CSCI 2824: Discrete Structures

Lecture 27: Probability Theory and its Applications

Rachel Cox

Department of Computer Science

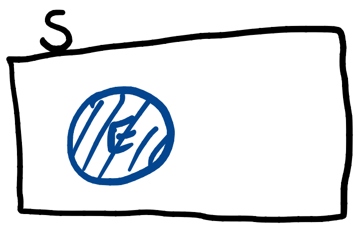
Quizlet 09 - Due at 8pm today!

Written HW10 - Due Friday at noon to Gradescope

Discrete Probability

Theorem: Let E be an event in a sample space S. The probability of the event $\bar{E} = S - E$, the complement of E, is given by

$$p(\bar{E}) = 1 - p(E)$$



Example: A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is a 1?

10 bits: how many possible bit strings? 2^{10} size of sample space.

number of ways to have no 1's = 1

$P(\text{At least one } 1) = 1 - P(\text{no } 1\text{'s}) = 1 - \frac{1}{2^{10}}$

$\boxed{= \frac{1023}{1024}}$

Discrete Probability

Recall: $p(E) = \dfrac{|E|}{|S|}$

$\boxed{\textbf{Theorem: } \text{Let } E_1 \text{ and } E_2 \text{ be two events in the sample space } S \quad p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)}$

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

$$p(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|} = \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|}$$

Example: Suppose you draw one card from a standard deck. What is the probability that it is an Ace or a Diamond?

$$P(\text{ACE} \cup \blacklozenge) = P(\text{ACE}) + P(\blacklozenge) - P(A\blacklozenge)$$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$$

$$= \frac{16}{52}$$

$$\boxed{= 4/13}$$

Probability Theory

Let S be the sample space of an experiment with a finite number of outcomes. We assign a probability $p(s)$ to each outcome $s \in S$. We require two conditions be met:

1. $0 \leq p(s) \leq 1$ for any $s \in S$
2. $\sum_{s \in S} p(s) = 1$

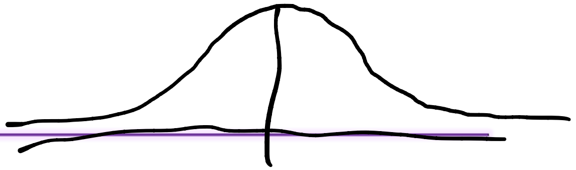
- Since E is a set of possible outcomes, we compute the probability of E by summing the probabilities of the outcomes in E.

The probability of an event E is the sum of the probabilities of the outcomes in E.

$$p(E) = \sum_{s \in E} p(s)$$

- Before we thought of $p(E)$ as a number, but now we think of p as a function that maps outcomes to a value between 0 and 1.

Probability Theory



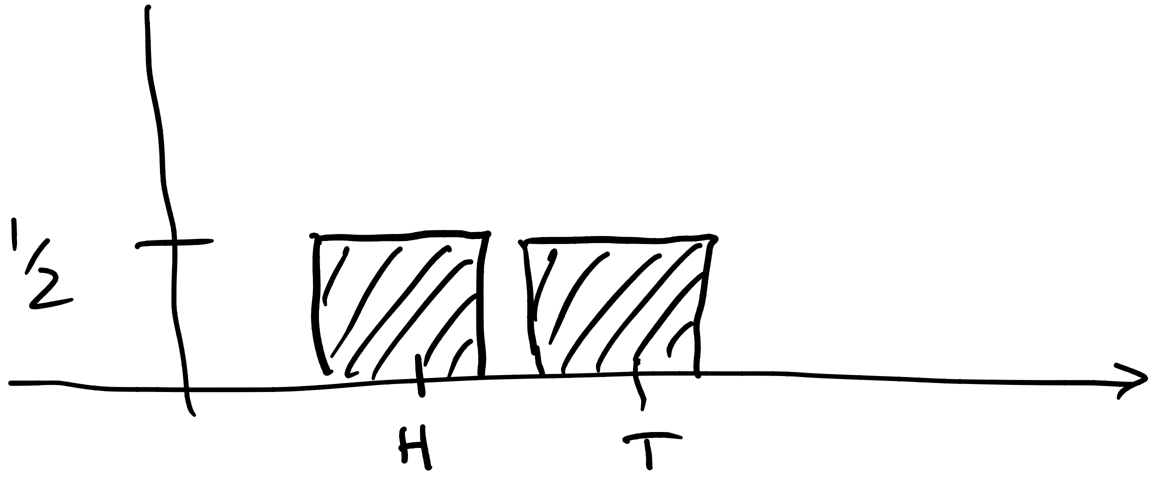
The function p from the set of all outcomes of the sample space S is called a ***probability distribution***.

