APMA4302 Homework 2

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1

1.1

Given $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{b}$ and $\boldsymbol{A}\hat{\boldsymbol{u}} = \hat{\boldsymbol{b}}$, we have

$$\boldsymbol{A}(\boldsymbol{u} - \hat{\boldsymbol{u}}) = \boldsymbol{b} - \hat{\boldsymbol{b}} \tag{1}$$

Multiply both sides by $\boldsymbol{A}^{-1}$ ($\boldsymbol{A}$ is invertible)

$$(\boldsymbol{u} - \hat{\boldsymbol{u}}) = \boldsymbol{A}^{-1}(\boldsymbol{b} - \hat{\boldsymbol{b}}) \tag{2}$$

Take the norm of both sides

$$\|\boldsymbol{u} - \hat{\boldsymbol{u}}\| = \|\boldsymbol{A}^{-1}(\boldsymbol{b} - \hat{\boldsymbol{b}})\| \tag{3}$$

Employ the given inequality $\|\boldsymbol{A}\boldsymbol{x}\| \leq \|\boldsymbol{A}\|\,\|\boldsymbol{x}\|$

$$\|\boldsymbol{u} - \hat{\boldsymbol{u}}\| \leq \|\boldsymbol{A}^{-1}\|\,\|\boldsymbol{b} - \hat{\boldsymbol{b}}\| \tag{4}$$

Multiply both sides by $\|\boldsymbol{b}\|$ (and note that $\|\boldsymbol{A}\boldsymbol{u}\| = \|\boldsymbol{b}\|$)

$$\|\boldsymbol{b}\|\,\|\boldsymbol{u} - \hat{\boldsymbol{u}}\| \leq \|\boldsymbol{A}\boldsymbol{u}\|\,\|\boldsymbol{A}^{-1}\|\,\|\boldsymbol{b} - \hat{\boldsymbol{b}}\| \tag{5}$$

Again employ the given inequality $\|\boldsymbol{A}\boldsymbol{x}\| \leq \|\boldsymbol{A}\|\,\|\boldsymbol{x}\|$

$$\|\boldsymbol{b}\|\,\|\boldsymbol{u} - \hat{\boldsymbol{u}}\| \leq \|\boldsymbol{A}\|\,\|\boldsymbol{A}^{-1}\|\,\|\boldsymbol{u}\|\,\|\boldsymbol{b} - \hat{\boldsymbol{b}}\| \tag{6}$$

Substitute $\kappa(\boldsymbol{A}) = \|\boldsymbol{A}\|\,\|\boldsymbol{A}^{-1}\|$ and divide both sides by $\|\boldsymbol{u}\|\,\|\boldsymbol{b}\|$

$$\frac{\|\boldsymbol{u} - \hat{\boldsymbol{u}}\|}{\|\boldsymbol{u}\|} \leq \kappa(\boldsymbol{A})\frac{\|\boldsymbol{b} - \hat{\boldsymbol{b}}\|}{\|\boldsymbol{b}\|} \tag{7}$$

Eq. (7) above is the final result.

1.2

Starting from $\boldsymbol{A}\boldsymbol{e} = \boldsymbol{r}$, multiply both sides by $\boldsymbol{A}^{-1}$ and take the norm of each side

$$\|\boldsymbol{e}\| = \|\boldsymbol{A}^{-1}\boldsymbol{r}\| \tag{8}$$

Multiply both sides by $\|\boldsymbol{b}\|$ (and note that $\|\boldsymbol{A}\boldsymbol{u}\| = \|\boldsymbol{b}\|$

$$\|\boldsymbol{b}\|\|\boldsymbol{e}\| = \|\boldsymbol{A}\boldsymbol{u}\|\|\boldsymbol{A}^{-1}\boldsymbol{r}\| \tag{9}$$

Employ the given inequality $\|\boldsymbol{A}\boldsymbol{x}\| \leq \|\boldsymbol{A}\|\,\|\boldsymbol{x}\|$

$$\|\boldsymbol{b}\|\|\boldsymbol{e}\| \leq \|\boldsymbol{A}\|\|\boldsymbol{u}\|\|\boldsymbol{A}^{-1}\|\|\boldsymbol{r}\| \tag{10}$$

Substitute $\kappa(\boldsymbol{A}) = \|\boldsymbol{A}\|\,\|\boldsymbol{A}^{-1}\|$ and divide both sides by $\|\boldsymbol{u}\|\,\|\boldsymbol{b}\|$

$$\frac{\|\boldsymbol{e}\|}{\|\boldsymbol{u}\|} \le \kappa(\boldsymbol{A})\frac{\|\boldsymbol{r}\|}{\|\boldsymbol{b}\|} \tag{11}$$

For the left side of the inequality, follow the same steps as above, starting again from $\boldsymbol{Ae} = \boldsymbol{r}$. First, take the norm of both sides and multiply each side by $\|\boldsymbol{u}\|$ (and note that $\|\boldsymbol{A}^{-1}\boldsymbol{b}\| = \|\boldsymbol{u}\|$)

$$\|\boldsymbol{A}^{-1}\boldsymbol{b}\|\|\boldsymbol{Ae}\| = \|\boldsymbol{u}\|\|\boldsymbol{r}\| \tag{12}$$

