MAL 206

Linear Algebra & Applications:-

→ G. Strang, Linear Algebra

→ Hoffmann & Kanje, Linear Algebra

Matrix representation

$$A : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{F}$$

$\mathbb{F}$ = Field ($\mathbb{R}$ (or) $\mathbb{C}$)

Elementary row operations)

1) Multiplication of any row by non-zero scalar c.

2) ~~(Addition of)~~ Replacement of r^{th} row by row r + (c * row s), where c is a non zero scalar.

3) Interchange of any two rows.

First definition:-

→ It says to every elementary row operation there corresponds an elementary row operation which is inverse of the given elementary row operation.

In other words, if e is given elementary row operation ∃ another elementary row operation e_1 such that

$$e(e_1(A)) = A = e_1(e(A))$$

Proof:

Let us assume a matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Let e be elementary row operation i.e multiplication with a scalar $c(\neq 0)$

So, $e(A) = \begin{bmatrix} c \cdot a_{11} & c \cdot a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Now, e_1 be a elementary row operation i.e multiplication with a scalar $\frac{1}{c}$.

So, Now,

$$e_1(e(A)) = e_1 \begin{bmatrix} c \cdot a_{11} & c \cdot a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$e_1(e(A)) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A$$

Definition:-

Let A and B be given $m \times n$ matrices we say A & B are row equivalent if one can be obtained from another by using finitely many elementary row operations.

ex: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & c \\ 1 & 0 \end{bmatrix}$

Here, A and B are row equivalent.

Row-Reduced Form:-

Let R be $m \times n$ matrix over the field, F (R or C)

we say R is in row-reduced form if

1) the first non-zero entry in each non zero rows of R is equal to 1.

2) Each column of R which contains the leading non zero entry of some row has all its other entries '0'.

Ex:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \checkmark \quad \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \times$$

$$I = (I_{ij}) \qquad I_{ij} = \begin{cases} 0 \ ; & i \neq j \\ 1 \ ; & i = j \end{cases}$$

$$\begin{bmatrix} 0 & 0 & 1 & 6 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \end{bmatrix} \checkmark \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \times$$

→ Every $m \times n$ matrix can be reduced to its row reduced form by using finitely many row operations.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row reduced echelon form:

Let R be given $m \times n$ matrix, we say R is in row-reduced echelon form, if it satisfies the following

1) R is row-reduced
2) every row of R which has all its entry '0' occurs below every row which has non zero entry.

3) If rows $1, 2, \ldots r$ are the non zero rows of R, and if the leading non zero entry of row i occurs in the column k_i, ~~where~~ then, $k_1 < k_2 < k_3 < \cdots < k_r$

ex:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark \qquad \begin{bmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 4 & 6 & 8 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \; -1 *$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; -1 *$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -4 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$-1 \times \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$(-1) \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

HW

1) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -5 & -11 & -13 \\ 0 & -3 & -3 & -6 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -5 & -11 & -13 \\ 0 & 1 & 1 & 2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 5 & 11 & 13 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 6 & 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$

2) $\begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 7 \\ 6 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & 1 \\ 0 & -10 & -11 \\ 0 & 1 & 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -10 & -11 \\ 0 & -4 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 19 \\ 0 & 0 & 13 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

* Any given $m \times n$ matrix is row equivalent to its row-reduced echelon form.

→ give examples of matrix A and B such that

(i) AB is defined, but BA is not defined.

(ii) give ex. of AB such AB, BA both defined but $AB \neq BA$.

(i) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$

①

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 6 \\ 8 & 9 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -6 \\ 0 & -7 & -21 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 7 & 21 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -9 \\ 0 & 1 & 6 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -9 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 7 & 6 & 4 \\ 5 & 8 & 9 \end{bmatrix} \xrightarrow{R_2 = R_2 - 7R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -17 \\ 5 & 8 & 9 \end{bmatrix}$$

same operation

perform of identity matrix. that will be the elementory matrix E.

So, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 = R_2 - 7R_1$

$$E \Rightarrow = \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 7 & 6 & 4 \\ 5 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -17 \\ 5 & 8 & 9 \end{bmatrix}$$

Rank: The no. of non zero rows in a row reduced echelon form.

Gauss elimination method to solve L.S. Eqn:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$Ax = B$

$\downarrow$

$Ux = B'$

$\downarrow$

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$\downarrow$

Augumented Matrix.

Now, we have to reduce to form,

$$\left[\begin{array}{ccc|c} u_{11} & u_{12} & u_{13} & b_1' \\ 0 & u_{22} & u_{23} & b_2' \\ 0 & 0 & u_{33} & b_3' \end{array}\right]$$

$$u_{33} \cdot x_3 = b_3'$$

$$x_3 = \frac{b_3'}{u_{33}}.$$

Now,

$$u_{22} x_2 + u_{23} x_3 = b_2'$$

$$\Rightarrow \quad x_2 = \frac{b_2' - u_{23} x_3}{u_{22}}$$

IIIy, $$x_1 = \frac{b_1' - u_{12} x_2 - u_{13} x_3}{u_{11}}$$

Ex

$$x + 2y - 4z = -4$$

$$5x + 11y - 21z = -22$$

$$3x - 2y + 3z = 11$$

Solve using Gauss elimination method.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 5 & 11 & -21 & -22 \\ 3 & -2 & 3 & 11 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 0 & 1 & -1 & -2 \\ 0 & -8 & 15 & 23 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 7 & 7 \end{array}\right]$$

$x_3 = 1$

$x_2 - x_3 = -2$

$\Rightarrow x_2 = -1$

$\Rightarrow x_1 = -4 - 2x_2 + 4x_3$

$x_1 = -4 + 2 + 4$

$x_1 = 2$

$\therefore x = 2$

$y = -1$

$z = 1$

Trace of matrix = sum of principle diagonal element of a square matrix.

$tr(A+B) = tr(A) + tr(B)$

$tr(c(A)) = c\, tr(A)$

$(A+B)^T = A^T + B^T$

$(AB)^T = B^T A^T$

$AB \neq BA$ in general

when; $AB = BA$?

① $AB = (AB)^T = B^T A^T = BA$

② $(AB)^T = (BA)^T = A^T B^T$

If, $AB = BA$ then
A, B are symmetric.

Determinants

* Any square matrix A can be expressed as a sum of a symmetric & skew symmetric.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$= B + C$$

$B^T = B$, $C^T = -C$

Note:- $A + A^T \to$ symmetric
$A - A^T \to$ skew symmetric

Minor and cofactor of an element a_{ij}:

Suppose,
$$A = (a_{ij})_{n \times n}$$

Minor of a_{ij} can be obtained by eliminating i^{th} row and j^{th} column from $A = (a_{ij})$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

cofactor of $A_{ij} = (-1)^{i+j} M_{ij}$

M = set of all square matrix.

Now, $f : M \longrightarrow \mathbb{R}$

$A \in M$

$$f(A) = |A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

Properties:-

① $|A| = |A^T|$

② $|cA| = c^n |A|$

③ $|A+B| \neq |A| + |B|$

④ $|AB| = |A||B|$

⑤ Interchanging row will give same value of determinant except the sign.

j^{th} $(-1)^j |A|$

⑥ if $|A| = 0$, then $|A| = 0$ not vice-versa.

Rank of a matrix:-

→ Rank (r) of a matrix A is a smallest positive integer r such that the matrix A has a submatrix of $r \times r$, whose determinant is non-zero.

→ Suppose $A_{m\times n}$, then rank $\leq \min(m,n)$

→ $A = 0 \iff$ rank $= 0$

ex:

$$A = \begin{bmatrix} 1 & 4 & 9 \\ 3 & 6 & 2 \\ 4 & 1 & 3 \end{bmatrix} = 1(16) - 4(1) + 9(-21)$$

$= 16 - 4 - 189$

$= -177 \neq 0$

rank $= 3$

(2) Find the value of μ for which rank of matrix A is 3.

$$A = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$$

$\mu(\mu - 6) + 1(11) \neq 0$

$\Rightarrow \mu^2 - 6\mu + 11 \neq 0$

$-\mu(6\mu) - 1(0) \neq 0$

$\Rightarrow \mu \neq 0$

$\cancel{\mu(\mu^2) + 1(0) = 0}$

$\mu[\mu(\mu - 6) + 11] - 1[6] = 0$

$\Rightarrow \mu[\mu(\mu - 6) + 11] - 6 = 0$

$\Rightarrow \mu^3 - 6\mu^2 + 11\mu - 6 = 0$

So, $\mu = 1$ (or) 2 (or) 3 $\quad r(A) = 3$

otherwise, $r(A) = 4$.

<u>Inverse:-</u>

$|A| = 0 \Rightarrow$ singular

$|A| \neq 0 \Rightarrow$ nonsingular

$AB = BA = I$

then, $A^{-1} = B$ (or) $B^{-1} = A$

We get inverse for a non-singular matrix.

Adjoint of a matrix:

adj (A) = Transpose of the matrix whose entities are cofactors of each element.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}^T$$

$$\therefore A^{-1} = \frac{\text{adj}(A)}{|A|}.$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|A| = 1(1+3) + 1(5) + 1(1)$$
$$= 4 + 5 + 1$$
$$= 10$$

$$\text{adj}(A) = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \Big/ 10.$$

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Solution of system of $n \times n$ linear equations :-

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

$AX = b$

if $|A| \neq 0$

$\Rightarrow A^{-1}AX = A^{-1}b$

$\Rightarrow IX = A^{-1}b$

$\Rightarrow X = A^{-1}b.$

Cramer's rule:

$$x_i = \frac{|A_i|}{|A|}$$

A_i = i^{th} column replaced by b.

when, $|A| = 0$, and $|A_i| \neq 0$ (no solution).

$|A| \neq 0$ (unique solution)

$|A| = 0$ and all $|A_i| = 0$ (infinitely many sd).

consistent → If A system has a sd^n.

inconsistent → system doesn't have sd^n.

Vector spaces:- / Linear spaces

V = set of elements usually called vector

F = Field

vector space has to satisfy some properties

adding (+) satisfies

1. for any $v_1, v_2 \in V$, $v_1 + v_2 \in V$
2. $v_1 + v_2 = v_2 + v_1 \quad \forall v_1, v_2 \in V$
3. $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$
4. $0 \in V$ such that $v + 0 = v \quad \forall v \in V$.
5. for any,

$v \in V \; \exists \; (-v)$ s.t $v + (-v) = 0$

scalar multiplication (·)

1. for any scalar $c \in F$, $v \in V$, $c.v \in V$.
2. $c(v_1 + v_2) = cv_1 + cv_2 \quad \forall v_1, v_2 \in V$
3. $(c_1 + c_2)v = c_1 v + c_2 v \quad \forall c_1, c_2 \in F$ and $v \in V$.
4. $\exists \; 1 \in F$ such that $1.v = v \quad \forall v \in V$.
5. $(c_1 c_2)v = c_1(c_2 v) \quad \forall c_1, c_2 \in F$, $v \in V$.

ex:

1) $V = \mathbb{R} = F$

Then $V(\mathbb{R})$ is a vector space

2) $V = \mathbb{R}^2$

$F = \mathbb{R}$

$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad cv = \begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix}$

$v + w = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$

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Here it is not a vector space, because we don't have $-v$ as $v \in \mathbb{R}^2$. So, it is not a vector space.

3) $V = \mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R} \right\}$

4) $V = \mathbb{R}^{m\times n} = \left\{ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} : a_{ij} \in \mathbb{R} \right\}$

This forms a vector space

5) $V = \{0\}$ $0+0=0$; $0 \cdot 0 = 0$

This is also vector space

6) $V =$ set of all polynomials of degree $\leq n$

$F = \mathbb{R}$ $p, q \in V$

$p(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$

$q(x) = q_0 + q_1 x + q_2 x^2 + \cdots + q_n x^n$

① $(p+q)(x) = p(x) + q(x)$

② $c\,(p(x)) = c \cdot p(x)$

This is a vector space.

If $V =$ set of polynomial of degree n.

Then not a vector space.

because, if, $p(x) = p_0 + p_1 x + \cdots + p_n x^n$

$q(x) = q_0 + q_1 x + \cdots - p_n x^n$

then $(p+q)(x) \notin V$.

So, not a vector space.

7) $V = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_2 = x_1 + 1 \right\}$

$F = \mathbb{R}$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

Here, $x_2 + y_2 = x_1 + y_1 + 2$

So, $(x_2 + y_2) \neq (x_1 + y_1) + 1$

So, not a vector space.

8) $V = \{ x : Ax = b \}$

$x \in \mathbb{R}^n \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$A_{n \times n}$

not a vector space

$A(x+y) \neq b$; if $Ax = 0$

then a vector space.

<u>Subspace:</u>

Suppose V is a vector space over the field $\mathbb{R}$.

Let $W \subset V$, we call W is a vector subspace of V if W is a vector space with respect to the same operation in V.

We need to check the distributive properties,

$\Rightarrow v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$

$\Rightarrow c(v_1 + v_2) = cv_1 + cv_2$.

$\Rightarrow v_1 + v_2 \in V$

① $v_1 + v_2 \in W \ \forall v_1, v_2 \in W$

② $0 \in W$

③ $cv \in W \ \forall v \in W, c \in F$

more simplified,

$$c \cdot v + w \in W \text{ (or) not}$$

<u>Theorem:</u>

A ~~[illegible]~~ non empty subset W of a vector space V is a subspace of V iff for any two vector $v, w \in W$

$$c \cdot v + w \in W, c \in F$$

<u>ex:</u>

$V = \mathbb{R}^n = \{$

$$V = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1 = 0 \right\} \subseteq \mathbb{R}^n$$

vector space / linear space:

<u>Theorem:</u> Let V_1 and V_2 are two subspace of a vector space V. $V_1 \cap V_2$ is also a subspace.

<u>Proof:</u> $v_1, v_2 \in V_1 \cap V_2 = W$

$$c_1 v_1 + c_2 v_2 \in W$$

$W = V_1 \cup V_2$ may not be holding the above theorem.

$$W = V_1 \cup V_2 = \left\{ \begin{matrix} (x,0)\,(or) \\ (0,y) \end{matrix} \right\}$$

$$V_1 = \{ (x,0) : x \in R$$

$$V_2 = \{ (0,y) : y \in R$$

Now, $v_1 + v_2 = (x,y) \notin W$.

So, union of two subspaces is not a subspace.

Linear combination of vector:

Suppose, $v_1, v_2, v_3 \ldots v_n$ are a set of vectors, then the linear combination of these would be

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

where,

$$c_i \in F = \mathbb{R}$$

Spanning Set:- Let V be a vector space and $S \subseteq V$. Then set S is called spanning set of V or S spans V if every element of V can be expressed as a linear combination of elements of S.

ex: Let V be the vector space of 2x2 matrices.

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a(\;) + b(\;) + c(\;) + d(\;)$$

So, S spans V.

ex: let V be the vector space of all polynomial of degree ≤ 3.

$$S = \{ t^3, t^2+t, t^3+t+1 \}$$

check whether S spans V or not.

Sol:-

$$c_1 t^3 + c_2 (t^2+t) + c_3 (t^3+t+1)$$

$$= p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3.$$

Here, it can't be expressed as linear comb.

so, S doesn't spans V. degree "0"

Linearly dependent set of vectors:

suppose $S = \{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vectors from a vector space V. we call the set S is linearly dependent if there exists scalars $c_1, c_2, \ldots, c_n$ (not all zero) such that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_n v_n = 0$$

(At least there should be one non zero c_i).

if all c_i's are zero, "Linearly independent"

Suppose, $c_i \neq 0$,

$$v_i = -\frac{c_1}{c_i} v_1 - \frac{c_2}{c_i} v_2 - \frac{c_3}{c_i} v_3 \cdots - \frac{c_n}{c_i} v_n$$

$$\cdots\left(-\frac{c_{i-1}}{c_i} v_{i-1} - \frac{c_{i+1}}{c_i} v_{i+1}\right)\cdots$$

if $\{v_1, v_2\}$ given,

if, $v_2 = c_1 v_1$, then linearly dependent

Properties :

① Any set which contains linearly dependent set is linearly dependent.

② Any subset of a linearly independent set is linearly independent.

③ Any set containing zero vector is always linearly dependent.

Ex:-

$V = \mathbb{R}^2$

$v_1 = (1,1)$ $v_2 = (-3,2)$ $v_3 = (2,4)$

$c_1 v_1 + c_2 v_2 + c_3 v_3$

Dependent.

$c_1 - 3c_2 + 2c_3 = 0$

$c_1 + 2c_2 + 4c_3 = 0$

$5c_2 + 2c_3 = 0$

$c_2 = -\frac{2}{5} c_3.$

so, c_1 can be arbitrary.

(2) $V = \mathbb{R}^3$

$v_1 = (1, -1, 0)$

$v_2 = (0, 1, -1)$

$v_3 = (0, 0, 1)$

$c_1 - c_2 + \cancel{0} = 0$

$c_2 - c_3 \neq 0$

$c_3 = 0$

$\Rightarrow c_1 = c_2 = c_3 = 0.$

"Independent."

$c_1 = 0$

$-c_1 + c_2 = 0$

$-c_2 + c_3 = 0.$

$\Rightarrow c_1 = c_2 = c_3 = 0.$

zero vector $\Rightarrow \theta$

zero scalar $\Rightarrow 0$

Theorem:-

Any set containing zero vector is linearly dependent set.

Pf Let,

$v_1, v_2, v_3, \dots v_n, \theta$ are the set of vectors from a vector space V over $\mathbb{R}$

$0v_1 + 0v_2 + \dots + 0v_n + e\theta = 0 + 0 + 0 + \dots + 0 = \theta,$

$e \neq 0$

So, there exists atleast one non zero scalar, $e \neq 0$. So, the set containing zero vector is linearly dependent.

Theorem:-

A collection of vectors which contains a collection of linearly dependent vectors is linearly dependent i.e Any super set of a linearly dependent set of vectors is linearly dependent.

Proof:-

Let $v_1, v_2, \dots, v_{p-1}, v_p, v_{p+1}, v_{p+2}, \dots, v_n$ are set of vectors from a vector space $V(\mathbb{R})$ in which $\{v_1, v_2, \dots, v_p\}$ is linearly dependent.

$c_1v_1 + c_2v_2 + \dots + c_pv_p = \theta \qquad c_i \neq 0 \ \forall i.$

$c_1v_1 + c_2v_2 + \dots + c_pv_p + 0v_{p+1} + 0v_{p+2} + \dots + 0 \cdot v_n = \theta$

So, the set containing $v_1, v_2, v_3, \dots, v_p, v_{p+1}, \dots, v_n$ is linearly dependent.

$\because$ atleast one $c_i \neq 0 \ \forall i$

Theorem: Any ~~set~~ subset of linearly independent set is linearly independent.

Proof:-

Same as previous theorem.

$AX = 0$ (or) $AX = b$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mxn} \end{bmatrix}_{m \times n}$$

$v_1 = (a_{11} \quad a_{12} \quad \cdots \quad a_{1n})$

$v_2 = (a_{21} \quad a_{22} \quad \cdots \quad a_{2n})$

$\vdots$

$v_m = (a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn})$

Here, $v_i \in \mathbb{R}^m$

rank $\leq \min\{m, n\}$

if, $r(A) = m$, then the set is linearly independent.

if $r(A) < m$, then linearly dependent.

Ex:

$$A = \begin{bmatrix} 2 & 4 & 8 & 12 & 8 \\ 1 & 2 & 4 & 6 & 4 \\ 1 & 2 & 2 & 2 & 2 \\ -1 & 0 & 2 & 4 & 2 \end{bmatrix}_{\substack{4 \times 5 \\ m \times n}}$$

$m = 4$

$n = 5$

$r = \min(4, 5)$

$r = 4$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 & 4 \\ 1 & 2 & 4 & 6 & 4 \\ 1 & 2 & 2 & 2 & 2 \\ -1 & 0 & 2 & 4 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 2 \\ -1 & 0 & 2 & 4 & 2 \end{bmatrix}$$

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$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 & 4 \\ -1 & 0 & 2 & 4 & 2 \\ 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 & 4 \\ 0 & 2 & 6 & 10 & 6 \\ 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 & 4 \\ 0 & 1 & 3 & 5 & 3 \\ 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 & 4 \\ 0 & 1 & 3 & 5 & 3 \\ 0 & 0 & -2 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 & 4 \\ 0 & 1 & 3 & 5 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 & -4 & -2 \\ 0 & 1 & 3 & 5 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(A) = 3 < (m = 4)$

So, A is linearly Dependent.

Replacement theorems:-

If the set $\{v_1, v_2, \ldots, v_n\}$

Problem:

Let $V_1, V_2, V_3 \ldots V_n$ are subsets of a vector space V.

$W = V_1 + V_2 + \ldots + V_n = \{v : v = v_1 + v_2 + v_3 + \ldots + v_n$, where $v_i \in V_i\}$

where, W is a vector subspace of V

$V = \{1, 2, 3, 4\}$

$V_1 = \{1, 2\}$

$V_2 = \{3, 4\}$

$V_1 + V_2 = \{1+3, 1+4, 2+3, 2+4\}$

Basis and dimensions of vector space:-

Suppose V is a vector space over $\mathbb{R}$. A basis for V is a linearly independent set of vectors in V which spans the space V.

Suppose B is basis then ① B is linearly independent
② B spans V.

→ Dimension is the total number of vectors in a basis.

→ Basis is not unique for a vector space.

Ex:

$V = \mathbb{R}^3$

$B = \{(1, -1, 0), (0, 1, -1), (0, 0, 1)\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$[I]$

Ex: Find the dimension of the vector subspace of $\mathbb{R}^4$ spanned by the set $\{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,1,1)\}$. Hence, find its Basis.

Sol:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$c_1(1,0,0,0) + c_2(0,1,0,0) + c_3(1,2,0,1) + c_4(0,0,1,1) = 0$

$\Rightarrow (c_1+c_3, c_2+2c_3, c_4, c_3+c_4) = (0,0,0,0)$

$\Rightarrow c_4 = 0$

$c_3 + c_4 = 0$

$\Rightarrow c_3 = 0$

$\Rightarrow c_2 = c_1 = 0$

So, linearly independent.