Example: Specify the probability distribution that models the outcomes H and T when a fair coin is flipped.

1. $0 \leq p(s) \leq 1$ for any $s \in S$
2. $\sum_{s \in S} p(s) = 1$

2 key properties

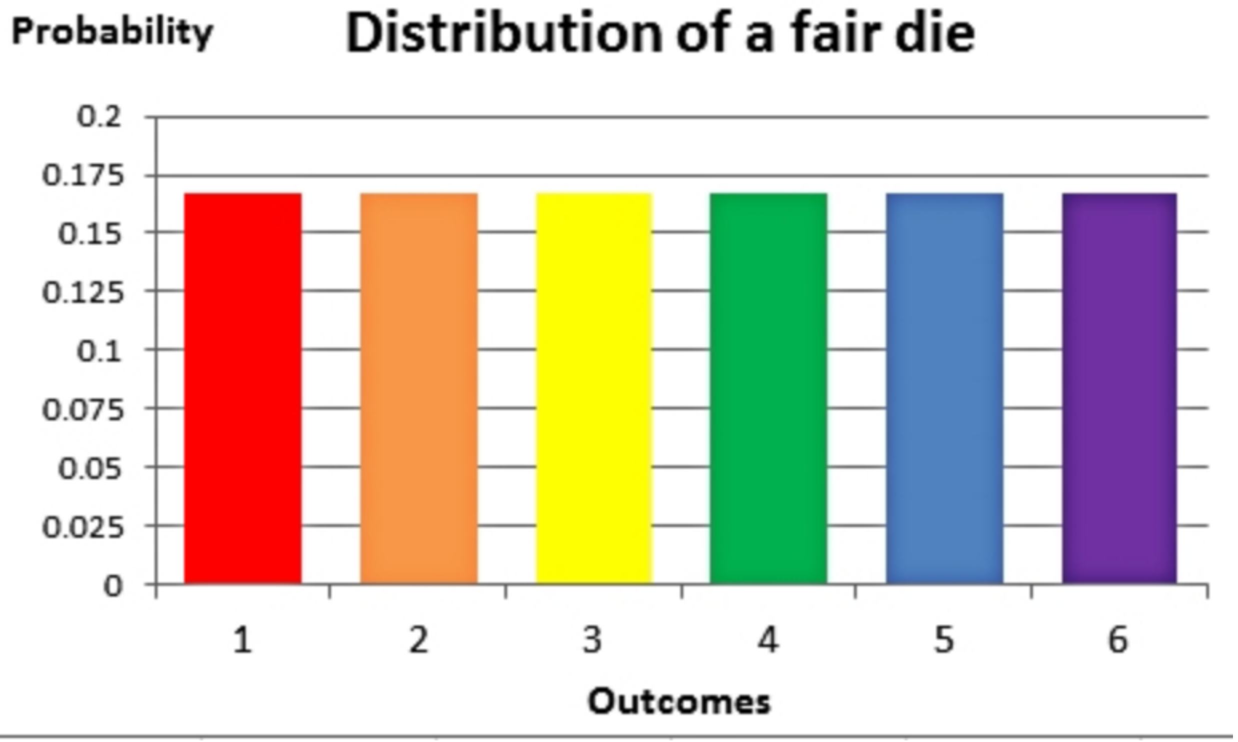
1. $p(H) = \frac{1}{2}, \quad p(T) = \frac{1}{2}$
2. $p(H) + p(T) = \frac{1}{2} + \frac{1}{2} = 1$



Probability Theory

Suppose that S is a set with n elements. The ***uniform distribution*** assigns the probability $1/n$ to each element of S.

e.g. The results of rolling a single fair die is a uniform distribution with $n = 6$.



