Visvesvaraya National Institute of Technology, Nagpur
Department of Mathematics
Sessional-I

Subject: Linear Algebra and Applications (MAL206)

Max. Marks: 15 **Time: 1:00 hrs**

Answer any five Questions. Marks are indicated against each Question.

1. A function $f : \mathbb{R} \to \mathbb{R}$ (reals) is an even function if $f(-x) = f(x)$ for each real number x. Prove that the set E of all even functions with the operations of addition and scalar multiplication defined by $(f + g)(x) = f(x) + g(x)$, and $kf(x) = (kf)(x)$, where $f, g \in E$; $k \in \mathbb{R}$ is a vector space. [3]

2. Define a subspace of a vector space. Hence prove that the intersection of any number of subspaces of a vector space is a subspace. [3]

3. If V is a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, ..., \beta_m$ then show that any independent set of vectors in V is finite and contains no more than m elements. [3]

4. Let W_1 and W_2 be finite dimensional subspaces of a vector space V. Show that W_1+W_2 is finite dimensional and [3]

$$\dim W_1 + \dim W_2 = \dim (W_1 + W_2) + \dim (W_1 \cap W_2).$$

5. Consider the polynomial space $P_2(t)$ over $\mathbb{R}$. Show that the set $S = \{1, t-1, (t-1)^2\}$ is a basis for $P_2(t)$. Find the coordinate vector of $v = 2t^2 - 5t + 6$ relative to S. [3]

6. Solve the system of equations

$$\begin{aligned} 2x_1 + x_2 + x_3 - 2x_4 &= -10 \\ 4x_1 + 2x_3 + x_4 &= 8 \\ 3x_1 + 2x_2 + 2x_3 &= 7 \\ x_1 + 3x_2 + 2x_3 - x_4 &= -5 \end{aligned}$$

using the Gauss elimination with partial pivoting. [3]

Q. No	Answers
1.	

Proof. **(C1)** Let f, g be even functions. Then $f(-x) = f(x)$ and $g(-x) = g(x)$. ***Is*** $f+g$ ***even?***

$$(f+g)(-x) \overset{\text{by 1.}}{=} f(-x)+g(-x) \overset{f,g \text{ even}}{=} f(x)+g(x) \overset{\text{by 1.}}{=} (f+g)(x)$$

Thus $f+g$ is even.

(C2) Let f be even and $c \in \mathbb{R}$. ***Is*** cf ***even?***

$$(cf)(-x) \overset{\text{by 1.}}{=} c[f(-x)] \overset{f \text{ even}}{=} c[f(x)] \overset{\text{by 1.}}{=} (cf)(x)$$

Thus cf is even.

(V1) Let f, g be even functions. ***Does*** $f+g=g+f$?

$$(f+g)(x) \overset{\text{by 1.}}{=} f(x)+g(x) \overset{\text{by 3.}}{=} g(x)+f(x) \overset{\text{by 1.}}{=} (g+f)(x)$$

Thus $f+g=g+f$.

(V2) Let f, g, h be even functions. ***Does*** $(f+g)+h = f+(g+h)$?

$$[(f+g)+h](x) \overset{\text{by 1.}}{=} (f+g)(x)+h(x) \overset{\text{by 1.}}{=} [f(x)+g(x)]+h(x)$$
$$\overset{\text{by 3.}}{=} f(x)+[g(x)+h(x)] \overset{\text{by 1.}}{=} f(x)+(g+h)(x) \overset{\text{by 1.}}{=} [f+(g+h)](x)$$

Thus $(f+g)+h = f+(g+h)$.

(V3) ***Is there a zero vector?*** We know the constant function $z(x)=0$ for all x is the additive identity, we just have to verify that it is even (so that it will be in this set of even functions.)

$$z(-x) \overset{\text{by def}}{=} 0 \overset{\text{by def}}{=} z(x)$$

Thus $z(x)=0$ is even.

(V4) Let f be even. ***Is*** $-f$ ***even?***

$$(-f)(-x) \overset{\text{by 1.}}{=} -[f(-x)] \overset{f \text{ even}}{=} -[f(x)] \overset{\text{by 1.}}{=} (-f)(x)$$

Thus $-f$ is even.

(V5) Let f be even. ***Does*** $1f = f$?

$$(1f)(x) \overset{\text{by 1.}}{=} 1[f(x)] \overset{\text{by 3.}}{=} f(x)$$

Thus $1f = f$.

(V6) Let f be even and $a, b \in \mathbb{R}$. ***Does*** $(ab)f = a(bf)$?

$$[(ab)f](x) \overset{\text{by 1.}}{=} (ab)[f(x)] \overset{\text{by 3.}}{=} a(b[f(x)]) \overset{\text{by 1.}}{=} a[(bf)(x)] \overset{\text{by 1.}}{=} [a(bf)](x)$$

Thus $(ab)f = a(bf)$.

