathbf{u}^T P^T P \mathbf{v}$$

$$= \mathbf{u}^T \mathbf{v}$$

$$= \langle \mathbf{u}, \mathbf{v} \rangle$$

$$\textcircled{2} \quad \|P\mathbf{u}\|^2 = \langle P\mathbf{u}, P\mathbf{u} \rangle$$

$$= (P\mathbf{u})^T P \mathbf{u}$$

$$= \mathbf{u}^T P^T P \mathbf{u}$$

$$= \mathbf{u}^T \mathbf{u}$$

$$= \langle \mathbf{u}, \mathbf{u} \rangle$$

$$= \|\mathbf{u}\|^2$$

$$\therefore \|P\mathbf{u}\| = \|\mathbf{u}\|.$$

Ex Suppose $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$ be two orthonormal basis of V . Let P be change of basis matrix from basis β to β' , then P is a orthogonal matrix.

Sol:

Proof: As β is basis, we can write

$$e_i' = b_{i1} e_1 + b_{i2} e_2 + \dots + b_{in} e_n$$

$$i = 1, 2, \dots, n$$

NOW,

$$S_{ij} = \langle e_i' | e_j' \rangle$$

$$= \langle b_{i1} e_1 + b_{i2} e_2 + \dots + b_{in} e_n | b_{j1} e_1 + b_{j2} e_2 + \dots + b_{jn} e_n \rangle$$

$$S_{ij} = b_{i1} b_{j1} + b_{i2} b_{j2} + \dots + b_{in} b_{jn}$$

Let $B = [b_{ij}]$ then $P = B^T$

Now, suppose $BB^T = [C_{ij}]$

$$C_{ij} = b_{i1} b_{j1} + b_{i2} b_{j2} + \dots + b_{in} b_{jn}$$

$$C_{ij} = S_{ij}$$

$$\Rightarrow BB^T = I = B^T B$$

$\Rightarrow B$ is orthogonal and

$B^T = P$ is also orthogonal

Suppose $\{e_1, e_2, \dots, e_n\}$ be orthonormal basis of \mathbb{R}^n

v. Let P be a orthogonal matrix, then

show that the following n vectors

given as

$$e_i' = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \dots + \alpha_n e_n$$
$$i = 1, 2, \dots, n$$

Show that these n vectors form an orthonormal basis on V .

Sol: $\delta_{ij} = \langle e_i' e_j' \rangle$

$$= \alpha_1 \alpha_1 + \alpha_2 \alpha_2 + \dots + \alpha_n \alpha_n$$

$$P = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix}$$

$$= \langle c_i c_j \rangle$$

where, c_i and c_j are i^{th} and j^{th} column of P .

$$= \delta_{ij}$$

So, these n vectors would form a orthonormal basis on V .

Defn: A symmetric matrix A is said to be positive definite if for every nonzero vector $v \in \mathbb{R}^n$ (A -real symmetric matrix, $(n \times n)$)

$$\langle v^T A v \rangle = v^T A v > 0$$

Result:

Let A be the (2×2) real symmetric matrix,

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \quad (\because \text{symmetric})$$

Then A is positive definite iff $ad - b^2 > 0$
and $a > 0, d > 0$.

$$u = (x, y)^T$$

$$\text{then, } f(u) = u^T A u$$

$$= (x, y) \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= (x, y) \begin{bmatrix} ax + by \\ bx + dy \end{bmatrix}$$

$$= ax^2 + bxy + bxy + dy^2$$

$$= ax^2 + 2bxy + dy^2$$

Let A is positive definite, then $f(u) > 0 \quad \forall u \neq 0$
 $= (x, y)^T \in \mathbb{R}^2$

$$\text{Take } u = (1, 0)^T$$

$$\text{then, } a > 0$$

$$\text{Take } u = (0, 1)^T$$

$$\text{then, } d > 0$$

Take $u = (b, -a)^T$

$$\text{then, } f(u) = ab^2 - 2ab^2 + da^2 \\ = da^2 - ab^2 \\ = a(ad - b^2)$$

$$\text{then, } ad - b^2 > 0$$

conversely,

$$\text{let } A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

$$\text{and } a > 0, d > 0, ad - b^2 > 0,$$

$$u^T A u = f(u) = ax^2 + 2bxy + dy^2 \\ = a \left[x^2 + \frac{2bxy}{a} + \frac{b^2}{a^2} y^2 \right] \\ + dy^2 - \frac{b^2}{a} y^2 \\ = a \left[x + \frac{b}{a} \right]^2 + dy^2 - \frac{b^2}{a} y^2$$

$$u^T A u = a \left(x + \frac{b}{a} \right)^2 + \frac{(ad - b^2)}{a} y^2 > 0$$

So, A is a positive definite matrix.

Hence, proved.

Let A be real positive definite matrix,
then show that $\langle u, v \rangle := u^T A v$ defines
inner product on \mathbb{R}^n .