Probability Theory

Example: Specify the probability distribution that models the outcomes H and T when a biased coin is flipped where the coin is twice as likely to come up H as T.

1. $0 \le p(s) \le 1$
2. $\sum p(s) = 1$

$\Omega = \{H, T\}$

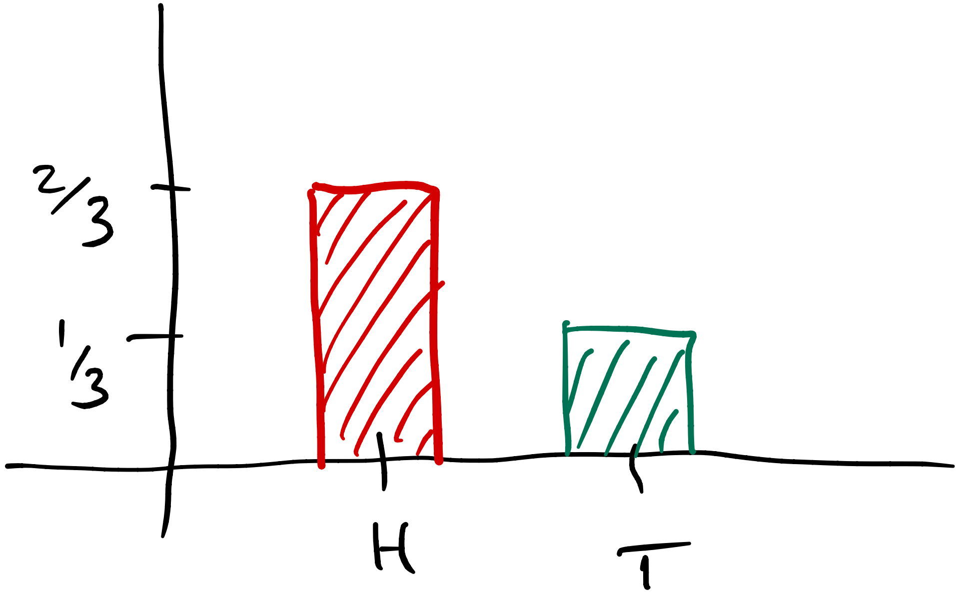
- $p(H) + p(T) = 1$

$\Rightarrow p(H) = 2p(T)$

$2p(T) + p(T) = 1$

$3p(T) = 1$

$p(T) = 1/3 \Rightarrow p(H) = 2/3$



Probability Theory

Example: Suppose that a die is biased so that 3 appears twice as often as each other number but the other five outcomes are equally likely. What is the probability that an odd number is rolled?

What do we know?

- $p(1) = p(2) = p(4) = p(5) = p(6)$, $\boxed{p(3) = 2p(k)}$ where $k = 1, 2, 4, 5, 6$

★ $p(1) + p(2) + p(3) + p(4) + p(5) + p(6) = 1$ ⟵ Recall $\sum_{s \in S} p(s) = 1$

Therefore,

★ $p(1) + p(1) + 2p(1) + p(1) + p(1) + p(1) = 1 \rightarrow 7p(1) = 1$

$\Rightarrow \boxed{p(1) = \frac{1}{7}}$ $[= p(2) = p(4) = p(5) = p(6)] \Rightarrow \boxed{p(3) = \frac{2}{7}}$

