CSCI 2824: Discrete Structures

Lecture 32: More Recurrence Relations

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HW11 - Due today at noon

HW12 - posted today

- Due Dec. 6th (Friday after break)

Solving Recurrence Relations – Linear, homogeneous

The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $a_n = r^n$, where r is a constant.

- $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ (recurrence relation)
  $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$ (let $a_n = r^n$)

- $$\Rightarrow \boldsymbol{r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0}$$

characteristic polynomial

❖ The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution of this last equation.

We call this the ***characteristic equation*** of the recurrence relation.

❖ These characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation.

Solving Recurrence Relations – Linear, Non–Homogeneous

Solution Process for Non-Homogeneous Case:

- Ignore the Non-Homogeneous term, solve the homogeneous case. $a_n^{(h)}$
- Guess a solution that fits the Non-Homogeneous term and solve again. $a_n^{(p)}$
- Combine the two solutions and solve for constants.

$$a_n = a_n^{(h)} + a_n^{(p)}$$

Theorem 5: If $\left\{a_n^{(p)}\right\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$, then every solution is of the form $\left\{a_n^{(p)} + a_n^{(h)}\right\}$, where $\left\{a_n^{(h)}\right\}$ is a solution of the associated homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$.

Solving Recurrence Relations – Linear, Non–Homogeneous

Example: Find a solution to the following non-homogeneous linear recurrence relation

$a_1 = 3$
$a_n = 3a_{n-1} + 2n$

this term makes this non-homogeneous.

Step 1: Solve $a_n = 3a_{n-1}$

we are solving the homogeneous piece first.

$r^n = 3r^{n-1}$
$r = 3$

$r - 3 = 0$

general solution for the homogeneous piece: $a_n = A \cdot 3^n$

Solving Recurrence Relations – Linear, Non–Homogeneous

e.g. $3x+5 = 3x+5$

Example: Find a solution to the following non-homogeneous linear recurrence relation

$$a_1 = 3$$
$$a_n = 3a_{n-1} + 2n$$

Step 2: Guess a solution for NH term.

$$a_n^{(p)} = Bn + C$$

We plug this in to the recursion:

$$a_n = 3\underbrace{a_{n-1}}_{a_{n-1}^{(p)}} + 2n$$

$$Bn + C = 3\big(B(n-1) + C\big) + 2n$$

$$Bn + C = 3Bn - 3B + 3C + 2n$$

$$Bn + C = (3B+2)n + -3B + 3C$$

- since $2n$ is a linear polynomial in n, we guess a generic linear polynomial for our particular solution guess.
- if we had $a_n = 3a_{n-1} + 2n^2$

guess: $An^2 + Bn + C$

Solving Recurrence Relations – Linear, Non–Homogeneous

Example: Find a solution to the following non-homogeneous linear recurrence relation

$a_1 = 3$
$a_n = 3a_{n-1} + 2n$

Step 2: Guess a solution for NH term.

$$Bn + C = (3B+2)n - 3B + 3C$$

Recall: $a_n^{(p)} = Bn + C$

$B = 3B + 2$
$-2B = 2$
$B = -1$

$C = -3B + 3C$
$C = -3(-1) + 3C$
$C = 3 + 3C$
$-2C = 3$
$C = -\frac{3}{2}$

$$a_n^{(p)} = -n - \frac{3}{2}.$$

Solving Recurrence Relations – Linear, Non–Homogeneous

Example: Find a solution to the following non-homogeneous linear recurrence relation

$a_1 = 3$
$a_n = 3a_{n-1} + 2n$

Step 3: Combine the two solutions.

general solution: $a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = A \cdot 3^n - n - \frac{3}{2}$$

Solving Recurrence Relations – Linear, Non–Homogeneous

Example: Find a solution to the following non-homogeneous linear recurrence relation

check: $a_1 = 3$ $a_3 = 45$

$a_2 = 13$

$$a_1 = 3$$
$$a_n = 3a_{n-1} + 2n$$

Step 4: Plug in initial value and solve for constant(s).

$$a_n = A \cdot 3^n - n - 3/2$$

$$a_1 = A \cdot 3^1 - 1 - 3/2 = 3$$

$$3A - 5/2 = 3$$

$$3A = 11/2$$

$$A = 11/6$$

Final solution:

$$\boxed{a_n = \frac{11}{6} \cdot 3^n - n - 3/2}$$

$$a_2 = \frac{11}{6} \cdot 3^2 - 2 - 3/2$$

$$= \frac{33}{2} - \frac{7}{2} = \frac{26}{2} = 13 \checkmark$$

Solving Recurrence Relations – Linear, Non–Homogeneous

Example: Find all solutions to $\underbrace{a_n = 5a_{n-1} - 6a_{n-2}} + 7^n$

non-homogeneous part

① Find $a_n^{(h)}$

$a_n = 5a_{n-1} - 6a_{n-2}$

$r^n = 5r^{n-1} - 6r^{n-2}$

$r^2 = 5r - 6$

$r^2 - 5r + 6 = 0$ characteristic eq.

