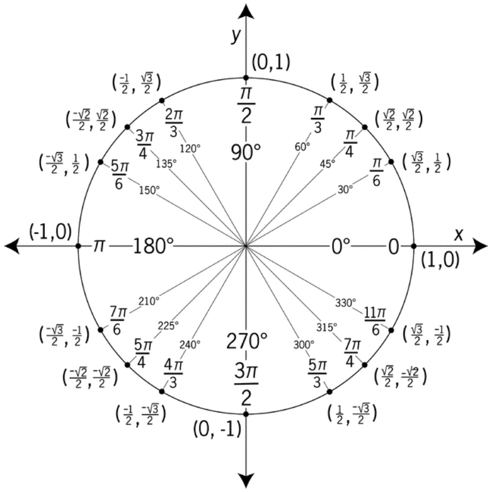
Miscellaneous



Nomenclature

Terms & Definitions

Term	Meaning
σ	Sidewash Angle
β	Sideslip Angle
$\vec{V}$	Wind Vector
p	Roll Rate
q	Pitch Rate
r	Yaw Rate
L	Roll Moment
M	Pitching Moment
N	Yaw Moment
u	Body-frame forward velocity
v	Body-frame lateral velocity
w	Body-frame upward velocity
X	Body-frame forward force
Y	Body-frame lateral force
Z	Body-frame upward force
ψ	Azimuth Angle
ϕ	Bank Angle
θ	Elevation Angle

Stability Derivatives

$$\begin{bmatrix}\dot v\\ \dot p\\ \dot r\\ \dot\phi\end{bmatrix}=\begin{bmatrix}\frac{Y_v}{m} & \frac{Y_p}{m} & \left(\frac{Y_r}{m}-u_o\right) & g\cos\theta_0\\ \left(\frac{L_v}{I'_x}+I'_{zx}N_v\right) & \left(\frac{L_p}{I'_x}+I'_{zx}N_p\right) & \left(\frac{L_r}{I'_x}+I'_{zx}N_r\right) & 0\\ \left(I'_{zx}L_v+\frac{N_v}{I'_z}\right) & \left(I'_{zx}L_p+\frac{N_p}{I'_z}\right) & \left(I'_{zx}L_r+\frac{N_r}{I'_z}\right) & 0\\ 0 & 1 & \tan\theta_0 & 0\end{bmatrix}\begin{bmatrix}v\\ p\\ r\\ \phi\end{bmatrix}+\begin{bmatrix}\frac{\Delta Y_c}{m}\\ \frac{\Delta L_c}{I'_x}+I'_{zx}N_c\\ I'_{zx}\Delta L_c+\frac{\Delta N_c}{I'_z}\\ 0\end{bmatrix}$$

$$\dot\psi = r\sec\theta_o$$
$$\Delta\dot y_E = u_o\psi\cos\theta_o + v$$
$$I'_x = (I_xI_z - I_{zx}{}^2)/I_z$$
$$I'_z = (I_xI_z - I_{zx}{}^2)/I_x$$
$$I'_{zx} = I_{zx}/(I_xI_z - I_{xz}{}^2)$$

	C_x	C_z	C_m
$\hat u$†	$\mathbf{M}_0\left(\frac{\partial C_T}{\partial \mathbf{M}}-\frac{\partial C_D}{\partial \mathbf{M}}\right)-\rho u_0^2\frac{\partial C_D}{\partial p_d}+C_{T_u}\left(1-\frac{\partial C_D}{\partial C_T}\right)$	$-\mathbf{M}_0\frac{\partial C_L}{\partial \mathbf{M}}-\rho u_0^2\frac{\partial C_L}{\partial p_d}-C_{T_u}\frac{\partial C_L}{\partial C_T}$	$\mathbf{M}_0\frac{\partial C_m}{\partial \mathbf{M}}+\rho u_0^2\frac{\partial C_m}{\partial p_d}+C_{T_u}\frac{\partial C_m}{\partial C_T}$
α	$C_{l_0}-C_{D_\alpha}$	$-(C_{L_\alpha}+C_{D0})$	$-a(h_n-h)$
$\hat{\dot\alpha}$	Neg.	$*-2a_tV_H\frac{\partial\epsilon}{\partial\alpha}$	$*-2a_tV_H\frac{l_t}{\bar c}\frac{\partial\epsilon}{\partial\alpha}$
$\hat q$	Neg.	$*-2a_tV_H$	$*-2a_tV_H\frac{l_t}{\bar c}$

Neg. means usually negligible.
*means contribution of the *tail only*, formula for wing-body not available.

†$C_{T_u}=\frac{(\partial T/\partial u)_0}{\frac{1}{2}\rho u_0 S}-2C_{T_0}$; $C_{T_0}=C_{D_0}+C_{w_0}\sin\theta_0$

	C_y	C_l	C_n
β	$*-a_F\frac{S_F}{S}\left(1-\frac{\partial\sigma}{\partial\beta}\right)$	N.A.	$*a_FV_V\left(1-\frac{\partial\sigma}{\partial\beta}\right)$
$\hat p$	$*-a_F\frac{S_F}{S}\left(2\frac{z_F}{b}-\frac{\partial\sigma}{\partial\hat p}\right)$	N.A.	$*a_FV_V\left(2\frac{z_F}{b}-\frac{\partial\sigma}{\partial\hat p}\right)$
$\hat r$	$*a_F\frac{S_F}{S}\left(2\frac{l_F}{b}+\frac{\partial\sigma}{\partial\hat r}\right)$	$*a_F\frac{S_F}{S}\frac{z_F}{b}\left(2\frac{l_F}{b}+\frac{\partial\sigma}{\partial\hat r}\right)$	$*-a_FV_V\left(2\frac{l_F}{b}+\frac{\partial\sigma}{\partial\hat r}\right)$

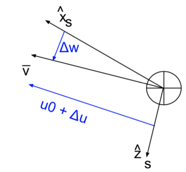
*means contribution of the *tail only*, formula for wing-body not available; $V_F/V=1$.