$$p(E) = \sum_{s \in E} p(s)$$

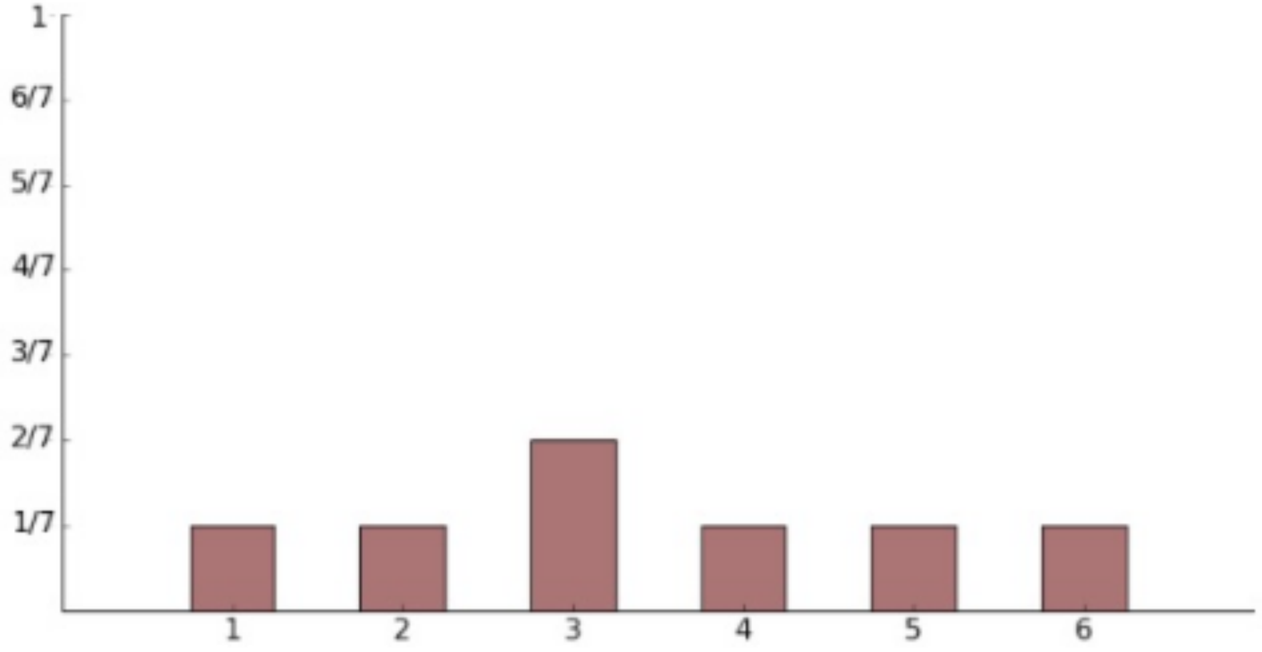
- $\boldsymbol{p(odd) = p(1) + p(3) + p(5) = \frac{1}{7} + \frac{2}{7} + \frac{1}{7} = \frac{4}{7}}$

Probability Theory

Example: (from previously) Suppose that a die is biased (or loaded) so that 3 appears twice as often as each other number but the other five outcomes are equally likely. What is the probability distribution?

Two common ways to write down distribution p

s	1	2	3	4	5	6
$p(s)$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$



Probability Theory

Complement: $p(\bar{E}) = 1 - p(E)$

Union: $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

Example: For the loaded die in the previous example, what is the probability that the die comes up even?

$$p(\text{even}) = 1 - p(\text{odd})$$

$$= 1 - 4/7$$

$$\boxed{= \frac{3}{7}}$$

Probability Theory – Conditional Probability

Example: Suppose that we flip a fair coin three times. Now suppose that the first coin flip is Tails. ***Given this information***, what is the probability that an odd number of coins comes up Tails?

$S = \{HHH, HHT, HTH, HTT, TTT, TTH, THT, THH\}$

$2/4 \boxed{= 1/2}$

P(odd # Tails "Given" First Tail)

P(odd # Tails | First Tails) =

Probability Theory – Conditional Probability

The probability that E occurs given that F occurred is called the ***conditional probability*** of E given F and denoted $\boldsymbol{p(E|F)}$.