Sol:-

$$\textcircled{1} \quad \langle u, u \rangle = u^T A u > 0 \quad \forall u \neq 0$$

Now, $\langle u, u \rangle = 0 \Leftrightarrow u^T A u = 0 \Leftrightarrow u = 0$

$$\textcircled{2} \quad \langle u, v \rangle = u^T A v$$

$$= (u^T A v)^T$$

$$= v^T A^T u$$

$$= v^T A u \quad (\because A^T = A)$$

$$= \langle v, u \rangle$$

$$\textcircled{3} \quad \langle cu, v \rangle = (cu)^T A v$$

$$= c(u^T A v)$$

$$= c \langle u, v \rangle$$

$$\textcircled{4} \quad \langle u+v, w \rangle = (u+v)^T A w$$

$$= (u^T + v^T) A w$$

$$= u^T A w + v^T A w$$

$$= \langle u, w \rangle + \langle v, w \rangle$$

Check the positive definiteness of the following matrices,

$$\textcircled{1} \quad \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

(x) (v)

$$\begin{bmatrix} 2 & -4 \\ -4 & k \end{bmatrix} \Rightarrow k > 8$$

$$\begin{bmatrix} k & 5 \\ 5 & 2 \end{bmatrix} \rightarrow (\underline{x}) \text{ since } d < 0$$

$$\begin{bmatrix} 6 & k \\ k & 4 \end{bmatrix} \Rightarrow 24 - k^2 > 0$$

$$\Rightarrow |k| < \sqrt{24}$$

$$\therefore -\sqrt{24} < k < \sqrt{24}$$

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1. Find 3×3 orthogonal matrix P whose first two rows are multiples of $u = (-1, 1, 1)$ and $v = (1, -2, 3)$ respectively.

Sol:-

$$\begin{bmatrix} k(-1, 1, 1) \\ p(1, -2, 3) \\ x \quad y \quad z \end{bmatrix}$$

third row is,

$$\left[\frac{5}{\sqrt{42}}, \frac{4}{\sqrt{42}}, \frac{1}{\sqrt{42}} \right]$$

$$\begin{aligned} k + p &= 0 \\ k - 2p &= 0 \\ k + 3p &= 0 \end{aligned} \Rightarrow 3k + 2p = 0$$

$$k = -\frac{2}{3}p$$

$$\begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{4}} & -\frac{2}{\sqrt{4}} & \frac{3}{\sqrt{4}} \\ x & y & z \end{bmatrix}$$

so,

$$P = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{4}} & -\frac{2}{\sqrt{4}} & \frac{3}{\sqrt{4}} \\ \frac{5}{\sqrt{42}} & \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{42}} \end{bmatrix}$$

$$\begin{aligned} \Rightarrow -x + y + z &= 0 \\ x - 2y + 3z &= 0 \\ \hline y &= 4z \end{aligned}$$

$$\Rightarrow k[5, 4, 1]$$

2. Find the numbers & exhibits all 2×2 orthogonal
matrices of the form

$$\begin{bmatrix} \frac{1}{3} & x \\ y & z \end{bmatrix}$$

Sol. $\left(\frac{1}{3}\right)^2 + x^2 = 1$

$$x^2 = 1 - \left(\frac{1}{3}\right)^2$$

$$x^2 = 1 - \frac{1}{9}$$

$$x^2 = \frac{8}{9}$$

$$x = \pm \frac{2\sqrt{2}}{3}$$

$$y^2 + z^2 = 1$$

$$\left(\frac{1}{3}\right)^2 + y^2 = 1$$

$$y = \pm \frac{2\sqrt{2}}{3}$$

$$\Rightarrow z^2 = \frac{1}{9}$$

$$\Rightarrow z = \pm \frac{1}{3}$$

$$\begin{bmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{3} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}$$

4 possible.

3. Definition: A linear transformation $T: V \rightarrow V$ (V i.p.s) is said to be self adjoint if $\langle Tx, y \rangle = \langle x, Ty \rangle \quad \forall x, y \in V$.

Suppose, V is finite dimensional inner product space $E = \{v_1, v_2, \dots, v_n\}$ be basis of V .

$T: V \rightarrow V$ be self adjoint then,

$$\langle Tv_i, v_i \rangle = \langle v_i, T v_i \rangle \quad i=1, 2, \dots, n$$

$\rightarrow F = \mathbb{C} \quad V = \mathbb{C}^n, T$ is self adjoint,

$[T]_{EE}$ = Hermitian matrix

$\rightarrow F = \mathbb{R} \quad V = \mathbb{R}^n, T$ is self adjoint,

$[T]_{EE}$ = Symmetric matrix.

Q- Show that eigen values of self-adjoint matrix are real numbers.

Sol: Proof:

Let λ be eigen value of T i.e. $\exists x \neq 0 \in V$ such that $Tx = \lambda x$

Also, note that, if T is self adjoint

$$\text{then } \langle Tx, x \rangle = \langle x, Tx \rangle$$

$$\Rightarrow \langle Tx, x \rangle \in \mathbb{R}$$

Take,

$$\begin{aligned}\langle Tx, x \rangle &= \langle \lambda x, x \rangle \\ &= \lambda \langle x, x \rangle\end{aligned}$$

$$\Rightarrow \lambda = \frac{\langle Tx, x \rangle}{\|x\|^2} \quad \text{and } \langle Tx, x \rangle \in \mathbb{R}$$

$\& \|x\|^2 > 0$

$\Rightarrow \lambda$ is a real number.