Derivative	Meaning
C_{n_β}	Yaw Stiffness
C_{n_p}	Yaw Damping
C_{ℓ_β}	Roll Stiffness
	(Dihedral Effect)
C_{ℓ_p}	Roll Damping
C_{y_β}	Side Force from Sideslip
C_{y_p}	Side Force from Roll Rate
C_{y_r}	Side Force from Yaw Rate
	Cross Derivatives
C_{ℓ_r}	Roll Moment from Yaw Rate
C_{n_p}	Yaw Moment from Roll Rate

Longitudinal Dynamics

Short Period Mode

$\Delta\dot q \approx \frac{1}{I_y}(M_w\Delta w + M_q\Delta q), \Delta\dot\theta \approx \Delta q$



$\Delta w \approx u_0\Delta\theta$

$$\begin{bmatrix}\Delta\dot q\\ \Delta\dot\theta\end{bmatrix}=\begin{bmatrix}\frac{M_q}{I_y} & \frac{u_0M_w}{I_y}\\ 1 & 0\end{bmatrix}\begin{bmatrix}\Delta q\\ \Delta\theta\end{bmatrix}$$

$|A-\lambda I| = 0 \Rightarrow \lambda = \frac{M_q}{2I_y} \pm \frac{1}{2I_y}\sqrt{M_q^2 + 4I_yu_0M_w}$

$I_y\Delta\dot q \cong M_q\Delta q + u_0M_w\Delta\theta$

[MOI $\cong$ damper + spring]

Effect of Airframe Design on $\mathbf{M}_w$,$\mathbf{M}_q$

$M_w = \frac{1}{2}\rho S\bar c(u_0^2\frac{\partial C_m}{\partial w}) = \frac{1}{2}u_0\rho S\bar c C_{m_\alpha}$

$C_{m_\alpha} = -C_{L_\alpha}(h_n - h)$

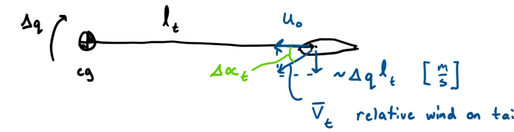
u_0M_w =neg. pitch stiffness

$M_q = \frac{\partial M}{\partial q} = \frac{1}{2}\rho S\bar c u_0^2\frac{\partial C_m}{\partial q} = \frac{1}{4}\rho S\bar c u_0\bar c C_{m_q}$

$\frac{\partial C_m}{\partial q} = \frac{\partial C_m}{\partial \hat q}\cdot\frac{\partial \hat q}{\partial q} = C_{m_q}\cdot\frac{\bar c}{2u_0}$, $\hat q = \frac{q}{2u_0/\bar c}$

C_m changing w/ q

$\Delta C_m \quad -V_HC_{L_t}$ due to Δq



$\Delta\alpha_t \approx \frac{\Delta q l_t}{u_0}$ ($\Delta\alpha_t$ in rad)

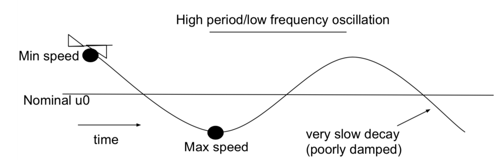
$\Delta C_m \cong -V_Ha_t\Delta\alpha_t = -V_H\frac{a_tl_t}{u_0}\cdot\Delta q = \frac{\partial C_m}{\partial q}\cdot\Delta q$

damper: $M_q = -\frac{1}{2}\rho S\bar c u_0a_tV_Hl_t$

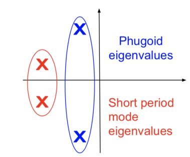
spring: $u_0M_w = -\frac{1}{2}\rho S\bar c u_0C_{L_\alpha}K_nu_0 \Rightarrow$ tail adds spring stiffness and damper

Longitudinal Control

Phugoid Mode



Eigenvalues of 4x4 Long. A-matrix:



Phugoid mode: very easy to excite
Goal: "better behaved/damped/stable"

Approximating Phugoid Characteristics

-isolate dynamics of phugoid mode using 2nd order approximation dynamics
-difficult to approx. damping ratio of $_{phugoid}$, but natural frequency ω_n can be using Lanchester Approximation

Lanchester Approximation
$\omega_{n,phugoid} \approx \sqrt{\frac{-gz_u}{mu_0}}$, $z_u = -\rho u_0 s c_{w0}$
$C_{w_0} = \frac{mg}{\frac{1}{2}\rho u_0^2 S}$, $T_{phugoid} \approx \pi\sqrt{2}\frac{u_0}{g}$

Control Derivatives:
$\triangle X_c = \frac{\partial X}{\partial \delta_e}\triangle\delta_e + \frac{\partial X}{\partial \delta_p}\triangle\delta_p$
$\triangle Z_c = \frac{\partial Z}{\partial \delta_e}\triangle\delta_e + \frac{\partial Z}{\partial \delta_p}\triangle\delta_p$
$\triangle M_c = \frac{\partial M}{\partial \delta_e}\triangle\delta_e + \frac{\partial M}{\partial \delta_p}\triangle\delta_p$
$\Rightarrow$ so $\triangle\dot{y} = A\triangle y \Rightarrow \triangle\dot{y} = A\triangle y + B\triangle\underline{U}$
for $\triangle\underline{U} = [\triangle\delta_e, \triangle\delta_p]^T \Leftarrow$ Control surface inputs

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix} A \\ 4x4 \\ stability \\ deriv\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{bmatrix} + \begin{bmatrix} B \\ 4x2 \\ control \\ deriv\end{bmatrix}\begin{bmatrix}\Delta\delta_e\\ \Delta\delta_p\end{bmatrix}$$

A is state matrix, B is input matrix

Feedback Controller Design
$\Delta\underline{U}$ should move eigenvalues of A
State feedback control law:
$\Delta\underline{U} = -\underline{K} * \Delta y$
(2x1) (2x4) (4x1)
$\Rightarrow$ Control gain matrix

$$\underline{K} = \begin{bmatrix}K_{11} & K_{12} & K_{13} & K_{14}\\ K_{21} & K_{22} & K_{23} & K_{24}\end{bmatrix}$$

$\Rightarrow$ "close the loop" w/ control law:
$\Delta\dot{y} = A\Delta y + B\cdot(-\underline{K}\Delta y) = A\cdot\Delta y - B\underline{K}\Delta y = (A - Bk)\Delta y$
$\Rightarrow \Delta\dot{y} = A_{CL}\cdot\Delta y$, where $A_{CL} = (A - Bk)$

Short Period Mode Control Phugoid Stability Augmentation

$$\begin{bmatrix}\Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix}\frac{M_q}{I_y} & \frac{u_0 M_w}{I_y}\\ 1 & 0\end{bmatrix}\begin{bmatrix}\Delta q\\ \Delta\theta\end{bmatrix} + \begin{bmatrix}\frac{M\delta_e}{I_y}\\ 0\end{bmatrix}\Delta\delta_e$$