Let E and F be events with $p(F) > 0$. The conditional probability of E given F is defined as

$$p(E|F) = \frac{p(E \cap F)}{p(F)}.$$

❖ In general, to compute the conditional probability of E given F we use F as the sample space. For E to occur it must be the case that E belongs to $E \cap F$.

Probability Theory – Conditional Probability

Let's check this definition with the coin flip example

$F = \{THH, THT, TTH, TTT\}$

F = "first flip is Tails"

$E = \{HHT, HTH, THH, TTT, \}$

E = "odd # of Tails"

$E \cap F = \{THH, TTT\}$

Recalling that the original sample space S has size 8, we have

$$P(E \cap F) = \frac{2}{8} = \frac{1}{4} \quad \text{and} \quad p(F) = \frac{4}{8} = \frac{1}{2}$$

Thus, by the definition of conditional probability

$$p(E \mid F) = \frac{1/4}{1/2} = \frac{1}{2} \quad \checkmark$$

Probability Theory – Conditional Probability

Example: A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

$$P(2 \text{ consecutive } 0\text{'s} \mid \text{first bit} = 0) = \frac{P(2 \text{ con. } 0\text{'s} \cap 1^{st} \text{bit} = 0)}{P(1^{st} \text{ bit} = 0)}$$

0 0 0 0
0 0 0 1
0 0 1 1
0 0 1 0
0 1 0 0

$$= \frac{5/16}{8/16}$$

$$\boxed{= \frac{5}{8}}$$

Probability Theory – Conditional Probability

Example: Suppose you flip a coin two times. If the first flip is Heads, what is the probability that the second flip is Heads as well?

$$P(2^{nd}\text{ flip} = H \mid 1^{st}\text{ flip} = H) = \frac{P(2^{nd}\text{ flip} = H \cap 1^{st}\text{ flip} = H)}{P(1^{st}\text{ flip} = H)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2}}$$

$$= \frac{1}{2}$$

$$= P(2^{nd}\text{ flip} = H)$$

$$P(A \mid B) = P(A)$$

* coin flips are independent!

Probability Theory

Events E and F are ***independent*** when $p(E|F) = p(E)$ ★

without replacement.

Example: Suppose you draw two cards from a deck. Are the events that the first card is Red and that the second card is Red independent?

$$P(2^{nd} \text{ card is Red} \mid 1^{st} \text{ card is Red}) = \frac{25}{51} = \frac{P(2^{nd} \text{ red} \cap 1^{st} \text{ Red})}{P(1^{st} \text{ red})}$$

$$P(\text{Red card}) = \frac{1}{2}$$

Since $P(2^{nd} \text{ card red} \mid 1^{st} \text{ card Red}) \neq P(\text{red card})$

these events are not independent.

When two events are independent, something nice happens

Assume that E and F are independent, then

$$\underbrace{p(E \mid F) = \frac{p(E \cap F)}{p(F)}}_{\text{conditional prob.}} = p(E) \quad \Rightarrow \quad \frac{p(E \cap F)}{p(F)} = p(E)$$

If we solve the second equality for $p(E \cap F)$ we find that

$$p(E \cap F) = p(E)p(F)$$

So if two events are independent, we can compute the probability of their intersection (i.e. that they both occur) by simply multiplying their individual probabilities!

Probability Theory

Example: Suppose you roll a fair die and flip a coin. What is the probability that you roll a 4 and that the coin lands Tails?

$$P(\text{Roll a 4} \cap \text{flipping Tails}) = P(\text{Roll a 4}) \cdot P(\text{flipping Tails})$$

$$= \frac{1}{6} \cdot \frac{1}{2}$$

$$\boxed{= \frac{1}{12}}$$

Probability Theory

Example: Suppose you draw two cards from a deck. What is the probability that the 1st card is a club and the 2nd card is a heart?