Q-show that eigen-vectors associated with distinct eigen values of a self-adjoint operator are orthogonal.

Proof: Let λ_1, λ_2 are two distinct eigen values of self-adjoint operator T and corresponding eigen vectors are x_1, x_2 respectively.

i.e $Tx_1 = \lambda_1 x_1$ and $Tx_2 = \lambda_2 x_2$

Take,

$$\begin{aligned}&\lambda_1 \langle x_1, x_2 \rangle \\ &= \langle \lambda_1 x_1, x_2 \rangle \\ &= \langle Tx_1, x_2 \rangle\end{aligned}$$

$$\begin{aligned}
 &= \langle x_1, Tx_2 \rangle \\
 &= \langle x_1, \lambda_2 x_2 \rangle \\
 &= \bar{\lambda}_2 \langle x_1, x_2 \rangle \\
 &= \lambda_2 \langle x_1, x_2 \rangle \\
 \Rightarrow (\lambda_1 - \lambda_2) \langle x_1, x_2 \rangle &= 0
 \end{aligned}$$

Since, $\lambda_1 \neq \lambda_2$

So,

$$\langle x_1, x_2 \rangle = 0$$

Hence, proved.

Remark:

Every self adjoint operator on a finite dimensional inner product space has an eigen value.

* Def: Let $T: V \rightarrow V$ be a self adjoint operator on finite dimensional inner product space. Then there exists an orthonormal basis of V consists of eigen vectors of T . \square

Remark: This theorem also says, if A is a symmetric matrix, then there exist orthogonal matrix P such that $P^T A P = D$.

→ if symmetric, then diagonalisable.

Lemma:

Let $T: V \rightarrow V$ be a self-adjoint operator on a finite dimensional inner product space V and V_0 is a subspace of V .

Then,

$$T(V_0) \subseteq V_0 \Rightarrow T(V_0^\perp) \subseteq V_0^\perp$$

$$(TV_0 = \{Tv \mid v \in V_0\})$$

If $TV_0 \subseteq V_0$ then we say that the subspace V_0 is invariant under T .

Proof

$$\text{Let } x \in V_0^\perp \quad \text{To show: } Tx \in V_0^\perp$$

$$\langle x, u \rangle = 0 \quad \forall u \in V_0$$

Let $u \in V_0$. Then $Tu \in V_0$

$$\langle Tx, u \rangle = \langle x, Tu \rangle = 0$$

$$\Rightarrow Tx \in V_0^\perp$$

Ex Let V be a ips (f.dim) and $T: V \rightarrow V$ be a linear operator (any). Show that if λ is a eigen value of T , then

$N(T - \lambda I)$ is invariant under T .

Sol: $x \in N(T - \lambda I)$

$$\text{then, } Tx = \lambda x$$

$$T(Tx) = T(\lambda x)$$

$$\Rightarrow T(\lambda x) = \lambda(T(x))$$

$$\Rightarrow T(\underbrace{Tx}) = \lambda \cdot \underbrace{T(x)}$$

Hence, $Tx \in N(T - \lambda I)$.

so, $N(T - \lambda I)$ is invariant under T .

Hence, proved.

Proof of 0:

Let $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigen values of T .

$$V_0 = N(T - \lambda_1 I) + N(T - \lambda_2 I) + \dots + N(T - \lambda_k I)$$

If $V_0 = V$, then union of orthonormal basis of null space of $T - \lambda_i I$ ie $N(T - \lambda_i I)$ $i=1, 2, \dots, k$

will orthonormal basis of V consist of eigen vectors of T .

Suppose, $V_0 \neq V$, then

$$V_0^\perp \neq 0$$

Now,

$$T(V_0) \subseteq V_0$$

$$\Rightarrow T(V_0^\perp) \subseteq V_0^\perp$$

Define, $T_1 : V_e^\perp \rightarrow V_e^\perp$

$$T_1 v = T v, \forall v \in V_e^\perp$$

Note T_1 is self adjoint (as T is self adjoint)

so, it will have eigen value say $\lambda \exists x_0 \in V_e^\perp$ and

$$\text{S.t } T_1 x = \lambda x$$

$$\Rightarrow T x = \lambda x$$

$\Rightarrow \lambda$ is eigen value of $T + \lambda_i I$
 $i \leq i \leq K$

$$\text{Say, } \lambda = \lambda_i$$

$$\Rightarrow x \in N(T - \lambda_i I) \in V_e$$

$$\Rightarrow (V_e \cap V_e^\perp) \neq \{0\}$$

which is a contradiction to $V_e \neq V$.

$$\text{So, } V_e = V.$$

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Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$T(x, y) = (7x + 3y, 3x - y)$$

Find orthogonal matrix P such that $\tilde{P}^{-1}[T]_B P$ is a diagonal matrix.