Dimension = 4

Replacement theorem:- If the set $\{v_1, v_2, \ldots, v_n\}$ be a basis of a vector space V over $\mathbb{R}$. Suppose, $v \neq 0 \in V$ such that v can be expressed as a linear combination

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \quad \text{where } c_j \neq 0 \text{ for}$$

some j.

Then if v_j is replaced by v then

$\{v_1, v_2, v_3, \ldots, v_{j-1}, v, v_{j+1}, \ldots, v_n\}$ is also a basis.

Theorem:- Suppose a basis of a vector space V contain n number of elements and if another linearly independent set contains m elements then $m \leq n$.

Proof:

$A = \{v_1, v_2, \ldots, v_n\}$ is a basis of a vector space V.

$B = \{w_1, w_2, \ldots, w_m\}$ is a linearly independent set. Here, $m \leq n$

$$w_1 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

claim $c_i \neq 0$ for atleast one i

Assume, $c_1 \neq 0$

$V = \{v_1, v_2, \ldots v_n\}$

By replacement theorem,

$V = \{w_1, v_2, \ldots v_n\}$

Now,

$w_2 = c_1 w_1 + c_2 v_2 + c_3 v_3 + \cdots + c_n v_n$

Say, $c_2 \neq 0$

By replacement theorem,

$w_3 = c_1 w_1 + c_2 w_2 + c_3 v_3 + \cdots + c_n v_n.$

after m steps, $\{w_1, w_2, w_3, \ldots w_m, v_{m+1}, v_{m+2}, \ldots v_n\}$

Here, we can conclude that $m \leq n$.

Extension theorem:- A linearly independent set of vectors can be extended to a basis.

In the last theorem if $m = n$, then it will be basis or else we have to extend w.

Theorem:- Any two basis of a vector space have same number of elements.

Proof: $A = \{v_1, v_2, \ldots v_n\}$

$B = \{w_1, w_2, \ldots, w_m\}$ are two bases.

Here, A is linearly independent and B is a basis. $n \leq m$.

B is linearly independent and A is a basis, $m \leq n$.

$$n \leq m \text{ and } m \leq n$$

$$\Rightarrow \quad n = m.$$

Theorem:- If $\{v_1, v_2, \dots, v_n\}$ be a basis of a vector space V then any vector V can be expressed uniquely as a linear combination of $v_1, v_2, \dots, v_n$.

if $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

$v = d_1 v_1 + d_2 v_2 + \dots + d_n v_n$

$$\theta = (c_1 - d_1) v_1 + (c_2 - d_2) v_2 + \dots + (c_n - d_n) v_n$$

$$\Rightarrow \quad c_i = d_i \ \forall i$$

Linear Transformation:-

$T : V \longrightarrow W$. $\quad T(c_1 v_1 + c_2 v_2) = c_1 T(v_1) + c_2 T(v_2)$.

any two $v_1, v_2 \in V$

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$

$T(cv) = c\,T(v)$, for any scalar c

and $\forall v \in V$.

$\text{Trace}(A+B) = Tr(A) + Tr(B)$

$tr(cA) = c\, tr(A)$

$M_{n\times n}$

$T : M \to R$

$T(A) = \text{trace } A.$

Sum and direct sum of vector spaces:-

Let $W_1, W_2, \ldots, W_n$ are subspace of a vector space V, Then

$W = W_1 + W_2 + W_3 + \cdots + W_n$ is a vector subspace of V.

Proof: let $u, v \in W$, $c_1 u + c_2 v \in W$

$cu + v \in W$

$u = w_1 + w_2 + \cdots + w_n \quad w_i \in W$
$v = w_1' + w_2' + \cdots + w_n' \quad w_i' \in W$

$c(w_1 + w_2 + \cdots + w_n) + (w_1' + w_2' + \cdots + w_n')$

$= (cw_1 + w_1') + (cw_2 + w_2') + \cdots + (cw_n + w_n')$

Here, $cw_1 + w_1' \in W_1$
$cw_2 + w_2' \in W_2$

So, W is a subspace of V.

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V is a vector space $\dim(V) = n$

W is a vector subspace $\dim(W) = m$

So, always $\dim(W) \leq \dim(V)$.

$W = W_1 + W_2$

$W_1 \cap W_2 \longrightarrow$ vector space

$W_1 \cap W_2 \subset W_1 + W_2$

Theorem:

If W_1 and W_2 are finite dimensional vector subspaces of a vector space V, Then $W_1 + W_2$ is finite dimensional and

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Proof:

$W_1 \cap W_2 \subset W_1$	$\dim(W_1 + W_2) \leq \dim(W_1)$
$W_1 \cap W_2 \subset W_2$	$\dim(W_1 + W_2) \leq \dim(W_2)$

Since, $W_1 \cap W_2$ is finite dimensional, it has a finite basis.

Say, $\beta_1 = \{v_1, v_2, \ldots, v_k\}$

β_1 is a part of $\{v_1, v_2, v_3, \ldots, v_k, u_1, u_2, u_3, \ldots, u_n\}$ of some basis of W_1. $\dim(W_1) = k + n$

lly, β_2 will be a part of $\{v_1, v_2, \ldots v_k, w_1, w_2, \ldots w_m\}$ of some basis of W_2.

$\dim(W_2) = k + m$

$\dim(W_1+W_2) = k+n+m$

$\because W_1+W_2 = \{v_1, v_2, v_3, \dots, v_k, u_1, u_2, \dots u_n, w_1, w_2, \dots, w_m$

Now,

$\dim(W_1) = k+n$

$\dim(W_2) = k+m$

$\dim(W_1 \cap W_2) = k$

So, $\dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

$= k+n+k+m-k$

$= k+n+m$

$= \dim(W_1+W_2)$.

$\therefore \dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

above we have to prove that (W_1+W_2) is a basis.

$\{v_1, v_2, \dots v_k, u_1, u_2, \dots, u_n, w_1, w_2, \dots, w_m\}$

Spans (W_1+W_2).

$$\sum_{i=1}^{k} x_i v_i + \sum_{j=1}^{n} y_j u_j + \sum_{k=1}^{m} z_k w_k = \theta$$

For it to be L.I, then $x_i = y_j = z_k = 0 \ \forall i, j, k$.

$$-\sum z_k w_k = \sum x_i v_i + \sum y_j u_j \in W_1$$
$$\in W_2$$

$$\sum z_k w_k = \sum c_i v_i \Rightarrow z_k = 0 \ \forall k.$$

$$\Rightarrow \quad 0 = \sum x_i v_i + \sum y_j u_j \in W_1$$

$$\Rightarrow \quad x_i = y_j = 0 \quad \forall i, j$$

Here,
$\{v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_n, w_1, w_2, \ldots, w_m\}$

spans $W_1 + W_2$ and is linearly independent.

So, <u>basis</u>

$$\dim(W_1 + W_2) = k + n + m$$
$$\dim(W_1) = k + n$$
$$\dim(W_2) = k + m$$
$$\dim(W_1 \cap W_2) = k$$
$$\Rightarrow \dim(W_1 + W_2) = (k+n) + (m+k) - (k)$$
$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

<u>direct sum:</u>

notation of direct sum.

$$W = W_1 \oplus W_2 \quad ; \quad W_1 + W_2 = \{0\}$$

$$W_1, W_2, W_3, \ldots, W_k$$

$$W = W_1 \oplus W_2 \oplus \cdots \oplus W_k \quad , \quad W_j \cap (W_1 + W_2 + \cdots + W_{j-1}) = \{0\}$$
for, $2 \le j \le k$

$$\rightarrow \quad w_1 + w_2 + \cdots + w_k = 0 \Rightarrow w_i = 0 \ \forall i.$$

Ex: $M_{n\times n}$ = set of all $n\times n$ matrixes

W_1 = set of all symmetric matrix

W_2 = set of skew symmetric matrix

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$= B + C$$

$$\Rightarrow \begin{cases} B \in W_1 \\ C \in W_2 \end{cases}$$

$$V = W_1 + W_2$$

Here, $W_1 \cap W_2 = \{0\}$

So, the above example is of direct sum.

(2) $M_{n\times n}$ = set of all $n\times n$ matrix

W_1 = set of all upper triangular matrix

W_2 = set of all lower

$$V = W_1 + W_2$$

But, $W_1 \cap W_2 = ?$

Linear transformation:-

$T(v) = 0$ trivial linear transformation

$T(v) = v$ non trivial.

ex:1) Let A be a fixed $m \times n$ matrix, Define

$T: F^{n\times 1} \longrightarrow F^{m\times 1}$

$T(X) = AX$, then T is a linear transformation.

Take, $X_1, X_2 \in F^{n\times 1}$, $c \in F$

then,

$T(cX_1 + X_2) = cT(X_1) + T(X_2)$

$= A(cX_1 + X_2)$

$= A(cX_1) + A(X_2)$

$= c\,A(x_1) + A(x_2)$

$= c\,T(x_1) + T(x_2)$

2) Let P be a fixed $m\times m$ matrix over the field F and Q be a fixed $n \times n$ matrix.

Define $T: F^{m\times n} \longrightarrow F^{m\times n}$ by $T(A) = PAQ$ $A \in F^{m\times n}$

Take, $A, B \in F^{m\times n}$, $c \in F$

then,

$T(cA + B) = c\,T(A) + T(B)$

$= P(cA + B)Q$

$= P(cA)Q + PBQ$

$= c\,PAQ + PBQ$ $= c\,T(A) + T(B)$

Let $T: V \to W$ be a linear transformation

Show, $T(\theta_v) = \cancel{T(}\theta_w)$

$$T(\theta + \theta) = T(\theta) + T(\theta)$$

$$T(\theta) = 2T(\theta)$$

$$\Rightarrow T(\theta) = \theta \in W$$

→ Image of a zero vector is a zero vector.

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Theorem:-

Let V be a finite dimensional vector space over the field F and let $\{v_1, v_2, \dots, v_n\}$ be an ordered basis of V.

Suppose W is another vector space over the same field F and

$\{w_1, w_2, \dots, w_n\}$ are any vector in W.

Then there is a precisely one linear transformation from V to W such that

$$T(v_j) = w_j \quad ; j = 1, 2, 3, \dots, n$$

Proof:

We prove that there is a linear transformation $T: V \to W$ such that

$$T(v_j) = w_j \quad , j = 1, 2, 3, \dots, n$$

$\{v_1, v_2, v_3, \ldots, v_n\}$ is a ordered basis of V, so $\exists$ scalars $x_1, x_2, \ldots, x_n$ such that any vector $v \in V$ can be written as

$$v = x_1 v_1 + x_2 v_2 + x_3 v_3 + \cdots + x_n v_n \quad \text{—(1)}$$

Let us define $T: V \to W$ by

$$T(v) = x_1 w_1 + x_2 w_2 + \cdots + x_n w_n$$

$$v = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$$

$$v' = x_1' v_1' + x_2' v_2' + \cdots + x_n' v_n'$$

$$T(cv + v') = \sum_{i=1}^{n} (c x_i + x_i') W_i$$

$$= c x_1 W_1 + x_1' W_1 + c x_2 W_2 + x_2' W_2 + \cdots$$
$$\cdots + c x_n W_n + x_n' W_n$$

$$= c(x_1 W_1 + x_2 W_2 + \cdots + x_n W_n)$$
$$+ (x_1' W_1 + x_2' W_2 + \cdots + x_n' W_n)$$

$$= c\,T(v) + T(v')$$

et us define $T: V \to W$ by

$$T_1 : V \to W \qquad T_1(V_j) = W_j$$

Suppose, $V = \sum x_i V_i$

$$T_1(V) = T_1\left(\sum x_i V_i\right)$$

$$= \sum x_i \, T_1(V_i)$$

$$= \sum x_i W_i$$

$$= T(v) \Rightarrow \underline{T_1 = T'}$$

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Theorem:

Def^n: $V = \mathbb{R}^2$, $W = \mathbb{R}^3$

(1) $v_1 = (1,2)$ $v_2 = (3,4)$

$T : \mathbb{R}^2 \to \mathbb{R}^3$

$T(v_1) = (3,2,1)$

$T(v_2) = (6,5,4)$

$T(1,0) = ?$

Def'n: (Range of T): Suppose $T : V \to W$ then

$R(T) = \{T(v) : v \in V\}$

Theorem: $R(T)$ forms a vector space and subspace of W (Range space of W).

Proof.

$R(T) = \{T(v) : v \in V\}$

$= \{w : w = T(v) \text{ for some } v\}$

$w_1, w_2 \in R(T)$, $c \in F$ then

$\exists$ some $v_1, v_2 \in V$ such that

$T(v_1) = w_1$, $T(v_2) = w_2$

$T(cv_1 + v_2) = cT(v_1) + T(v_2)$

$= c w_1 + w_2$

So, $cw_1 + w_2 \in R(T)$

Hence, $R(T)$ is a vector space

Def^n Null Space: $N(T) = \{v : T(v) = 0\}$ is a subspace of V.

Proof: $v_1, v_2 \in N(T)$, $c \in F$, $T(v_1) = 0$, $T(v_2) = 0$

$T(cv_1 + v_2) = cT(v_1) + T(v_2) = 0$

$\Rightarrow cv_1 + v_2 \in N(T)$.

Def^n Suppose V is a finite dimensional vector space, dimension of $R(T)$ is known as the rank of T.

dimension of $N(T)$ is called nullity of T.

Theorem:-

Let V and W be two vector spaces over some field F

$T : V \to W$ be a linear transformation. Then

$$\boxed{\text{rank}(T) + \text{nulity}(T) = \dim(V).}$$

Proof:

Suppose $\{v_1, v_2, \ldots, v_k\}$ is a basis of $N(T)$.

Then there exists vectors $v_{k+1}, v_{k+2}, \ldots, v_n$ such that $\{v_1, v_2, \ldots, v_n\}$ is a basis of V ($\dim V = n$)

Claim: $\{T(v_{k+1}), T(v_{k+2}), \ldots T(v_n)\}$ forms a basis of $R(T)$

Clearly $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ spans $R(T)$

To show,

$T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)$ spans $R(T)$

Since, $T(v_1) = T(v_2) = \ldots = T(v_k) = 0$

then,

$T(v_{k+1}), T(v_{k+2}) \ldots, T(v_n)$ spans $R(T)$.

Now, we have to show that the set,

$\{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}$ is L-Independent.

let, $\sum_{i=k+1}^{n} c_i T(v_i) = 0$

$\Rightarrow \sum T(c_i v_i) = 0$

$\Rightarrow T(\sum c_i v_i) = 0$

$\Rightarrow \sum c_i v_i \in NT$

Since, $\{v_1, v_2, \ldots, v_k\}$ is a basis of $N(T)$

and $\sum_{i=k+1}^{n} c_i v_i \in N(T)$

$$\sum_{i=k+1}^{n} c_i v_i = \sum_{i=1}^{k} d_i v_i$$

$$\Rightarrow \sum_{i=k+1}^{n} c_i v_i - \sum_{i=1}^{k} d_i v_i = 0$$

$\Rightarrow c_i = d_i = 0 \ \forall i$

$\{T(v_{k+1}), T(v_{k+2}), \dots T(v_n)\}$ is L.I.

Hence $\{T(v_{k+1}), T(v_{k+2}), \dots, T(v_n)\}$ is a basis of $R(T)$.

Therefore, $\dim(R(T)) = n-k$

$\Rightarrow$ Rank $(T) = n-k$

and Rank of $N(T)$

$\Rightarrow$ nulity $(T) = k$

Rank (T) + nulity (T)

$= n-k+k$

$= n$

$= \dim(V)$

$\therefore \boxed{\text{Rank}(T) + \text{nulity}(T) = \dim(V)}$

Def^n:- Let A be a $m \times n$ matrix over the field F

The row vector of A are $V_1, V_2, \dots, V_m$ in F^n defined by

$v_i = (a_{i1}, a_{i2}, \dots, a_{in}) \quad i = 1, 2, \dots, m$

Row space is a vector space stand by the row ~~values~~. vectors.

11y, column space is defined.

Theorem:-

If A is a $m\times n$ matrix over the Field F then

row rank (A) = column rank (A)

Define:

$T: F^{n\times 1} \longrightarrow F^{m\times 1}$ by $T(x) = AX$

$N(T) = \{ X : TX = 0 \Rightarrow AX = 0 \}$

$R(T) = \{ Y_{m\times 1} : AX = Y$ for some $X \in F^{n\times 1} \}$

Suppose, $A_1, A_2, \dots, A_n$ be the columns of A

$T(x) = AX = x_1 A_1 + x_2 A_2 + \dots\dots + x_n A_n$

$R(T)$ is spanned by the columns of A.

$\Rightarrow R(T)$ is a column space.

rank (T) = column rank of A.

$\dim(R(T)) + \dim N(T) = \dim V$

$\dim N(T)$ + column rank of $A = \dim V = n$

Suppose r is the dimension of row space of A

then, $N(T)$ has a basis with $(n-r)$ elements

$\Rightarrow$ $n - r$ + column rank $= n$

$\Rightarrow$ column rank $= r$ = row rank of A.

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Algebra of Linear transformation :-

Theorem:- Let V and W are two vector spaces over the field F. suppose T_1 and T_2 are two linear transformation from V to W. Then the function (T_1+T_2) is defined by

$$(T_1+T_2)(v) = T_1(v) + T_2(v) \quad \forall v \in V, \text{ is a}$$

linear transformation.

If $c \in F$, the function $(cT_1)v$ is defined by

$(cT_1)(v) = cT_1(v) \; \forall v \in V$ is also a L.T.

→ The set of all linear transformation from V to W, together with the above two operation forms a vector space over the field.

Proof:

Let A be the set of all linear transformation from V to W

Let $T_1, T_2 \in A$

$\Rightarrow T_1 + T_2 \in A$

Take $v_1, v_2 \in V$ and $c \in F$

To show,

$$(T_1+T_2)(cv_1+v_2) = c(T_1+T_2)v_1 + (T_1+T_2)v_2$$

$$= T_1(cv_1+v_2) + T_2(cv_1+v_2)$$

$$= cT_1v_1 + T_1v_2 + T_2cv_1 + T_2v_2$$

$$= c((T_1+T_2)v_1) + (T_1+T_2)v_2$$

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$A = L(V, W)$ = set of all L.T from $V \to W$

__Note__

① Suppose V be n-dimensional vector space, W be a m-dimensional vector space. $L(V, W)$ is finite dimensional vector space and $\dim(L(V, W)) = mn$.

② Suppose U, V, W are three vector spaces over some field F

$T_1 : U \to V$ be a L.T

$T_2 : V \to W$ be a L.T

Define $(T_2 T_1)v = T_2(T_1(v))$

Show that $T_2 T_1 : U \to W$ is L.T

$(T_2 T_1)(cv_1 + v_2) = c(T_2 T_1)v_1 + (T_2 T_1)(v_2)$

$\Rightarrow T_2 T_1 (cv_1) + T_2 T_1 (v_2)$

$= c T_2 T_1 (v_1) + T_2 T_1 (v_2)$

$T_2 T_1 (cv_1 + v_2)$

$= T_2 (T_1 (cv_1 + v_2))$

$= T_2 (c T_1 v_1 + T_1 v_2)$

$= c T_2 T_1 v_1 + T_2 T_1 v_2$

$\Rightarrow T_2 T_1 : U \to W$ is a linear transformation

③ There can be a L.T such that $T: V \to V$.

$T: V \to V$ is called linear transformation on V.

Example:-

$A \in F^{m \times n}$

$T_1 : F^{n\times 1} \longrightarrow F^{m \times 1}$ by $T(X) = AX$ $A \in F^{m \times n}$

$T_2 : F^{m\times 1} \longrightarrow F^{p \times 1}$ by $T(Y) = BY$ $B \in F^{p \times m}$

$T_2 T_1 : F^{n\times 1} \longrightarrow F^{p\times 1}$

$T_2 T_1 (X)$

$= T_2 (T_1 (X))$

$= T_2 (A (X))$

$= \cancel{A T_2 (X)}$ $= B(AX)$

$= \cancel{AB(X)}$ $= (BA)X$

$= \cancel{(AB)X}$ $\therefore T_2 T_1 (X) = (BA)X$

Def:
~~Proof:~~

Let $T: V \to W$ be a function. T is called invertible if there exists a function $T_1 : W \to V$ such that $T_1 T$ is a identity ($T_1 T = I$) on V.

$T T_1 = I$ on W

T invertible we use ($T^{-1} = T_1$)

iff, ① T is 1-1

② T is onto $R(T) = W$

Suppose T is a L.T, from $V \to W$ then $T^{-1}: W \to V$ is also a L.T? (yes)

Proof of above prev page dbt:-

$T(x) = T(y) \Rightarrow x = y$ for one-one func

Suppose, T in L.T,

$T(x-y) = T(x) - T(y)$

$T(x) = T(y)$

$\Rightarrow T(x-y) = 0$

As per nulity, $N(T) = \{v : T(v) = 0\}$

$= \{0\}$

$\Rightarrow x - y = 0$

$\Rightarrow x = y$.

A linear transformation is 1-1 iff $N(T) = \{0\}$

Example: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$T(x_1, x_2) = (x_1 + x_2, x_1)$$

Show that T is non singular L.T and compute T^{-1}.

Sol:- $T(x_1, x_2) = 0$

$\Rightarrow T(x_1, x_2) = 0$

$x_1 + x_2 = 0 ; x_1 = 0$

$\Rightarrow x_2 = 0$

$\Rightarrow x_1 = x_2 = 0.$

$\Rightarrow N(T) = \{0\}$ So, T is 1-1.

$z_1, z_2 \in \mathbb{R}^2$

$z_1 = x_1 + x_2$

$z_2 = x_1$

So, $T(z_1, z_2) = T(z_2, z_1 - z_2)$

<u>Theorem:</u> Let T be a linear transformation from $V \to W$, Then T is non-singular iff T carries each linearly independent subset of V onto a linearly independent subset of W.

