Visvesvaraya National Institute of Technology
Department of Mathematics
Re-Examination -May 2018
Linear Algebra and its Applications-MAL206

Time: 3 hours Marks: 60

i) Section A is compulsory. ii) Attempt any **Five** questions from section B.

Section A

Q.1 Answer any five questions. $5 \times 2 = 10$

(a) Show that if square matrix A has a left inverse B and a right inverse C, then $B = C$.
(b) Is $W = \{(a, b, c) \in \mathbb{R}^3 : a^2 + b^2 + c^2 \leq 1\}$ is subspace of V? Give explanation for your answer.
(c) Let T be a linear transformation on $\mathbb{R}^2$ defined by $T(x, y) = (x - y, x - 2y)$. Show that operator T is invertible. Also find T^{-1}.
(d) Suppose matrix B is similar to matrix A then show that matrix B^n is similar to matrix A^n.

(e) Let $w = (1, 2, 3, 1)$ be a vector in $\mathbb{R}^4$. Find an orthogonal basis for $w^\perp$.
(f) Determine whether matrix $A = \begin{pmatrix} 1 & i \\ 1 & 2+i \end{pmatrix}$ is normal or not?

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Section B

- Answer any five questions.

Q.2 (a) Let $A = A_1 A_2 ... A_k$, where $A_1, ..A_k$ are $n \times n$ matrices. Then show that A is invertible if and only if each A_j is invertible. [3]

(b) Express the vector $v = (2, -5, 3)$ in $\mathbb{R}^3$ as a linear combination of the vectors $u_1 = (1, -3, 2)$, $u_2 = (2, -4, -1)$ and $u_3 = (1, -5, 7)$. [3]

(c) Let V be a vector space of 2×2 matrices over field F. Let W be subspace of symmetric matrices. Show that dim $W = 3$, by finding a basis of W. [4]

Q.3 (a) Let $F : V \to W$ be a linear transformation between vector space V and W. Show that the kernel of F is a subspace of V and the image of F is a subspace of W. [3]

(b) Let $F : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$F(x, y, z, t) = (x + 2z - t, 2x + 3y - z + t, -2x - 5z + 3t).$$

Find a basis and the dimension of: (i) the image of F, (ii) the kernel of F. [3]

(c) Consider the following bases of $\mathbb{R}^2$:

$$S = \{u_1, u_2\} = \{(1, -2), (3, -4)\} \quad and \quad S' = \{v_1, v_2\} = \{(1, 3), (3, 8)\}$$

(i) Find the coordinates of $v = (a, b)$ relative to the basis S.
(ii) Find the change-of-basis matrix P from S to S'.
(iii) Find the change-of-basis matrix Q from S' back to S . [4]

Q.4 (a) Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$F(x, y, z) = (3x + 2z - 4z, x - 5y + 3z).$$

(i) Find the matrix of F in the following bases of $\mathbb{R}^3$ and $\mathbb{R}^2$:

$$S = \{(1, 1, 1), (1, 1, 0)\,(1, 0, 0)\} \text{ and } \quad S' = \{(1, 3), (2, 5)\}.$$

(ii) Verify $[F]_{S,S'}[v]_S = [F(v)]_{S'}$ for $v = (x, y, z) \in \mathbb{R}^3$. [3]

(b) Let A be real symmetric matrix then show that each root of its characteristic polynomial is real. [3]

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (y + z, x + z, x + y)$.
(i) Find all eigenvalues of T.
(ii) Find a maximal set S of linearly independent eigenvectors of T.
(iii) Is T diagonalizable? If yes, find matrix P such that $D = P^{-1}[T]_\beta P$ is a diagonal matrix. [4]

Q.5 (a) Find the orthogonal transforms which transform the quadratic form $2x^2 - 4xy + 5y^2$ to canonical form. [3]

(b) Show that a 2×2 real symmetric matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

is positive definite if and only if the diagonal entries a and d are positive and the determinant $|A| = ad - bc = ad - b^2$ is positive. [3]

(c) Let U be the subspace of $\mathbb{R}^4$ spanned by

$$v_1 = (1, 1, 1, 1), \qquad v_2 = (1, -1, 2, 2) \qquad v_3 = (1, 2, -3, -4).$$

Apply the Gram-Schmidt algorithm to find an orthogonal and an orthonormal basis for U. [4]

Q.6 (a) Suppose $E = \{e_i\}$ and $E' = \{e_i'\}$ are orthogonal bases of V. Let P be the change of basis matrix E to E'. Then show that P is an orthogonal matrix. [3]

(b) Suppose V is an inner product space and $\{u_1, u_2, ...u_n\}$ is an orthonormal basis of V. Then show that for every $x = \sum_{j=1}^{n} \langle x, u_j \rangle u_j$, $\|x\|^2 = \sum_{j=1}^{n} |\langle x, u_j \rangle|^2$. [4]

(c) Let $V = \mathbb{R}^2$ with the usual inner product and let $V_0 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$ Find the best approximation of $x = (1, 2)$ from the space V_0. [3]

Q.7 (a) Let T be a linear operator on vector space V. Show that each of the following implies $T = 0$:
(i) $\langle Tu, v \rangle = 0$ for every $u, v \in V$.
(ii) V is a complex space and $\langle Tu, u \rangle = 0$ for every $u \in V$.
(iii) T is self adjoint and $\langle Tu, u \rangle = 0$ for every $u \in V$. [5]

(b) Let λ be an eigen value of a linear operator T on vector space V.
(i) If $T^* = T^{-1}$ then show that $|\lambda| = 1$.
(ii) If $T^* = T$ then show that λ is real.
(iii) If $T = S^*S$ with S is non-singular then show that λ is real and positive. [5]

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