Sol: $[T]_B = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

λ are real for symmetric.

$$| [T]_B - \lambda I | = 0$$

$$\Rightarrow (7-\lambda)(1+\lambda) + 9 = 0$$

$$\Rightarrow -\lambda^2 + 6\lambda + 16 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda - 16 = 0$$

$$\Rightarrow (\lambda - 8)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = -2, 8$$

for $\lambda = 8$,

$$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x + 3y = 0$$

$$x = 3y$$

$$\text{Take } y = 1$$

$$x = 3$$

So, $(3, 1)$ is eigen vector.

$$\lambda = -2$$

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x + y = 0$$

$$\Rightarrow y = 3 \Rightarrow x = -y$$

$$\text{So, } v_2 = (-1, 3) = (-1, 3)$$

$$\text{Now, } B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

Let $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$. A real quadratic form is an expression of the form

$$Q = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$$

$$Q = a_{11} x_1^2 + (a_{12} + a_{21}) x_1 x_2 + \dots + (a_{1n} + a_n) x_1 x_n \\ + \dots + \dots + a_{nn} x_n^2$$

$$A = (a_{ij})$$

$$Q = x^T A x$$

$= x^T B x$ where B is a symmetric matrix.

$B = (b_{ij})$ is defined as

$$b_{ii} = a_{ii} \quad \& \quad b_{ij} = \frac{a_{ij} + a_{ji}}{2}$$

Ex Obtain the symmetric matrix B from the quadratic forms.

$$1) Q = 2x_1^2 + 3x_1 x_2 + x_2^2$$

$$2) Q = x_1^2 + 2x_1 x_2 - 4x_1 x_3 + 6x_2 x_3 \\ - 5x_2^2 + 4x_3^2$$

1)

$$A = \begin{bmatrix} 2 & 3/2 \\ 3/2 & 1 \end{bmatrix}$$

2)

$$B = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -5 & 3 \\ -2 & 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3/2 \\ 3/2 & 1 \end{bmatrix}$$

We can reduce the quadratic form to a canonical form by Principal axis theorem.

Principal axis theorem:

Let B be symmetric matrix with eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$. Let P be the orthogonal matrix which diagonalizes B .

Then, $X = PY$ (which is a linear transformation) transforms quadratic form $\sum_{i=1}^n \sum_{j=1}^n b_{ij}x_i x_j$

to the form

$\sum_{i=1}^n \lambda_i (y_i)^2$ which is called as canonical form / standard form.

Here,

$x = PY$ is the "orthogonal transformation" bcz
 P is an orthogonal matrix.

Proof:-

Note that, $\sum_{i=1}^n \sum_{j=1}^n b_{ij} x_i x_j = x^T B x$

$$= (PY)^T B (PY)$$

$$= Y^T P^T B P Y$$

$$= Y^T (\overset{\leftrightarrow}{P} B P) Y$$

$$= Y^T D Y$$

Here, P is orthogonal

$$\text{so, } P^T = P^{-1}$$

$$\text{and } D = \overset{\leftrightarrow}{P} B P.$$

where, D is diag $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$$

$$= \sum_{i=1}^n \lambda_i y_i^2$$

Ex:- Find the orthogonal transformation
which converts the given quadratic form
to the canonical form.

$$Q(x_1, x_2) = x_1^2 + 6x_1 x_2 - 7x_2^2$$

Sol:

$$B = \begin{bmatrix} 1 & 3 \\ 3 & -7 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & -(7+\lambda) \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(7+\lambda) + 9 = 0$$

$$\Rightarrow -\lambda^2 - 6\lambda + 16 = 0$$

$$\Rightarrow \lambda^2 + 6\lambda - 16 = 0$$

$$\Rightarrow (\lambda + 8)(\lambda - 2) = 0$$

$$\Rightarrow \lambda = +2, \lambda = -8$$

for $\lambda = 2$,

$\lambda = -8$,

$$\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} = 0 \quad \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x + 3y = 0 \quad \Rightarrow 3x + y = 0$$

$$x = 3y$$

$$x = -1$$

$$y = 1 \Rightarrow x = 3 \quad \text{Take, } y = 3$$

$$v_1 = (3, 1)^T$$

$$v_2 = (-1, 3)^T$$

$$\text{So, } P = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

$$X = PY$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow x_1 = (3y_1 - y_2)/\sqrt{10}$$

$$x_2 = (y_1 + 3y_2)/\sqrt{10}$$

So,

$$\Omega = \frac{(3y_1 - y_2)^2}{10} + \frac{6(3y_1 - y_2)(y_1 + 3y_2)}{10}$$

$$\bullet -7 \frac{(y_1 + 3y_2)^2}{10}$$

$$= \frac{1}{10} \left[9y_1^2 + y_2^2 - 6y_1y_2 + 18y_1^2 + 4y_1y_2 - 12y_2^2 \right] \\ - 7y_1^2 - 63y_2^2 - 42y_1y_2$$

$$= \frac{1}{10} \left[20y_1^2 - 80y_2^2 \right]$$

$$= 2y_1^2 - 8y_2^2$$

$$= \sum \lambda_i y_i^2$$

Find the orthogonal trans X_m which transform quad to canonical form.