<u>Proof:</u> Suppose T is non-singular

To prove T carries each of linearly independent set of V to a L.I set of W.

Let $S = \{v_1, v_2, \ldots v_k\}$ is a linearly independent set.

$\{T(v_1), T(v_2), \ldots, T(v_k)\}$ is Linearly independent.

Let us consider,

$c_1 T(v_1) + c_2 T(v_2) + \ldots + c_k T(v_k) = 0$
$\forall c_i \in F$

$$T(c_1v_1 + c_2v_2 + \cdots + c_kv_k) = 0 \qquad N(T) = \{0\}$$

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

$$\Rightarrow c_1 = c_2 = \cdots = c_k = 0$$

Now, the converse,

Let us assume T carries each of the linearly independent subset of V onto a linearly independent subset of W.

To prove T is non singular

Suppose, $v \neq 0$ is a vector in V

$$S = \{v\}$$

$$S = \{T(v)\} \Rightarrow T(v) \neq 0 \quad [\because T(v) \text{ is L.I}]$$

$$v \neq 0$$

$$\Rightarrow T(v) \neq 0$$

$$\Rightarrow N(T) = 0$$

Hence, T is non-singular

Matrix representation of linear transformation:

Let V be a n dimensional vector space over the field F. Let W be an m-dimensional vector space over the same field F.

Let $B = \{v_1, v_2, \ldots, v_n\}$ is a basis of V

$B' = \{w_1, w_2, \ldots, w_m\}$ is a basis of W

$T: V \rightarrow W$ is a linear transformation.

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \ , \quad j = 1, 2, \ldots, n$$

$T(v_j) \in W$

$T(v_1) = a_{11} w_1 + a_{21} w_2 + a_{31} w_3 + \cdots + a_{m1} w_m$

The scalars $a_{1j}, a_{2j}, \ldots, a_{mj}$ are the coordinates of $T(v_j)$ with respect to B'.

$$\begin{array}{cc} & T(v_1) \\ A = & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

This $m \times n$ matrix $A = (a_{ij})$ is called matrix of T relative to the pair of B, B'.

How the matrix A determines Linear transformation?

$$x = x_1 v_1 + x_2 v_2 + \dots + x_n v_n$$

$$T(x) = x_1 T(v_1) + x_2 (T(v_2)) + \dots + x_n T(v_n)$$

$$= \sum_{i=1}^{n} x_i T(v_i)$$

Here, $T(v_i) \in W$

So,

$$= \sum_{i=1}^{n} x_i \sum_{j=1}^{m} a_{ij} w_j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i a_{ij}) w_j$$

Coordinates of $v = \{x_1, x_2, \dots, x_n\} = X$

coordinates of $T(v) = AX$

→ representation $[v]_B = \{x_1, x_2, \dots, x_n\}$

$V \rightarrow n$ dimensional vector space

$W \rightarrow m$ dimensional vector space

$$[T(v)]_{B'} = A[v]_B$$

$T: V \rightarrow M$

$A_{m \times n}$. but not $A_{n \times m}$.

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$ (6)

$T(x_1, x_2) = (x_1, 0)$

check it is L.T. Then find the matrix of T relative to standard basis.

$B = \{(1,0), (0,1)\}$

$B' = \{(1,0), (0,1)\}$

$T(1,0) = (1,0) = \underline{1}(1,0) + \underline{0}(0,1)$

$T(0,1) = (0,0) = \underline{0}(1,0) + \underline{0}(0,1)$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

(2) $T: \mathbb{R}^3 \to \mathbb{R}^2$

$T(x, y, z) = (y - z, y - z)$

w.r.t S.Basis.

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$

$B' = \{(1,0), (0,1)\}$

$T(1,0,0) = (0,0)$

$= 0(1,0) + 0(0,1)$

$T(0,1,0) = (1,1)$

$= 1(1,0) + 1(0,1)$

$T(0,0,1) = (-1,-1)$

$= -1(1,0) + -1(0,1)$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

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3

$V =$ space of all polynomial of degree less than or equal to 3.

$T(f) = f'(x)$

$T : V \longrightarrow V$

$\{f : f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3\}$.

$B = \{1, x, x^2, x^3\}$ $\quad B' = \{1, x, x^2\}$

$T(1) = 0$

$= 0(1) + 0(x) + 0(x^2) + \cancel{0(x^3)}$

$T(x) = 1$

$= 1(1) + 0(x) + 0(x^2) + \cancel{0(x^3)}$

$T(x^2) = 2x$

$= 0(1) + 2(x) + 0(x^2) + \cancel{0(x^3)}$

$T(x^3) = 3x^2$

$= 0(1) + 0(x) + 3(x^2) + \cancel{0(x^3)}$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

suppose U, V, and W are vector space over the field F with respective dimension n, m, and p.

$$B = \{u_1, u_2, \dots, u_n\}$$

$$B' = \{v_1, v_2, \dots, v_m\}$$

$$B'' = \{w_1, w_2, \dots, w_p\}$$

$T_1 : U \longrightarrow V$ be a L.T

$T_2 : V \rightarrow W$ be a L.T

Let A be the matrix of T_1 relative to the basis B, B'.

Let B be the matrix of T_2 relative to the basis B', B''.

$T_2 T_1 : U \longrightarrow W$ is a linear transformation

C is the matrix of $T_2 T_1$ relative to the basis B, B''

$$C = BA$$

$$[T_1(u)]_{B'} = A[u]_B$$

$$[T_2(v)]_{B''} = B[v]_{B'}$$

$$[T_2 T_1(u)]_{B''} = T_2[T_1(u)] = B[T_1(u)]$$

$$= BA[u]_B$$

$A = (a_{ij})_{m \times n}$

$B = (b_{ij})_{n \times p}$

$$AB = \sum_{k=1}^{m} (a_{ik} b_{kj})_{m \times p}$$

Matrix multiplication using LT:

$$(T_2 T_1)(u_j) = T_2(T_1(u_j)) \quad j = 1, 2, \ldots, n$$

$$= T_2\left(\sum_{k=1}^{m} a_{kj} v_k\right)$$

$$= \sum_{k=1}^{m} a_{kj} T_2(v_k)$$

$$= \sum_{k=1}^{m} a_{kj} \sum_{i=1}^{p} b_{ik} w_i$$

$$= \sum_{k=1}^{m} \sum_{i=1}^{p} a_{kj} b_{jk} w_i$$

~~The/ U,V,W / be~~

Theorem:

Let U, V, W be the finite dimensional spaces over the field F. Let T_1 be a linear transformation from U to V and T_2 be a linear transformation from V to W.

If B, B', B'' are bases of U, V and W, respectively and A is the matrix of T_1, relative to the bases B, B' and B is the matrix of T_2 relative to B', B'' then $C = BA$ is the matrix of $T_1 T_2$ relative to basis B, B''.

Change of ordered Basis:-

Suppose $T: V \to V$, V is finite dimensional

$$B = \{ v_1, v_2, \ldots v_n \} \quad B' = \{ v_1', v_2', \ldots, v_n' \}$$

A is the matrix of T relative to basis B

$$T[(v)]_B = A[v]_B$$

$$T[(v')]_{B'} = B[v']_{B'}$$

Def^n (similar matrix)

A matrix A is said to be similar to B if there exists a non singular matrix P such that $B = P^{-1}AP$

$$[T]_{B'} = P^{-1}[T]_B P$$

$$P = [P_1, P_2 \ldots P_n]_{n \times n}$$

$$P_j = [v_j]_B$$

$T: R^2 \to R^2$

$T(x_1, x_2) = (x_1, 0)$

$$[T]_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$B' = \{(1,2), (2,1)\}$

$(1,2) = 1(1,0) + 2(0,1)$

$(2,1) = 2(1,0) + 1(0,1)$

$$P = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}^T}{|P|}$$

$B = P^{-1} A P$

$$B = -\frac{1}{3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \frac{\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}}{-3}$$

$$B = -\frac{1}{3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$B = -\frac{1}{3}\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$$

$$B = \begin{bmatrix} -1/3 & -2/3 \\ 2/3 & 4/3 \end{bmatrix}$$

26/08

A similar to B

B similar to C

$C = Q^{-1} B Q$

$= Q^{-1} P^{-1} A P Q$

$= (PQ)^{-1} A (PQ)$

Hence, C similar to A.

Def^n

(Linear functional)

If V is a vector space over the field F, a linear transformation $T : V \to F$ is called a linear function.

Ex:- 1) $T : F^{n \times n} \to \mathbb{R}$

$T(A) = \text{Trace of } A$

2) $T(f) = \int_a^b f \quad f \in V = C[a,b]$

3) $T : F^n \to F$

$T(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$

$$[T]_B = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & \cdots & 0 \\ \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

$[T]_B = A =$ ~~diag~~ $[a_1 \; a_2, \ldots \; a_n]$

$$L(V, W) = mn$$

$L(V, W) \rightarrow$ set of all L.T from $V \rightarrow W$. which would be a vector Space.

$L(V, F) \rightarrow$ collection of all L.F from $V \rightarrow W$.

$$L(V, F) = m \times 1$$

$$V^* = L(V, F) = \{ f : f\ V \rightarrow \cancel{X}F \text{ is a L.T} \}$$

$$= \text{Dual space of } V.$$

$$\dim(V^*) = \dim(V)$$

$$B = \{v_1, v_2, \ldots, v_n\}$$

we need to find the basis of V^*,

So, take

$$f_i(v_j) = \delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

So, $\{f_1, f_2, \ldots, f_n\}$

Eigen vector:

$$AX = \lambda X$$

$\downarrow$

Eigen vector.

Defn.

Suppose, $T: V \to V$ be a linear ~~transformation~~ operator and V is finite dimensional. A non-zero vector ($v \neq 0$) and a scalar ($\lambda \in F$) are called eigenvectors and eigenvalue of the linear operator T if $T(v) = \lambda v$.

λ = eigenvalue

v = eigen vector

λ-eigen space :- Collection of all eigen vectors corresponding to the eigenvalue, λ.

spectrum:- Collection of all eigen values of a linear operator is called spectrum of the linear operator.

Algebraic multiplicity:- No. of repitition of eigen value.

Geometric multiplicity: Number of linearly independent eigen vectors corresponding to eigen value λ is called geometric multiplicity.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 4 & 6-\lambda & 7 \\ 2 & 4 & 6-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\underline{|A - \lambda I| = 0}}$$

$\cancel{(1+\lambda/\lambda)}$

$$(1-\lambda)\left[(6-\lambda)(6-\lambda) - 28\right]$$

$$-2\left[4(6-\lambda) - 14\right]$$

$$+3\left[16 - 2(6-\lambda)\right] = 0$$

$$\Rightarrow (1-\lambda)\left[36 + \lambda^2 - 12\lambda - 28\right] - 2\left[10 - 4\lambda\right]$$

$$+3\left[4 + 2\lambda\right] = 0$$

$$\Rightarrow \lambda^2 - 12\lambda + 8 - \lambda^3 + 12\lambda^2 - \cancel{8\lambda} - 20 + \cancel{8\lambda}$$
$$+12 + 6\lambda = 0$$

$$\Rightarrow -\lambda^3 + 13\lambda^2 - 6\lambda - 14 = 0$$

$$\Rightarrow \lambda^3 - 13\lambda^2 + 6\lambda + 14 = 0$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & 6 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 7 \\ 0 & 0 & 6-\lambda \end{bmatrix}$$

$$\Rightarrow (1-\lambda)\left[(1-\lambda)(6-\lambda)\right] = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda)(6-\lambda) = 0$$

$$\lambda = 1, 1, 6$$

$\cancel{\gamma \neq y}$

$\cancel{T(v) = 1(v)}$

$\cancel{(1,0,0) = 1(1,0,0)}$

$\cancel{T(v_1) = (1,0,0)}$

$\cancel{T(v_2) = (2,1,0)}$

$\cancel{T(v_3) = (3,7,6)}$

for $\lambda = 1$,

algebraic multiplicity $= 2$

Geometric multiplicity $= 1$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 7 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5z = 0$

$\Rightarrow z = 0$

$\Rightarrow y = 0$

So, $x = k$ (So, k is a scalar)

$$v = k\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} -5 & 2 & 3 \\ 0 & -5 & 7 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$z = k$

$-5y + 7z = 0$

$5y = 7k$

$y = 7k/5$

$-5x + 2y + 3z = 0$

$5x = \frac{14k}{5} + 3k$ (4)

$5x = 29k/5$ wrong

$x = \frac{29k}{25}$

$-5y + 7z = 0$

$-5x + 2y + 3z = 0$

$z = k$

$y = 7k/5$

$x = \frac{29}{25}k$

$y = l$

$z = 5l/7$

$x = \frac{29l}{7\times5}$

$$k\begin{bmatrix} 29/25 \\ 7/5 \\ 1 \end{bmatrix} \quad p\begin{bmatrix} 1 \\ 35/29 \\ 25/29 \end{bmatrix}$$

$$l\begin{bmatrix} 29/35 \\ 1 \\ 5/7 \end{bmatrix}$$

All these are same

So, $v = k\begin{bmatrix} 29/25 \\ 7/5 \\ 1 \end{bmatrix}$

$x = p$

$2\times(7z = 5y) = 10y - 14z = 0$

$5(x\ 2y + 3z = 5p) = 10y + 15z = 25p$

$29z = 25p$

$y = 35/29\,p \quad z = \frac{25p}{29}$

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Theorem: Each λ-eigenspace is a subspace of V.

Proof:

Let $v_1, v_2, \in V$ be two eigenvectors corresponding to λ

$$cv_1 + v_2 \in \lambda\text{-eigenspace}$$

$$T(cv_1 + v_2) = T(cv_1) + T(v_2)$$

$$= c\,T(v_1) + T(v_2)$$

$$= c\lambda v_1 + \lambda v_2$$

$$= \lambda(cv_1 + v_2)$$

so, λ-eigenspace is subspace of V.

Hence proved

09/09/19

Q- Whether eigen value and eigenvector exists for every L.T

$$T: R^2 \to R^2$$

$$T(x_1, x_2) = (-x_2, x_1)$$

$$T(x_1, x_2) = \lambda(x_1, x_2) \quad , \; x \neq 0$$

$$(-x_2, x_1) = (\lambda x_1, \lambda x_2)$$

$$x_1 = \lambda x_2$$

$$x_2 = -\lambda x_1$$

$$\Rightarrow x_1 = -\lambda^2 x_1 \quad \Rightarrow x_1(1+\lambda^2) = 0$$

So, since $x \neq 0$

$\cancel{\lambda^2+1=0}$

$\cancel{\lambda = i, -i}$

Here, $(1+\lambda^2)$ is a positive quantity. So,

$x_1 = 0 \Rightarrow x_2 = 0$

So, there exists no non zero x.

$T(1,0) = (0,1)$
$T(0,1) = (-1,0)$

So, this is rotational operator $(90°)$

But, $T = \lambda x$ is translational operator.

So, there are no non zero "x".

$A = [T]_B$

$B = [T]_{B'}$

Whether A, B have same eigenvalues or not?

$AX = \lambda X$ $\quad\quad$ $BX = \lambda_1 X$

$|A - \lambda I| = 0$ $\quad\quad$ $|B - \lambda_1 I| = 0$

$A - \lambda I = B - \lambda_1 I = 0$

$A_{n\times n}$,

then $|A-\lambda I| = 0$ would given us a n-degree polynomial called the characteristic polynomial.

we know,

$$B = P^{-1}AP$$

$$\Rightarrow |B| = |P^{-1}P|\,|A|$$

$$\Rightarrow |B| = |I|\,|A|$$

$$\Rightarrow |B| = |A|$$

$$B-\lambda I = P^{-1}AP - \lambda I$$

$$= P^{-1}AP - \lambda P^{-1}P$$

$$= P^{-1}(A-\lambda I)P$$

So, $B-\lambda I$ is equivalent to $A-\lambda I$.

So, $|B-\lambda I| = |A-\lambda I|$

So, since the characteristic polynomials are same for A and B, so the eigenvalues are also the same.

Ex:

Def^n

$f: L(V) \to F$ by

$f(T) = \det(T) = |[T]_B|$

for some basis B.

② $A \in \mathbb{R}^{3\times 3}$

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$

eigenvalue, vectors, G.M, A.M

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} (3-\lambda) & 1 & -1 \\ 2 & (2-\lambda) & -1 \\ 2 & 2 & -\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[-2\lambda + \lambda^2 + 2\right] - 1\left[-2\lambda + 2\right] - 1\left[\cancel{4} - \cancel{4} + 2\lambda\right] = 0$$

$$\Rightarrow (3-\lambda)\left[\lambda^2 - 2\lambda + 2\right] + \cancel{2\lambda} - 2 - \cancel{2\lambda} = 0$$

$$\Rightarrow 3\lambda^2 - 6\lambda + 6 - \lambda^3 + 2\lambda^2 - 2\lambda - 2 = 0$$

$$\Rightarrow -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$\lambda = 1$

$$
\begin{array}{r|rrrr}
1 & 1 & -5 & 8 & -4 \\
  & 0 & 1 & -4 & 4 \\
\hline
  & 1 & -4 & 4 & 0
\end{array}
$$

$$(\lambda-1)(\lambda^2 - 4\lambda + 4) = 0$$

$$(\lambda-1)(\lambda-2)^2 = 0$$

$$\lambda = 1, 2, 2$$

Eigen vectors:

$\lambda = 1$

$$\begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y = 0$$

$$2x + y - z = 0$$

$$2x = z$$

if $z = k$

$$x = k/2$$

So, $k \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$

So, $\lambda = 1$

A.M $= 1$

G.M $= 1$

$\lambda = 2$

$$\begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{vmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + y - z = 0$

$x - y = 0$

If $y = k$

$x = k$

$z = 2k$

$\Rightarrow k\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

for $\lambda = 2$,

A.M = 2

G.M = 1

for a linear transformation,

then $[T]_B = D$

$\downarrow$

Diagonal matrix $\rightarrow$ condition is

A.M = G.M

L.T is Diagonalisable.

Symmetric and Skew symmetric:-

$A = A^T$ (field is real) $A = -A^T$ (skew symmetric) (Real)

$A = (\bar{A})^T$ (Hermitian) $A = -(\bar{A})^T$ (complex) (S.S) (S. Hermitian)

→ Symmetric matrices have real eigen value.

$\bar{a}_{ij} = a_{ji}$

→ Diagonal elements of a hermitian matrix are real.

for skew hermitian,

$(\bar{A})^T = -A$

$\bar{a}_{ij} = -a_{ji}$

$\Rightarrow \bar{a}_{ij} + a_{ji} = 0$

for diagonal elements,

$+\bar{a}_{ii} + a_{ii} = 0$

$\Rightarrow$ diagonal elements are either purely imaginary (or) zero.

Theorem:-

If $H = P + iQ$ be a hermitian matrix then the diagonal elements of H are all real number and P is symmetric, Q is skew-symmetric.

Proof:

$$(\bar{H})^T = H$$

$$(\overline{P+iQ})^T = H$$

$$\Rightarrow (\overline{P+iQ})^T = P + iQ$$

$$\Rightarrow (\bar{P})^T + (\overline{iQ})^T = P + iQ$$

$$\Rightarrow (\bar{P})^T - i(\bar{Q})^T = P + iQ$$

$$\Rightarrow \boxed{\begin{array}{l} (\bar{P})^T = P \\ +(\bar{Q})^T = -Q \end{array}}$$

Suppose, $H = P + iQ$

if diagonal elements of H are purely imaginary or zero, then P is skew symmetric and Q is symmetric.

Proof:

$$(\bar{H})^T = -H$$

$$\Rightarrow \left(\overline{P+iQ}\right)^T = -P - iQ$$

$$\Rightarrow (\bar{P})^T + (\overline{iQ})^T = -P - iQ$$

$$\Rightarrow (\bar{P})^T - i(\bar{Q})^T = -P - iQ$$

$$\Rightarrow (\bar{P})^T = -P \quad \text{and} \quad (\bar{Q})^T = Q$$

Note: If A is any complex matrix then

$A + (\bar{A})^T$ is Hermitian and

$A - (\bar{A})^T$ is Skew-Hermitian

and $A = \frac{1}{2}\left[(A + (\bar{A})^T) + (A - (\bar{A})^T)\right]$

Def^n: A matrix A is called orthogonal if $AA^T = I$

A complex matrix A is called unitary matrix if $A(\bar{A})^T = I$ (or) $AA^* = I$, where $A^* = (\bar{A})^T$

Theorem: If A is an unitary matrix, then A is non-singular and $|A| = 1$.

Proof: Given A is unitary matrix,

So, $AA^* = I$

$\Rightarrow A(\bar{A})^T = I$

Taking determinant on both sides,

$$|A| \cdot |(\bar{A})^T| = |I|$$

$$\Rightarrow |A||\bar{A}| = |I|$$

$$\Rightarrow |A|^2 = |I|$$

$$\Rightarrow |A|^2 = 1$$

$$\Rightarrow |A| = \pm 1$$

So, mod $(|A|) = 1$.

- Inverse of a unitary matrix, A^{-1} is ?

$$A^{-1} = (\bar{A})^T.$$

$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$ verify unitary or not?

$$(\bar{A})^T = \begin{bmatrix} +\frac{1-i}{2} & +\frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}^T = \begin{bmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

Theorem: Eigenvalues of a Hermitian matrix are all real.

Proof: Suppose λ be a eigenvalue of Hermitian matrix A.

$\Rightarrow \quad |A - \lambda I| = 0$

$\Rightarrow \quad (A - \lambda I)X = 0$ for some $X \neq 0$

Suppose $X_1 \neq 0$ is the eigen vector to λ.