Employ the given inequality $\|\boldsymbol{Ax}\| \le \|\boldsymbol{A}\|\,\|\boldsymbol{x}\|$

$$\|\boldsymbol{A}^{-1}\|\|\boldsymbol{b}\|\|\boldsymbol{A}\|\|\boldsymbol{e}\| \ge \|\boldsymbol{u}\|\|\boldsymbol{r}\| \tag{13}$$

Substitute $\kappa(\boldsymbol{A}) = \|\boldsymbol{A}\|\,\|\boldsymbol{A}^{-1}\|$ and divide both sides by $\|\boldsymbol{u}\|\,\|\boldsymbol{b}\|$

$$\frac{\|\boldsymbol{e}\|}{\|\boldsymbol{u}\|} \ge \frac{1}{\kappa(\boldsymbol{A})}\frac{\|\boldsymbol{r}\|}{\|\boldsymbol{b}\|} \tag{14}$$

Eqs. (11) and (14) above are the final result.

2

2.1

Finding the eigenvalues and eigenvectors of $\boldsymbol{A}$ amounts to solving the problem

$$\boldsymbol{Av} = \lambda\boldsymbol{v} \tag{15}$$

Which can be rewritten as

$$\big(\boldsymbol{A} - \boldsymbol{I}\lambda\big)\boldsymbol{v} = \boldsymbol{0} \tag{16}$$

The following relation results from each row of the above linear equation

$$-\boldsymbol{v}_{i-1} + (2 - h^2\lambda)\boldsymbol{v}_i - \boldsymbol{v}_{i+1} = \boldsymbol{0} \tag{17}$$

Given the nature of the problem, it is reasonable to assume a sine-wave solution, $\boldsymbol{v} = \sin(i\theta)$. The homogeneous boundary condition for $i = 0$ is satisfied automatically

$$\boldsymbol{v}_0 = \sin(0 \cdot \theta) = 0 \tag{18}$$

And the homogeneous boundary condition for $i = n$ can be expressed as

$$\boldsymbol{v}_n = \sin(n \cdot \theta) = 0 \tag{19}$$

For Eq. (19) to be true, we must have $\theta = \frac{j\pi}{n}$. Therefore, we get the final definition for the eigenvectors of $\boldsymbol{A}$

$$\boldsymbol{v}_j = \sin(\frac{j\pi}{n}i) \tag{20}$$

Which is the solution we set out to show.

2.2

Plugging the sinusoid solution $\boldsymbol{v} = \sin(i\theta)$ into Eq. (17) gives

$$-\sin(\theta i - \theta) + (2 - h^2\lambda)\sin(i\theta) - \sin(\theta i + \theta) = \boldsymbol{0} \tag{21}$$

Simplify and employ the trigonometric identity $\sin(A + B) + \sin(A - B) = 2\sin(A)\cos(B)$

$$-2\sin(i\theta)\cos(\theta) + (2 - h^2\lambda)\sin(i\theta) = 0 \tag{22}$$

Pull the $\sin(i\theta)$ out of the expression

$$\sin(i\theta)\Big[-2\cos(\theta)+(2-h^2\lambda)\Big]=0 \tag{23}$$

For this expression to be true for all i, we have

$$-2\cos(\theta)+2-h^2\lambda=0 \tag{24}$$

Simplify again and employ the trigonometric identity $1-\cos(2\alpha)=2\sin^2(\alpha)$

$$\lambda=\frac{4}{h^2}\sin^2(\frac{\theta}{2}) \tag{25}$$

Since we know that $\theta=\frac{j\pi}{n}$, we have

$$\lambda_j=\frac{4}{h^2}\sin^2(\frac{j\pi}{2n}) \tag{26}$$

Which is the final eigenvalue expression.

2.3

The condition number of the matrix $\boldsymbol{A}$ can be expressed as the ratio of its largest eigenvalue divided by its smallest eigenvalue. For the smallest eigenvalue, we have

$$\lambda_1=\frac{4}{h^2}\sin^2(\frac{1\cdot\pi}{2n}) \tag{27}$$

As $n\to\infty$, we can use the small angle approximation $\sin(\alpha)=\alpha$

$$\lambda_1\approx\frac{4}{h^2}\left(\frac{\pi}{2n}\right)^2 \tag{28}$$

Which can be simplified as

$$\lambda_1\approx\frac{\pi^2}{(hn)^2} \tag{29}$$

For the largest eigenvalue, we have

$$\lambda_{n-1}=\frac{4}{h^2}\sin^2\left[\frac{(n-1)\cdot\pi}{2n}\right] \tag{30}$$

Which simplifies to

$$\lambda_{n-1}=\frac{4}{h^2}\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2n}\right) \tag{31}$$

As $n\to\infty$, $\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)\to\sin^2(\frac{\pi}{2})=1$. Therefore, we have

$$\lambda_{n-1}\approx\frac{4}{h^2} \tag{32}$$

After dividing the largest eigenvalue by the smallest eigenvalue, we have

$$\kappa(\boldsymbol{A})\approx\frac{\frac{4}{h^2}}{\frac{\pi^2}{h^2n^2}}=\frac{4n^2}{\pi^2}=O(n^2) \tag{33}$$

Which is the behavior we expected.