$$Q = x_1^2 - 2x_1x_2 + x_2^2$$

$$B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} (1-\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 = 1$$

$$\begin{array}{c|c} \Rightarrow 1-\lambda = 1 & 1-\lambda = -1 \\ \Rightarrow \lambda = 0 & \lambda = 2 \end{array}$$

$$\lambda = 0,$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x = y$$

$$\Rightarrow y = 1$$

$$\text{so, } x = 1$$

$$v_1 = (1, 1)$$

$$\lambda = 2$$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x + y = 0$$

$$y = 1$$

$$\Rightarrow x = -1$$

$$v_2 = (-1, 1)$$

$$\text{so, } P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$X = PY$$

$$x_1 = \frac{y_1}{\sqrt{2}} - \frac{y_2}{\sqrt{2}}$$

$$x_2 = \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}}$$

Now, on taking the orthogonal transformation,

$$\mathcal{G} = \sum \lambda_i y_i^2$$

$$\mathcal{G} = 0(y_1)^2 + (2)y_2^2$$

$$\mathcal{G} = 2y_2^2$$

Analyse the chaotic, $4x_1^2 - 3x_1x_2 + 2x_2^2 = 8$,
using principal axes theorem.

Sol:- $4x_1^2 - 3x_1x_2 + 2x_2^2 = 8$

$$B = \begin{bmatrix} 4 & -3/2 \\ -3/2 & 2 \end{bmatrix}$$

$$\Rightarrow (4-\lambda)(2-\lambda) - \frac{9}{4} = 0$$

$$\Rightarrow 4[\lambda^2 - 6\lambda + 8] - 9 = 0$$

$$\Rightarrow 4\lambda^2 - 24\lambda + 23 = 0$$

$$\frac{24 \pm \sqrt{576 - (23)(16)}}{8}$$

$$= \frac{6 \pm \sqrt{13}}{2}$$

$$Q = \left[\frac{6 + \sqrt{13}}{2} \right] y_1^2 + \left[\frac{6 - \sqrt{13}}{2} \right] y_2^2 = 8$$

solution

Best approximation:

Def Let V be ips and V_0 be subspace of V . Let $x \in V$.

We say $x_0 \in V_0$ is best approximation for x from V_0 if

$$\|x - x_0\| \leq \|x - v\| \quad \forall v \in V_0$$

$$V = C[a, b] \quad f \in C[a, b] \quad P \in P[a, b]$$

$$V_0 = P[a, b]$$

$$\|f - p\| \leq \|f - q\| \quad \forall q \in P[a, b]$$

Theorem: Let V be ips and V_0 be subspace of V . If $x_0 \in V_0$ is such that $x - x_0 \perp V_0$, then x_0 is the best approximation for the element x and also it is the unique best approximation from V_0 for x .

Conversely, if x_0 is the best approximation for $x_0 \in V$ then $x - x_0 \perp V_0$.

Proof:

Suppose, $x_0 \in V_0$ is such that $x - x_0 \perp V_0$,
for every $u \in V_0$,

$$\|x - u\|^2 = \|x - x_0\|^2 + \|x_0 - u\|^2 \quad \forall u \in V_0.$$

$$\Rightarrow \|x - x_0\| \leq \|x - u\| \quad \forall u \in V_0.$$

$\therefore x_0$ is the best approximation for x from V_0 .

Uniqueness:

Let v_0 be another best approximation
then we have $\|x - v_0\| \leq \|x - x_0\|$

Also,

$$\|x - x_0\| \leq \|x - v_0\|$$

$$\Rightarrow \|x - x_0\| = \|x - v_0\| \quad \text{--- } \textcircled{1}$$

We know that,

$$\langle x - x_0, x_0 - v_0 \rangle = 0$$

So we have,

$$\|x - v_0\|^2 = \|x - x_0\|^2 + \|x_0 - v_0\|^2$$

from $\textcircled{1}$,

$$\|x_0 - v_0\| = 0$$

$$\Rightarrow \boxed{x_0 = v_0}$$

for Converse part,

Let x_0 be the best approximation for $x \in E$
then $\|x - x_0\| \leq \|x - u\| \forall u \in V_0$.

We take a particular u , $u = x_0 + \alpha v$ where
 $v \in V_0$ and $\alpha \in \mathbb{R}$

$$\Rightarrow \|x - x_0\|^2 \leq \|x - (x_0 + \alpha v)\|^2$$

$$\Rightarrow \|x - x_0\|^2 \leq \|x - x_0 - \alpha v\|^2 = \|x - x_0 - \alpha v\|$$

$$\Rightarrow \|x - x_0\|^2 \leq \|x - x_0\|^2 - \langle x - x_0, x - x_0 \rangle - \langle x - x_0, \alpha v \rangle \\ - \langle \alpha v, x - x_0 \rangle + \alpha^2 \|v\|^2$$

$$0 \stackrel{?}{\leq} \|x - x_0\|^2 \leq \|x - x_0\|^2 - 2 \operatorname{Re} \langle x - x_0, \alpha v \rangle + \alpha^2 \|v\|^2$$

