# Gaussian Elimination

P. Danziger

## 1 $m$ Equations in $n$ Unknowns

Given $n$ variables $x_1, x_2, \ldots, x_n$ and $n+1$ constants $a_1, a_2, \ldots, a_n, b$ the equation

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

represents an $n-1$ dimensional object in $n$-space, called a hyperplane.
We want to consider the situation where we have $m$ such equations

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m \end{array}$$

This is called a system of $m$ (linear) equations in $n$ unknowns (or variables).
We want to find solutions of this system of equations.

**Theorem 1** *Given a system of $m$ equations in $n$ unknowns:*

- *If $m < n$ then the number of parameters in the solution will be* at least $n - m$.

  *(Thus if there is a unique solution we must have $m \geq n$.)*

- *If $m > n$ the system is called* overprescribed.

  *Overprescribed systems either have no solution or they contain reduncancy. redundancy means that we can find $(m - n)$ equations which can be dropped without affecting the solution.*

If a system of equations has no solution it is called *inconsistent*
If a system of equations has at least one solution it is called *consistent*

### 1.1 Coefficient Matrices and Augmented Matrices

The $x_i$ actually carry no information, the system is completely described by the $a_{ij}$ and $b_i$, $i = 1, \ldots m$, $j = 1, \ldots, n$.
We thus use the *matrix of coefficients*, wich is an $m \times n$ array containing the coefficients of the equations.

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

We also have the *Augmented Matrix*, which includes the $b_i$ on the right:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

The augmented matrix contains all the information necessary to solve the system.

1. Find the matrix of coefficients and the augmented matrix for the following system.

$$\begin{array}{ccccccc} x & + & 2y & - & 3z & = & 1 \\ & + & y & + & z & = & 1 \\ x & + & y & + & z & = & 0 \end{array}$$

   This system of equations has coefficient matrix:

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

   and Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

2. Find the augmented matrix for the following system.

$$\begin{array}{ccccccc} x & + & & - & 2z & = & 1 \\ & + & y & - & z & = & 0 \end{array}$$

   This system of equations has Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

3. Given the following augmented matrix find the original system of equations.

$$\left(\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{array}\right)$$

   The system is

$$\begin{array}{ccccc} x & + & 2y & = & -3 \\ & & y & = & 1 \\ x & + & y & = & 0 \end{array}$$

This is a system of 3 equations in 2 unknowns.
It is inconsistent (no solution), since by the second equation $y = 1$, the third equation then tells us that $x = -1$, but then the first equation states (substituting in $x = -1$ and $y = 1$): $-1 + 2 = 3$, which is not true.
Note that each ow of the augmented matrix corresponds to one of the original equations.
Each column contains the all the coefficients of a given variable in the system. We say that this column *corresponds* to this variable.

**Example 2**

$$\begin{array}{ccccc} x & + & 2y & = & -3 \\ & & y & = & 1 \\ x & + & y & = & 0 \end{array} \qquad \left(\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{array}\right)$$

The first row corresponds to $x$, the second corresponds to $y$ and the third corresponds to the constants.

# 2 Elementary Row Operations

There are three basic operations we can preform on equations, these correspond to *Row Operations* on the corresponding matrices.

1. We can multiply an equation by a constant $\equiv$ Multiply a row by a constant.
2. Add a multiple of one equation to another $\equiv$ replace a row by itself plus a multiple of another row.
3. Interchange the order of equations $\equiv$ Interchange two rows.

**Notation** We generally denote the $i^{\text{th}}$ row of the matrix by $R_i$. Let $c$ be a constant, and $1 \leq i, j \leq m$ then
$R_i \to R_i + cR_j$ means replace Row $i$ by row $i$ plus $c$ times row $j$.
$R_i \to cR_i$ means replace row $i$ with $c$ times row $i$.
$R_i \leftrightarrow R_j$ means interchange row $i$ with row $j$.
Note that preforming any of these operations does not change the solution to the original system of equations.
**When using row operations always indicate the operation you have used!**

**Example 3**

1.

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right) \quad \begin{array}{ccc} R_2 & \to & R_2 - 2R_1 \\ R_3 & \to & R_3 - R_1 \end{array} \quad \longrightarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

2.

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right) \quad R_1 \leftrightarrow R_2 \longrightarrow \quad \left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right)$$

3.

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right) \quad R_2 \rightarrow 2R_2 \longrightarrow \quad \left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 4 & 4 & 6 & 6 \\ 1 & 1 & 1 & 1 \end{array}\right)$$

Never operate on the same row twice in one step.