$\Delta\delta_e \Rightarrow \Delta\underline{U}_s p \Rightarrow$ only elevator has significant effect
$\Rightarrow$ control law : $\Delta\delta_e = -K_1\Delta q - K_2\Delta\theta$
$\Rightarrow$ look @ closed loop dynamics:
$A_{CL,sp} = (A_{sp} - B_{sp}\cdot\underline{K}_{sp})$, $\underline{K}_{sp} = [K_1, K_2]$

$$\begin{bmatrix}\Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix}\frac{M_q}{I_y} - K_1\frac{M\delta_e}{I_y} & \frac{u_0 M_w}{I_y} - K_2\frac{M\delta_e}{I_y}\\ 1 & 0\end{bmatrix}\begin{bmatrix}\Delta q\\ \Delta\theta\end{bmatrix}$$

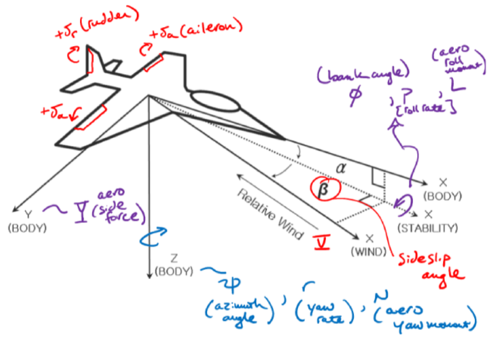
K_1 changes damping of λ_{sp}
K_2 changes stiffness of λ_{sp}
The closed-loop short period eigenvalues:
$\det(\lambda I - A_{sp,CL}) =$
$\det(\lambda I - [A_{sp} - B_{sp}\underline{K}_{sp}]) = 0$
$\Rightarrow 0 =$
$\lambda^2 + \lambda(\frac{-M_q}{I_y} + \frac{M_{\delta_e}}{I_y}\cdot K_1) + (\frac{-u_0 M_w}{I_y} + \frac{M_{\delta_e}K_2}{I_y})$
for $K_1 \neq 0 K_2 \neq 0$: proportional derivative control (PD)
Closed-loop control law on ENTIRE set of long dynamics:

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\\ \Delta theta\end{bmatrix} = \begin{bmatrix}(...) & (...)^{*2}\\ (...)^{*1} & "A_{sp}"\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{bmatrix} + \begin{bmatrix}(...)\\ "B_{sp}"\end{bmatrix}\begin{bmatrix}0 & 0 & -K_1 & -K_2\\ 0 & 0 & 0 & 0\end{bmatrix}\Delta y$$

$\Rightarrow \Delta\dot{y} = A\Delta y + B\overline{U} = (A - B\overline{K})\Delta y$
Expanding -B$\underline{K}$ term for full long. dynamics:

$$-B\underline{K} = \begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\\ B_{31} & B_{32}\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & -K_1 & -K_2\\ 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & -B_{11}K_1 & -B_{11}K_2\\ 0 & 0 & -B_{21}K_1 & -B_{21}K_2\\ 0 & 0 & -B_{31}K_1 & -B_{31}K_2\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Lateral Dynamics



$+\delta_r \rightarrow -$yaw moment, $+\delta_a \rightarrow -$roll moment
$\alpha = tan^{-1}(\frac{w}{u})$, $\beta = sin^{-1}(\frac{v}{V}) \Rightarrow v = V sin\beta$
for small β (near trim, $V \approx u_0$): $\beta = \frac{v}{u_0}$
$V = |\overline{V}_B| = \sqrt{u^2 + v^2 + w^2}$

Lat. State Variables	Lat. Aero Forces/Moments
v (or β)	L (aero roll moment)
p, r	N (aero yaw moment)
ϕ, ψ	Y (aero side force)

1 trans. DOF, 2 rot. DOFs $\rightarrow$ all coupled

Linearized Lateral EOMs
$\Delta\dot{v} = \frac{1}{m}Y + gcos\theta_0\cdot\Delta\phi - u_0\Delta r$
$\Delta\dot{p} = (I_x I_z = I_{xz}^2)^{-1}\cdot[I_z\Delta L + I_{xz}\Delta N]$
$\Delta\dot{r} = (I_x I_z - I_{xz}^2)^{-1}\cdot[I_{zx}\Delta L + I_x\Delta N]$
$\Delta\dot{\phi} = \Delta p + tan(\theta_0)\cdot\Delta r$
Decoupled from long. states (Δu, Δw, etc.)

Lateral Stability:

Express lat. aero F&M in terms of stability/control derivatives, where stab. derivs can be obtained from relevant stability coeffs. [then stab. derivs. to get $\Delta\dot{\underline{y}}_{lat} = [A_{lat}]\cdot\Delta\underline{y}_lat]$

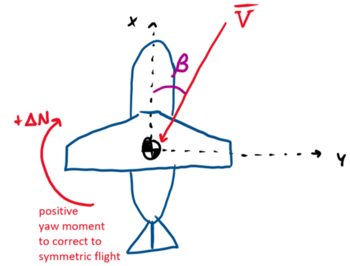
$$\underline{y}_{lat} = \begin{bmatrix}\Delta v\\ \Delta p\\ \Delta r\\ \Delta\phi\end{bmatrix}$$ In general,

$\Delta Y = Y_v\Delta v + Y_p\Delta p + Y_r\Delta r(\Delta Y_c)$
$\Delta L = L_v\Delta v + L_p\Delta p + L_r\Delta r(\Delta L_c)$
$\Delta M = M_v\Delta v + M_p\Delta p + M_r\Delta r(\Delta M_c)$
$\rightarrow$ lateral stability derivatives (derive from partials of nondim. C_y, C_l, C_n forces/moments w.r.t. v, p, r @ trim)

Ex. 1
$Y_v = \frac{\partial Y}{\partial v}|_0 = \frac{1}{2}\rho V^2 S C_{Y_v} = \frac{1}{2}\rho V^2 S(\frac{\partial C_y}{\partial v}|_0)$
$\beta = \frac{v}{u_0} \rightarrow \frac{\partial\beta}{\partial v} = \frac{1}{u_0}$ & $V = u_0$
$\therefore Y_v = \frac{1}{2}\rho V^2 S(\frac{\partial C_y}{\partial v}|_0)$
$\xrightarrow[rule]{chain} = \frac{1}{2}\rho V^2 S(\frac{\partial C_y}{\partial\beta}\cdot\frac{\partial\beta}{\partial v})_0$
$\xrightarrow{V=u_0} = \frac{1}{2}\rho u_0^2 S(\frac{\partial C_y}{\partial\beta}|_0)\cdot\frac{1}{u_0} = \frac{1}{2}\rho u_0 S C_{y_\beta}$
C_{y_β} = non-dim. change in aero side force due to change in β [stab. coeff.]