L ⊕

$$\text{Taking } \alpha = \frac{\langle x - x_0, v \rangle}{\|v\|^2}$$

$$\operatorname{Re}(\langle x - x_0, \alpha v \rangle)$$

$$= \langle x - x_0, \frac{\langle x - x_0, v \rangle v}{\|v\|^2} \rangle$$

$$= \frac{|\langle x - x_0, v \rangle|^2}{\|v\|^2}$$

$$|\alpha|^2 = \frac{|\langle x - x_0, v \rangle|^2}{\|v\|^4}$$

from *

from * in ②,

$$\Rightarrow \|x - x_0\|^2 \leq \|x - x_0\|^2 - \frac{|\langle x - x_0, v \rangle|^2}{\|v\|^2}$$

$$\Rightarrow \langle x - x_0, v \rangle = 0 \text{ for all } v \in V_0$$

$$\therefore x - x_0 \perp V_0$$

Ex: Let V be a ips, V_0 be a finite dimensional
vector subspace of V and $x \in V$. Let
 u_1, u_2, \dots, u_n be an orthonormal basis of V_0 .
then show that the element x_0 given as,
 $x_0 = \sum_{i=1}^n \langle x, u_i \rangle u_i$ will be unique

best approximation for x .

Sol: Here we need to show that, $x - x_0 \perp V_0$.

so, it is sufficient to show that,

$$\langle x - x_0, u_j \rangle = 0 \text{ for } j=1, 2, \dots, n$$

$$\Rightarrow \langle x - \sum_{i=1}^n \langle x, u_i \rangle u_i, u_j \rangle = 0$$

$$= \langle x, u_j \rangle - \sum_{i=1}^n \langle x, u_i \rangle \langle u_i, u_j \rangle$$

$$= 0$$

$$\langle x - x_0, u_j \rangle = \langle x, u_j \rangle - \langle x_0, u_j \rangle = 0$$

$$x - x_0 \perp V_0$$

Here, it will be 1
for $i=j$
 $= 0$, elsewhere

Remark:

Suppose, V is inner product space and V_0 be finite dimensional vector subspace of V . Let $x \in V$ we are interested in finding best approximation for x i.e we find $x_0 \in V_0$.

s.t $\langle x - x_0, v \rangle = 0$ for all $v \in V_0$.

Let $\{u_1, u_2, \dots, u_n\}$ basis of V_0 .

$$\text{then, } x_0 = \sum_{i=1}^n \alpha_i u_i$$

so, we find scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ by the relation,

$$\langle x - x_0, u_j \rangle = 0 \quad j = 1, 2, \dots, n$$

$$\Rightarrow \langle x - \sum_{i=1}^n \alpha_i u_i, u_j \rangle = 0$$

Ex Let $V = \mathbb{R}^2$ be usual inner product,

$$V_0 = \{ (x_1, x_2) \mid x_1 = x_2, x_1, x_2 \in V \}$$

Find the best approximation for the following elements,

- (a) (6, 1) (b) (1, 2)

Sol:- we note that $\{(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix})\}$ be the basis of V_0 .

find $x_0 \in V_0$ such that,

$$\langle x - x_0, u_1 \rangle = 0$$

as $x_0 \in V_0$, so x_0 will be of form (α, α)

$$\langle x - (\alpha, \alpha), u_1 \rangle = 0$$

$$\langle (x_1, x_2) - (\alpha, \alpha), (1, 1) \rangle = 0$$

$$\cancel{\langle x_1, x_2 (1, 1) \rangle} - \cancel{\langle (\alpha, \alpha) (1, 1) \rangle} = 0$$

(a)

$$\cancel{\langle (0, 1) (1, 1) \rangle} = \langle \cancel{\alpha}, \cancel{\alpha} \rangle$$

$$\cancel{\langle 1, 2 \rangle}$$

$$\langle -\alpha, 1 - \alpha - (1, 1) \rangle = 0$$

$$-\alpha + 1 - \alpha = 0$$

$$2\alpha = 1$$

$$\alpha = 1/2$$

$$(1/2, 1/2)$$

(b) (112)

$$\langle 1 - \alpha, 2 - \alpha - (1, 1) \rangle = 0$$

$$1 - \alpha + 2 - \alpha = 0$$

$$2\alpha = 3$$

$$\alpha = 3/2$$

$$\text{so, } (3/2, 3/2)$$

2) let V be vector space, $V = C[0,1]$ over \mathbb{R}
 with $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$. Let V_0 is
 space of polynomials of degree 1, find
 the best approximation for the function,

$$f(t) = t^2.$$

Sol: $\beta\{1, t\}^{u_1, u_2}$ is a basis of V_0 .

$$\text{we find } f_0(t) = a_0 + a_1 t$$

such that,

$$\langle f - f_0, u_i \rangle = 0 \quad (i=1, 2)$$

$$\langle t^2 - a_1 t - a_0, 1 \rangle = 0$$

$$\langle t^2 - a_1 t - a_0, t \rangle = 0$$

$$\Rightarrow \int_0^1 (t^2 - a_1 t - a_0) dt = 0$$

$$\frac{1}{3} - \frac{a_1}{2} - a_0 = 0 \quad \text{--- (1)} \Rightarrow a_0 = \frac{1}{3} - \frac{a_1}{2}.$$

$$\int_0^1 (t^3 - a_1 t^2 - a_0 t) dt = 0$$

$$\Rightarrow \frac{1}{4} - \frac{a_1}{3} - \frac{a_0}{2} = 0 \quad \text{--- (2)}$$

$$\Rightarrow \frac{1}{4} - \frac{a_1}{3} - \frac{1}{6} + \frac{a_1}{4} = 0$$

$$\Rightarrow \frac{a_1}{12} = -\frac{2}{24} \Rightarrow a_1 = 1$$

$$\text{So, } a_0 = \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = -\frac{1}{6}$$