(V7) Let f, g be even and $a \in \mathbb{R}$. ***Does*** $a(f+g) = af+ag$?

$$[a(f+g)](x) \overset{\text{by 1.}}{=} a[(f+g)(x)] \overset{\text{by 1.}}{=} a[f(x)+g(x)] \overset{\text{by 3.}}{=} a[f(x)]+a[g(x)] \overset{\text{by 1.}}{=} (af)(x)+(ag)(x) \overset{\text{by 1.}}{=} [af+ag](x)$$

Thus $a(f+g) = af+ag$.

(V8) Let f be even and $a, b \in \mathbb{R}$. ***Does*** $(a+b)f = af+bf$?

$$[(a+b)f](x) \overset{\text{by 1.}}{=} (a+b)[f(x)] \overset{\text{by 3.}}{=} a[f(x)]+b[f(x)] \overset{\text{by 1.}}{=} (af)(x)+(bf)(x) \overset{\text{by 1.}}{=} [af+bf](x)$$

Thus $(a+b)f = af+bf$.

We have shown that all of the properties of a vector space are true for the set of even functions. Therefore, this set is a vector space. □

2. ***Definition.*** *Let V be a vector space over the field F. A* **subspace** *of V is a subset W of V which is itself a vector space over F with the operations of vector addition and scalar multiplication on V.*

Proof. Let $\{W_a\}$ be a collection of subspaces of V, and let $W = \bigcap_a W_a$ be their intersection. Recall that W is defined as the set of all elements belonging to every W_a (see Appendix). Since each W_a is a subspace, each contains the zero vector. Thus the zero vector is in the intersection W, and W is non-empty. Let α and β be vectors in W and let c be a scalar. By definition of W, both α and β belong to each W_a, and because each W_a is a subspace, the vector $(c\alpha + \beta)$ is in every W_a. Thus $(c\alpha + \beta)$ is again in W. By Theorem 1, W is a subspace of V. ∎

3. *Proof.* To prove the theorem it suffices to show that every subset S of V which contains more than m vectors is linearly dependent. Let S be such a set. In S there are distinct vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ where $n > m$. Since $\beta_1, \ldots, \beta_m$ span V, there exist scalars A_{ij} in F such that

$$\alpha_j = \sum_{i=1}^{m} A_{ij}\beta_i.$$

For any n scalars $x_1, x_2, \ldots, x_n$ we have

$$\begin{aligned} x_1\alpha_1 + \cdots + x_n\alpha_n &= \sum_{j=1}^{n} x_j\alpha_j \\ &= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij}\beta_i \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} (A_{ij}x_j)\beta_i \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij}x_j\right)\beta_i. \end{aligned}$$

Since $n > m$, Theorem 6 of Chapter 1 implies that there exist scalars $x_1, x_2, \ldots, x_n$ not all 0 such that

$$\sum_{j=1}^{n} A_{ij}x_j = 0, \qquad 1 \le i \le m.$$

Hence $x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = 0$. This shows that S is a linearly dependent set. ∎

4. *Proof.* By Theorem 5 and its corollaries, $W_1 \cap W_2$ has a finite basis $\{\alpha_1, \ldots, \alpha_k\}$ which is part of a basis

$$\{\alpha_1, \ldots, \alpha_k, \quad \beta_1, \ldots, \beta_m\} \quad \text{for} \quad W_1$$

and part of a basis

$$\{\alpha_1, \ldots, \alpha_k, \quad \gamma_1, \ldots, \gamma_n\} \quad \text{for} \quad W_2.$$

The subspace $W_1 + W_2$ is spanned by the vectors

$$\alpha_1, \ldots, \alpha_k, \qquad \beta_1, \ldots, \beta_m, \qquad \gamma_1, \ldots, \gamma_n$$

and these vectors form an independent set. For suppose

$$\Sigma\, x_i\alpha_i + \Sigma\, y_j\beta_j + \Sigma\, z_r\gamma_r = 0.$$

Then

$$-\Sigma\, z_r\gamma_r = \Sigma\, x_i\alpha_i + \Sigma\, y_j\beta_j$$

which shows that $\Sigma\, z_r\gamma_r$ belongs to W_1. As $\Sigma\, z_r\gamma_r$ also belongs to W_2 it follows that