Note: This is NOT independent because you are more likely to draw a heart after one of the clubs is removed.

$$\frac{P(2^{nd}\text{ card }\heartsuit \cap 1^{st}\text{ card }\clubsuit)}{P(1^{st}\text{ card }\clubsuit)} = P(2nd\text{ card }\heartsuit \mid 1^{st}\text{ card }\clubsuit)$$

$$P(2^{nd}\text{ card }\heartsuit \cap 1^{st}\text{ card }\clubsuit) = P(2nd\text{ card }\heartsuit \mid 1^{st}\text{ card }\clubsuit) \cdot P(1^{st}\text{ card }\clubsuit)$$

$$\boxed{= \frac{13}{51} \cdot \frac{13}{52}}$$

Probability Theory

def. of conditional prob.

Example: Suppose you draw two cards from a deck. What is the probability that the 1st card is a club and the 2nd card is an Ace?

$$P(2^{nd}\text{ card Ace} \cap 1^{st}\text{ card } \clubsuit) = P(\text{2nd card Ace} \mid 1^{st}\text{ card } \clubsuit) \cdot P(1^{st}\text{ card } \clubsuit)$$

Case 1: First card is not the $A\clubsuit$

$$P(\text{2nd card Ace} \mid 1^{st}\text{ card is } \clubsuit \text{ (not } A\clubsuit)) \cdot P(1^{st}\text{ card } \clubsuit \text{ not } A\clubsuit) = \frac{4}{51}\cdot\frac{12}{52}$$

Case 2: First card is the $A\clubsuit$

$$P(\text{2nd card Ace} \mid 1^{st}\text{ card is } A\clubsuit) \cdot P(1^{st}\text{ card } A\clubsuit) = \frac{3}{51}\cdot\frac{1}{52}$$

$$P(2^{nd}\text{ card Ace} \wedge 1^{st}\text{ card } \clubsuit) = \frac{4}{51}\cdot\frac{12}{52} + \frac{3}{51}\cdot\frac{1}{52}$$

Probability Theory

Example: What is the minimum number of people who need to be in a room so that the probability that at least two of them have the same birthday is greater than ½ ?

First we make some assumptions:

1. Birthdays of any two people are independent
2. Being born on any day of the year is equally likely
3. There are 366 days in the year

Restatement: The probability that two people have the same birthday is 1 minus the probability that no two people have the same birthday

Probability Theory

Now let's compute the probability that with n people in the room no two people have the same birthday

Think of asking people their birthdays 1 at a time

1. When the first person enters the room, the probability is 1 that they do not have the same birthday as anyone else

$n=2$ 2. When the second person enters the room, the probability is $\frac{365}{366}$ that they do not have the same birthday as the first person $\frac{366-1}{366}$

$n=3$ 3. When the third person enters the room, the probability is $\frac{364}{366}$ that they do not have the same birthday as the first two people $\frac{366-2}{366}$

n. When the n^{th} person enters the room, the probability is $\frac{366-(n-1)}{366}$ that they do not have the same birthday as the first two people $= \frac{366-n+1}{366}$

$= \frac{367-n}{366}$

Probability Theory

Since birthdays are independent, the probability that no two people have the same birthday is

$$p_n = \frac{365}{366} \cdot \frac{364}{366} \cdots \frac{367-n}{366}$$

And so the probability that two of the n people have the same birthday is

$$\underbrace{1 - p_n}\ = \underbrace{1 - \frac{365}{366} \cdot \frac{364}{366} \cdots \frac{367-n}{366}}$$

We want to know what is the first value of n such that $1 - p_n \geq \frac{1}{2}$

We could find n with Calculus, but let's just plug in some numbers

Probability Theory

When $n = 3$ the probability is $1 - p_n \approx 0.027$

When $n = 10$ the probability is $1 - p_n \approx 0.117$

When $n = 22$ the probability is $1 - p_n \approx 0.475$

When $n = 23$ **the probability is** $1 - p_n \approx 0.506$

When $n = 50$ the probability is $1 - p_n \approx 0.970$

When $n = 100$ the probability is $1 - p_n \approx 0.9999996$

Next Week: Bayes' Theorem!