Ex. 2
$N_v = \frac{\partial N}{\partial v|_0} = \rho V^2 S\frac{b}{2}(\frac{\partial C_N}{\partial v}|_0) \xrightarrow[rule]{chain} \xrightarrow{V=u_0}$
$N_v = \rho u_0^2 S\frac{b}{2}(\frac{\partial C_N}{\partial\beta}|_0)\frac{1}{u_0} = \rho u_0 S\frac{b}{2}\cdot C_{N_\beta}$
C_{N_β} = yaw stiffness, non-dim. change in aero yaw moment due to change in β [stab. coeff.]

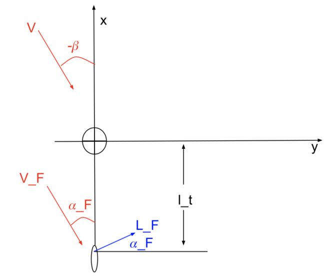
Yaw Stiffness - "Weathervane Stability"



- need change in lift @ vertical tail to get positive ΔN yaw moment to restore symm. flight
- if $C_N = \frac{N}{\frac{1}{2}\rho V^2 Sb}$, then $\frac{\partial C_n}{\partial\beta} = C_{n_\beta} > 0$ for static yaw stability (+ yaw stiffness)

aircraft will (initially) naturally want to restore direction & remove β by turning into wind
Analogous to C_{M_α} EXCEPT:
- depends on vert. tail fin, not horizontal
- $m_\alpha < 0$ for static pitch stability (+ pitch stiffness)

Estimate of Yaw Stiffness Stab. Coeff.

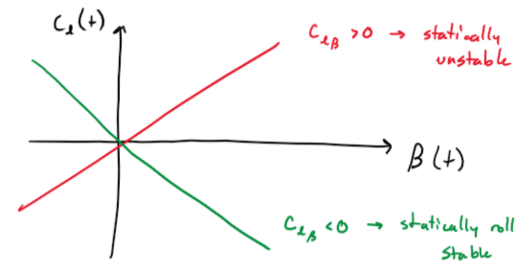


Primarily consider lifting effect of vert. fin & rudder
- produces "fin side lift" L_F due to induced AoA, α_F - L_F conventionally + when pointing along a/c $+y$ axis
- so in case of figure: $\alpha_F = -\beta + \sigma$ where σ is sidewash angle: local flow distortion due to wing/fuselage & propeller wash in yaw
- σ usually negligible ($\ll \beta$); $\sigma > 0 \rightarrow$ flow in $+y$ dir. (increases α_F)

Lift coeff. of vertical tail surface:
$C_{L_F} = -a_F(-\beta + \sigma) + a_r\delta_r$
Dimensionalized Lift force: $L_F = C_{L_F}\frac{1}{2}\rho V_F^2 S_F$
where V_F = tail fin vol., S_F = tail fin area
Dimensionalized Yaw Moment:
$N_F = -L_F l_f cos(\alpha_F) =$
$-C_{L_F}\frac{1}{2}\rho V_F^2 S_F l_f cos(\alpha_F)$
non-dim:
$C_{n_f} = \frac{N_F}{\frac{1}{2}\rho V^2 Sb} = -C_{L_F}V_V(\frac{V_F}{V})^2 cos(\alpha_F)$
where $V_V = \frac{S_F l_F}{Sb}$ = vert. tail ratio
$C_{n_\beta} \approx \frac{\partial C_{n_F}}{\partial\beta}|_0 \xrightarrow[\text{for } C_{L_F}]{\text{plug in}} = V_V(\frac{V_F}{V})^2 a_F(1 - \frac{\partial\sigma}{\partial\beta})|_0$
$V_V = \frac{S_f l_f}{Sb}$ where $(\frac{V_F}{V})^2 \approx 1$ if not in propulsion slipstream

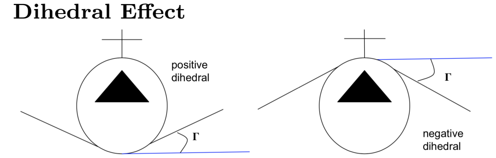
Roll Stability/Dihedral Effect

Roll Stiffness a.k.a. static roll stability,
$C_{l_\beta} = \frac{\partial C_l}{\partial\beta}|_0$
$\Rightarrow$ one of the most important a/c characteristics
[l = roll moment, not lift!]
$\Rightarrow$ Airplane has static roll stability if it has an initial tendency to develop a restoring roll moment when disturbed from wings level symmetric flight (if a sideslip β develops)

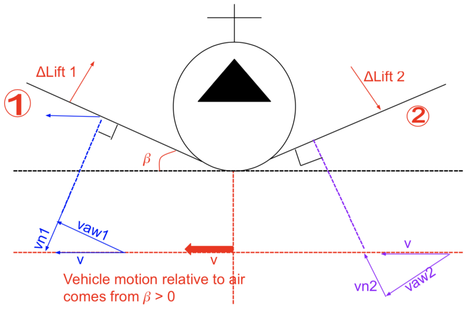


$C_{l_\beta} < 0$ for static roll stability
$\Rightarrow C_{l\beta}$ tricky to estimate, but wing dihedral angle Γ is a major contributor
$\Rightarrow$ restoring roll moment due to β often called...