# 3 Row Echelon Form

**Definition 4** *1. A matrix is in* <u>*Row Echelon Form*</u> *(REF) if all of the following hold:*

*(a) Any rows consisting entirely of 0's appear at the bottom.*

*(b) In any non-zero row the first number, from the left, is a one. Called the* leading one *or* pivot.

*(c) In any two successive non-zero rows the leading one on top is to the left of the one on the bottom.*

*2. A matrix is in* <u>*Reduced Row Echelon Form*</u> *(RREF) if it is in REF (all of the above hold) and any column containing a leading one is zero in all other entries.*

**Example 5**

1. The following are in REF

$$\begin{pmatrix} \boxed{1} & 1 & 3 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \end{pmatrix} \quad \begin{pmatrix} 0 & \boxed{1} & 3 & 3 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix} \quad \begin{pmatrix} \boxed{1} & 2 \\ 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$$

$\boxed{1}$ indicates a pivot.

2. The following are **NOT** in REF

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 3 \\ 3 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

3. The following are in RREF

$$\begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix} \quad \begin{pmatrix} 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 2 & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix} \quad \begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$$

$\boxed{1}$ indicates a pivot. All of the 0's in these examples are forced.

4. The following are **NOT** in RREF

$$\begin{pmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix} \quad \begin{pmatrix} 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{1} \end{pmatrix} \quad \begin{pmatrix} \boxed{1} & 2 \\ 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$$

# 4 The Gaussian Algorithm

The following Algorithm reduces an $m \times n$ matrix to REF by means of elementary row operations alone.

1. For Each row $i$ ($R_i$) from 1 to $m$
   (a) If any row $j$ below row $i$ has non zero entries to the left of the first non zero entry in row $i$ exchange row $i$ and $j$ ($R_i \leftrightarrow R_j$) [Ensure We are working on the leftmost nonzero entry.]
   (b) Preform $R_i \to \frac{1}{c} R_i$ where $c =$ the first non-zero entry of row $i$. [This ensures that row $i$ starts with a one.]
   (c) For each row $j$ ($R_j$) below row $i$ (Each $j > i$)
      i. Preform $R_j \to R_j - dR_i$ where
      $d =$ the entry in row $j$ which is directly below the pivot in row $i$. [This ensures that row $j$ has a 0 below the pivot of row $i$.]
   (d) If any 0 rows have appeared exchange them to the bottom of the matrix.

# 5 The Gaussian-Jordan Algorithm

The following Algorithm reduces an $n \times m$ matrix to RREF by means of elementary row operations alone.

1. Preform Gaussian elimination to get the matrix in REF
2. For each non zero row $i$ ($R_i$) from $n$ to 1 (bottom to top)
   (a) For each row $j$ ($R_j$) above row $i$ (Each $j < i$)
      i. Preform $R_j \to R_j - bR_i$ where
         $b =$ the value in row $j$ directly above the pivot in row $i$. [This ensures that row $j$ has a zero above the pivot in row $i$]

## 5.1 Gaussian Elimination

To Solve a system of equations we preform the following steps:

1. Translate the system to its augmented matrix $A$.
2. Use Gaussian elimination to reduce $A$ to REF. Note that the REF form of $A$ has the same solution set.
3. For each column which does **not** contain a pivot introduce a parameter and set the corresponding variable equal to that parameter.
4. Substitute the parameters back into the remaining non zero equations, this will produce a solution for the remaining variables.

The number of pivots in the REF of a matrix $A$ is called the *rank of* $A$ and is denoted by $r$ or $r(A)$. Note that the number of parameters in the solution is equal to $n - r$.

**Example 6** *Solve the following system of equations.*

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 3 \\ x_1 & + & 3x_2 & + & 2x_3 & = & 5 \\ & & 2x_2 & + & x_3 & = & 6 \end{array}$$

Row reduce augmented matrix to REF

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 6 \end{array}\right) \quad R_2 \to R_2 - R_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & 6 \end{array}\right) \quad R_3 \to R_3 - 2R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

For Gaussian elimination use back substitution:

$$\begin{array}{rcll} x_1 + 2x_2 + x_3 & = & 3 & (1) \\ x_2 + x_3 & = & 2 & (2) \\ x_3 & = & -2 & (3) \end{array}$$

From (3) $x_3 = -2$,
From (2) $x_2 = 2 - x_3 = 2 - (-2) = 4$ and
From (1) $x_1 = 3 - 2x_2 - x_3 = 3 - 2(4) - (-2) = -3$.

