Visvesvaraya National Institute of Technology
Department of Mathematics
End Semester Examination -April 2018
Linear Algebra and its Applications-MAL206

Time: 3 hours Marks: 60

i) Section A is compulsory. ii) Attempt any **Five** questions from section B.

Section A

Q.1 Answer any five questions. $5 \times 2 = 10$

(a) Show that square matrix with either a left or right inverse in invertible.

(b) Find a basis and dimension of the subspace $W = \{(a, b, c) : a + b + c = 0\}$ of $\mathbb{R}^3$.

(c) Let T be a linear transformation on $\mathbb{R}^2$ defined by $T(x, y) = (2x + 3y, 4x - 5y)$. Find matrix representation $[T]_S$ of T relative to the basis $S = \{(1, 2), (2, 5)\}$.

(d) Let A be a real positive definite matrix. Then show that the function $\langle u, v\rangle := u^T Av$ is an inner product on $\mathbb{R}^n$.

(e) Let T be a linear transformation on $\mathbb{R}^2$ that reflects each point P across the line $y = kx$, where $k > 0$. Then show that
(i) $v_1 = (1, k)$ and $v_2 = (1, -k)$ are eigen vectors of T.
(ii) T is diagonaizable and find its diagonal representation.

(f) Find the adjoint of the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x+2y, 3x-4z, y)$.

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Section B

- Answer any five questions.

Q.2 (a) Prove that if A is an $m \times n$ matrix and $m < n$, then the homogeneous system of linear equations $AX = 0$ has a non-trivial solution. [3]

(b) Show that $U = W$, where U and W are the following subspaces of $\mathbb{R}^3$:
$U = \text{span}\{(1, 1, -1), (2, 3, -1), (3, 1, -5)\}$ and $W = \text{span}\{(1, -1, -3), (3, -2, -8), (2, 1, -3)\}$ [3]

(c) Suppose U and W are finite dimentionsal subspaces of a vector space V. Then show that $U + W$ is finite dimensional and [4]

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

Q.3 (a) Let $F : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation defined by

$$F(x, y, z, s, t) = (x + 2y + 2z + s + t,\ x + 2y + 3z + 2s - t,\ 3x + 6y + 8z + 5s - t).$$

Find a basis and the dimension of: (i) the image of F, (ii) the kernel of F. [3]

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(x, y) = (x+2y, 3x+4y)$. Find the formula for $f(T)$, where $f(t) = t^2 + 2t - 3$. [3]

(c) Consider the following linear operator G on $\mathbb{R}^3$ and basis S:

$$G(x, y, z) = (2y + z, x - 4y, 3x) \quad \text{and} \quad S = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$$

(a) Find the matrix representation $[G]_S$ of G relative to S
(b) Verify $[G]_S[v]_S = [G(v)]_S$ for any vector $v = (a, b, c)$ in $\mathbb{R}^3$. [4]

Q.4 (a) Suppose $v_1, v_2, ..., v_n$ are nonzero eigenvectors of a linear operator T belonging to distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Then show that $v_1, v_2, ..., v_n$ are linearly independent. [3]

(b) Find the minimal polynomial of the matrix $A = \begin{pmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{pmatrix}$. [3]

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (4x + y - z, 2x + 5y - 2z, x + y + 2z)$.
(i) Find all eigenvalues of T.
(ii) Find a maximal set S of linearly independent eigenvectors of T.
(iii) Is T diagonalizable? If yes, find matrix P such that $D = P^{-1}[T]_\beta P$ is a diagonal matrix. [4]

Q.5 (a) Let W be the subspace of $\mathbb{R}^5$ spanned by $u = (1, 2, 3, -1, 2)$ and $v = (2, 4, 7, 2, -1)$. Find a basis of the orthogonal complement $W^\perp$ of W. [3]

(b) Find a symmetric orthogonal matrix P whose first row is $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$. [3]

(c) Let U be the subspace of $\mathbb{R}^4$ spanned by

$$v_1 = (1, 1, 1, 1), \qquad v_2 = (1, 1, 2, 4) \qquad v_3 = (1, 2, -4, -3).$$

Apply the Gram-Schmidt algorithm to find an orthogonal and an orthonormal basis for U. [4]

Q.6 (a) Let A be a self-adjoint matrix. Then show that there exists a unitary matrix U such that $U^{-1}AU$ is a diagonal matrix. [3]

(b) Let $V = C[0, 1]$ over $\mathbb{R}$ with the inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)dt$ and let $V_0 = P_1$, i.e the space of polynomials of degree less than or equal to 1. Find the best approximation of x defined by $x(t) = e^t$ from the space V_0. [2]

(c) Find singular value decoposition (SVD) of the matrix $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$. [5]

Q.7 (a) Show that any operator T is the sum of a selfadjoint operator and a skew-adjoint operator. [3]

(b) Let T be a normal operator. Then show that eigen vectors of T belonging to distinct eignvalues are orthogonal. [2]

(c) Show that the following conditions on an operator $P : V \to V$ are equivalent:
(i) $P = T^2$ for some self-adjoint operator T.
(ii) $P = S^*S$ for some operator S, i.e, P is positive.
(iii) P is self-adjoint and $\langle P(u), u\rangle \geq 0$ for every u in V. [5]

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