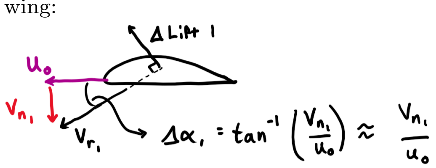
Dihedral Effect



Additional lift on one wing and decreased lift on the other wing created by dihedral Γ whenever β develops from asymmetric/non-wings level flight
$\Rightarrow$ = difference in lift across both ways $\Rightarrow$ restoring roll moment for $\Gamma > 0$



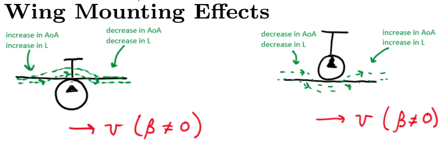
$v_{n1} = v sin\gamma \Rightarrow v_{n1} \approx v\Gamma$ (for smaller Γ)
$v = v_{n1} + v_{aw1}$
$v = v_{n2} + v_{aw2}$
$\Rightarrow v_{aw1}$ = side velocity v resolved (projected) along to wing 1
$\Rightarrow v_{n1}$ = side velocity v projected natural to wing 1
$\Rightarrow v_{aw2}$ = v projected along wing 2
$\Rightarrow v_{n2}$ = v projected perpendicular to wing 2
Look at wing 1 from the side at some slice of wing:



$\Rightarrow \approx \frac{v}{u_0} v_{n1} \approx v\Gamma$
$\Rightarrow v_{n1} = (\beta u_0) \cdot \Gamma$
$\Rightarrow$ so $\delta\alpha_1 \approx \frac{\beta u_0 \cdot \Gamma}{u_0} = \boxed{\beta\Gamma \approx \Delta\alpha_1}$
$\Rightarrow$ So since Δ Lift 1 proportional to $\Delta\alpha_1 : \beta > 0 \Gamma > 0$
$\Rightarrow$ increase in lift as wing 1 because of v_{n1} induced by $\beta > 0 \Gamma > 0$
$\Delta\alpha_2 = -\beta\Gamma$
$\Rightarrow \Gamma > 0 \beta > 0 \Rightarrow \Delta$ Lift 1 > 0 and Δ Lift 2 < 0
$\Rightarrow$ negative roll moment!

Wing Sweep Effects
(C_{l_β} also dependent on wing pos. w.r.t. fuselage, vert. & horiz. tail designs)
- swept back wings ($\Lambda > 0$) : enhanced C_{l_β}; in side slip motion, downward/windward wing has effective decrease in $\Lambda \rightarrow$ higher lift than 'trailing' wing - forward sweep ($\Lambda > 0$) diminishes C_{l_β}

Wing Mounting Effects

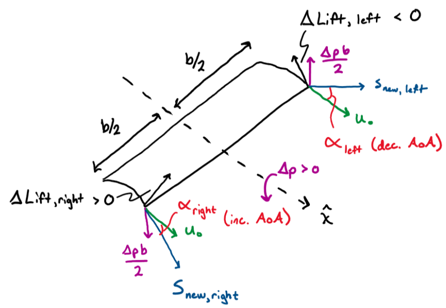


C_{l_β} effect comes from roll moment due to sideflow around fuselage (different for top/bottom-mounted wings)
Tail Effects - large horiz. tail can provide roll moment response to $\beta \neq 0$ in same way as a wing (tail pos. on fuselage & tail dihedral Γ_t matter)

- vert. tail fin contributes most to C_{l_β}, MAC pos. above cg determines roll moment effect

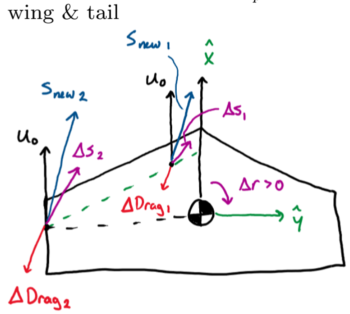
Lateral Damping

Damping in Roll $C_{l_p} = \frac{\partial C_l}{\partial \hat{p}}|_0$
- resistance of airplane to a pure "body axis" (stability axis) roll motion [assuming $\beta = r = \phi = 0$]
Main effect: due to change in lift distrib. across wing, $\Delta\alpha$ changes along wing due to local increases in linear velocity normal to wing (induced by pure RB roll rotation)



$\Delta\alpha \approx \frac{pb}{2u_0}$ (small Δp)
*All valid as long as α_0 of wing is below stall angle
ΔLift on both wings leads to neg. induced roll moment, $L_p < 0$

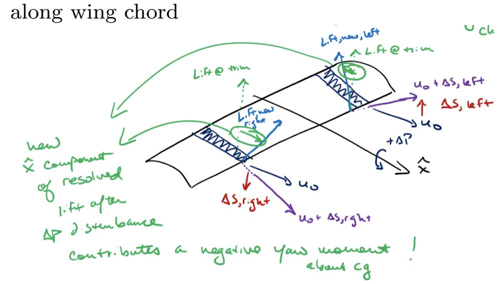
Damping in Yaw $C_{n_r} = \frac{\partial C_m}{\partial \hat{r}}|_0$
Always < 0, (just like C_{l_p}) w/ main effects from wing & tail



- if plane undergoes sudden pure yaw motion w/ rate r, then velocity field over wing is significantly altered [also gives asymm. trailing vortex sheet & sidewash @ tail]
- key effect: velocity of $\frac{1}{4}$ chord wing line normal to itself increases over left wing, decreases over right wing (for $\Delta r > 0$)
$\rightarrow$ increases drag on left wing, decreases drag on right wing
$\rightarrow$ negative yaw moment (for $\Delta r > 0$)

Lateral Cross Derivatives and Linear SS

Important difference between lateral aircraft dynamics and longitudinal
$\Rightarrow$ Cross-coupling of aero rolling moment L to yaw rate r (Δr) and of aero yawing moment to roll rate p (Δp)
$\rightarrow C_{lr} and C_{np}$ generally non-negligible!
(1) $C_{np} = \frac{\partial C_n}{\partial \hat{p}}|_0 \Rightarrow$ change in yawing moment due to roll rate Δp
$\rightarrow$ wing and tail primarily contribute
<u>2 effects due to wing</u> (i) right wing in (+) roll has increased velocity profile (left wing decreased) $\rightarrow$ more lift and drag on right wing (less for left wing) $\rightarrow$ drag icreasecan induce a positive yaw moment
(ii) subsonic force-aft inclination of lift vector along wing chord