So, $f_0 = -\frac{1}{6} + t$ is the best approximation for

3) Let $V = C[0,1]$ with $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$
 $V_0 = P_2$ find best approximation for the
 following;

a) e^t b) $\sin t$ c) $\cos t$ in V_0 .

$$f_0(t) = a_0 + a_1 t + a_2 t^2$$

July 10/19

Some special operators on inner product space.

Adjoint operator: A linear operator T on inner product space V is said to have an adjoint operator T^* on V

$$\text{if } \langle Tu, v \rangle = \langle u, Tv \rangle \quad \forall u, v \in V$$

Ex: Let A be a real n square matrix

Note: A can be seen operator on space \mathbb{R}^n

Then,

$$\begin{aligned} \langle Au, v \rangle &= (Au)^T v \\ &= u^T A^T v \\ &= \langle u, A^T v \rangle \end{aligned}$$

Thus, the transpose of A is, A^T is the adjoint of a real matrix A .

Ex Let A be square complex matrix of n ,

$$\begin{aligned}\langle Au, v \rangle &= (Au)^T v \\ &= u^T \bar{A}^T v \\ &= \langle u, \bar{A}^T v \rangle\end{aligned}$$

In case of a complex matrix the conjugate transpose of A is adjoint of A .

Ex Find the adjoint of L.T, $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, defined by

$$T(x, y, z) = (3x+4y-5z, 2x-6y+7z, 5x-9y+z)$$

$$[T]_B = A = \begin{bmatrix} 3 & 4 & -5 \\ 2 & -6 & 7 \\ 5 & -9 & 1 \end{bmatrix}$$

$$\text{Adj}(A) = A^T = \begin{bmatrix} 3 & 2 & 5 \\ 4 & -6 & -9 \\ -5 & 7 & 1 \end{bmatrix}$$

$$T^*(x, y, z) = (3x+2y+5z, 4x-6y-9z, -5x+7y+z)$$

is the adjoint operator of T .

2) Find adjoint of $L(T)$, $T: \mathbb{C}^3 \rightarrow \mathbb{C}^3$ defined as

$$T(x, y, z) = (2x + (1-i)y, (3+2i)x - 4iz, 2ix + (4-3i)y - 3z)$$

$$A = \begin{bmatrix} 2 & 1-i & 0 \\ 3+2i & 0 & -4i \\ 2i & 4-3i & -3 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 2 & 1+i & 0 \\ 3-2i & 0 & 4i \\ -2i & 4+3i & -3 \end{bmatrix}$$

$$(\bar{A})^T = \begin{bmatrix} 2 & 3-2i & -2i \\ 1+i & 0 & 4+3i \\ 0 & 4i & -3 \end{bmatrix}$$

$$T^*(x, y, z) = (2x + (3-2i)y - 2iz, (1+i)x + (4+3i)z, 4iy - 3z),$$

is the adjoint operator of T .

Ex Let λ be the eigen value of linear operator T on ips V , then show the following,

① if $T^* = T^{-1}$ (i.e T is orthogonal or unitary operator) then $|\lambda| = 1$.

- ② IF $T^* = T$ (T is self adjoint) then λ is real.
- ③ IF $T^* = -T$ (T is skew-adjoint) then λ is purely imaginary.
- ④ IF $T = S^*S$ with S non singular
(T is positive definite) then λ is real
and positive.

Proof:

$$\begin{aligned} \textcircled{1} \quad \bar{\lambda}\bar{\lambda} < v, v > &= < \lambda v, \lambda v > \\ &= < Tv, Tv > \\ &= < v, T^*Tv > \\ &= < v, v > \end{aligned}$$

$$\text{then, } \bar{\lambda}\bar{\lambda} = 1 \quad (\because v \neq 0)$$

$$\Rightarrow |\lambda|^2 = 1$$

$$\Rightarrow \lambda = 1$$

$$\begin{aligned} \textcircled{2} \quad \lambda < v, v > &= < \lambda v, v > \\ &= < Tv, v > \\ &= < v, T^*v > \\ &= < v, Tv > \end{aligned}$$

$$= \langle v, \lambda v \rangle$$

$$= \bar{\lambda} \langle v, v \rangle$$

$$\Rightarrow (\lambda - \bar{\lambda}) \langle v, v \rangle = 0$$

$$\therefore v \neq 0 \Rightarrow \langle v, v \rangle \neq 0$$

so, $\boxed{\lambda = \bar{\lambda}}$ Hence, λ is real.

$$\begin{aligned} \textcircled{3} \quad \lambda \langle v, v \rangle &= \langle \lambda v, v \rangle \\ &= \langle Tv, v \rangle = \langle v, T^* v \rangle \\ &= \langle v, -T v \rangle \\ &= \langle v, -\lambda v \rangle \\ &= -\bar{\lambda} \langle v, v \rangle \end{aligned}$$

$$\Rightarrow (\lambda + \bar{\lambda}) \langle v, v \rangle = 0$$

$$\therefore v \neq 0 \Rightarrow \langle v, v \rangle \neq 0$$

so, $\lambda + \bar{\lambda} = 0$ Hence λ is purely imaginary.