$\Rightarrow \quad (A - \lambda I)X_1 = 0$

$\Rightarrow \quad (A - \lambda I)X_1 = 0 \quad \forall X_1 \neq 0$

$\Rightarrow \quad AX_1 = \lambda X_1$

$\Rightarrow \quad (\bar{A})(\bar{X}_1) = (\bar{\lambda})(\bar{X}_1)$

$\Rightarrow \quad (\bar{A})^T(\bar{X}_1)^T = (\bar{\lambda})^T(\bar{X}_1)^T \qquad (\overline{AX_1})^T = (\overline{\lambda X_1})^T$

$\Rightarrow \quad A(\bar{X}_1)^T = (\bar{\lambda})^T(\bar{X}_1)^T \qquad (\bar{X}_1)^T(\bar{A})^T = (\bar{X}_1)^T(\bar{\lambda})^T$

$\Rightarrow \quad (\bar{X}_1)^T A = (\bar{X}_1)^T(\bar{\lambda})^T$

λ is some scalar, so $(\bar{\lambda})^T = (\bar{\lambda})$

$\Rightarrow \quad (\bar{X}_1)^T A = (\bar{X}_1)^T(\bar{\lambda})$

$\Rightarrow \quad (\bar{X}_1)^T A \cdot (X_1) = (\bar{X}_1)^T(\bar{\lambda})X_1$

$\Rightarrow A X_1 = \bar{\lambda} X_1$

and $A X_1 = \lambda X_1$

So, $\boxed{\lambda = \bar{\lambda}}$

Hence, the eigen values are real for Hermitian matrix.

For skew-Harmitian,

$$A X_1 = \lambda X_1$$

$$\Rightarrow (\overline{A X_1})^T = (\overline{\lambda X_1})^T$$

$$\Rightarrow (\bar{X_1})^T (\bar{A})^T = (\bar{X_1})^T (\bar{\lambda})^T$$

$$\Rightarrow (\bar{X_1})^T (\bar{A})^T = (\bar{X_1})^T (\bar{\lambda})$$

$$\Rightarrow -(\bar{X_1})^T A = (\bar{X_1})^T \bar{\lambda}$$

$$\Rightarrow (\bar{X_1})^T A X_1 + (\bar{X_1})^T \bar{\lambda} X_1 = 0$$

$$\Rightarrow (\bar{X_1})^T \lambda X_1 + (\bar{X_1})^T \bar{\lambda} X_1 = 0$$

$$\Rightarrow (\bar{X_1})^T (\lambda + \bar{\lambda}) X_1 = 0$$

$$\Rightarrow \boxed{\lambda + \bar{\lambda} = 0}$$

Hence, the eigen values are purely imaginary for skew-harmitian matrix.

Theorem:- The eigenvector corresponding to two distinct eigen value of symmetric matrix are orthogonal.

Proof:

Let λ_1, λ_2 are two distinct eigen-values

$$AX_1 = \lambda_1 X_1$$

$$AX_2 = \lambda_2 X_2$$

$$(\overline{AX_1})^T = (\overline{\lambda_1 X_1})^T$$

$$\Rightarrow (\bar{X}_1)^T (\bar{A})^T = (\bar{X}_1)^T (\bar{\lambda}_1)^T$$

$$\Rightarrow (\bar{X}_1)^T A = (\bar{X}_1)^T (\bar{\lambda}_1)$$

$$\Rightarrow (\bar{X}_1)^T A X_2 = (\bar{X}_1)^T (\bar{\lambda}_1) X_2$$

$$\Rightarrow (\bar{X}_1)^T \lambda_2 X_2 = (\bar{X}_1)^T \lambda_1 X_2$$ (Since for Herm λ are real)

$$\Rightarrow (\bar{X}_1)^T (\lambda_2 - \lambda_1) X_2 = 0$$

$$\Rightarrow (\lambda_2 - \lambda_1)(\bar{X}_1)^T X_2 = 0$$

$$\lambda_2 \neq \lambda_1$$

So, $(\bar{X}_1)^T X_2 = 0$

Hence, X_1, X_2 are orthogonal.

Eigen value and eigen vector of linear transformation:

$T: V \to V$ V = vector space over field F ($\mathbb{R}$ or $\mathbb{C}$)

we say $\lambda \in \mathbb{F}$ is a eigen value of T if $\exists$ $v \neq 0 \in V$

such that $Tv = \lambda v$, then v is called the eigen vector corresponding to eigen value λ.

$$E_\lambda = N(T - \lambda I) = \{ v \in V \mid (T - \lambda I)v = 0 \}$$

$$= \{ v \in V \mid Tv = \lambda v \}$$

E_λ is called as eigen space of T.

$\to$ λ is eigenvalue of $E_\lambda \neq \{0\}$

$\to$ E_λ is subspace of V.

Proof:

Let $v_1, v_2 \in E_\lambda$

$$Tv_1 = \lambda v_1 \quad Tv_2 = \lambda v_2$$

$$T(c_1 v_1 + c_2 v_2) = c_1 T(v_1) + c_2 T(v_2)$$

$$= c_1 \lambda v_1 + c_2 \lambda v_2$$

$$= \lambda (c_1 v_1 + c_2 v_2)$$

$\Rightarrow$ $c_1 v_1 + c_2 v_2$ also belongs to E_λ.

$$c_1 v_1 + c_2 v_2 \in E_\lambda.$$

So, E_λ is a subspace of V.

① Show that if T is Linear transformation on a finite dimensional space V, then show the following are equivalent.

(i) λ is eigen value of T

(ii) The operator $T-\lambda I$ is singular

(iii) $\det([T]_B - \lambda I) = 0$

where $[T]_B$ is the matrix repprsentation of T w.r.t basis B.

$T: V \rightarrow V$

$Tv = 0 \Rightarrow v = 0$ (non singular)

$Tv = 0$, $v \neq 0$ then (singular)

① → λ is eigen value of T

$\Rightarrow \exists\, v \neq 0 \in V$ such that $Tv = \lambda v$.

$\Rightarrow (T - \lambda I) v = 0$

$\Rightarrow T - \lambda I = 0$ i.e is singular $(\because v \neq 0)$

4②

③

equation ③ is called as "characteristic equation" of a linear operator.

Find the eigen value and corresponding eigen vector of a L.T $T: R^3 \to R^3$. $T(x,y,z) = (3x+2y+2z,\ x+2y+2z,\ -x-2y)$.

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$

$$[T]_B = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$$

$$[T]_B - \lambda I = \begin{bmatrix} 3-\lambda & 2 & 2 \\ 1 & 2-\lambda & 2 \\ -1 & -1 & -\lambda \end{bmatrix}$$

$$|[T]_B - \lambda I| = 0$$

$$\Rightarrow (3-\lambda)[-2\lambda+\lambda^2+2] - 2[-\lambda+2] + 2[-1+2-\lambda] = 0$$

$$3\lambda^2 - 6\lambda + 6 - \lambda^3 + 2\lambda^2 - 2\lambda + \cancel{2\lambda} - 4 + 2 - \cancel{2\lambda} = 0$$

$$\Rightarrow -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$\lambda = 1$

1	1	-5	8	-4
	0	1	-4	4
	1	-4	4	0

$$(\lambda-1)(\lambda^2-4\lambda+4) = 0$$

$$\Rightarrow \lambda = 1, 2, 2$$

for $\lambda = 1$,

$$v = (x \quad y \quad z)^T$$

s.t

$$[T]_\beta v = I v$$

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Rank $([T]_\beta - I) = 2$

$N([T]_\beta - I) = 3 - 2 = 1$ = eigen space $(\lambda = 1)$

$$x + y + z = 0$$
$$z = 0$$

$\Rightarrow (-1, 1, 0)$

So, null space of $([T]_\beta - I)$ has dimension 1 i.e the eigen space corresponding to eigen value 1 has dimension 1.

$\lambda = 2$;

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x + 2z = 0 ;\ y = 0$

$x + 2z = 0 ;\ y = 0$

$(-2, 0, 1)$

$N(T - \lambda I) = 3 - 2 = 1$

→ Whether every linear operator on a finite dimensional space has a eigen value or not.

$T : R^2 \to R^2$

$T(x, y) = (-y, x)$

$T(1, 0) = (0, 1)$

$T(0, 1) = (-1, 0)$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$|T - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 + 1 = 0$

It does not have root in R.

But over the complex field it has an eigen value.

(2) Let λ is the eigen value of a linear operator T and the corresponding eigen vector is V, then show that kv is also a eigen vector corresponding to the same eigen value.

$\exists\ v \neq 0$, s.t

$Tv = \lambda v$

$T(kv) = kT(v)$

$= k\lambda v$

$= \lambda(kv)$

So, kv is also an eigen-vector.

→ Find the characteristic polynomial of the following L. operator.

(a) $T: \mathbb{R}^2 \to \mathbb{R}^2$; $T(x,y) = (3x+4y, 2x-7y)$

(b) $D: V \to V$

V = space of all func whose basis is

$\beta = \{\sin t, \cos t\}$

$D(f) = f' = \frac{d(f)}{dt}$

$D(\sin t) = \cos t$

$= 0(\sin t) + 1(\cos t)$

$D(\cos t) = -\sin t$

$= -1(\sin t) + 0(\cos t)$.

$[D]_\beta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$[D]_\beta - \lambda I = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$

$\therefore\ \lambda^2 + 1$ is the characteristic polynomial.

(a) $\begin{bmatrix} 3 & 4 \\ 2 & -7 \end{bmatrix}$

$$\begin{bmatrix} 3-\lambda & 4 \\ 2 & -\lambda-7 \end{bmatrix} = 0$$

$(3-\lambda)(\lambda+7) + 8 = 0$

$= \underline{-\lambda^2 - 4\lambda + 29}$

- Show that if T is investible, and λ is the eigen value of T, then λ^{-1} is a eigen value of T^{-1}.

Sol: T is investible and λ = eigenvalue

if $\lambda = 0$, then

$T(v) = \lambda v = 0$

$\Rightarrow Tv = 0$

$\Rightarrow v = 0$ which is a contradiction.

So, $\lambda \neq 0$ and

$Tv = \lambda v$

$T^{-1}(Tv) = T^{-1}(v\lambda)$

$Iv = \lambda T^{-1}(v)$

$$\boxed{T^{-1}(v) = \frac{1}{\lambda}(v)}$$

$\Rightarrow \frac{1}{\lambda}$ is the eigen value corresponding to T^{-1}.

Similar matrices:

A & B are two matrices we say that A is similar to B if $\exists$ investible matrix P such that $B = P^{-1}AP$.

→ Similar matrices have same eigen value.

Note that $\det(B-\lambda I) = \det(P^{-1}AP - \lambda I)$

$= \det(P^{-1}AP - \lambda P^{-1}P)$

$= \det(P^{-1}(A-\lambda I)P)$

$= \det P^{-1} \det(A-\lambda I) \det$

$= \det(A-\lambda I)$

$\therefore \det(B-\lambda I) = \det(A-\lambda I)$

⇒ characteristic equation of matrix B is same as characteristic equation of A.

→ Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigen values of operator T and $v_1, v_2, \dots, v_k$ be corresp-onding vectors then $\{v_1, v_2, \dots, v_k\}$ are L.I.

Proof: Suppose the results is not true.

Let $\{v_1, v_2, \dots, v_s\}$ be a minimal set of vect for which theorem is not true.

We have $s > 1$, $v_1 \neq 0$

Then by assumption $\{v_2, v_3, \dots, v_s\}$ wi be linearly independent.

So, $\exists$, $c_2, c_3, \dots, c_s$ not all zero such tha

$v_1 = c_2 v_2 + \dots + c_s v_s$

$$V_1 = c_2V_2 + \cdots + c_sV_s \quad -① $$

$$T(v_1) = T(c_2V_2 + c_3V_3 + \cdots + c_sv_s)$$

$$(Tv_i = \lambda_i v_i \ ; \ i = 1, 2, \ldots, s)$$

$$\lambda_1 V_1 = c_2\lambda_2 V_2 + c_3\lambda_3 V_3 + \cdots + c_s\lambda_s V_s \quad - ②$$

$\lambda_1 ① - ② \Rightarrow$

$$0 = c_2(\lambda_1 - \lambda_2)V_2 + c_3(\lambda_1 - \lambda_3)V_3 + \cdots + c_s(\lambda_1 - \lambda_s)V_s.$$

if $\{v_2, v_3, \ldots, v_s\}$ is L.I, then

$$c_2(\lambda_1 - \lambda_2) = 0, \ c_3(\lambda_1 - \lambda_3) = 0 \ \cdots\cdots \ c_s(\lambda_1 - \lambda_s) = 0$$

$$\Rightarrow c_2 = c_3 = \cdots\cdots = c_s = 0.$$

which is a contradiction.

→ Show that if λ is eigen value of operator, T, then λ^k is the eigen value of T^k; $k > 0$

Sol:

$$T\upsilon = \lambda\upsilon$$

$$T(T\upsilon) = T(\lambda\upsilon)$$

$$T^2(\upsilon) = \lambda T(\upsilon) = \lambda(\lambda\upsilon)$$

$$T^2(\upsilon) = \lambda^2\upsilon$$

Say, $T^{n-1}\upsilon = \lambda^{n-1}\upsilon$

$$T(T^{n-1}\upsilon) = T(\lambda^{n-1}\upsilon)$$

By induction,

$$T^n(\upsilon) = \lambda^{n-1}(\lambda\upsilon)$$

$$\boxed{T^n(\upsilon) = \lambda^n\upsilon}$$

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Let λ be eigen value of T, ~~The a polynomial,~~

$P(t) = a_0 + a_1 t + \dots + a_n t^n$ is poly. of degree n.

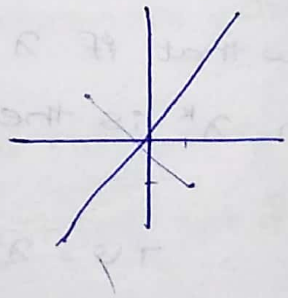
Show that $P(\lambda)$ is eigen value of $P(T)$.

Sol: $P(T) = a_0 + a_1 T + \dots + a_n T^n$

→ We have to prove the previous theorem and then we should conclude this one for exam.

Q Let n be the linear Transformation on $\mathbb{R}^2$ that reflects $T: \mathbb{R}^2 \to \mathbb{R}^2$, which reflects each points across the line, $y = kx$ where k is considered (> 0). Show that $(1, k)$ is a eigen vector of T.

$T(1, k) = (1, k)$



Here, $(1, k)$ lies on $y = kx$,

Hence, $Tv = \lambda v$

So, $\lambda = 1$ and $v = (1, k)$.

$T(1, -k) = -1\,(1, -k)$

$\lambda = -1$, $v = (1, -k)$

Def: Let T be linea[...]
vector space V [...]
$\exists$ a basis of [...]
an eigen vect[...]
diagonasible [...]
spans V.

Proof: Let dim [...]
Suppose $\exists$ [...]
$\beta = \{v_1, v_2, [...]$
is an eigen [...]
$Tv_i = \lambda_i v$[...]

Then,

$[T]_\beta =$ [...]

Ex Show that [...]
operator T [...]
$T(x, y, z)$ [...]

is diagonisa[...]

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Def:
Let T be linear operator on finite dimensional vector space V, we say T is diagonasible if $\exists$ a basis of V, in which each vector is an eigen vector of T. In otherswords, T is diagonasible if characteristic vectors of T spans V.

Proof: Let $\dim V = n$

Suppose $\exists$ basis of V

$\beta = \{v_1, v_2, \dots, v_n\}$ in which each vector is an eigen vector

$Tv_i = \lambda_i v_i$, $i = 1, 2, 3, \dots, n$

Then,

$$[T]_\beta = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & c_4 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & & \cdots & c_n \end{bmatrix} = \text{diagonal matrix.}$$

Ex Show that check whether the linear operator T, from $\mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (2x + y - 2z,\ 2x + 3y - 4z,\ x + y - z)$$

is diagonisable or not?

$T(1,0,0) = (2,2,1)$

$T(0,1,0) = (1,3,1)$

$T(0,0,1) = (-2,-4,-1)$

$$[T]_\beta = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{bmatrix}$$

$$|[T]_\beta - \lambda I| = 0$$

$$\Rightarrow \begin{bmatrix} (2-\lambda) & 1 & -2 \\ 2 & (3-\lambda) & -4 \\ 1 & 1 & (-1-\lambda) \end{bmatrix} = 0$$

$$(2-\lambda)\left[(3-\lambda)(-1-\lambda)+4\right]$$
$$-1\left[2(-1-\lambda)+4\right]$$
$$-2\left[2-3+\lambda\right] = 0$$

$$(2-\lambda)\left[\lambda^2-2\lambda+1\right] - 1\left[-2\lambda+2\right]$$
$$-2\left[\lambda-1\right] = 0$$

$$(2-\lambda)(\lambda-1)^2 + 2\cancel{(\lambda-1)} - 2\cancel{(\lambda-1)} = 0$$

$$\Rightarrow \lambda = 1, 1, 2$$

$\lambda = 1$

$$\begin{bmatrix} 1 & 1 & -2 \\ 2 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\text{Rank}(A) = 1$

so, $N(T) = 3 - 1 = 2$

$x + y - 2z = 0$

$x = 0, \; y = 1 \Rightarrow z = 1/2$

$x = 1, \; y = 0 \Rightarrow z = 1/2$

In general, for L.I, we take like this.

$(1, 0, 1/2)$

$(0, 1, 1/2)$

for $\lambda = 2$,

$$\begin{bmatrix} 0 & 1 & -2 \\ 2 & 1 & -4 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -3 \\ 2 & 1 & -4 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -3 \\ 0 & -1 & -2 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R(A) = 2 \Rightarrow N(T) = 1.$

$x - z = 0$

$y - 2z = 0$

$z = k,$

$x = k$

$y = 2k$

$\Rightarrow v = k\,(1, 2, 1)$

Here, we got in total 3 L.I vectors which can span $\mathbb{R}^3$. So, diagonasible.

$$\beta' = \{ \overset{v_1}{(1, 0, 1/2)}, \overset{v_2}{(0, 1, 1/2)}, \overset{v_3}{(1, 2, 1)} \}.$$ spans R

Here, we keep the... in col.wi... that gi...

$T(1, 0, 1/2) = (1, 0, 1/2)$

$= 1 v_1 + 0 v_2 + 0 v_3$

$T(0, 1, 1/2) = 0 v_1 + 1 v_2 + 0 v_3$

$T(1, 2, 1) = 0 v_1 + 0 v_2 + 2 v_3$

$$[T]_{\beta'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

T is diagonasible, then $\exists$ P (inverstible) such th...

$D = P^{-1} [T]_\beta . P$

where D is diagonasible matrix

$D = \text{diag}\{ \lambda_1, \lambda_2, \ldots, \lambda_n \}.$

Q- Let T be, $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x, y) = (7x + 3y, 3x - y)$

$T(1, 0) = (7, 3)$

$T(0, 1) = (3, -1)$

$$\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$$

$(7-\lambda)(-1-\lambda) - 9 = 0$

$(7-\lambda)(1+\lambda) + 9 = 0$

$-\lambda^2 + 6\lambda + 16 = 0$

$\lambda^2 - 6\lambda - 16 = 0$

$\lambda^2 - 8\lambda + 2\lambda - 16 = 0$

$\lambda(\lambda - 8) + 2(\lambda - 8) = 0$

$(\lambda + 2)(\lambda - 8) = 0$

$\lambda = -2, 8$

$\lambda = -2,$

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$N(T) = 1$

$9x + 3y = 0 \Rightarrow K(1, -3)$

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$\lambda = 3$

$$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x + 3y = 0$$

$$3y = x$$

$y = 1/3$, $\Rightarrow x = 1$

$$P = \begin{bmatrix} 1 & 1 \\ 1/3 & -3 \end{bmatrix}$$

$k(1, 1/3)$

So, $\{(1,-3), (1, 1/3)\}$ spans R^2.

Hence, T is diagonasible.

$$D = P^{-1}[T]_B P$$

$$\Rightarrow \begin{bmatrix} 3 & -1 \\ -1/3 & -1 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1/3 & -3 \end{bmatrix} / \text{Det}(P)$$

$$\Rightarrow \begin{bmatrix} 18 & 10 \\ -16/3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1/3 & -3 \end{bmatrix}$$

$\Rightarrow [$

Q1) Show that the L.T, $T: R^2 \to R^2$, that reflects each point across the line $y = kx$, where $k > 0$, is diagonasible, find its diagonal representation.

2) Let T be the L.T, $T: R^3 \to R^3$, $T(x, y, z)$

$= (4x + y - z, 2x + 5y - 2z, x + y + 2z)$

Check whether the L.T is diagonasible or not and if diagonasible find the matrix P, s.t

$D = P^{-1}[T]_\beta P$.

<u>Remarks:</u> Let T be a linear operator on finite dimensional vector space V. Let $\lambda_1, \lambda_2, \dots \lambda_k$ be distinct eigen values of T and let $W_i = N(T - \lambda_i I)$ (eigen space corresponding to eigen value λ_i)

Then the following are equivalent,

(i) T is diagonlizable

(ii) The characteristic polynomial of T is,

$f = (x - \lambda_1)^{d_1} * (x - \lambda_2)^{d_2} \dots\dots (x - \lambda_k)^{d_k}$, where

(d_i = multiplicity of eigen value λ_i)

(iii) $\dim W_1 + \dim W_2 + \dots + \dim W_k = \dim V$.

① ⇒ ②

T is diagonalizable ⇒ ∃ basis in each element is characteristic vector.

Let $\dim V = n$

$\beta = \{v_1, v_2, \dots, v_n\}$

$Tv_i = \lambda v_i$

Suppose λ_i is occuring d_i times,

$$[T]_\beta = \begin{bmatrix} \lambda_1 I_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 I_2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \lambda_k I_k \end{bmatrix}$$

I_i = identity matrix of order $d_i \times d_i$.

$$f = (x-\lambda_1)^{d_1}(x-\lambda_2)^{d_2} \cdots (x-\lambda_k)^{d_k}$$

Def^n: A polynomial $p(t)$ is said to be Annihilating polynomial of the operator $T: V \to V$, if

$$p(T) = 0 \Rightarrow p(T)v = 0 \ \forall \ v \in V.$$

Minimal polynomial: A polynomial, $p(t)$ is said of operator T, to be minimal polynomial, if it satisfies the following three conditions.