<u>vertical fin contribution to C_np</u>:
roll rate Δp $\rightarrow \Delta\alpha_{fin}$ increase $\rightarrow$ negative side lift force at fin $\rightarrow$ positive yaw moment N_{fin}
(2) $C_{lr} = \frac{\partial C_l}{\partial \hat{r}}|_0$ = Change in aero rolling moment due to change in body yaw rate
$\rightarrow$ For a sudden pure positive Δr (yaw rate), get increased lift on left wing and decreased lift on right wing
$\rightarrow$ net change in lift forces across both wings leads to a positive rolling moment (proportional to C_{L0} origional lift coefficient near trim)
$\rightarrow$ largest effect at low speeds
$\rightarrow$ wing contribution to C_{lr} depends on : aspect ratio, taper ratio, sweepback angle
$\rightarrow$ tail also contributes via side force
<u>Aerodynamic Side Force Derivatives</u>
In general, also get $\Delta Y = \underset{C_{y\beta}}{Y_\beta}\Delta\beta + \underset{C_{yp}}{Y_p}\Delta p + \underset{C_{yr}}{Y_r}\Delta r$
$\Rightarrow$ usually only $Y_r(C_{yr})$ particularly significant due to vertical tailfin side force, but influence of other derivatives may vary depending on vehicle and flight conditions [wing can contribute to C_{yp}]
$(C_{y\beta})_{tail} = -a_F(1 - \frac{\partial\sigma}{\partial\beta})\frac{S_F}{S}$
$\Delta C_{yF} = a_F \Delta\alpha_F = -a_F \hat{p}(2\frac{z_F}{b} - \frac{\partial\sigma}{\partial\hat{p}})$
$(C_{yr})_{tail} = a_F \frac{S_F}{S}(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}})$
<u>Linearized Lateral Dynamics Models</u>
Using stability coefficients:
$C_{lp}, C_{lr}, C_{l\beta}, C_{np}, C_{nr}, C_{n\beta}, C_{yp}, C_{yr}, C_{y\beta}$
$\rightarrow$ we can now express lateral [dimensional] stability derivatives for lateral aero forces and moments:
$\Delta Y = Y_v\Delta v + Y_p\Delta p + Y_r\Delta r + \Delta Y_c$
$\Delta L = L_v\Delta v + L_p\Delta p + L_r\Delta r + \Delta L_c$
$\Delta N = N_v\Delta v + N_p\Delta p + N_r\Delta r + \Delta N_c$

Where we previously defined and showed that:
$Y_v = (\frac{\partial Y}{\partial v}|_0 = \frac{1}{2}\rho\overline{v}^2 S\frac{\partial C_y}{\partial v}|_0 = \frac{1}{2}\rho u_0 s C_{y\beta}$
$N_v = (\frac{\partial N}{\partial v}|_0 = \frac{1}{2}\rho\overline{v}^2 S\frac{b}{2}\frac{\partial C_n}{\partial v}|_0 = \rho u_0 s\frac{b}{2}C_{y\beta}$
Longitudinal Dimensional Derivatives
All multiplied by $\frac{1}{2}\rho u_0 S$
(note $\dot{\alpha}$ subscripts in last row)

$$\begin{array}{c} \\ u \\ \\ w \\ q \\ \dot{w} \end{array}\begin{bmatrix} & X & Z & M \\ u & 2C_{w0}sin\theta_0 & -2C_{w0}cos\theta_0 & \bar{c}C_{mu} \\ & +C_{x_u} & +C_{z_u} & \\ w & C_{x_\alpha} & C_{z_\alpha} & \bar{c}C_{m_\alpha} \\ q & \frac{1}{2}\bar{c}C_{x_q} & \frac{1}{2}\bar{c}C_{z_q} & \frac{1}{2}\bar{c}^2C_{m_q} \\ \dot{w} & \frac{1}{2u_0}\bar{c}C_{x_{\dot{\alpha}}} & \frac{1}{2u_0}\bar{c}C_{z_{\dot{\alpha}}} & \frac{1}{2u_0}\bar{c}^2C_{m_{\dot{\alpha}}} \end{bmatrix}$$

Linearized equations of motion for lateral aero forces and moments are:
$\Delta\dot{v} = \frac{1}{m}\Delta Y + gcos\theta_0 \cdot \Delta\phi - u_0\Delta r$
$\Delta\dot{p} = (I_xI_z - I_{xz}^2)^{-1}[I_z\Delta L + I_{xz}\Delta N]$
$\Delta\dot{r} = (I_xI_z - I_{xz}^2)^{-1}[I_{zx}\Delta L + I_x\Delta N]$
$\Delta\dot{\phi} = \Delta p + tan\theta_0 \cdot \Delta r$
Now plug in the expressions for $\Delta Y, \Delta L, \Delta N$ listed i the lin lat dynamics models section
$\rightarrow$ get linear ODEs for lateral state variables
$\underline{y}_{lat} = [\Delta v, \Delta p, \Delta r, \Delta\phi]^T \leftarrow$ which describes lateral a/c dynamics near trim
$\Delta\dot{v} = \frac{Y_v}{m}\Delta v + \frac{Y_p}{m}\Delta p + (\frac{Y_r}{m} - u_0)\Delta r + gcos\theta_0\Delta\phi$
$\Delta\dot{p} = (\frac{L_v}{I_x'} + I_{zx}'N_v)\Delta v + (\frac{L_p}{I_x'} + I_{zx}'n_p)\Delta p + (\frac{L_r}{I_x'} + I_{zx}'N_r)\Delta r$
$\Delta\dot{r} = (I_{zx}'L_v + \frac{N_v}{I_z'})\Delta v + (...)\Delta p + (...)\Delta r$
$\Delta\dot{\phi} = \Delta p + tan\theta_0\Delta r$
$I_x' = (\frac{I_xI_z - I_{xz}^2}{I_z})$
$I_z' = (\frac{I_xI_z - I_{xz}^2}{I_z})$
$I_zx' = (\frac{I_{zx}}{I_xI_z - I_{xz}^2})$
State space model for $\underline{y}_{lat} = [\Delta v, \Delta p, \Delta r, \Delta\phi]^T$
$\rightarrow \frac{d}{dt}(\underline{y}_{lat}) = \dot{\underline{y}}_{lat} = A_{lat}\underline{y}_{lat}$

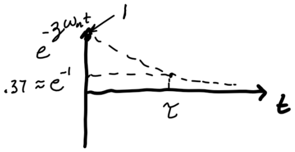
$$\begin{bmatrix} \Delta\dot{v} \\ \Delta\dot{r} \\ \Delta\dot{r} \\ \Delta\dot{\phi} \end{bmatrix} = \begin{bmatrix} 4x4 \\ matrix \\ 4.9, 19 \end{bmatrix}\begin{bmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{bmatrix}$$

A_{lat} has 4 eigenvalues corresponding to 3 distinct dynamical modes
$\rightarrow$ 2 lightly damped by stable complex conjugate eigenvalues - dutch roll mode
$\rightarrow$ 1 slow real eigenvalue - spiral mode (stable or unstable)
$\rightarrow$ 1 fast real eigenvalue - roll convergence mode (stable)