$(r-3)(r-2) = 0$

$r = 2, 3$

$\Rightarrow a_n^{(h)} = A \cdot 2^n + B \cdot 3^n$

② Find the particular solution, $a_n^{(p)}$

$a_n^{(p)} = C \cdot 7^n$

$C \cdot 7^n = 5(C \cdot 7^{n-1}) - 6(C \cdot 7^{n-2}) + 7^n$

$C \cdot 7^2 = 5 \cdot C \cdot 7 - 6C + 7^2$

$49C = 35C - 6C + 49$

$49C = 29C + 49$

$20C = 49 \longrightarrow C = 49/20$

Solving Recurrence Relations – Linear, Non–Homogeneous

Example (continued): Find all solutions to $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

$a_n^{(p)} = C \cdot 7^n \qquad C = 49/20$

$a_n^{(p)} = \frac{49}{20} \cdot 7^n$

3 general solution: $a_n = a_n^{(p)} + a_n^{(h)}$

$$\boxed{a_n = A \cdot 2^n + B \cdot 3^n + \frac{49}{20} \cdot 7^n}$$

If we had been given initial conditions, we would now plug them in.

e.g. $a_1 = 7 \qquad A \cdot 2^1 + B \cdot 3^1 + \frac{49}{20} \cdot 7^1 = 7$

$a_2 = 8 \qquad \ldots$

Solving Recurrence Relations – Linear, Non–Homogeneous

Example: Solve the handshake problem

$$a_n = \frac{n(n-1)}{2}$$ ← from before.

$a_2 = 1$
$a_n = a_{n-1} + (n-1)$ for $n > 2$

$a_n = a_{n-1}$

$r^n = r^{n-1}$

$r = 1$

$a_n^{(h)} = A \cdot 1^n = A$

$a_n^{(h)} = A$

particular solution guess:

$$a_n^{(p)} = Bn + C$$

Trouble! overlaps with homogeneous solution

e.g. general solution

$a_n = A + Bn + C$

$= D + Bn$

$Bn + C = B(n-1) + C + n - 1$

$\cancel{Bn + C = Bn - B + C + n - 1}$

$\cancel{Bn + C = (B+1)n - B + C - 1}$

$B = B + 1$

WRONG guess!

Solving Recurrence Relations – Linear, Non–Homogeneous

Example: Solve the handshake problem

$a_2 = 1$
$a_n = a_{n-1} + (n-1)$ for $n > 2$

So we alter our ~~guess~~ to be $a_n^{(p)} = n(Bn + C) = Bn^2 + Cn$

$Bn^2 + Cn = B(n-1)^2 + C(n-1) + (n-1)$

$Bn^2 + Cn = B(n^2 - 2n + 1) + Cn - C + n - 1$

$Bn^2 + Cn = Bn^2 - 2Bn + B + Cn - C + n - 1$

$Bn^2 + Cn = Bn^2 + (-2B + C + 1)n + B - C - 1$

$B = B$ ✓ $\quad C = -2B + C + 1 \quad 0 = B - C - 1$

Solving Recurrence Relations – Linear, Non–Homogeneous

Example: Find all solutions to $a_n = 3a_{n-1} + \underbrace{2^n}$, then solve when $a_0 = 1$

- First we solve the homogenous piece

$$a_n = 3a_{n-1}$$

$$r^n = 3r^{n-1}$$

$$r = 3$$

$$a_n^{(h)} = A \cdot 3^n$$

- Second we make a particular solution ~~guess~~ guess.

$$a_n^{(p)} = B \cdot 2^n$$

Plug "$B \cdot 2^n$" into the recursion:

$$B \cdot 2^n = 3(B \cdot 2^{n-1}) + 2^n$$

$$2B = 3B + 2$$

$$-B = 2$$

$$B = -2$$

general solution: $a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = A \cdot 3^n - 2 \cdot 2^n$$

- Last we plug in $a_0 = 1$ to solve for A

Solving Recurrence Relations – Linear, Non–Homogeneous

Example: Find all solutions to $a_n = 3a_{n-1} + 2^n$, then solve when $a_0 = 1$

$a_0 = 1$

$a_n = A \cdot 3^n - 2 \cdot 2^n$

$a_0 = A \cdot 3^0 - 2 \cdot 2^0 = 1$

$A - 2 = 1$

$A = 3$

$a_n = 3 \cdot 3^n - 2 \cdot 2^n$

$\boxed{a_n = 3^{n+1} - 2^{n+1}}$