(i) $p(T) = 0$

(ii) $p(t)$ is a monic polynomial (monic polynomial means the highest power term's coefficient should be 1).

(iii) no polynomial whose degree is less than the degree of $p(t)$ can annihiliate T.

If $\deg q(t) \leq \deg p(t)$

then, $q(T) \neq 0$

→Q. Show that if V is finite dimensional vector space and $T: V \to V$, then there exists annihiliating polynomial for T.

Sol: Say, $\dim V = n$

Take space $L(V,V)$

we know that $\dim(L(V,V)) = n^2$.

Now, we take set $\{I, T, T^2, \ldots, T^{n^2}, T^{n^2+1}\} = S$

As, S has n^2+1 elements, ⇒ S is L.D

$\exists$ $c_0, c_1, c_2, \ldots c_{n^2}$ not all 0, such that

$$c_0 I + c_1 T + c_2 T^2 + \cdots + c_{n^2} T^{n^2} = 0$$

$$p(t) = c_0 I + c_1 T + c_2 T^2 + \cdots + c_{n^2} T^{n^2}$$

then, p(t) is a annihiliating polynomial of T.

Q- Let T be a linear operator on finite dimensional space V, $T: V \to V$ and $\dim(V) = n$. Then show that the characteristic and minimal polynomial of T will have same roots, except for multiplicity.

Sol: Let λ be root of characteristic eqn of T.

i.e λ is an eigen value.

Let p(t) be minimal polynomial of T

We show $p(\lambda) = 0$

As λ is eigen value, $\exists$ a non zero vector v, such that $Tv = \lambda v$. ⇒ $p(T)v = p(\lambda)v$.

⇒ As p(t) is a minimal polynomial,

$P(T) = 0$ ⇒ $p(\lambda) v = 0$

$\Rightarrow p(\lambda) = 0 \quad (\because v \neq 0)$

Hence. $\boxed{p(\lambda) = 0}$

Converse, suppose c is a root of $p(t)$ (minimal polynomial)

i.e $p(c) = 0$

we can write,

$$p(t) = (t-c)\, q(t)$$

As, deg. $q(t) \leq$ deg. $p(t)$

So, $q(T) \neq 0$

i.e $\exists\ \beta \in V$

s.t $q(T)\beta \neq 0$

Take, $v = q(T)\beta$

we have,

$P(T)\beta = 0$ ($\because P(t)$ is minimal polynomial)

$$\Rightarrow (T - cI)\underbrace{q(t)\beta}_{v \neq 0} = 0$$

So, $(T - cI) = 0 \Rightarrow \exists\ v \neq 0$ s.t $Tv = cv$

$\Rightarrow c$ is a eigen value.

Remark:

The Cayley-Hamilton theorem, says

Let T be a linear operator on a finite dimensional vector space V, $T: V \to V$. and if $f(t)$ is a characteristic polynomi of T, then $f(T) = 0$. Also, minimal poly-nomial of T, divides characteristic polynom

We say that, poly. g divides poly. f, if

$f = hg$, where poly. h has degree < degree(f).

Q. Find characteristic and minimal polynomial of a linear operator $T: R^3 \to R^3$ and

$T(x, y, z) = (2x + 3y + z, 2x + 7y + 2z, -5x - 15y - 4z)$

$T(1,0,0) = (2, 2, -5)$
$T(0,1,0) = (3, 7, -15)$
$T(0,0,1) = (1, 2, -4)$

$$\begin{bmatrix} 2-\lambda & 2 & -5 \\ 3 & 7-\lambda & -15 \\ 1 & 2 & -4-\lambda \end{bmatrix} = 0$$

$\Rightarrow (2-\lambda)[(7-\lambda)(-4-\lambda) + 30]$

$-2[3(-4-\lambda)+15] - 5[6-7+\lambda] = 0$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 2 & 7 & 2 \\ -5 & -15 & -4 \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & 3 & 1 \\ 2 & 7-\lambda & 2 \\ -5 & -15 & -4-\lambda \end{vmatrix} = 0$$

$(2-\lambda)[(7-\lambda)(-4-\lambda) + 30]$

$-3[-8-2\lambda+10]$

$+1[-30+35-5\lambda] = 0$

$(2-\lambda)[\lambda^2-3\lambda+2]+6[\lambda-1]-5[\lambda-1]=0$

$(2-\lambda)(\lambda-2)(\lambda-1)+(\lambda-1)=0$

$\Rightarrow (\lambda-1)[1-(\lambda-2)^2]=0$

$(\lambda-2)^2=1$

$\lambda-2=1$	$\lambda-2=-1$	$\lambda-1=0$
$\lambda=3$	$\lambda=1$	$\lambda=1$

$\Rightarrow (\lambda-1)^2(\lambda-3)=0$

$f(t)=(t-1)^2(t-3)$

So, $p(t)$ can be,

$$\left.\begin{aligned} P_1(t) &= (t-1)^2(t-3) \\ &\text{(or)} \\ P_2(t) &= (t-1)(t-3) \end{aligned}\right\}\ \text{because roots should be same}$$

Now,

$(T-I)(T-3I)=0$

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 2 \\ -5 & -15 & -5 \end{bmatrix}\begin{bmatrix} -1 & 3 & 1 \\ 2 & 4 & 2 \\ -5 & -15 & -7 \end{bmatrix}$$

$=0$

Here, $P_2(t)$ chosen, because degree ≤ 3.

Q- Find the minimal polynomial of a matrix

$$A=\begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$$

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$$\begin{bmatrix} (4-\lambda) & 1 & -1 \\ 2 & (5-\lambda) & -2 \\ 1 & 1 & (2-\lambda) \end{bmatrix} = 0$$

$$(4-\lambda)\left[(5-\lambda)(2-\lambda)+2\right] - 1\left[2(2-\lambda)+2\right] - 1\left[2-5+\lambda\right] = 0$$

$$\Rightarrow (4-\lambda)\left[\lambda^2 - 7\lambda + 12\right] - 1\left[-2\lambda+6\right] - 1\left[\lambda-3\right] = 0$$

$$\Rightarrow (4-\lambda)(\lambda-4)(\lambda-3) + 2(\lambda-3) - 1(\lambda-3) = 0$$

$$\Rightarrow -(\lambda-3)(\lambda-4)^2 + (\lambda-3) = 0$$

$$\Rightarrow (\lambda-3)\left[1-(\lambda-4)^2\right] = 0$$

$$\Rightarrow (\lambda-3)\left[\lambda^2+16-8\lambda-1\right] = 0$$

$$\Rightarrow (\lambda-3)\left[\lambda^2-8\lambda+15\right] = 0$$

$$\Rightarrow (\lambda-3)^2(\lambda-5) = 0$$

→ minimal polynomial can be,

$(t-3)^2(t-5)$ (or) $(t-3)(t-5)$.

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Suppose matrix is block triangular matrix.

$$M = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix}$$

where A_1 and A_2 are square matrix

Then characteristic polynomial of M is product of characteristic polynomials of the diagonal block.

Ex: Find the characteristic polynomial of

$$M = \begin{bmatrix} \begin{pmatrix} 9 & -1 \\ 8 & 3 \end{pmatrix} & \begin{pmatrix} 5 & 7 \\ 2 & -4 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 3 & 6 \\ 1 & 8 \end{pmatrix} \end{bmatrix}$$

$$= \left[(9-\lambda)(3-\lambda)+(8)\right]\left[(3-\lambda)(8-\lambda)-6\right]$$

$$= \left[(9-\lambda)(3-\lambda)+8\right]\left[(3-\lambda)(8-\lambda)-6\right]$$

Here, M is a block triangular form

$$\det(M-\lambda I) = \det(A_1-\lambda I)\cdot\det(A_2-\lambda I)$$

$$= (\lambda^2-12\lambda+35)(\lambda^2-11\lambda+18)$$

$$= (\lambda-5)(\lambda-7)(\lambda-9)(\lambda-2)$$

(2) Find, for

$$M = \begin{bmatrix} 2 & 5 & 1 & 1 \\ 1 & 4 & 2 & 2 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

$$\text{ch. poly} = \left[(2-\lambda)(4-\lambda)-5\right]\left[(6-\lambda)(3-\lambda)+10\right]$$

$$= \left[\lambda^2-6\lambda+3\right]\left[\lambda^2-9\lambda+28\right]$$

Remark:

Suppose the matrix M is a block diagonal matrix, with diagonal blocks $A_1, A_2, \dots, A_r$, then minimal polynomial of M is equal to least common multiplier (LCM) of the minimal polynomials of the diagonal blocks.

ex Find characteristic polynomial and minimal polynomial of

$$\begin{bmatrix} \begin{pmatrix} 2 & 5 \\ 0 & 2 \end{pmatrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} \begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & [7] \end{matrix} \end{bmatrix}$$

C. Poly,

$$= ((2-\lambda)^2)((4-\lambda)(5-\lambda)-6)(7-\lambda)$$

$$= (\lambda-2)^2[\lambda^2 - 9\lambda + 14][(7-\lambda)]$$

$$= (\lambda-2)^2[(\lambda-7)][\lambda-2][7-\lambda]$$

$$= -(\lambda-2)^3(\lambda-7)^2$$

minimal polynomial,

$$A_1 = (2-\lambda)^2$$

$$A_2 = (\lambda-2)(\lambda-7)$$

$$A_3 = (7-\lambda)$$

So, L.C.M $= +(\lambda-2)^2(\lambda-7)$. is the minimal polynomial of M.

Q- Find characteristic and minimal polynomial of

$$\begin{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} \end{bmatrix}$$

Q- Show that if T is a linear transformation from $V \to V$, V is finite dimensional, Let $\lambda_1, \lambda_2, \dots \lambda_k$ be distinct eigen values of T and T is diagonalizable show that the minimal polynomial of T will be of the form

$$P = (t-\lambda_1)(t-\lambda_2)\dots(t-\lambda_k)$$

Inner Product on vector space :-

Let V be a vector space over the field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$).

Inner product on V is function $\langle\ \rangle : V \times V \to \mathbb{F}$ satisfying following conditions

① $\langle v, v \rangle \geq 0 \ \forall\ v \in V$

& $\langle v, v \rangle = 0$ iff $v = 0$

② $\langle v, w \rangle = \overline{\langle w, v \rangle} \ \forall\ v, w \in V$

③ $\langle cv, w \rangle = c \langle v, w \rangle$

$\forall\ v, w \in V$

and $c \in \mathbb{F}$

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(4) $\langle v+u, w\rangle = \langle u, w\rangle + \langle v, w\rangle \ \forall v, u, w \in V$

→ V together with inner product i.e $(V\langle\cdot\rangle)$ is called as inner product space.

Ex: 1) Show that inner product of $\langle v, cw\rangle = \bar{c}\langle v, w\rangle \ \forall \ v, w \in V$ and $c \in \mathbb{F}$

$\langle v, cw\rangle$

$= \overline{\langle cw, v\rangle}$

$= \langle \overline{cw}, \bar{v}\rangle$

$= \bar{c}\langle \bar{w}, \bar{v}\rangle$

$= \bar{c}\ \overline{\langle w, v\rangle}$

$= \bar{c}\langle v, w\rangle$

2) $V = \mathbb{R}^n$ and $\mathbb{F} = \mathbb{R}$

$\langle\cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{F}$

$x = (x_1, x_2, \ldots, x_n)$

$y = (y_1, y_2, \ldots, y_n)$

we defined it

$\langle x, y\rangle := \sum_{i=1}^{n} x_i y_i = x^T y$

Show that the $\langle\ \rangle$ defines inner product on $\mathbb{R}^n$.

Sol: 1) $\langle x\ x\rangle = \sum_{i=1}^{n} x_i^2 \geq 0 \Rightarrow x_i = 0 \ \forall i$

$\Rightarrow \bar{x} = \bar{0}$

2) $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$

$\langle y, x\rangle = \sum_{i=1}^{n} y_i x_i$

$\Rightarrow \langle x, y\rangle = \langle y, x\rangle$ (real field)

3) $< cx, y > = \sum_{i=1}^{n} c x_i y_i$

$= c < x, y >$

4) $< x+y, y > = \sum x_i y_i + \sum y_i^2$

$= < x, y > + < y, y >$

3) Try above for $F = \mathbb{C}$ and $V = \mathbb{R}^n$

$< x, y > := \sum_{i=1}^{n} x_i \bar{y_i}$

Ex 4) Let $u = (1, 3, -4, 2)$, $v = (4, -2, 2, 1)$

$V = \mathbb{R}^4$. find $< u, v >$

$< u, v > = (4 + -6 + -8 + 2)$

$\therefore < u, v > = -8$

$w = (5, -1, -2, 6)$

$< 3u - 2v, w >$ $(-5, 13, -16, 4)$

$= -25 - 13 + 32 + 24$

$= 18$

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Inner product space:-

Ex: $C[a, b]$ $F = \mathbb{R}$

$(f+g)x = f(x) + g(x)$

$(\alpha f)(x) = \alpha \cdot f(x)$

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

if $F = \mathbb{C}$

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, dx$$

① $\langle f \cdot f \rangle = \int_a^b f^2(x)\, dx \geq 0$

$\langle f \cdot f \rangle = 0 \iff \int_a^b f^2(x)\, dx$

$\iff f(x) = 0 \ \forall x$

$\iff f = 0$

② $\langle f \cdot g \rangle = \int_a^b f(x)\, g(x)\, dx$

$= \int_a^b g(x)\, f(x)\, dx$

$= \langle g, f \rangle$

③ $\langle cf \cdot g \rangle = \int_a^b c \cdot f(x)\, g(x)\, dx$

$= c \langle f \cdot g \rangle$

④ $\langle f+g, h \rangle = \int_a^b \left(f(x) + g(x)\right) h(x)\, dx$

$= \int_a^b f(x) h(x)\, dx + \int_a^b g(x) h(x)\, dx$

$= \langle f \cdot h \rangle + \langle g \cdot h \rangle$

Let, $f(t) = 3t - 5$ and $g(t) = t^2$ be the elements of polynomial space $p(t)$. Find the following, $P[0, 1]$

① $\langle f.g \rangle$

② $\langle f.f \rangle$

③ $\langle g.g \rangle$

① $\int_0^1 (3t-5)t^2\,dt = \int_0^1 (3t^3-5t^2)\,dt = \frac{3}{4}-\frac{5}{3} = \frac{-11}{12}$

② $\int_0^1 (3t-5)^2\,dt = \frac{(3t-5)^3}{(3)3} = \frac{-8+125}{9} = \frac{+117}{9}$

$= 13$

③ $\int_0^1 t^4\,dt = \frac{1}{5}$

Ex $X = M_{m\times n}$

$\langle A, B \rangle = \text{trace}(B^T A)$

Verify defines inner product on $M_{m\times n}$.

V - vector space over field $\mathbb{F}$, Norm of V is function $\|\cdot\| : V \to \mathbb{R}$ satisfying following properties

1) $\|v\| \geq 0 \ \forall \ v \in V$ if $\|v\| = 0$ iff $v = 0$

2) $\|cv\| = \|v\|\,|c| \ \forall v \in V$ and $c \in \mathbb{F}$

3) $\|v + w\| \leq \|v\| + \|w\|$

Ex : 1) $V = \mathbb{R}$; $\mathbb{F} = \mathbb{R}$

$\|\cdot\| : V \to \mathbb{R}$

$\|v\| = |v|$

2) $V = \mathbb{R}^n$, $\mathbb{F} = \mathbb{R}$

$\|\cdot\| : V \to \mathbb{R}$

$\|v\|_1 = \underbrace{|x_1| + |x_2| + |x_3| + \dots + |x_n|}_{1\text{-norm}}$

Verify is this a norm ?

i) $\|v\| \geq 0$ (True)

ii) $\|cv\| = c|x_1| + c|x_2| + \dots + c|x_n|$

$= c(|x_1| + |x_2| + \dots + |x_n|)$ (True)

iii) $\|x + y\| =$ ~~$|x| + |y|$~~

$|x_1 + y_1| + |x_2 + y_2| + \dots + |x_n + y_n|$

$\leq |x_1| + |y_1| + |x_2| + |y_2| + \dots + |x_n| + |y_n|$

$\leq (|x_1| + |x_2| + \dots + |x_n|) + (|y_1| + |y_2| + \dots + |y_n|)$

$\leq \|x\| + \|y\|$ (True)

Hence, it is a Norm.

3) $V : v$

$\underbrace{\|\cdot\|_\infty}_{\infty\text{ norm}} = \text{Max}\{|\alpha_1|, |\alpha_2|, \dots, |\alpha_n|\}$

Verify $\|\cdot\|_\infty$ defines norm on $\mathbb{R}^n$.

Sol:- i) $\|\cdot\|_\infty \geq 0$ (✓)

ii) $\|c \cdot v\| = |c| \cdot \|v\|$ (✓)

iii) $|x_i + y_i| \leq |x_i| + |y_i|$ $\forall i$

$\text{Max}(|x_i + y_i|) \leq \text{Max}(|x_i| + |y_i|)$

$\Rightarrow \|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$.

3) $X : \mathbb{R}^n$

$$\|\cdot\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

$$\|\cdot\|_p = \left(|x_1|^p + |x_2|^p + \dots + |x_n|^p\right)^{1/p} \quad p = 1, 2, \dots$$

Ex: $X = \mathbb{R}^5 \quad x = (1, 1, -2, 3, -1)$

$$\|x\|_1 = 1 + 1 + 2 + 3 + 1 = 8$$

$$\|x\|_\infty = 3$$

$$\|x\|_2 = \sqrt{1 + 1 + 4 + 9 + 1} = 4$$

Note: $y = \dfrac{x}{\|x\|}$ → can be any norm.

then, $\|y\|_1 = 1$
↳ same norm as x.

V - inner product space

$$\|v\|_i = \langle v . v \rangle^{1/2}$$

Schwartz inequality :- Let V be inner product space and $v, w \in V$ then

$$|\langle v, w \rangle| \leq \|v\| \cdot \|w\|$$

where, $\|v\| = \langle v, v \rangle^{1/2}$

$\|w\| = \langle w, w \rangle^{1/2}$

$v, w \in V$ and $c \in \mathbb{F}$

Proof $\quad \| v - cw \|^2 = \langle v - cw, v - cw \rangle$

$$= \langle v, v \rangle - \langle v, cw \rangle - \langle cw, v \rangle + \langle cw, cw \rangle$$

$$= \| v \|^2 - \langle v, cw \rangle - \overline{\langle v, cw \rangle} + c\bar{c} \| w \|^2$$

$$0 \leq \| v - cw \|^2 = \| v \|^2 - 2 \, \mathrm{Re} \langle v, cw \rangle + |c|^2 \| w \|^2 \quad \text{— (1)}$$

Take, $\quad c = \dfrac{\langle v, w \rangle}{\| w \|^2}$

$$\langle v, cw \rangle = \left\langle v, \frac{\langle v, w \rangle}{\| w \|^2} w \right\rangle$$

$$= \frac{\langle v, w \rangle}{\| w \|^2} \langle v, w \rangle$$

$$= \frac{| \langle v, w \rangle |^2}{\| w \|^2} \quad ; \quad \mathrm{Re}(\langle v, cw \rangle) = \frac{| \langle v, w \rangle |^2}{\| w \|^2}$$

Now, $\quad c^2 = \dfrac{| \langle v, w \rangle |^2}{\| w \|^4}$

Now in (1),

$$0 \leq \| v \|^2 - 2 \frac{| \langle v, w \rangle |^2}{\| w \|^2} + \frac{| \langle v, w \rangle |^2}{\| w \|^4} \| w \|^2$$

$$0 \leq \| v \|^2 - \frac{| \langle v, w \rangle |^2}{\| w \|^2}$$

$$\Rightarrow | \langle v, w \rangle | \leq \| v \| \cdot \| w \| .$$

Remark: If v and w are linearly dependent, then equality occurs in a Schwartz inequality

Q- $\|v\| = \langle v, v\rangle^{1/2}$ defines norm on inner product space V. This norm is called as "Norm induced by inner product."

Properties

1) $\langle v, v\rangle \geq 0 \Rightarrow \langle v, v\rangle^{1/2} \geq 0 \Rightarrow \|v\| \geq 0$

Now if,

$\|v\| = 0 \Rightarrow \langle v, v\rangle = 0 \Leftrightarrow v = 0$

2) $\|cv\|^2 = \langle cv, cv\rangle = c\bar{c}\langle v, v\rangle$

$= c^2 \langle v, v\rangle$

$= c^2 \|v\|^2$

$\Rightarrow \|cv\| = c\|v\|$

3) $\|v+w\|^2 = \langle v+w, v+w\rangle$

$= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle$

$= \|v\|^2 + 2\,\mathrm{Re}\langle v, w\rangle + \|w\|^2$

(By Schwartz ineq) $\leq \|v\|^2 + 2\|v\|\,\|w\| + \|w\|^2$

$\Rightarrow \|v+w\| \leq \|v\| + \|w\|$

$\rightarrow$ $\|x\| = \langle x \cdot x \rangle^{1/2} = \left(\sum_{i=0}^{n} x_i^2 \right)^{1/2}$

$\uparrow$

Norm induced by inner product (i.e 2 norm)

$\rightarrow$ Suppose v, w are two vectors in a inner product space V and θ is the angle b/w v and w. s.t by Cauchy Schwartz inequality

$$\cos\theta = \frac{|\langle v, w \rangle|}{\|v\| \cdot \|w\|}$$

Ex Find the angle b/w the vectors, $v = (2,3,5)$ and $w = (1,-4,3)$ in $\mathbb{R}^3$.

$$\cos\theta = \frac{|\langle v, w \rangle|}{\|v\| \cdot \|w\|}$$

$|\langle v, w \rangle| = |2 + (-12) + (15)| = |3+2| = 5$

$\|v\|^2 = \langle v, v \rangle = 4 + 9 + 25 = 38$

$\|w\|^2 = \langle w, w \rangle = 1 + 16 + 9 = 26$

$\Rightarrow \cos\theta = \frac{5}{\sqrt{38} \cdot \sqrt{26}}$

$\Rightarrow \theta = \cos^{-1}\left[\frac{5}{\sqrt{38} \cdot \sqrt{26}} \right]$

② Let $f(t)$ be $3t-5$ and $g(t) = t^2$ and on $P(t)$ where $\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$.

find θ b/w f and g.

$$\cos\theta = \frac{|\langle f, g\rangle|}{\|f\| \, \|g\|} = \frac{11/12 \sqrt{5}}{\sqrt{13}} = \frac{11\sqrt{5}}{12\sqrt{13}}$$

So, $\theta = \cos^{-1}\left[\frac{11\sqrt{5}}{12\sqrt{13}}\right]$.