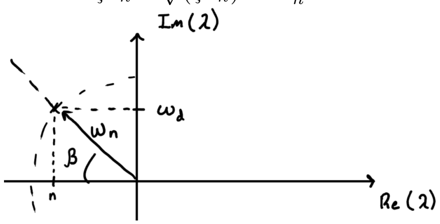
Lateral Dynamical Modes

<u>Mode</u>: state variables for system maintain a particular (eigen) relationship to each other in terms of magnitude & phase over time, dictated by v_i&λ_i
solution to linear ODE system is $y(t) = cv_ie^{\lambda_i t}$, excites mode represented by λ_i
T = period, time to complete a full oscillation cycle [s]

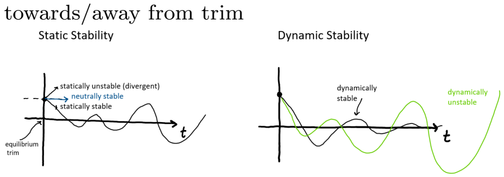
τ = time constant, time to settle to 0.37 times the initial value [s]



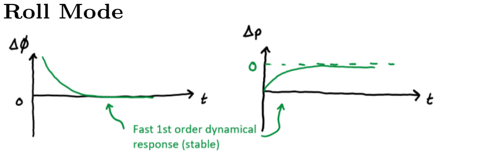
$\omega_n = -\frac{n}{\zeta} = \sqrt{n^2+\sigma^2}$ for any
$\lambda = n \pm \sigma i = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}i$
$\omega_d = \sqrt{\omega_n^2 - n^2}$ $\zeta = -\frac{n}{\omega_n} = cos(\beta)$
$\tau = -\frac{1}{n} = \frac{1}{\zeta\omega_n}$
char. eqn: $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$
$\Rightarrow \lambda = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2}$

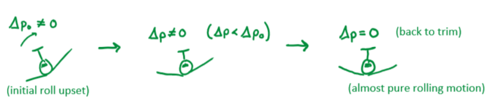


Dynamic stability is where the motion settles, static stability is initial tendency to move towards/away from trim



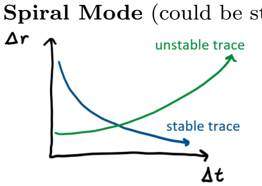
Roll Mode





Roll Mode Approximation
lock in $\Delta v = 0$, $\Delta r = 0$ in $\underline{\dot{y}}_{lat} = A_{lat}\underline{y}_{lat}$
$\Rightarrow$ left w/ $\Delta\dot{p} = (\frac{L_p}{I'_{zx}} + I'_{zx}N_p)\Delta p = \mathcal{L}_p\Delta p$
$\rightarrow \Delta p(t) \approx \Delta p(0)\cdot e^{\lambda_{roll}t}$, where $\lambda_{roll} = \mathcal{L}_p$
$\rightarrow \lambda_{roll}$ largely influenced by
L_p & $N_p \rightarrow C_{l_p}$ & C_{n_p} (damping in roll & yaw coupling) excludes $L_v(C_{l_\beta})$ (usually important) by setting $\Delta v = 0$

Spiral Mode (could be stable or unstable)



If stable, disturbance gives new course heading $\Delta\psi \neq 0$ but back to trim w/ $\underline{y}_{lat} = [\bar{0}]$
If unstable, slow growing Δr, Δv, Δp, & $\Delta\phi$ response (inc. yaw & roll rate, sideslip, bank angle)

Approximating Spiral and DR

Spiral Approx Last 2 term approximation:
$d\lambda + e \approx 0$
Thus: $\lambda_{spiral} \approx -e/d$
where: $e =$
$g[(\mathcal{L}_v N_r - \mathcal{L}_r\mathcal{N}_v)cos\theta_0 + (\mathcal{L}_p\mathcal{N}_v - \mathcal{L}_v\mathcal{N}_p)sin\theta_0]$
and: $d = -g(\mathcal{L}_v cos\theta_0 + \mathcal{N}_v sin\theta_0) + \mathcal{Y}_v(\mathcal{L}_v\mathcal{N}_p - \mathcal{L}_p\mathcal{N}_v) + \mathcal{Y}_r(\mathcal{L}_p\mathcal{N}_v - \mathcal{L}_v\mathcal{N}_p)$
so for θ_0 :$\mathrm{L}_v\mathcal{N}_r > \mathcal{L}_r\mathcal{N}_v$
λ_{spiral} stability is dominated between yaw stiffness and roll stiffness ("dihedral effect"), stable if "dihedral effect wins" and unstable if lift forces from vertical tail fin producing yaw moments due to sideslip β dominate. A smaller vertical fin will lead to stable spiral mode.

DR Approx $\zeta_{DR} \approx \frac{-(\mathcal{Y}_v + \mathcal{N}_r)}{2\sqrt{\mathcal{Y}_v\mathcal{N}_r + u_0\mathcal{N}_v}}$
Primary effects from vertical tail fin: large tail fin (big V_v) gives well camped DR (makes ζ_{DR} bigger. However, this is the *opposite* of whats needed for stable λ_{spiral}

Lateral Control Derivatives

Lateral Control Inputs
3 principal functions of lateral control:
1) provide trim in presence of asymm. thrust (e.g. engine failure on multi-engine a/c)
2) correct undesired motion due to turbulence or other events (e.g. "crabbing" for cross-wind landing)
3) enable turning maneuvers (must bank to turn)

Lateral State and Stability Augmentation

Influence of Aileron and rudder deflections on lateral aero forces and moments near trim can be captured by Control Derivatives in linearized model.
$\Rightarrow \underline{\dot{y}}_{lat} = A_{lat}\underline{y}_{lat} + B_{lat}\underline{u}_{lat}$, $\underline{u}_{lat} = \begin{bmatrix}\Delta\delta_a \\ \Delta\delta_r\end{bmatrix}$
(deflections from trim)

Dimensional Control Derivatives

$$\begin{bmatrix}\Delta Y_c \\ \Delta L_c \\ \Delta N_c\end{bmatrix} = \begin{bmatrix}Y_{\delta a} & Y_{\delta r} \\ L_{\delta a} & L_{\delta r} \\ N_{\delta a} & N_{\delta r}\end{bmatrix}\begin{bmatrix}\Delta\delta a \\ \Delta\delta r\end{bmatrix}$$