## 5.2 Gaussian-Jordan

Instead of using back substitution as in Gaussian elimination, we can continue reducing until $A$ is in RREF.
As before, for each column which does **not** contain a pivot introduce a parameter and set the corresponding variable equal to that parameter.
But now we may read off the other variables with no further work.

**Example 7** *Solve the following system of equations.*

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & 3x_3 & = & 3 \\ 2x_1 & + & 2x_2 & + & 3x_3 & = & 3 \\ x_1 & + & x_2 & + & x_3 & = & 1 \end{array}$$

We write out the Augmented matrix and use Gaussian-Jordan to reduce it to RREF.

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right) \quad \begin{array}{ccc} R_2 & \to & R_2 - 2R_1 \\ R_3 & \to & R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -2 & -2 \end{array}\right) \quad R_2 \to -\tfrac{1}{3}R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right) \quad R_3 \to R_3 + 2R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_1 \to R_1 - 3R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We let the variable corresponding to the column not containing a pivot (the second column which corresponds to $x_2$) be the free variable.
Let $t \in \mathbb{R}$, set $x_2 = t$, then $x_3 = 1$ (from row 2) and
$x_1 = -x_2 = -t$ (from row 1).
Or $(x_1, x_2, x_3) = (-t, t, 1)$

**Example 8** *Solve the following system of equations.*

$$\begin{array}{rcl} x_1 + 3x_2 - 2x_3 + 2x_5 & = & 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 & = & -1 \\ 5x_3 + 10x_4 + 15x_6 & = & 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 & = & 6 \end{array}$$

The Augmented Matrix is:

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right)$$

First leading 1 is in the 1,1 position, already 1.
Get all 0's below this leading 1 position.

$$\begin{array}{c} R_2 \longrightarrow R_2 - 2R_1 \\ R_4 \longrightarrow R_4 - 2R_1 \end{array} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

Get leading 1 in second row.

$$R_2 \longrightarrow -R_2 \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

Get all 0's below second leading 1.

$$\begin{array}{c} R_3 \longrightarrow R_3 - 5R_2 \\ R_4 \longrightarrow R_4 - 4R_2 \end{array} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right)$$

Move row of 0's to bottom:

$$R_3 \leftrightarrow R_4 \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Get next leading 1.

$$R_3 \longrightarrow \frac{1}{6}R_3 \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Matrix is now in Row Echelon Form.

## Gauss Elimination

We now use back substitution. The Matrix translates to the following system of equations:

$$\begin{array}{rcl} x_1 + 3x_2 - 2x_3 + 2x_5 & = & 0 \\ x_3 + 2x_4 + 3x_6 & = & 1 \\ x_6 & = & \frac{1}{3} \end{array}$$

For each variable corresponding to a column not containing a leading 1, we assign a free variable.
Let $s, t, r \in \mathbb{R}$.
Let $x_2 = s, x_4 = t, x_5 = r$.
Then the equations imply: $x_6 = \frac{1}{3}$
$x_3 = 1 - 2x_4 - 3x_6 = 1 - 2t - 1 = -2t$ So $x_3 = -2t$.
$x_1 = -3x_2 + 2x_3 - 2x_5 = -3s + 2(-2t) - 2r$. So $x_1 = -3s - 4t - 2r$.
Thus the final solution is:
$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3s - 4t - 2r, s, -2t, t, r, \frac{1}{3})$

## Gauss-Jordan

We continue the algorithm to get the matrix in Reduced Row Echelon Form.
Get 0's above rightmost leading 1 (in column 6).

$$R_2 \longrightarrow R_2 - 3R_3 \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Get 0's above next leading 1 (in column 3).

$$R_1 \longrightarrow R_1 + 2R_2 \left(\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The Matrix is now in Reduced Row Echelon Form.
The Matrix translates to the following system of equations:

$$\begin{array}{rcl} x_1 + 3x_2 + 4x_4 + 2x_5 & = & 0 \\ x_3 + 2x_4 & = & 0 \\ x_6 & = & \frac{1}{3} \end{array}$$

For each variable corresponding to a column not containing a leading 1, we assign a free variable.
Let $s, t, r \in \mathbb{R}$.
Let $x_2 = s, x_4 = t, x_5 = r$.
Then the matrix implies: $x_6 = \frac{1}{3}$
$x_3 = -2t$
$x_1 = -3x_2 - 4x_4 - 2x_5 = -3s - 4t - 2r$.
Thus the final solution is
$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3s - 4t - 2r, s, -2t, t, r, \frac{1}{3})$.