3) Find the angle θ b/w the matrix

$$A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$B^T A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2} \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 33 & 28 & 23 \\ 48 & 41 & 34 \\ 63 & 54 & 45 \end{bmatrix}$$

$\langle A, B\rangle = 119$

$\langle A, A\rangle =$

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→ Let V be inner product space we say $v, w \in V$ are orthogonal vectors if $\langle v, w \rangle = 0$.

$\langle 0, v \rangle = 0 \ \forall \ v \in V$

so 0 is orthogonal to each element of V.

→ Let $S \subseteq V$ (inner product space = ips)

$S = \{v_1, v_2, \dots, v_n\}$

we say S is orthogonal subset of V if

$$\langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ c \neq 0 & \text{if } i = j \end{cases}$$

→ A subset $S = \{v_1, v_2, \dots v_n\}$ is said to be orthonormal if

$$\langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

→ 1) Show that the vector $u = (1,1,1)$ and $v = (1,2,-3)$ are orthogonal to each other.

Sol:- $\langle u, v \rangle = (1)(1) + (1)(2) + (1)(-3)$

$= 0$

Hence, u, v are orthogonal to each other.

2) $C[-\pi, \pi]$, Show that the $f(t) = \sin t$ and $g(t) = \cos t$ are orthogonal.

Sol:- $\langle f, g \rangle = \int_{-\pi}^{\pi} \sin t \cos t \, dt$

$= \frac{1}{2}\left[\frac{-\cos 2t}{2}\right]_{-\pi}^{\pi} = -\frac{1}{4}[1-1] = 0$

3. Find a non-zero vector which is orthogonal to $u(1,2,1)$ and $w(2,5,4)$ in $\mathbb{R}^3$.

Sol:- Let non zero vector be, $v = (x, y, z)$

Now,

$\langle u \cdot v \rangle = 0 \Rightarrow x + 2y + z = 0$

$\langle w \cdot v \rangle = 0 \Rightarrow 2x + 5y + 4z = 0$

$\Rightarrow y + 2z = 0$

take, $z = 1$

$\Rightarrow y = -2$

$\Rightarrow x = 3$

So, $v = (3, -2, 1)$ is orthogonal to u, w.

4. Find the value of k such that the vector $(1, 2, k, 3)u$ and $v(3, k, 7, -5)$ are orthogonal

Sol:- $\langle u \cdot v \rangle = 0$

$\Rightarrow 3 + 2k + 7k - 15 = 0$

$\Rightarrow 9k = 12$

$\Rightarrow k = 4/3$

5. Show that orthogonal subset of a inner product space containing non zero element is linearly independent.

Sol:- Let V be the inner product space.

$S = \{v_1, v_2, \ldots, v_n\}$ be orthogonal subset of and $v_i \neq 0 \quad i = 1, 2, \ldots, n$

By taking linear combination,

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

$$\Rightarrow \langle c_1 v_1 + c_2 v_2 + \cdots + c_n v_n , v_i \rangle = 0$$

$$\Rightarrow c_1 \langle v_1 , v_i \rangle + c_2 \langle v_2 , v_i \rangle + \cdots + c_n \langle v_n , v_i \rangle = 0$$

$$\Rightarrow c_1 \langle v_1 , v_i \rangle + c_2 \langle v_2 , v_i \rangle + \cdots + c_i \langle v_i , v_i \rangle + \cdots + c_n \langle v_n , v_i \rangle = 0$$

$$\Rightarrow c_i \| v_i \|^2 = 0$$

$$\Rightarrow c_i = 0 , \forall i$$

So, they are linearly independent.

Similar proof, would work for the orthonormal subset.

Re-Mark: Orthonormal subset of inner product space containing nonzero elements is L.I.

Definition: Orthogonal subset S of inner product space V, which spans V is called as the "orthogonal basis of V."

Similarly, orthonormal set which spans inner product space V is called as the orthonormal basis of V.

ex: $\{ (1,0,0), (0,1,0), (0,0,1) \}$

Here, this is orthogonal and orthonormal basis of $\mathbb{R}^3$.

$\{ (1,1,0), (0,1,1), (1,1,1) \}$ is neither orthogonal nor orthonormal basis.

→

V - i.p.s

$S \subseteq V$

$S^{\perp} = \{ v \in V \mid \langle v,u \rangle = 0 \ \forall u \in S \}$

Now,

$V = \mathbb{R}^2$

$S = \{ (x,0) \mid x \in \mathbb{R} \}$

$S^{\perp} = \{ (0,y) \mid y \in \mathbb{R} \}$

1) Show that $S^{\perp}$ is a subspace of V.

Sol:

$\langle 0,u \rangle = 0 \quad \forall \ u \in S$

$\Rightarrow 0 \in S^{\perp}$

Let v, w be $\in S^{\perp}$

$\langle c_1 v + c_2 w \ \ u \rangle \qquad u \in S$

$= \langle c_1 v \ u \rangle + \langle c_2 w \ u \rangle$

$= c_1 \langle v,u \rangle + c_2 \langle w,u \rangle$

$= c_1 (0) + c_2 (0)$

$= 0$

$\Rightarrow c_1 v + c_2 w \in S^{\perp}$

Hence, $S^{\perp}$ is a subspace of V.

Note: $S^{\perp}$ is also called as orthogonal complement of S.

Q- Let W be a subspace of $\mathbb{R}^5$ spanned by $u = (1,2,3,1,-2)$ and $v = (2,4,7,2,-1)$. Find orthogonal compliment of W.

Sol:
$$W^\perp = \{ w \in \mathbb{R}^5 \mid \langle w, u \rangle = 0 \ \forall \ u \in W \}$$

we find $w(x,y,z,s,t) \in \mathbb{R}^5$ such that

$\langle w, u \rangle = 0$

$\langle w, v \rangle = 0$

$\Rightarrow \quad x + 2y + 3z + s - 2t = 0$

$\Rightarrow \quad 2x + 4y + 7z + 2s - t = 0$

$\cancel{z - 5t = 0}$

$z + 3t = 0$

Take y, s, t, $\quad y = 1, \ s = 0, \ t = 0$

$\Rightarrow \quad z = 0$

$\Rightarrow \quad x = -2$

So, $w = (-2, 1, 0, 0, 0)$

if $y = 0, \ s = 1, \ t = 0$

$\Rightarrow \quad z = 0$

$\Rightarrow \quad x = -1 \qquad w = (-1, 0, 0, 1, 0)$

if $y = 0, \ s = 0, \ t = 1$

$\Rightarrow \quad z = -3$

$\Rightarrow \quad x = 11 \qquad w = (11, 0, -3, 0, 1)$

So,
$$W^\perp = \{ (-2,1,0,0,0), (-1,0,0,1,0), (11,0,-3,0,1) \}.$$
forms basis of $W^\perp$.

$\dim(\mathbb{R}^5) = \dim(W) + \dim(W^\perp)$

Let $v = (1,2,3,1) \in \mathbb{R}^4$ find orthogonal basis of $v^\perp$.

Sol:- Let $w = (x, y, z, t)$

$$\Rightarrow x + 2y + 3z + t = 0$$

$x=1, y=0, z=0, t=-1$

$x=0, y=1, z=0, t=-2$

$x=0, y=0, z=1, t=-3$

So, $\cancel{W^\perp} = \{(1,0,0,-1), (0,1,0,-2), (0,0,1,-3)\}$

is the subspace of v.

Now, $\left.\begin{array}{l}(1,0,0,-1)\\(x,y,z,t)\end{array}\right]$ orthogonal

$$x - t = 0$$

and $x + 2y + 3z + t = 0$

$\Rightarrow$ $y = 1$ $z = 0$ $\Rightarrow$ $x = -1$ $t = -1$ $(-1, 1, 0, -1)$	~~$y = 0$~~ ~~$z = 1$~~ ~~$x = -3/2$~~ ~~$t = -3/2$~~ ~~$(-3/2, 0, 1, -3/2)$~~

(x, y, z, t) such that

$x - t = 0$

$x - y + t = 0$

and $x + 2y + 3z + t = 0$

$\Rightarrow$ $x = t$

$y = 2t$

$\Rightarrow$ $t + 4t + 3z + t = 0$

$\Rightarrow$ $z = -2t$

$t = 1$

$\Rightarrow$ $x = 1$

$y = 2$

$z = -2$

So, $(1, 2, -2, 1)$

So, the elements of orthogonal basis are

$\{(-1, 0, 0, -1), (-1, 1, 0, -1), (1, 2, -2, 1)\}$.

2/10/19

Let x, y be orthogonal vectors in inner product space V, then show that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

(Pythagoras Theorem)

Sol:- $\|x+y\|^2 = \langle x+y, x+y \rangle$

$= \langle x \cdot x \rangle + \underbrace{\langle x \cdot y \rangle + \langle y \cdot x \rangle}_{=0} + \langle y \cdot y \rangle$

$\because$ orthogonal.

$\therefore \|x+y\|^2 = \|x\|^2 + \|y\|^2$

Hence, proved.

$\rightarrow$ X ips

$x, y \in X$

Then verify that,

$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

Sol: $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

$\|x-y\|^2 = \langle x-y, x-y \rangle$

$= \langle x, x \rangle - \underbrace{\langle x, y \rangle}_{} - \underbrace{\langle y, x \rangle}_{=0} + \langle y, y \rangle$

$= \|x\|^2 + \|y\|^2$

$\Rightarrow \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

Hence, proved.

Let V be a ips and the set consists of $v_1, v_2, \ldots, v_n$ be orthogonal basis of V

$S = \{v_1, v_2, \ldots, v_n\}$ be orthogonal basis of V then show that any $x \in V$ can be written

as $$x = \sum_{i=1}^{n} \frac{\langle x, v_i \rangle v_i}{\langle v_i, v_i \rangle}$$

$$= \frac{\langle x, v_1 \rangle v_1}{\langle v_1, v_1 \rangle} + \frac{\langle x, v_2 \rangle v_2}{\langle v_2, v_2 \rangle} + \cdots + \frac{\langle x, v_n \rangle v_n}{\langle v_n, v_n \rangle}$$

Here, $\dfrac{\langle x, v_i \rangle}{\langle v_i, v_i \rangle}$ $i = 1, 2, \ldots, n$ are called as Fourier coefficients.

Proof: As S is a basis of V

$x \in V$ can be written as

$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

$$x = \sum_{i=1}^{n} c_i v_i \quad \text{— (1)}$$

$$\langle x, v_j \rangle = \sum_{i=1}^{n} \langle c_i v_i, v_j \rangle$$

$$\langle x\, v_j \rangle = c_j \langle v_j\, v_j \rangle$$

$$c_j = \frac{\langle x\, v_j \rangle}{\langle v_j \cdot v_j \rangle}$$

so, ① $\Rightarrow$

$$x = \sum_{i=1}^{n} \frac{\langle x\, v_i \rangle}{\langle v_i\, v_i \rangle} v_i$$ Hence, proved.

* Suppose, S is orthonormal basis, then just the denominator i.e $\langle v_i\, v_i \rangle$ would be 1.

These two expressions i.e for the above two expressions are called Fourier Expansions.

Let S $\{v_1, v_2, \ldots v_n\}$ be orthonormal basis of V

Then $\|x\|^2 = \sum_{i=1}^{n} |\langle x, v_i \rangle|^2$

Proof: $\|x\|^2 = \langle x \cdot x \rangle$

$$= \left\langle \sum_{i=1}^{n} \frac{\langle x\, v_i \rangle}{\langle v_i \cdot v_i \rangle} v_i \quad x \right\rangle$$

$$= \left\langle \sum_{i=1}^{n} \langle x\, v_i \rangle v_i \quad x \right\rangle$$

$$= \sum_{i=1}^{n} \langle x\, v_i \rangle \langle v_i\, x \rangle$$

$$= \sum_{i=1}^{n} \langle x\, v_i \rangle \overline{\langle x\, v_i \rangle}$$

$$= \sum_{i=1}^{n} |\langle x\, v_i \rangle|^2 .$$

Let $W = \mathbb{R}^4$ (ips) and $S = \{ \frac{1}{2}(1,1,1,1), \frac{1}{\sqrt{6}}(-1,-1,0,2), \frac{1}{5\sqrt{2}}(1,3,-6,2) \}$ which is a orthonormal basis of W. Find Fourier coefficient of the element

$(1,1,2,4)$.

Sol:-

$$c_1 = \frac{\langle (1\ 1\ 2\ 4)\ \frac{1}{2}(1,1,1,1) \rangle}{\langle \frac{1}{2}(1,1,1,1)\ \frac{1}{2}(1,1,1,1) \rangle}$$

$$= \frac{\frac{1}{2} + \frac{1}{2} + \frac{5}{2} + \frac{9}{2}}{\frac{1}{2}[4]}$$

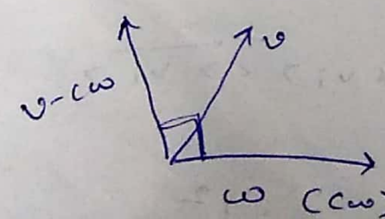
$$= \frac{8}{2} = 4$$

$c_2 = \sqrt{6}$

$c_3 = 0$

Remark: Let V be ips and w belongs to V then $w \neq 0$ projection of v (v is another vector in V) along vector w

$$\text{proj}(v, w) = \frac{\langle v, w \rangle}{\langle w, w \rangle} w$$



$v' = v - cw$

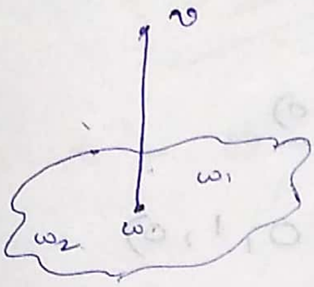
$$\langle v - cw, w \rangle \Rightarrow c = \frac{\langle v, w \rangle}{\langle w, w \rangle}$$

Let v be a vector given as $v = (1,3,5,7)$ and $W = \text{span}\{\underset{w_1}{(1,1,1,1)}, \underset{w_2}{(1,-3,4,-2)}\}$. Find proj. of v onto W.

Sol:- // $\text{proj}(v, w_1) = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$

$$= \left(\frac{16}{4}\right)(1,1,1,1)$$

$$= (4,4,4,4) \;//$$



$$\boxed{\text{proj}(v, W) = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle v\, w_2 \rangle}{\langle w_2\, w_2 \rangle} w_2}$$

Gram - Schmidt orthogonalisation :-

Let V be inner product space and $u_1, u_2, \ldots u_n$ are linearly independent vectors in V. Then, there exists vectors $v_1, v_2, \ldots v_n$ in V which are orthogonal s.t,

$$\text{span}\{u_1, u_2, \ldots, u_k\} = \text{span}\{v_1, v_2, \ldots, v_k\}$$

for $k = 1, 2, \ldots, n$

In fact the vectors are defined as

$$v_1 = u_1$$

$$v_{k+1} = u_{k+1} - \sum_{i=1}^{k} \frac{\langle u_{k+1}, v_i \rangle}{\langle v_i, v_i \rangle} v_i$$

$k = 1, 2, \ldots, n-1$

Let $V = \mathbb{R}^3$ and $\beta = \{(1,0,0), (1,1,0), (1,1,1)\}$ using Gram-schmidt orthogonalisation process find orthogonal basis of $\mathbb{R}^3$.

Sol:-

$$V_1 = U_1 = (1,0,0)$$

$$V_2 = U_2 - \frac{\langle U_2\ V_1 \rangle}{\langle V_1\ V_1 \rangle} V_1$$

$$= (1,1,0) - \frac{1}{1}(1,0,0)$$

$$= (1,1,0) - (1,0,0) = (0,1,0)$$

$$V_3 = U_3 - \frac{\langle U_3\ V_1 \rangle}{\langle V_1, V_1 \rangle} V_1 - \frac{\langle U_3\ V_2 \rangle}{\langle V_2\ V_2 \rangle} V_2$$

$$= (1,1,1) - 1(1,0,0) - 1(0,1,0)$$

$$= (0,0,1)$$

So, orthogonal basis $= \{(1,0,0), (0,1,0), (0,0,1)\}$

Let v be a vector space of polynomials with inner product $\langle f, g \rangle = \int_{-1}^{1} f(t) g(t)\, dt$. Apply Gram-schmidt orth... process to set $\{1, t, t^2\}$ to get orthogonal basis for space $P_2(t)$.

$$V_1 = U_1 = 1$$

$$V_2 = U_2 - \frac{\langle U_2\ V_1 \rangle}{\langle V_1 V_1 \rangle} U_1$$

$$= t - \frac{\int_{-1}^{1} t \, dt}{\int_{-1}^{1} 1 \, dt} (1)$$

$$= t - 0$$

$$= t$$

$$v_3 = u_3 - \cancel{\frac{\langle u_2 \, v_1 \rangle}{\langle v_1 \, v_1 \rangle}} v_1 - \frac{\langle u_3 \, v_2 \rangle}{\langle v_2 \, v_2 \rangle} v_2$$

$$= t^2 - 0 - \frac{\int t^2 . t \, dt}{\int t^2 \, dt} (t)$$

$$= t^2 - 0 - 0 = t^2$$

Gram Schmidt Orthogonalisation:-

We construct $v_1, v_2, \ldots, v_n$ in V s.t

span $\{u_1, u_2, \ldots, u_n\}$ = span $\{v_1, v_2, \ldots, v_n\}$

Let $v_1 = u_1$

we define $u_2 = \alpha v_1 + v_2$

where α is such that $\langle v_2, v_1 \rangle = 0$

$$v_2 = u_2 - \alpha v_1$$

$$v_2 = u_2 - \alpha u_1$$

Now, $\langle v_2, v_1 \rangle = 0$

$$\Rightarrow \langle u_2 - \alpha u_1, v_1 \rangle = 0$$

$$\Rightarrow \langle u_2 \, v_1 \rangle - \alpha \langle u_1 v_1 \rangle = 0$$

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$\Rightarrow \alpha = \dfrac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle}$

So,

$$\boxed{v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1}$$

Now, write

$$u_3 = (\alpha_1 v_1 + \alpha_2 v_2) + v_3$$

$$v_3 = u_3 - (\alpha_1 v_1 + \alpha_2 v_2)$$

α_1, α_2 are s.t

$\langle v_1, v_3 \rangle = 0$ and $\langle v_2, v_3 \rangle = 0$

$\Rightarrow \langle v_3 \; v_1 \rangle = 0$

$$\langle u_3 - (\alpha_1 v_1 + \alpha_2 v_2), v_1 \rangle = 0$$

$$\langle u_3 \; v_1 \rangle - \alpha_1 \langle v_1, v_1 \rangle - \alpha_2 \underbrace{\langle v_2 \; v_1 \rangle}_{=0} =$$

$$\alpha_1 = \frac{\langle u_3 \; v_1 \rangle}{\langle v_1, v_1 \rangle}$$

$\Rightarrow \langle v_3 \; v_2 \rangle = 0$

$$\langle u_3 - (\alpha_1 v_1 + \alpha_2 v_2) \; v_2 \rangle = 0$$

$$\langle u_3 \; v_2 \rangle - \alpha_1 \underbrace{\langle v_1, v_2 \rangle}_{=0} - \alpha_2 \langle v_2 \; v_2 \rangle =$$

$\Rightarrow \langle u_3\, v_2 \rangle = \alpha_2 \langle v_2\, v_2 \rangle$

$$\alpha_2 = \frac{\langle u_3\, v_2 \rangle}{\langle v_2\, v_2 \rangle}$$

so,

$$v_3 = u_3 - \frac{\langle u_3\, v_1 \rangle v_1}{\langle v_1\, v_1 \rangle} - \frac{\langle u_3\, v_2 \rangle}{\langle v_2\, v_2 \rangle} v_2$$

similarly, $\operatorname{span}\{v_1, v_2, v_3\} = \operatorname{span}\{u_1, u_2, u_3\}$

$$u_{k+1} = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k + v_{k+1}$$

where,

$$v_{k+1} = u_{k+1} - (\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k)$$

where, $\alpha_1, \alpha_2, \ldots, \alpha_k$ are such that,

$$\langle v_{k+1}\, v_i \rangle = 0, \text{ for } i = 1, 2, \ldots, k$$

$$\Rightarrow \alpha_i = \frac{\langle u_{k+1}\, v_i \rangle}{\langle v_i\, v_i \rangle}$$

$$\Rightarrow v_{k+1} = u_{k+1} - \sum_{i=1}^{k} \frac{\langle u_{k+1}\, v_i \rangle}{\langle v_i\, v_i \rangle} v_i$$

$\operatorname{span}\{u_1, u_2, \ldots, u_{k+1}\} = \operatorname{span}\{v_1, v_2, \ldots, v_{k+1}\}$

Q- Consider the subspace U of $\mathbb{R}^4$ spanned by the vector $u_1 = (1,1,1,1)$ $u_2 = (1,1,2,4)$ $u_3 = (1,2,-4,-3)$.