$$\Sigma\, z_r\gamma_r = \Sigma\, c_i\alpha_i$$

for certain scalars $c_1, \ldots, c_k$. Because the set

$$\{\alpha_1, \ldots, \alpha_k, \quad \gamma_1, \ldots, \gamma_n\}$$

is independent, each of the scalars $z_r = 0$. Thus

$$\Sigma\, x_i\alpha_i + \Sigma\, y_j\beta_j = 0$$

and since

$$\{\alpha_1, \ldots, \alpha_k, \quad \beta_1, \ldots, \beta_m\}$$

is also an independent set, each $x_i = 0$ and each $y_j = 0$. Thus,

$$\{\alpha_1, \ldots, \alpha_k, \quad \beta_1, \ldots, \beta_m, \quad \gamma_1, \ldots, \gamma_n\}$$

is a basis for $W_1 + W_2$. Finally

$$\begin{aligned} \dim W_1 + \dim W_2 &= (k + m) + (k + n) \\ &= k + (m + k + n) \\ &= \dim (W_1 \cap W_2) + \dim (W_1 + W_2). \quad \blacksquare \end{aligned}$$

5. First show that the given set is a basis and then:

Consider the vector space $\mathbf{P}_2(t)$ of polynomials of degree ≤ 2. The polynomials

$$p_1 = 1 \qquad p_2 = t - 1 \qquad p_3 = (t-1)^2 = t^2 - 2t + 1$$

form a basis S of $\mathbf{P}_2(t)$. Let $v = 2t^2 - 5t + 6$. The coordinate vector of v relative to the basis S is obtained as follows.

Set $v = xp_1 + yp_2 + zp_3$ using unknown scalars x, y, z and simplify:

$$\begin{aligned} 2t^2 - 5t + 6 &= x(1) + y(t-1) + z(t^2 - 2t + 1) \\ &= x + yt - y + zt^2 - 2zt + z \\ &= zt^2 + (y - 2z)t + (x - y + z) \end{aligned}$$

Then set the coefficients of the same powers of t equal to each other:

$$\begin{aligned} x - y + z &= 6 \\ y - 2z &= -5 \\ z &= 2 \end{aligned}$$

The solution of the above system is $x = 3$, $y = -1$, $z = 2$. Thus

$$v = 3p_1 - p_2 + 2p_3 \qquad \text{and so} \qquad [v] = [3, -1, 2]$$

6. **Solution** The augmented matrix is given by

$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & -2 & -10 \\ 4 & 0 & 2 & 1 & 8 \\ 3 & 2 & 2 & 0 & 7 \\ 1 & 3 & 2 & -1 & -5 \end{array}\right].$$

We perform the following elementary row transformations and do the eliminations.

$$R_1 \leftrightarrow R_2 : \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 2 & 1 & 1 & -2 & -10 \\ 3 & 2 & 2 & 0 & 7 \\ 1 & 3 & 2 & -1 & -5 \end{array}\right]. \; R_2 - (1/2)\, R_1,\; R_3 - (3/4)\, R_1,\; R_4 - (1/4)\, R_1:$$

$$\left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 1 & 0 & -5/2 & -14 \\ 0 & 2 & 1/2 & -3/4 & 1 \\ 0 & 3 & 3/2 & -5/4 & -7 \end{array}\right]. \quad R_2 \leftrightarrow R_4: \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 3 & 3/2 & -5/4 & -7 \\ 0 & 2 & 1/2 & -3/4 & 1 \\ 0 & 1 & 0 & -5/2 & -14 \end{array}\right].$$

$$R_3 - (2/3)\,R_2,\ R_4 - (1/3)R_2: \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 3 & 3/2 & -5/4 & -7 \\ 0 & 0 & -1/2 & 1/12 & 17/3 \\ 0 & 0 & -1/2 & -25/12 & -35/3 \end{array}\right].\ R_4 - R_3:$$

$$\left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 3 & 3/2 & -5/4 & -7 \\ 0 & 0 & -1/2 & 1/12 & 17/3 \\ 0 & 0 & 0 & -13/6 & -52/3 \end{array}\right].$$

Using back substitution, we obtain

$$x_4 = \left(-\frac{52}{3}\right)\left(-\frac{6}{13}\right) = 8,\ x_3 = -2\left(\frac{17}{3} - \frac{1}{12}x_3\right) = -2\left(\frac{17}{3} - \frac{1}{12}(8)\right) = -10,$$

$$x_2 = \frac{1}{3}\left[-7 - \left(\frac{3}{2}\right)x_3 + \left(\frac{5}{4}\right)x_4\right] = \frac{1}{3}\left[-7 - \left(\frac{3}{2}\right)(-10) + \left(\frac{5}{4}\right)(8)\right] = 6,$$

$$x_1 = \frac{1}{4}[8 - 2x_3 - x_4] = \frac{1}{4}[8 - 2(-10) - 8] = 5.$$