\textcircled{4} Given, $T = S^* S$, S is non-singular,

$$\text{As } v \neq 0 \Rightarrow Sv \neq 0$$

$$\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Tv, v \rangle$$

$$= \langle S^* S v, v \rangle$$

$$= \langle Sv, S^* v \rangle$$

$$\text{As, } \langle v, v \rangle > 0 \Rightarrow \langle Sv, Sv \rangle > 0$$

so, λ is positive

Ex
 Let, T, T_1, T_2 be linear operators on V ,
 and $K \in \mathbb{F}$, then show the following

$$1) (T_1 + T_2)^* = T_1^* + T_2^*$$

$$2) (T_1 T_2)^* = T_2^* T_1^*$$

$$3) (K T)^* = (K)^T T^*$$

$$4) (T^*)^* = T$$

Sol:-

$$1) \langle (T_1 + T_2)u, v \rangle = \langle T_1 u + T_2 u, v \rangle$$

$$= \langle T_1 u, v \rangle + \langle T_2 u, v \rangle$$

$$= \langle u, T_1^* v \rangle + \langle u, T_2^* v \rangle$$

$$= \langle u, (T_1^* + T_2^*)v \rangle$$

$$\Rightarrow \langle u, (T_1 + T_2)^* v \rangle = \langle u, (T_1^* + T_2^*)v \rangle$$

$$\text{So, } (T_1 + T_2)^* = T_1^* + T_2^*$$

$$2) \langle (T_1 T_2)u, v \rangle = \langle T_2 u, T_1^* v \rangle$$

$$= \langle u, T_2^* T_1^* v \rangle$$

$$\langle u, (T_1 T_2)^* v \rangle = \langle u, T_2^* T_1^* v \rangle$$

$$\Rightarrow (T_1 T_2)^* = T_2^* T_1^*$$

$$4) \quad \langle T^*u, v \rangle = \overline{\langle v, T^*u \rangle} \\ = \overline{\langle Tu, v \rangle} \\ = \langle u, Tv \rangle$$

$$\Rightarrow (T^*)^* = T$$

Note.

$$\rightarrow O: V \rightarrow V$$

$$Ov = O$$

$$O^* = ?$$

$$\langle Ov, v \rangle = \langle O, v \rangle = O = \langle u, O \rangle = \langle u, O \cdot v \rangle$$

$$\therefore O^* = O$$

$$\rightarrow I: V \rightarrow V ; I^v = v$$

$$\text{so, } I^* = I.$$

Ex show that $(T^{-1})^* = (T^*)^{-1}$.

$$I = I^* = (TT^{-1})^* = (T^{-1})^* T^*$$

$$\Rightarrow (T^{-1})^* T^* = I$$

$$\Rightarrow (T^{-1})^* = (T^*)^{-1}$$

Hence proved.

2) Let T be a linear operator on ips V , then show that each of the following conditions implies, $T = 0^*$

$$1) \langle Tu, v \rangle = 0 \quad \forall u, v \in V \Rightarrow T = 0$$

$$2) V \text{ is complex ips and } \langle Tu, u \rangle = 0 \quad \forall u \in V \Rightarrow T = 0$$

$$3) T \text{ is self-adjoint and } \langle Tu, u \rangle = 0 \quad \forall u \in V \Rightarrow T = 0$$

Sol:

$$1) \text{ Given that } \langle Tu, v \rangle = 0 \quad \forall u, v \in V$$

$$\text{Take, } v = Tu$$

$$\Rightarrow \langle Tu, Tu \rangle = 0$$

$$\Rightarrow \|Tu\|^2 = 0 \quad \forall u \in V$$

$$\Rightarrow Tu = 0 \quad \forall u$$

$$\Rightarrow T = 0$$

$$2) \text{ Given that } \langle Tu, u \rangle = 0 \text{ and } V \text{ is complex ips}$$

$$\Rightarrow \langle T(v+w), (v+w) \rangle = 0$$

$$\Rightarrow \langle Tv, v \rangle + \langle Tv, w \rangle + \langle Tw, v \rangle +$$

$$+ \langle Tw, w \rangle = 0$$

$$\Rightarrow \langle Tv, w \rangle + \langle Tw, v \rangle = 0 \quad \text{---} ①$$

replace w by iw in ①,

$$\langle Tv, iw \rangle + \langle T(iw), v \rangle = 0$$

$$\Rightarrow -i \langle Tv, w \rangle + i \langle Tw, v \rangle = 0 - ②$$

Multiply $[-② \times i + ①]$

$$\Rightarrow \langle Tw, v \rangle = 0 \quad \forall v, w \in V$$

from ①,

$$\Rightarrow T = 0.$$