3

3.1

Given the manufactured solution $u(x)=\sin(k\pi x)+c(x-\frac{1}{2})^3$, we substitute into the BVP given and get

$$f(x)=\left[\gamma+(k\pi)^2\right]\sin(k\pi x)+c\gamma(x-\frac{1}{2})^3-6c(x-\frac{1}{2}) \tag{34}$$

3.2

The result is illustrated in Figure 1 below.

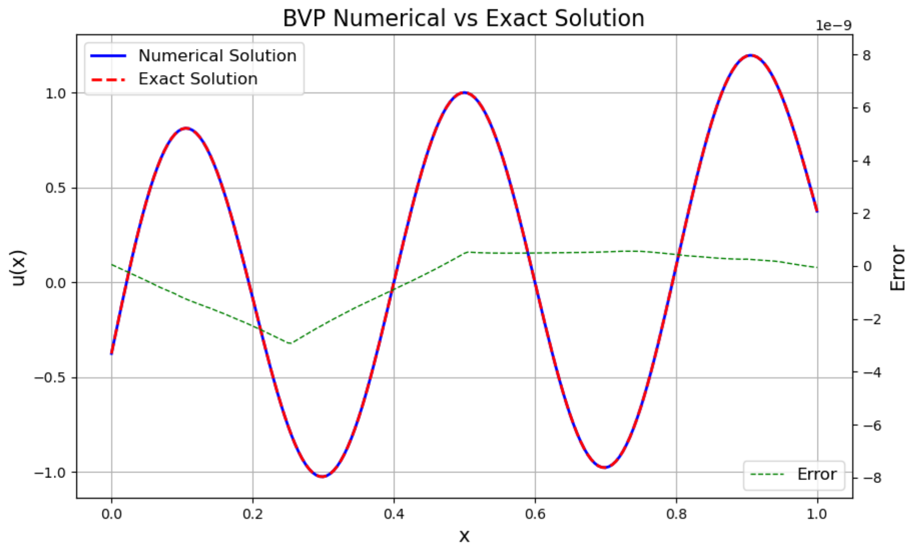

Figure 1: The exact (red) and numerical (blue) solutions to the given BVP, as well as the error between them (green). The error is on the order of 10^{-9}

3.3

The result is illustrated in Figure 2 below.

4

4.1

Jacobi pre-conditioned Richardson: 10,000 iterations to convergence.

4.2

Unconditioned conjugate gradient with 1 processor: 102 iterations to convergence.

4.3

Unconditioned conjugate gradient with c = 0 (explain your results): 1 iteration to convergence.

4.4

icc pre-conditioned conjugate gradient with 1 processor (explain this result): 1 iteration to convergence.

4.5

Block Jacobi pre-conditioned conjugate gradient with 4 processors: 2 iterations to convergence.

4.6

Mumps direct solver: 1 iteration to convergence (for both processor amounts).

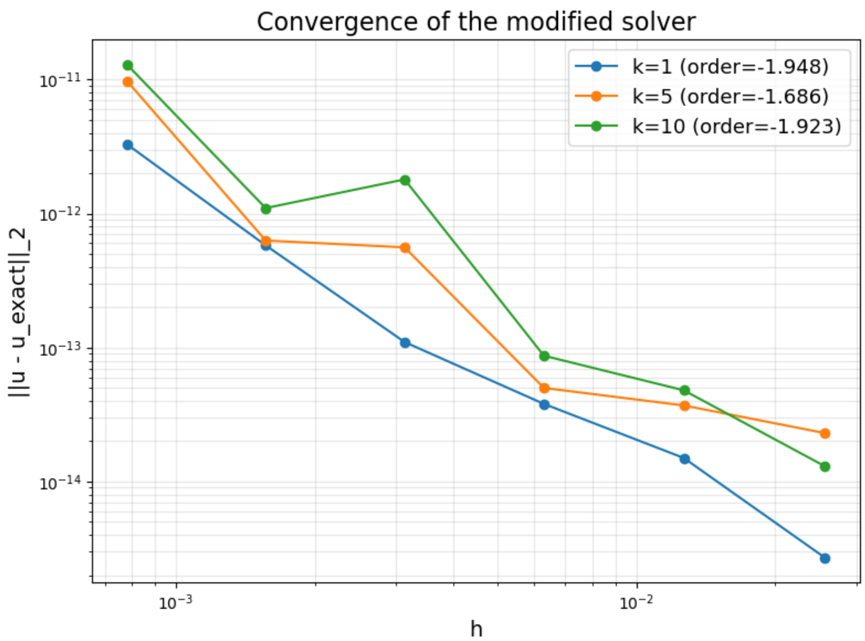

Figure 2: The convergence behavior as a function of h for $k = 1$, $k = 5$, and $k = 10$. The convergence rates are given in the legend.