Inserting these into linearized later dynamics model and rearranging:

$$\underline{\dot{y}}_{lat} = \begin{bmatrix}\Delta\dot{v} \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\phi}\end{bmatrix} =$$

$$\begin{bmatrix}Y_v & \mathcal{Y}_p & \mathcal{Y}_r & gcos\theta_0 \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 0 & 1 & tan\theta_0 & 0\end{bmatrix}\begin{bmatrix}\Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi\end{bmatrix} +$$

$$\begin{bmatrix}\frac{Y_{\delta_a}}{m} & \frac{Y_{\delta_r}}{m} \\ \frac{L_{\delta_a}}{I'_x} + I'_{zx}N_{\delta_a} & \frac{L_{\delta_r}}{I'_x}I'_{zx}N_{\delta_r} \\ I'_{zx}L_{\delta_a} + \frac{N_{\delta_a}}{I'_z} & I'_{zx}L_{\delta_r} + \frac{N_{\delta r}}{I'_z} \\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta\delta_a \\ \Delta\delta_r\end{bmatrix}$$

Dimensional Control Derivatives

$$\begin{bmatrix} & X & Z & M \\ \delta_e & C_{x\delta e}\frac{1}{2}\rho u_0^2 S & C_{z\delta e}\frac{1}{2}\rho u_0^2 S & C_{m\delta e}\frac{1}{2}\rho u_0^2 S\underline{\bar{c}} \\ \delta_p & C_{x\delta p}\frac{1}{2}\rho u_0^2 S & C_{z\delta p}\frac{1}{2}\rho u_0^2 S & C_{m\delta p}\frac{1}{2}\rho u_0^2 S\underline{\bar{c}}\end{bmatrix}$$

$$\begin{bmatrix} & Y & L & N \\ \delta_a & C_{y\delta a}\frac{1}{2}\rho u_0^2 S & C_{l\delta a}\frac{1}{2}\rho u_0^2 Sb & C_{n\delta a}\frac{1}{2}\rho u_0^2 Sb \\ \delta_r & C_{y\delta r}\frac{1}{2}\rho u_0^2 S & C_{l\delta r}\frac{1}{2}\rho u_0^2 Sb & C_{n\delta r}\frac{1}{2}\rho u_0^2 Sb\end{bmatrix}$$

Lateral State Augmentation
To assess performance of airplane, and/or introduce useful variables for automatic feedback control design, it is often useful to include the following " consequence" states to linearized model:
$\Delta\psi$ = total azimuth/course angle change from trim
Δy^E = total displacement along inertial y axis (ie total "sideways" displacement relative to the ground)
$\Delta\dot{\psi} = \Delta r sec\theta_0$
$\Delta\dot{y}^E = u_0\Delta\psi cos\theta_0 + \Delta v$
Augmented lateral State Vector

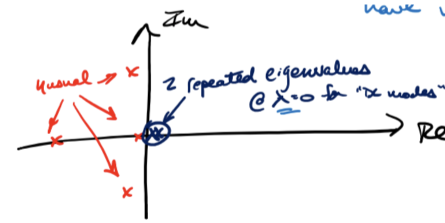
$$\underline{y}_{lat,aug} = \begin{bmatrix}\underline{y}_{lat}(4x1) \\ \Delta\psi \\ \Delta y^E\end{bmatrix} = \begin{bmatrix}\Delta v \\ \Delta p \\ \Delta r \\ \Delta\psi\end{bmatrix}$$

$$\begin{bmatrix}\Delta\dot{v} \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\phi} \\ \Delta\dot{\psi} \\ \Delta\dot{y}^E\end{bmatrix} = \underline{\dot{y}}_{lat,aug} =$$

$$\begin{bmatrix} & & & & 0 & 0 \\ & A_{lat}(4x4) & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ 0 & 0 & sec\theta_0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & u_0 cos\theta_0 & 0\end{bmatrix}\begin{bmatrix}\Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \\ \Delta y^E\end{bmatrix} +$$

$$\begin{bmatrix} & \\ B_{lat}(4x2) & \\ & \\ 0 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta\delta_a \\ \Delta\delta_r\end{bmatrix}$$

When $\underline{u}_{lat} = [0\ 0]^T$, eigenvalues of $A_{lat,aug}$ are the eigenvalues of A_{lat} [spiral, roll, dutchroll] plus two more eigenvalues at $\lambda = 0$ which correspond to "DC mode" for additional $\Delta\psi\Delta y^E$ states



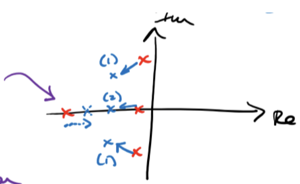
Feedback Control for Lateral Stability Augmentation
To improve the stability and maneuvering properties on aircraft in desired flight conditions, can use state feedback control to alter locations of eigenvalues of A_{lat}

$$\Rightarrow \underline{u}_{lat} = \begin{bmatrix}\Delta\delta_a \\ \Delta\delta_r\end{bmatrix} = -\underline{K}\underline{y}_{lat,aug} =$$

$$-\begin{bmatrix}K_{11} & K_{12} & ... & K_{16} \\ K_{21} & K_{22} & ... & K_{26}\end{bmatrix}\begin{bmatrix}\Delta v \\ \Delta p \\ ... \\ \Delta y^E\end{bmatrix}$$

$\rightarrow$ We can generally obtain closed loop augmented state space model:
$\underline{\dot{y}}_{lat,aug} = A_{lat,aug}\underline{y}_{lat,aug} + B_{lat,aug}\underline{u}_{lat,aug} = (A_{lat,aug} - B_{lat,aug}\cdot\underline{K})\underline{y}_{lat,aug}$
How to choose $\underline{K}$ gains:
(1) Improve damping of Dutch Roll eigenvalues
(2) Improve stability/decay rate of spiral mode eigenvalue (decrease time constant and stabilize $\dot{y}$ if possible)

Attempting to alter one set of eigenvalues will also introduce (possibley unintended) changes to other modal eigenvalues also!



Many different options for lateral feedback available $\rightarrow$ Example:
consider feeding back Δp to $\Delta\delta_a$ only:

$$\underline{u}_{lat} = \begin{bmatrix}\Delta\delta_a \\ \Delta\delta_r\end{bmatrix} = -\begin{bmatrix}K_{sp}\Delta p \\ 0\end{bmatrix} = -\underline{K}\cdot\underline{y}_{lat,aug} =$$

$$-\begin{bmatrix}0 & K_{sp} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \\ \Delta y^E\end{bmatrix}$$

$\rightarrow$ So $B_{lat,aug}\cdot\underline{y}_{lat,aug} =$

$$\begin{bmatrix}B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ 0 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & -K_{sp} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Lateral Stability Augmentation and State Feedback Control

Effect of $\Delta p \rightarrow \Delta\delta_a$: alters roll mode derivatives, primarily $L_p(C_{l,p})$ and cross terms.