Find ⓐ an orthogonal basis of U

ⓑ an orthonormal basis of U

Sol:-

$$\vartheta_1 = u_1 = (1,1,1,1)$$

$$v_2 = u_2 - \frac{\langle u_2\, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= (1,1,2,4) - \frac{8}{4}(1,1,1,1)$$

$$= (1,1,2,4) - (2,2,2,2)$$

$$v_2 = (-1,-1,0,2)$$

$$\vartheta_3 = u_3 - \frac{\langle u_3\, v_1 \rangle}{\langle v_1\, v_1 \rangle} v_1 - \frac{\langle u_3\, v_2 \rangle}{\langle v_2\, v_2 \rangle} v_2$$

$$= (1,2,-4,-3) + \frac{4}{4}(1,1,1,1) + \frac{\cancel{9}^{3}}{\cancel{6}\,2}(-1,-1,0,$$

$$= (1,2,-4,-3) + (1,1,1,1) + \cancel{\left(-\tfrac{5}{3}, -\tfrac{5}{3}, 0,\right.}$$

$$\left(-\tfrac{3}{2}, -\tfrac{3}{2}, 0, 3\right.$$

$$= \left(\tfrac{1}{2}, \tfrac{3}{2}, -3, 1\right)$$

$$S = \left\{ (1,1,1,1), (-1,-1,0,2), \left(\tfrac{1}{2}, \tfrac{3}{2}, -3, 1\right) \right.$$

$$S' = \left\{ \frac{\vartheta_1}{\|\vartheta_1\|}, \frac{\vartheta_2}{\|\vartheta_2\|}, \frac{v_3}{\|v_3\|} \right\}$$

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$\|v_1\| = (\langle v_1, v_1 \rangle)^{1/2}$

$\|v_1\| = 2$

$\|v_2\| = \sqrt{6}$

$\|v_3\| = \frac{5}{\sqrt{2}}$

Q- Show that every finite dimensional inner product space has an orthonormal basis.

→ we taken $\{u_1, u_2 \ldots, u_n\}$ be finite dim. ips and by Gram.... we would produce the orthogonal and orthonormal basis.

Q- Suppose V is a finite dimensional ips and V_0 is subspace of V, then show that $\exists$ a subspace W of V such that $V = V_0 + W$ and $V_0 \perp W$.

Sol:- case, $V_0 = V$

Take, $W = \{0\}$

then $V = V_0 + W$ & $V_0 \perp W$.

case, $V_0 \neq V$ i.e $V_0 \subset V$

Let $\dim V_0 = K < \dim V$

Let $\beta\{v_1, v_2, \ldots v_K\}$ basis of V_0.

So, β can be extended that as $\{v_1, v_2 \ldots, v_K, v_{K+1}, \ldots v_n\}$ to form basis of V.

we use G-S process to get $\beta = \{w_1, w_2, \ldots, w_n\}$ as orthogonal basis of V.

Take, $W = \text{span}\{w_{K+1}, \ldots, w_n\}$

then, $V = V_0 + W$

and $V_0 \perp W$

Ex: $V = \mathbb{R}^3$

$V_0 = \{(x, 0, 0), | x \in \mathbb{R}\}$

$\beta = \{(1,0,0)\}$

$\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$

$W = \text{span}\{(0,1,0), (0,0,1)\}$

$R^3 = V_0 \oplus W$

Def: A real matrix P is said to be orthogonal if $P^{-1} = P$

Remark: Let P be a real matrix then the following are equivalent.

① P is orthogonal

② The rows of P forms an orthonormal set

③ The columns of P forms an orthogonal set

Ex:- $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ \frac{2}{\sqrt{6}} & 1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$

Find the orthogonal matrix P whose first row is $(1/3, 2/3, 2/3)$.

Sol:

$$P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -4/\sqrt{18} & 1/\sqrt{18} & 1/\sqrt{18} \end{bmatrix}$$

$\langle (x,y,z)\ (1/3\ 2/3\ 2/3) \rangle = 0$

$$\frac{x}{3} + \frac{2y}{3} + \frac{2z}{3} = 0$$

$$x + 2y + 2z = 0$$

$(0,1,-1)$

$(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$

$x + 2y + 2z = 0$

$y - z = 0$

$y = z$

$z = 1$

$y = 1$ $\quad (-4, 1, 1)$

$x = -4$ $\quad (\frac{-4}{\sqrt{18}}, \frac{1}{\sqrt{18}}, \frac{1}{\sqrt{18}})$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{bmatrix}$$

we though here rows are orthogonal A is not orthogonal.
so, to make it ~~orthogonal~~ orthonormal divide each row with its norm.

$$A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{26} & \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ 7/\sqrt{78} & \frac{-5}{\sqrt{78}} & \frac{2}{\sqrt{78}} \end{bmatrix}$$

Q. show the following

(1) P is orthogonal iff P^T is orthogonal.

(2) P is orthogonal iff P^{-1} is orthogonal.

(3) if P and Q are orthogonal, if product exist, then PQ is also orthogonal.

Sol: (1) $PP^T = I = P^T P$

so, P^T is also orthogonal.

(2) $PP^{-1} = I = P^{-1}P$

(3) $(PQ)^{-1} = Q^{-1}P^{-1} = Q^T P^T = (PQ)^T$

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orthogonal matrices:-

$P^{-1} = P^T$

Show that P is orthogonal matrix, then

① $\langle Pu, Pv \rangle = \langle u, v \rangle \ \forall\ u, v \in V$

② $\| Pu \| = \| u \| \ \forall\ u \in V$

① $\langle Pu, Pv \rangle = (Pu)^T (Pv)$

$= u^T P^T P v$

$= u^T v$

$= \langle u, v \rangle$

② $\| Pu \|^2 = \langle Pu, Pu \rangle$

$= (Pu)^T Pu$

$= u^T P^T P u$

$= u^T u$

$= \langle u, u \rangle$

$= \| u \|^2$

$\therefore \| Pu \| = \| u \|.$

ex Suppose $\beta\{e_1, e_2, \ldots, e_n\}$ and $\beta'\{e_1', e_2', \ldots, e_n'\}$ be two orthonormal basis of ips V. let P be change of basis matrix from basis β to β', then P is a orthogonal matrix.

Sol:

Proof: As β is basis, we can write

$$e_i' = b_{i1} e_1 + b_{i2} e_2 + \cdots + b_{in} e_n$$

$$i = 1, 2, \ldots, n$$

Now,

$$\delta_{ij} = \langle e_i' \; e_j' \rangle$$

$$= \langle b_{i1} e_1 + b_{i2} e_2 \ldots + b_{in} e^n, \; b_{j1} e_1 + \ldots + b_{jn} e_n \rangle$$

$$\delta_{ij} = b_{i1} b_{j1} + b_{i2} b_{j2} + \cdots + b_{in} b_{jn}$$

Let $B = [b_{ij}]$ then $P = B^T$

Now, suppose $BB^T = [C_{ij}]$

$$C_{ij} = b_{i1} b_{j1} + b_{i2} b_{j2} + \cdots + b_{in} b_{jn}$$

$$C_{ij} = \delta_{ij}$$

$$\Rightarrow BB^T = I = B^T B$$

$\Rightarrow$ B is orthogonal and

$B^T = P$ is also orthogonal

Suppose $\{e_1 e_2 \ldots e_n\}$ be orthonormal basis of ips V. Let P be a orthogonal matrix, then show that the following n vectors given as

$$e_i' = a_{1i} e_1 + a_{2i} e_2 + a_{3i} e_3 + \dots + a_{ni} e_n$$

$i = 1, 2, \dots n$

Show that these n vectors form an orthonormal basis on V.

Sol: $\delta_{ij} = \langle e_i' \, e_j' \rangle$

$$= a_{1i} a_{1j} + a_{2i} a_{2j} + \dots + a_{ni} a_{nj}$$

$$P = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$= \langle c_i \; c_j \rangle$$

where, c_i and c_j are i^{th} and j^{th} column of P.

$$= \delta_{ij}$$

So, these n vectors would form a ortho-normal basis on V.

Defn: A symmetric matrix A is said to be positive definite if for every nonzero vector $v \in \mathbb{R}^n$ ($\underset{(n \times n)}{A}$ - real symmetric matrix)

$$\langle v \, A v \rangle = v^T A v > 0$$

Result:

Let A be the (2×2) real symmetric matrix,

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

($\because$ symmetric)

Then A is positive definite iff $ad-b^2>0$

and $a>0$, $d>0$.

$u=(x,y)^T$

then, $f(u) = u^T A u$

$$= (x, y)\begin{bmatrix} a & b \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= (x\ y)\begin{bmatrix} ax+by \\ bx+dy \end{bmatrix}$$

$$= ax^2+bxy+bxy+dy^2$$

$$= ax^2+2bxy+dy^2$$

Let A is positive definite, then $f(u)>0\ \forall\ u\neq 0$ $=(x\ y)^T\in\mathbb{R}^2$

Take $u=(1,0)^T$

then, $a>0$

Take $u=(0,1)^T$

then, $d>0$

Take $u = (b, -a)^T$

then, $f(u) = ab^2 - 2ab^2 + da^2$

$= da^2 - ab^2$

$= a(ad - b^2)$

then, $ad - b^2 > 0$

conversely,

Let $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

and $a > 0$, $d > 0$, $ad - b^2 > 0$,

$u^T A u = f(u) = ax^2 + 2bxy + dy^2$

$= a\left[x^2 + \frac{2bxy}{a} + \frac{b^2}{a^2}y^2\right]$

$+ dy^2 - \frac{b^2}{a}y^2$

$= a\left[x + \frac{b}{a}\right]^2 + dy^2 - \frac{b^2}{a}y^2$

$u^T A u = a\left(x + \frac{b}{a}\right)^2 + \frac{(ad - b^2)}{a}y^2 > 0$

So, A is a positive definite matrix.

Hence, proved.

Let A be real positive definite matrix, then show that $\langle u, v \rangle := u^T A v$ defines inner product on R^n.

Sol:-

① $\langle u, u \rangle = u^T A u > 0 \ \forall \ u \neq 0$

Now, $\langle u, u \rangle = 0 \Leftrightarrow u^T A u = 0 \Leftrightarrow u = 0$

② $\langle u, v \rangle = u^T A v$

$$= (u^T A v)^T$$

$$= v^T A^T u$$

$$= v^T A u \quad (\because A^T = A)$$

$$= \langle v, u \rangle$$

③ $\langle cu, v \rangle = (cu)^T A v$

$$= c(u^T A v)$$

$$= c\langle u, v \rangle$$

④ $\langle u+v, w \rangle = (u+v)^T A w$

$$= (u^T + v^T) A w$$

$$= u^T A w + v^T A w$$

$$= \langle u, w \rangle + \langle v, w \rangle$$

Check the positive definiteness of the following matrices,

① $\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$ (×) $\quad \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ (✓)

$\begin{bmatrix} 2 & -4 \\ -4 & k \end{bmatrix} \Rightarrow k > 8$

$\begin{bmatrix} k & 5 \\ 5 & -2 \end{bmatrix} \rightarrow (\times)$ since, $d < 0$

$\begin{bmatrix} 6 & k \\ k & 4 \end{bmatrix} \Rightarrow 24 - k^2 > 0$

$\Rightarrow |k| < \sqrt{24}$

$\Rightarrow -\sqrt{24} < k < \sqrt{24}$

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1. Find 3X3 orthogonal matrix P whose first two rows are multiples of $u = (-1,1,1)$ and $v = (1,-2,3)$ respectively.

Sol:-

$$\begin{bmatrix} k(-1, & 1, & 1) \\ p(1, & -2, & 3) \\ x & y & z \end{bmatrix}$$

$\cancel{k + p = 0}$
$\cancel{k - 2p = 0} \Rightarrow \cancel{3k + 2p = 0}$
$\cancel{k + 3p = 0} \quad \cancel{k = -\frac{2}{3}p}$

$$\begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{14}} & \frac{-2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ x & y & z \end{bmatrix}$$

$\Rightarrow -x + y + z = 0$

$x - 2y + 3z = 0$

$y = 4z$

$\Rightarrow k[5, 4, 1]$

third row is,

$$\left[\frac{5}{\sqrt{42}}, \frac{4}{\sqrt{42}}, \frac{1}{\sqrt{42}}\right]$$

so,

$$P = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{14}} & \frac{-2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ \frac{5}{\sqrt{42}} & \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{42}} \end{bmatrix}$$

2. Find the number & exhibits all 2x2 orthogonal matrices of the form

$$\begin{bmatrix} \frac{1}{3} & x \\ y & z \end{bmatrix}$$

Sol:

$$\left(\frac{1}{3}\right)^2 + x^2 = 1$$

$$x^2 = 1 - (1/3)^2$$

$$x^2 = 1 - 1/9$$

$$x^2 = 8/9$$

$$x = \pm\frac{2\sqrt{2}}{3}$$

$$y^2 + z^2 = 1$$

$$\left(\frac{1}{3}\right)^2 + y^2 = 1$$

$$y = \pm\frac{2\sqrt{2}}{3}$$

$$\Rightarrow z^2 = 1/9$$

$$\Rightarrow z = \pm\frac{1}{\sqrt{3}}$$

$$\begin{bmatrix} \frac{1}{3} & \frac{\sqrt{8}}{3} \\ \frac{\sqrt{8}}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{3} & -\frac{\sqrt{8}}{3} \\ \frac{\sqrt{8}}{3} & \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{3} & -\sqrt{8}/3 \\ -\sqrt{8}/3 & -1/3 \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & \sqrt{8}/3 \\ -\sqrt{8}/3 & 1/3 \end{bmatrix}$$

4 possible.

3. Definition: A linear transformation $T: V \to V$ (V i.p.s) is said to be self adjoint if

$< Tx, y > = < x, Ty > \quad \forall\, x, y \in V$.

Suppose, V is finite dimensional inner product space $E = \{v_1, v_2, \ldots, v_n\}$ be basis of V.

$T: V \to V$ be self adjoint then,

$< Tv_j, v_i > = < v_j, Tv_i > \quad i = 1, 2, \ldots, n$

$\to$ $\mathbb{F} = \mathbb{C}$ $\quad V = \mathbb{C}^n$, T is self adjoint,

$[T]_{EE}$ = Hermitian matrix

$\to$ $F = \mathbb{R}$ $\quad V = \mathbb{R}^n$, T is self adjoint,

$[T]_{EE}$ = Symmetric matrix.

Q- Show that eigen values of self-adjoint matrix are real numbers.

Sol: Proof:

Let λ be eigen value of T i.e $\exists$ $x \neq 0 \in V$ such that $Tx = \lambda x$

Also, note that, if T is self adjoint then $< Tx, x > = < x, Tx >$

$\Rightarrow < Tx, x > \in \mathbb{R}$

Take,

$$\langle Tx, x \rangle = \langle \lambda x, x \rangle$$

$$= \lambda \langle x, x \rangle$$

$$\Rightarrow \lambda = \frac{\langle Tx, x \rangle}{\|x\|^2} \quad \text{and } \langle Tx, x \rangle \in \mathbb{R}$$

$$\& \ \|x\|^2 > 0$$

$\Rightarrow \lambda$ is a real number.

Q- Show that eigen-vectors associated with distinct eigen values of a self-adjoint operator are orthogonal.

Proof: Let λ_1, λ_2 are two distinct eigen values of self-adjoint operator T and corresponding eigen vectors are x_1, x_2 respectively.

i.e $Tx_1 = \lambda_1 x_1$ and $Tx_2 = \lambda_2 x_2$

Take,

$$\lambda_1 \langle x_1, x_2 \rangle$$

$$= \langle \lambda_1 x_1, x_2 \rangle$$

$$= \langle Tx_1, x_2 \rangle$$

$= \langle x_1, Tx_2 \rangle$

$= \langle x_1, \lambda_2 x_2 \rangle$

$= \bar{\lambda}_2 \langle x_1, x_2 \rangle$

$= \lambda_2 \langle x_1, x_2 \rangle$

$\Rightarrow (\lambda_1 - \lambda_2) \langle x_1, x_2 \rangle = 0$

Since, $\lambda_1 \neq \lambda_2$

So,

$\langle x_1, x_2 \rangle = 0$

Hence proved.

Remark:

Every self adjoint operator on a finite dimensional inner product space has an eigen value.

*

Def: Let $T: V \to V$ be a self adjoint operator on finite dimensional inner product space. Then there exists an orthonormal basis of V consists of eigen vectors of T.

L ①

Remark: This theorem also says, if A is a symmetric matrix, then there exist orthogonal matrix P such that $P^{-1}AP = D$.

→ if symmetric, then diagonasible.

Lemma:

Let $T: V \to V$ be a self-adjoint operator on a finite dimensional inner product space V and V_0 is a subspace of V.

Then,

$$T(V_0) \subseteq V_0 \Rightarrow T(V_0^{\perp}) \subseteq V_0^{\perp}$$

$$(TV_0 = \{ Tv \mid v \in V_0 \})$$

if $TV_0 \subseteq V_0$ then we say that the subspace V_0 is invariant under T.

Proof

Let $x \in V_0^{\perp}$ To show: $Tx \in V_0^{\perp}$

$$\langle x, u \rangle = 0 \quad \forall \quad u \in V_0$$

Let $u \in V_0$ Then $Tu \in V_0$

$$\langle Tx, u \rangle = \langle x, Tu \rangle = 0$$

$$\Rightarrow Tx \in V_0^{\perp}$$

ex Let V be a ips (f.dim) and $T: V \to V$ be a linear operator (any). Show that, if λ is a eigen value of T, then

$N(T - \lambda I)$ is invariant under T.

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sol: $x \in N(T-\lambda I)$

then, $Tx = \lambda x$

$T(Tx) = T(\lambda x)$

$\Rightarrow T(\lambda x) = \lambda(T(x))$

$\Rightarrow T(Tx) = \lambda \cdot T(x)$

Hence, $Tx \in N(T-\lambda I)$.
So, $N(T-\lambda I)$ is invariant under T.
Hence, proved.

Proof of ①:

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigen values of T.

$$V_0 = N(T-\lambda_1 I) + N(T-\lambda_2 I) + \cdots + N(T-\lambda_k I)$$

If $V_0 = V$, then union of orthonormal basis of null space of $T-\lambda_i I$ i.e $N(T-\lambda_i I)$ $i = 1,2,\ldots K$ will orthonormal basis of V consist of eigen vectors of T.

Suppose, $V_0 \neq V$, then

$V_0^{\perp} \neq 0$

Now, $T(V_0) \subseteq V_0$

$\Rightarrow T(V_0^{\perp}) \subseteq V_0^{\perp}$

Define, $T_1 : V_0^{\perp} \to V_0^{\perp}$

$$T_1 v = Tv, \ \forall \ v \in V_0^{\perp}$$

Note T_1 is self adjoint (as T is self adjoint)

so, it will have eigen value say λ $\exists$ x $\neq$ 0 $\in V_0^{\perp}$ and

s.t $\quad T_1 x = \lambda x$

$\Rightarrow Tx = \lambda x$

$\Rightarrow \lambda$ is eigenvalue of T & $\lambda = \lambda_i$
$1 \leq i \leq k$

Say, $\lambda = \lambda_i$

$\Rightarrow x \in N(T - \lambda_i I) \in V_0$

$\Rightarrow (V_0 \cap V_0^{\perp}) \neq \{0\}$

which is a contradiction to $V_0 \neq V$.

So, $V_0 = V$.

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Let $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x, y) = (7x + 3y, 3x - y)$

Find orthogonal matrix P such that $P^{-1}[T]_\beta P$ is a diagonal matrix.

Sol: $[T]_\beta = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

λ are real for symmetric.

$|[T]_\beta - \lambda I| = 0$

$\Rightarrow (7-\lambda)[1+\lambda] + 9 = 0$

$\Rightarrow -\lambda^2 + 6\lambda + 16 = 0$

$\Rightarrow \lambda^2 - 6\lambda - 16 = 0$

$\Rightarrow (\lambda - 8)(\lambda + 2) = 0$

$\Rightarrow \lambda = -2; 8$

for $\lambda = 8$,

$$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x+3y=0$

$x=3y$

Take, $y=1$

$x=3$

So, $(3,1)$ is eigen vector.

$\lambda=-2$;

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 3x+y=0$

$\Rightarrow y=3 \Rightarrow x=-1$

So, $v_2=(-1,3)=(-1,3)$

Now, $Q=\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

$\therefore P=\begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$

Let $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$. A real quadratic form is an expression of the form

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

$$Q = a_{11} x_1^2 + (a_{12} + a_{21}) x_1 x_2 + \dots + (a_{1n} + a_{n1}) x_1 x_n$$

$$+ \dots + + \dots + a_{nn} x_n^2$$

$$A = (a_{ij})$$

$$Q = x^T A x$$

$= x^T B x$ where B is a symmetric matrix.

$B = (b_{ij})$ is defined as

$$b_{ii} = a_{ii} \quad \& \quad b_{ij} = \frac{a_{ij} + a_{ji}}{2}$$

Ex Obtain the symmetric matrix B from the quadratic forms.

1) $Q = 2x_1^2 + 3x_1 x_2 + x_2^2$

2) $Q = x_1^2 + 2x_1 x_2 - 4x_1 x_3 + 6x_2 x_3 - 5x_2^2 + 4x_3^2$.

1)

$$A = \cancel{\begin{bmatrix} 2 & 3/2 \\ 3/2 & 1 \end{bmatrix}}$$

2)

$$B = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -5 & 3 \\ -2 & 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3/2 \\ 3/2 & 1 \end{bmatrix}$$

We can reduce the quadratic form to a canonical form by Principal axis theorem.

Principal axis theorem:

Let B be symmetric matrix with eigen values $\lambda_1, \lambda_2 \dots \lambda_n$ Let P be the orthogonal matrix which diagonalizes B.

Then, $X = PY$ (which is a Linear transformation) transforms quadratic form $\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} x_i x_j$

to the form

$$\sum_{i=1}^{n} \lambda_i (y_i)^2$$ which is called as

canonical form/standard form.

Here,

$X = PY$ is the "orthogonal transformation" bcz P is an orthogonal matrix.

