CSCI 2824: Discrete Structures

Lecture 33: Recurrences and Induction

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- HW12 is posted
  - → Due Friday at noon to Gradescope.
- HW13 (not collected)

Recurrences Relations – Linear Non–Homogeneous

Guess a solution that fits the Non-homogeneous term. Your guess should look like the Non-Homogeneous term.

NH part of recurrence	Guess for $a_n^{(p)}$	Example
$F_n = c_k n^k + c_{k-1} n^{k-1} + \ldots + c_1 n + c_0$	$d_k n^k + d_{k-1} n^{k-1} + \ldots + d_1 n + d_0$	$F_n = n^2 \Rightarrow a_n^{(p)} = A \cdot n^2 + B \cdot n + C$
$F_n = b \cdot c^n$	$d \cdot c^n$	$F_n = 3 \cdot 2^n \Rightarrow a_n^{(p)} = A \cdot 2^n$
$F_n = (c_k n^k + \ldots c_1 n + c_0)(b \cdot e^n)$	$(d_k n^k + \ldots d_1 n + d_0)(e^n)$	$F_n = 3n \cdot 2^n \Rightarrow a_n^{(p)} = (An+B) \cdot 2^n$

$c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0$

$3 \cdot 4^n$

Recurrences and Induction

Example: Find all solutions (i.e. the general solution) to the recurrence relation:
$a_n = 4a_{n-1} - 4a_{n-2} + (n+1) \cdot 2^n$

- $a_n = 4a_{n-1} - 4a_{n-2}$

$r^n = 4r^{n-1} - 4r^{n-2}$

$r^2 = 4r - 4$

$r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0$

$a_n^{(h)} = A \cdot 2^n + Bn \cdot 2^n$

$= (A + Bn) \cdot 2^n$

Now we look at the non-homogeneous part:

$(n+1) \cdot 2^n$

guess? $(Cn + D) \cdot 2^n$ overlaps with homogeneous part.

guess? $n(Cn + D) \cdot 2^n = (Cn^2 + \underline{Dn}) \cdot 2^n$ still overlaps

final guess: $n^2(Cn + D) \cdot 2^n$

$a_n^{(p)} = (Cn^3 + Dn^2) \cdot 2^n$

Plug this in to the Recurrence:

$(Cn^3 + Dn^2) \cdot 2^n = 4(C(n-1)^3 + D(n-1)^2) \cdot 2^{n-1}$

$- 4(C(n-2)^3 + D(n-2)^2) \cdot 2^{n-2} + (n+1) \cdot 2^n$

Recurrences and Induction

Example: Find all solutions (i.e. the general solution) to the recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2} + (n+1)\cdot 2^n$$

From last slide:

$$(Cn^3+Dn^2)2^n = 4\left(C(n-1)^3 + D(n-1)^2\right)2^{n-1} - 4\left(C(n-2)^3 + D(n-2)^2\right)\cdot 2^{n-2} + (n+1)\cdot 2^n$$

$$4(Cn^3+Dn^2) = 8\left(C(n-1)^3 + D(n-1)^2\right) - 4\left(C(n-2)^3 + D(n-2)^2\right) + 4(n+1)$$

$$= 8C\left((n-1)(n^2-2n+1)\right) + 8D(n^2-2n+1)$$
$$-4C\left((n-2)(n^2-4n+4)\right) - 4D(n^2-4n+4) + 4n+4$$

$$n^3-3n^2+3n-1$$
$$= 8C(n^3-2n^2+n-n^2+2n-1) + 8Dn^2 - 16Dn + 8D$$
$$n^3-6n^2+12n-8$$
$$-4C(n^3-4n^2+4n-2n^2+8n-8) - 4Dn^2 + 16Dn - 16D + 4n+4$$

$$4Cn^3+4Dn^2 = \cancel{8Cn^3} \cancel{- 24Cn^2} \cancel{+ 24Cn} - 8C \cancel{+ 8Dn^2} \cancel{- 16Dn} + 8D$$
$$\cancel{-4Cn^3} \cancel{+ 24Cn^2} \cancel{- 48Cn} + 32C \cancel{- 4Dn^2} \cancel{+ 16Dn} - 16D \cancel{+ 4n} + 4$$
$$= n^3(8C - 4C) + n^2(-24C + 8D + 24C - 4D) + n(24C - 16D - 48C + 16D + 4) + \text{constants}$$

Recurrences and Induction

Example: Find all solutions (i.e. the general solution) to the recurrence relation:

$a_n = 4a_{n-1} - 4a_{n-2} + (n+1) \cdot 2^n$

$$4Cn^3 + 4Dn^2 = n^3(8C - 4C) + n^2(8D - 4D + 24C - 24C)$$
$$+ n\underbrace{(24C - 16D - 48C + 16D + 4)}_{=0} \underbrace{- 8C + 8D + 32C - 16D + 4}_{=0}$$

$4C = 4C$

$4D = 4D$

$$-24C + 4 = 0$$
$$-24C = -4$$
$$C = 1/6$$

$$24C - 8D + 4 = 0$$
$$24\left(\tfrac{1}{6}\right) - 8D + 4 = 0$$
$$4 - 8D + 4 = 0$$
$$8 = 8D$$
$$1 = D$$

$a_n^{(p)} = \left(\frac{1}{6}n^3 + n^2\right) \cdot 2^n \rightarrow$ general solution:

$$a_n = (A + Bn) \cdot 2^n + \left(n^2 + \tfrac{1}{6}n^3\right) \cdot 2^n$$
$$= \left(A + Bn + n^2 + \tfrac{1}{6}n^3\right) \cdot 2^n$$

Recurrences and Induction

Example: For a recurrence of the form $a_n = Ka_{n-1} + L$, with initial condition a_0 given, we found by matching patterns that a closed form solution is $\underbrace{a_n = K^n a_0 + L\left(\frac{K^n - 1}{K-1}\right)}$.
Let's prove this using induction!