Proof:-

Note that, $\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_i x_j = x^T B x$

$= (PY)^T B (PY)$

$= y^T P^T B P y$

$= y^T (P^{-1} B P) y$

$= y^T D y$

Here, P is orthogonal

So, $P^T = P^{-1}$

and $D = P^{-1} B P$.

where, D is $diag\{\lambda_1, \lambda_2, \dots, \lambda_n\}$

$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots \lambda_n y_n^2$

$= \sum_{i=1}^{n} \lambda_i y_i^2$

Ex:- Find the orthogonal transformation which converts the given quadratic form to the canonical form.

$Q(x_1, x_2) = x_1^2 + 6x_1x_2 - 7x_2^2$

Sol:

$$B = \begin{bmatrix} 1 & 3 \\ 3 & -7 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & -(7+\lambda) \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(7+\lambda)+9 = 0$

$\Rightarrow -\lambda^2-6\lambda+16=0$

$\Rightarrow \lambda^2+6\lambda-16=0$

$\Rightarrow (\lambda+8)(\lambda-2)=0$

$\Rightarrow \lambda=+2;\ \lambda=-8$

for $\lambda = 2$,

$$\begin{bmatrix} x \\ y \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} = 0$$

$\Rightarrow -x+3y=0$

$x = 3y$

$y=1 \Rightarrow x=3$

$V_1 = (3,1)^T$

$\lambda = -8$,

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 3x+y=0$

$x=-1$

Take, $y = 3$

$V_2 = (-1,3)^T$

So, $P = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$

$X = PY$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$\Rightarrow x_1 = (3y_1 - y_2)/\sqrt{10}$

$x_2 = (y_1 + 3y_2)/\sqrt{10}$.

So,

$$Q = \frac{(3y_1 - y_2)^2}{10} + \frac{6(3y_1 - y_2)(y_1 + 3y_2)}{10}$$

$$- 7\frac{(y_1 + 3y_2)^2}{10}$$

$$= \frac{1}{10}\left[9y_1^2 + y_2^2 - 6\cancel{y_1y_2} + 18y_1^2 + 4\cancel{8}y_1y_2 - 18y_2^2 - 7y_1^2 - 63y_2^2 - 42\cancel{y_1y_2} \right]$$

$$= \frac{1}{10}\left[20y_1^2 - 80y_2^2 \right]$$

$$= 2y_1^2 - 8y_2^2$$

$$= \Sigma \lambda_i y_i^2$$

Find the orthogonal tran Xm which transform quad to Canonical form.

$$Q = x_1^2 - 2x_1x_2 + x_2^2$$

$$B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} (1-\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)^2 = 1$

$\Rightarrow 1-\lambda = 1 \;\;|\;\; 1-\lambda = -1$

$\Rightarrow \lambda = 0 \;\;|\;\; \lambda = 2$

$\lambda = 0$,

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x = y$

$\Rightarrow y = 1$

so, $x = 1$

$v_1 = (1,1)$

$\lambda = 2$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x + y = 0$

$y = 1$

$\Rightarrow x = -1$

$v_2 = (-1,1)$

So, $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$X = PY$

$$x_1 = \frac{y_1}{\sqrt{2}} - \frac{y_2}{\sqrt{2}}$$

$$x_2 = \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}}$$

Now, on taking the orthogonal transformation,

$$Q = \Sigma \lambda_i y_i^2$$

$$Q = 0(y_1)^2 + (2)y_2^2$$

$$Q = 2y_2^2$$

Analyse the chronic, $4x_1^2 - 3x_1x_2 + 2x_2^2 = 8$, using principal axes theorem.

Sol :- $4x_1^2 - 3x_1x_2 + 2x_2^2 = 8$

$$B = \begin{bmatrix} 4 & -3/2 \\ -3/2 & 2 \end{bmatrix}$$

$$\Rightarrow (4-\lambda)(2-\lambda) - \frac{9}{4} = 0$$

$$\Rightarrow 4[\lambda^2 - 6\lambda + 8] - 9 = 0$$

$$\Rightarrow 4\lambda^2 - 24\lambda + 23 = 0$$

$$\frac{24 \pm \sqrt{576 - (23)(16)}}{8}$$

$$= \frac{6 \pm \sqrt{13}}{2}$$

$$Q = \left[\frac{6+\sqrt{13}}{2}\right]y_1^2 + \left[\frac{6-\sqrt{13}}{2}\right]y^2 = 8$$

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Best approximation:

Def Let V be ips and V_0 be subspace of V. Let $x \in V$

We say $x_0 \in V_0$ is best approximation for x, from V_0 if

$$\|x - x_0\| \leq \|x - v\| \quad \forall v \in V_0$$

$V = C[a,b]$ $f \in C[a,b]$ $p \in P[a,b]$

$V_0 = P[a,b]$

$$\|f - p\| \leq \|f - q\| \quad \forall q \in P[a,b]$$

Theorem: Let V be ips and V_0 be subspace of V If $x_0 \in V_0$ is such that $x - x_0 \perp V_0$ Then x_0 is the best approximation for the element x and also it is the unique best approximation from V_0 for x.

Conversely, if x_0 is the best approximation for $x_0 \in V$ then $x - x_0 \perp V_0$.

<u>Proof:</u>

Suppose, $x_0 \in V_0$ is such that $x - x_0 \perp V_0$,

for every $u \in V_0$,

$$\|x - u\|^2 = \|x - x_0\|^2 + \|x_0 - u\|^2 \quad \forall u \in V_0$$

$$\Rightarrow \|x - x_0\| \leq \|x - u\| \quad \forall u \in V_0$$

$\therefore$ x_0 is the best approximation for x from V_0.

<u>Uniqueness:</u>

Let v_0 be another best approximation

then we have $\|x - v_0\| \leq \|x - x_0\|$

Also,

$$\|x - x_0\| \leq \|x - v_0\|$$

$$\Rightarrow \|x - x_0\| = \|x - v_0\| \quad \text{—} \ (1)$$

We know that,

$$\langle x - x_0, x_0 - v_0 \rangle = 0$$

So we have,

$$\|x - v_0\|^2 = \|x - x_0\|^2 + \|x_0 - v_0\|^2$$

from (1),

$$\|x_0 - v_0\| = 0$$

$$\Rightarrow \boxed{x_0 = v_0}$$

for Converse part,

Let x_0 be the best approximation for $x \in$

then $\|x - x_0\| \leq \|x - u\| \;\forall\; u \in V_0$

We take a particular u, $u = x_0 + \alpha v$ where $v \in V_0$ and $\alpha \in \mathbb{F}$

$$\Rightarrow \|x - x_0\|^2 \leq \|x - (x_0 + \alpha v)\|^2$$

$$\Rightarrow \|x - x_0\|^2 \leq \langle x - x_0 - \alpha v \quad x - x_0 - \alpha v \rangle$$

$$\Rightarrow \|x - x_0\|^2 \leq \langle x - x_0 \quad x - x_0 \rangle - \langle x - x_0 \quad \alpha v \rangle$$

$$- \langle \alpha v \quad x - x_0 \rangle + \alpha\bar{\alpha} \langle v \quad v \rangle$$

$$\Rightarrow 0 \leq \|x - x_0\|^2 \leq \|x - x_0\|^2 - 2\,\mathrm{Re}\langle x - x_0 \quad \alpha v \rangle + |\alpha|^2 \|v\|^2 \quad \text{--- (1)}$$

Taking $\alpha = \dfrac{\langle x - x_0 \quad v \rangle}{\|v\|^2}$

$\mathrm{Re}(\langle x - x_0 \quad \alpha v \rangle)$

$$= \left\langle x - x_0 \quad \frac{\langle x - x_0 \quad v \rangle}{\|v\|^2} v \right\rangle$$

$$= \frac{|\langle x - x_0 \quad v \rangle|^2}{\|v\|^2}$$

$$|\alpha|^2 = \frac{|\langle x - x_0 \quad v \rangle|^2}{\|v\|^4}$$

from *

from * in (2),

$$\Rightarrow \|x-x_0\|^2 \leq \|x-x_0\|^2 - \frac{|\langle x-x_0, v\rangle|^2}{\|v\|^2}$$

$\Rightarrow$

$\langle x-x_0, v\rangle = 0$ for all, $v \in V_0$

$\therefore x-x_0 \perp V_0$

Ex Let V be a ips, V_0 be a finite dimensional ~~vector~~ subspace of V and $x \in V$. Let $u_1, u_2, \ldots, u_n$ be an orthonormal basis of V_0 then show that the element x_0 given as,

$$x_0 = \sum_{i=1}^{n} \langle x, u_i\rangle u_i$$ will be unique

best approximation for x.

Sol: Here we need to show that, $x-x_0 \perp V_0$.

so, it is sufficient to show that,

$\langle x-x_0, u_j\rangle = 0$ for $j=1,2,\ldots,n$

$$\Rightarrow \langle x - \sum_{i=1}^{n} \langle x, u_i\rangle u_i, u_j\rangle$$

$$= \langle x, u_j\rangle - \sum_{i=1}^{n} \langle x, u_i\rangle \langle u_i, u_j\rangle$$

$\Downarrow$

Here, it will be 1 for $i=j$

$=0$, elsewhere

$= 0$

$$\langle x-x_0, u_j\rangle = \langle x, u_j\rangle - \langle x, u_j\rangle = 0$$

$x-x_0 \perp V_0$

Remark:

Suppose, V is inner product space and V_0 be finite dimensional vector subspace of V, Let $x \in V$ we are interested in finding best approximation for x i.e we find $x_0 \in V_0$

s.t $\langle x - x_0, v \rangle = 0$ for all $v \in V_0$.

Let $\{u_1, u_2, \ldots, u_n\}$ basis of V_0

then, $x_0 = \sum_{i=1}^{n} \alpha_i u_i$

So, we find scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ by the relation,

$$\langle x - x_0, u_j \rangle = 0 \quad j = 1, 2, \ldots, n$$

$$\Rightarrow \langle x - \sum_{i=1}^{n} \alpha_i u_i, u_j \rangle = 0$$

Ex Let $V = \mathbb{R}^2$ be usual inner product,

$$V_0 = \{ (x_1, x_2) \mid x_1 = x_2, \; x_1, x_2 \in V \}$$

Find the best approximation for the following elements,

(a) $(0,1)$ (b) $(1,2)$

Sol:- we note that $\{\overset{u_1}{(1,1)}\}$ be the basis of V_0.

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find $x_0 \in V_0$ such that,

$$\langle x - x_0, u_1 \rangle = 0$$

as $x_0 \in V_0$, so x_0 will be of form (α, α)

$$\langle x - (\alpha, \alpha), u_1 \rangle = 0$$

$$\langle (x_1, x_2) - (\alpha, \alpha), (1,1) \rangle = 0$$

$$\cancel{\langle x_1, x_2 \; (1,1) \rangle - \langle (\alpha,\alpha) \; (1,1) \rangle = 0}$$

(a)

$$\cancel{\langle (0,1) \; (1,1) \rangle = \langle \alpha, \alpha \rangle}$$

$$\langle 1, 2$$

$$\langle -\alpha, 1-\alpha \; (1,1) \rangle = 0$$

$$-\alpha + 1 - \alpha = 0$$

$$2\alpha = 1$$

$$\alpha = 1/2$$

$$(1/2, 1/2)$$

(b) $(1,2)$

$$\langle 1-\alpha, 2-\alpha \; (1,1) \rangle = 0$$

$$1 - \alpha + 2 - \alpha = 0$$

$$2\alpha = 3$$

$$\alpha = 3/2$$

so, $(3/2, 3/2)$

2) Let V be vector space, $V = C[0,1]$ over $\mathbb{R}$ with $\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$. Let V_0 is space of polynomials of degree 1, find the best approximation for the function,

$f(t) = t^2.$

Sol: $\beta \{\overset{u_1}{1}, \overset{u_2}{t}\}$ is a basis of V_0

we find $f_0(t) = a_0 + a_1 t$

such that,

$\langle f - f_0, u_i \rangle = 0 \quad i = 1, 2,$

$\langle t^2 - a_1 t - a_0 \quad 1 \rangle = 0$

$\langle t^2 a_1 t - a_0 \quad t \rangle = 0$

$\Rightarrow \int_0^1 (t^2 - a_1 t - a_0)\,dt = 0$

$\frac{1}{3} - \frac{a_1}{2} - a_0 = 0 \quad - (1) \quad \Rightarrow a_0 = \frac{1}{3} - \frac{a_1}{2}.$

$\int_0^1 (t^3 - a_1 t^2 - a_0 t)\,dt = 0$

$\Rightarrow \frac{1}{4} - \frac{a_1}{3} - \frac{a_0}{2} = 0 \quad - (2)$

$\Rightarrow \frac{1}{4} - \frac{a_1}{3} - \frac{1}{6} + \frac{a_1}{4} = 0$

$\Rightarrow \frac{a_1}{12} = \frac{2}{24} \quad \Rightarrow a_1 = 1$

So, $a_0 = \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = -\frac{1}{6}$

So, $f_0 = -\frac{1}{6} + t$ is the best approximation for f

3) Let $V = C[0,1]$ with $\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$, for the $V_0 = P_2$ find best approximation for the following;

a) e^t b) $\sin t$ c) $\cos t$ in V_0.

$f_0(t) = a_0 + a_1 t + a_2 t^2$

24/10/19

Some special operators on inner product space.

<u>Adjoint operator</u>: A linear operator T on inner product space V is said to have an adjoint operator T on V

if $\langle Tu, v \rangle = \langle u, Tv \rangle \;\forall\; u, v \in V$

<u>Ex</u>: Let A be a real n square matrix

Note: A can be seen operator on space $\mathbb{R}^n$

Then,

$$\langle Au, v \rangle = (Au)^T v$$
$$= u^T A^T v$$
$$= \langle u, A^T v \rangle$$

Thus, the transpose of A i.e., A^T is the adjoint of a real matrix A.

Ex Let A be square complex matrix of n,

$$\langle Au, v \rangle = (Au)^T \bar{v}$$
$$= u^T \bar{A}^T v$$
$$= \langle u, \bar{A}^T v \rangle$$

In case of a complex matrix the conjugate transpose of A is adjoint of A.

Ex Find the adjoint of L.T, $T : R^3 \rightarrow R^3$, defined by

$$T(x, y, z) = (3x + 4y - 5z,\ 2x - 6y + 7z,\ 5x - 9y + z)$$

$$[T]_B = A = \begin{bmatrix} 3 & 4 & -5 \\ 2 & -6 & 7 \\ 5 & -9 & 1 \end{bmatrix}$$

$$\text{Adj}\,([T]_B) = A^T = \begin{bmatrix} 3 & 2 & 5 \\ 4 & -6 & -9 \\ -5 & 7 & 1 \end{bmatrix}$$

$$T^*(x, y, z) = (3x + 2y + 5z,\ 4x - 6y - 9z,\ -5x + 7y + z)$$

is the adjoint operator of T.

2) Find adjoint of L.T, $T : C^3 \to C^3$ defined as

$T(x, y, z) = (2x + (1-i)y,\ (3+2i)x - 4iz,$

$2ix + (4-3i)y - 3z)$

$$A = \begin{bmatrix} 2 & 1-i & 0 \\ 3+2i & 0 & -4i \\ 2i & 4-3i & -3 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 2 & 1+i & 0 \\ 3-2i & 0 & 4i \\ -2i & 4+3i & -3 \end{bmatrix}$$

$$(\bar{A})^T = \begin{bmatrix} 2 & 3-2i & -2i \\ 1+i & 0 & 4+3i \\ 0 & 4i & -3 \end{bmatrix}$$

$T^*(x, y, z) = (2x + (3-2i)y - 2iz,\ (1+i)x + (4+3i)z,$

$4iy - 3z)$.

is the adjoint operator of T.

Ex Let λ be the eigen value of linear operator T on ips V, then show the follaing,

① if $T^* = T^{-1}$ (i.e T is orthogonal or unitary operator) then $|\lambda| = 1$.

② If $T^* = T$ (T is self adjoint) then λ is re

③ If $T^* = -T$ (T is skew-adjoint) then λ is purely imaginary.

④ If $T = S^* S$ with S non singular (T is positive definite) then λ is real and positive.

Proof:

① $\lambda\bar{\lambda}\langle v, v\rangle = \langle \lambda v, \lambda v\rangle$

$= \langle Tv, Tv\rangle$

$= \langle v, T^*Tv\rangle$

$= \langle v, v\rangle$

then, $\lambda\bar{\lambda} = 1 \quad (\because v \neq 0)$

$\Rightarrow |\lambda|^2 = 1$

$\Rightarrow \lambda = 1$

② $\lambda\langle v, v\rangle = \langle \lambda v, v\rangle$

$= \langle Tv, v\rangle$

$= \langle v, T^* v\rangle$

$= \langle v, Tv\rangle$

$= \langle v, \lambda v \rangle$

$= \bar{\lambda} \langle v, v \rangle$

$\Rightarrow (\lambda - \bar{\lambda}) \langle v, v \rangle = 0$

$\because v \neq 0 \Rightarrow \langle v, v \rangle \neq 0$

So, $\boxed{\lambda = \bar{\lambda}}$ Hence, λ is real.

(3) $\lambda \langle v, v \rangle = \langle \lambda v, v \rangle$

$= \langle Tv, v \rangle = \langle v, T^* v \rangle$

$= \langle v, -Tv \rangle$

$= \langle v, -\lambda v \rangle$

$= -\bar{\lambda} \langle v, v \rangle$

$\Rightarrow (\lambda + \bar{\lambda}) \langle v, v \rangle = 0$

$\because v \neq 0 \Rightarrow \langle v, v \rangle \neq 0$

So, $\lambda + \bar{\lambda} = 0$ Hence λ is purely imaginary.

(4) Given, $T = S^* S$, S is non-singular,

As $v \neq 0 \Rightarrow Sv \neq 0$

$\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Tv, v \rangle$

$= \langle S^* S v, v \rangle$

$= \langle Sv, S^{\bullet} v \rangle$

As, $\langle v, v \rangle > 0 \Rightarrow \langle Sv, Sv \rangle > 0$

So, λ is positive

Ex

Let, T, T_1, T_2 be linear operator on V, and $K \in \mathbb{F}$, then show the following

1) $(T_1 + T_2)^* = T_1^* + T_2^*$

2) $(T_1 T_2)^* = T_2^* T_1^*$

3) $(KT)^* = (\bar{K})^T T^*$

4) $(T^*)^* = T$

Sol:-

1) $\langle (T_1 + T_2)u, v \rangle = \langle T_1 u + T_2 u, v \rangle$

$= \langle T_1 u\ v \rangle + \langle T_2 u\ v \rangle$

$= \langle u, T_1^* v \rangle + \langle u, T_2^* v \rangle$

$= \langle u, (T_1^* + T_2^*) v \rangle$

$\Rightarrow \langle u, (T_1 + T_2)^* v \rangle = \langle u, (T_1^* + T_2^*) v \rangle$

So, $(T_1 + T_2)^* = T_1^* + T_2^*$

2) $\langle (T_1 T_2) u, v \rangle = \langle T_2 u, T_1^* v \rangle$

$= \langle u, T_2^* T_1^* v \rangle$

$\langle u, (T_1 T_2)^* v \rangle = \langle u, T_2^* T_1^* v \rangle$

$\Rightarrow (T_1 T_2)^* = T_2^* T_1^*$

4) $\langle T^* u, v \rangle = \overline{\langle v, T^* u \rangle}$

$= \overline{\langle Tv, u \rangle}$

$= \langle u, Tv \rangle$

$\Rightarrow (T^*)^* = T$

Note.

$\rightarrow$ $O : V \longrightarrow V$

$Ov = 0$

$O^* = ?$

$\langle Ou, v \rangle = \langle 0, v \rangle = 0 = \langle u, 0 \rangle = \langle u, O \cdot v \rangle$

$\therefore O^* = O$

$\rightarrow$ $I : V \longrightarrow V$; $Iv = v$

So, $I^* = I$.

Ex Show that $(T^{-1})^* = (T^*)^{-1}$.

$I = I^* = (TT^{-1})^* = (T^{-1})^* T^*$

$\Rightarrow (T^{-1})^* T^* = I$

$\Rightarrow (T^{-1})^* = (T^*)^{-1}$

Hence, proved.

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2) Let T be a linear operator on ips V, then show that each of the following condition implies, $T = 0^*$

1) $\langle Tu, v \rangle = 0 \ \forall \ u, v \in V \Rightarrow T = 0$

2) V is complex ips and $\langle Tu, u \rangle = 0 \ \forall \ u \in$ $\Rightarrow T = 0$

3) T is self-adjoint and $\langle Tu, u \rangle = 0 \ \forall \ u \in$ $\Rightarrow T = 0$

Sol:

1) Given that $\langle Tu, v \rangle = 0 \ \forall \ u, v \in V$

Take, $v = Tu$

$\Rightarrow \langle Tu, Tu \rangle = 0$

$\Rightarrow \| Tu \|^2 = 0 \ \forall \ u \in V$

$\Rightarrow Tu = 0 \ \forall \ u$

$\Rightarrow T = 0$

2) Given that $\langle Tu, u \rangle = 0$ and V is complex ips

$\Rightarrow \langle T(v+w), (v+w) \rangle = 0$

$\Rightarrow \langle Tv, v \rangle + \langle Tv, w \rangle + \langle Tw, v \rangle + \langle Tw, w \rangle = 0$

$\Rightarrow \langle Tv, w \rangle + \langle Tw, v \rangle = 0$ — ①

Replace w by iw in ①,

$\langle Tv, iw \rangle + \langle T(iw), v \rangle = 0$

$\Rightarrow \quad -i \langle Tv, w \rangle + i \langle Tw, v \rangle = 0$ — ②

Multiply $[-② \times i + ①]$

$\Rightarrow \quad \langle Tw, v \rangle = 0 \quad \forall\, v, w \in V$

from ①,

$\Rightarrow T = 0.$