Base Case: $n = 0$

$$a_0 = K^0 a_0 + L\left(\frac{K^0 - 1}{K-1}\right)$$

$$= a_0 + L \cdot 0$$

$$= a_0 \checkmark$$

Inductive Step: Assume for some $l \geq 0$ that $a_l = K^l a_0 + L\left(\frac{K^l - 1}{K-1}\right)$

$$a_{l+1} = Ka_l + L \quad \text{by the recursive definition}$$

$$= K\left(K^l a_0 + L\left(\frac{K^l - 1}{K-1}\right)\right) + L \quad \text{by the inductive hypothesis}$$

$$= K^{l+1} a_0 + KL\left(\frac{K^l - 1}{K-1}\right) + L$$

Recurrences and Induction

Example: For a recurrence of the form $a_n = Ka_{n-1} + L$, with initial condition a_0 given, we found by matching patterns that a closed form solution is $a_n = K^n a_0 + L\left(\frac{K^n - 1}{K-1}\right)$.

Let's prove this using induction!

$$a_{l+1} = K^{l+1} a_0 + KL\left(\frac{K^l - 1}{K-1}\right) + L$$

$$= K^{l+1} a_0 + \frac{L(K^{l+1} - K) + L(K-1)}{K-1}$$

$$= K^{l+1} a_0 + \frac{L(K^{l+1} - K + K - 1)}{K-1}$$

$$= K^{l+1} a_0 + \frac{L(K^{l+1} - 1)}{K-1}$$

So by weak induction we've proven the closed solution.

Recurrences and Induction

Example: Given a recurrence: $a_n = 5a_{n-1} - 3$, with the initial condition $a_0 = 2$, prove that the closed form solution is $a_n = 2 \cdot 5^n - 3\left(\frac{5^n - 1}{4}\right)$ by using induction.

Base Case: let $n=0$

$$a_0 = 2 \cdot 5^0 - 3\left(\frac{5^0 - 1}{4}\right)$$

$$= 2 - 3\left(\frac{1-1}{4}\right)$$

$$= 2 - 3 \cdot 0$$

$$= 2 \checkmark$$

Induction Step: Assume for some $k \geq 0$ that $a_k = 2 \cdot 5^k - 3\left(\frac{5^k - 1}{4}\right)$.

$a_{k+1} = 5a_k - 3$ by the recursive definition.

$= 5\left(2 \cdot 5^k - 3\left(\frac{5^k - 1}{4}\right)\right) - 3$ by the induction hypothesis.

$$= 2 \cdot 5^{k+1} - \frac{3 \cdot 5}{4}\left(5^k - 1\right) - 3$$

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Recurrences and Induction

(continued)

Example: Given a recurrence: $a_n = 5a_{n-1} - 3$, with the initial condition $a_0 = 2$, prove that the closed form solution is $a_n = 2 \cdot 5^n - 3\left(\frac{5^n - 1}{4}\right)$ by using induction.

$$a_{k+1} = 2 \cdot 5^{k+1} + \frac{-3 \cdot 5^{k+1} + 15 - 12}{4}$$

$$= 2 \cdot 5^{k+1} - 3\left(\frac{5^{k+1} - 1}{4}\right)$$

Therefore, using weak induction we have proven that $a_n = 2 \cdot 5^n - 3\left(\frac{5^n - 1}{4}\right)$ is the closed form solution.

Extra Practice

Example: an old exam problem

a) Solve for the general solution $a_n^{(h)}$ to the homogeneous recurrence relation $a_n = 2a_{n-1} + 8a_{n-2}$

b) Find a particular solution $a_n^{(p)}$ to the nonhomogeneous recurrence relation $a_n = 2a_{n-1} + 8a_{n-2} + 3^n$

c) What is the full general solution to the nonhomogeneous recurrence relation in (b)?

d) Set up but **do not solve** the system of equations that would allow you to compute the values of any remaining unknown constants in your answer to part (c).

Solution

(a) $r^n = 2r^{n-1} + 8r^{n-2}$

$r^2 = 2r + 8$

$r^2 - 2r - 8 = 0$

$(r-4)(r+2) = 0$

$r = -2, 4$

$$\boxed{a_n^{(h)} = A(-2)^n + B \cdot 4^n}$$

(b) $a_n^{(p)} = C \cdot 3^n$

$C \cdot 3^n = 2(C \cdot 3^{n-1}) + 8(C \cdot 3^{n-2}) + 3^n$

Divide all terms by 3^{n-2}

$C \cdot 3^2 = 2C \cdot 3 + 8C + 3^2$

$9C = 6C + 8C + 9$

$9C = 14C + 9$

$-5C = 9$

$C = -9/5$

$$\boxed{a_n^{(p)} = -\frac{9}{5} \cdot 3^n}$$

(c)

$a_n = A(-2)^n + B \cdot 4^n - \frac{9}{5} \cdot 3^n$

(d) Assume we know a_0, a_1

$a_0 = A(-2)^0 + B \cdot 4^0 - \frac{9}{5} \cdot 3^0$

$a_0 = A + B - \frac{9}{5}$

$a_1 = A(-2) + B \cdot 4 - \frac{9}{5} \cdot 3$

$a_1 = -2A + 4B - 27/5$

system:

$a_0 = A + B - \frac{9}{5}$

$a_1 = -2A + 4B - 27/5$