Summary of: Stabilization of Planar Collective Motion: All-to-All Communication

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Introduction

How do we achieve synchrony with distributed dynamical systems? We often want measurements taken in certain spatial or temporal spread. The goal is to come up with a set of control primitives to create complex behavior.

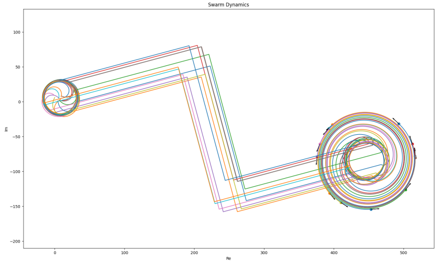


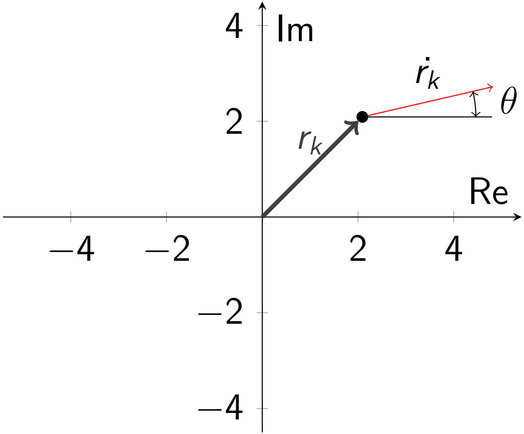
Figure: Demonstration of full set of primitives

Outline

- Model
- Shape Space
- Dynamic Propagation
- Review (Kuramoto Controller)
- Stabilizing Circular Formations
- Main Controller
- Control Primitives
- Results
- Stabilizing Isolated Circular Equilibria

Kinematic Model

We model N agents as particles of unit mass, moving at unit speed. Position of agent k is, $r_k \in \mathbb{C} \approx \mathbb{R}^2$, the heading, also called phase, of the particle is, $\theta \in S^1$.



Kinematic Model

The velocity of the particle is given by $\dot{r}_k$, and the change in heading, $\dot{\theta}_k$, is directly controlled (steering control) by u_k. This is taken from an earlier paper discussing the dynamics of UAVs.

$$\begin{aligned} \dot{r}_k &= e^{i\theta_k} \\ \dot{\theta}_k &= u_k, \quad k = 1, \ldots, N \end{aligned} \tag{1}$$

Bolded variables, i.e. $\mathbf{r}$, represent the column vector of states of all agents. The notation of something like $r_{kj} = r_k - r_j$ denotes differences.

Shape Space

Now lets just dip our toe into the group theory behind this...
"When the control law only depends on relative orientations and relative spacing... the closed-loop vector field is invariant under an action of the symmetry group $SE(2)$ and the closed-loop dynamics evolve on a reduced quotient manifold."

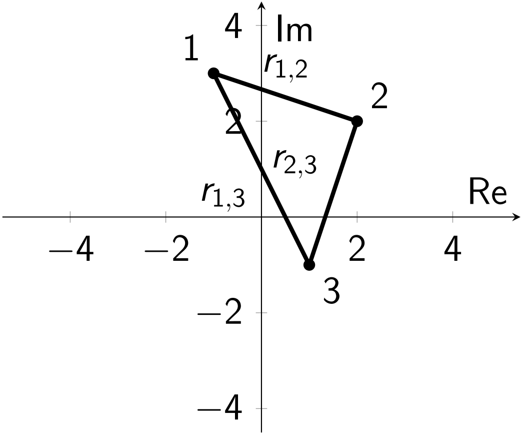
What in the world does this mean???

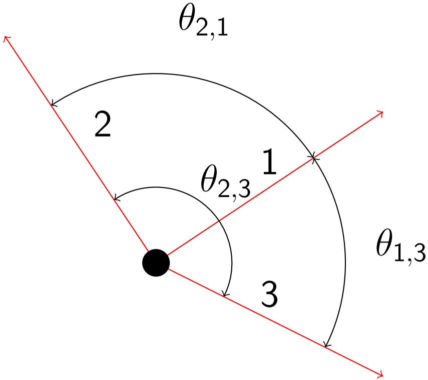
Shape Space

$SE(2)$ is the the group of transformations that are translations and rotations (rigid body transformations).

When the model is defined by the full configuration space the space is actually just N copies of $SE(2)$. But when we can reduce this to the relative orientations, this reduces the dimension of the space (reduced quotient manifold) to a $(3N - 3)$ dimensional space.

Shape Space





Shape Space

Now All equilibria are those that result in parallel or circular motion! If $\dot{\boldsymbol{\theta}}$ is zero and $\dot{\mathbf{r}}$ is zero then the particles (always moving at unit speed) are travelling in parallel and maintatining relative spacing. Conversely, if $\dot{\mathbf{r}}$ is zero and $\dot{\theta}$ is not, then the particles must be moving in a circle to maintain relative spacing.

Now lets apply one last simplification, our controller is solely based on phase. This means that $\boldsymbol{\theta} \in T^N$, the N dimensional torus.

Shape Space

This is an important result since models like this have been the study of many papers, it is called a phase model of a coupled oscillator. This research comes from neural research, chemical analysis and superconductors.

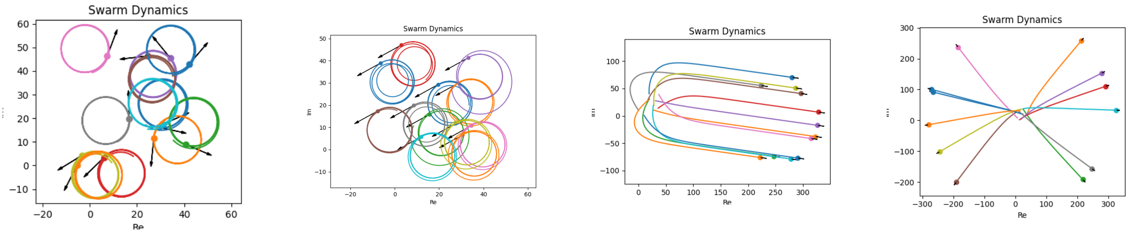
Dynamic Propagation

The paper uses a very simple Euler update:

$$\begin{aligned} r_k(t+1) &= e^{i\theta_k} + r_k(t) \\ \theta_k(t+1) &= u_k + \theta_k(t) \end{aligned} \tag{2}$$

Review

Using the Kuramoto model for our control in class we were able to obtain the following results:



Review

This was using a Kuramato type controller, but written as:

$$\begin{aligned} \dot{\theta} &= \omega_0 \mathbf{1} - K\,\mathbf{grad}\,U_1 \\ -K\,\mathbf{grad}\,U_1 &= -K\frac{\partial U_1}{\partial \theta_k} \end{aligned} \tag{3}$$

U_1 is the same potential that we were concerned about in the Kuramoto model we worked on in class. Namely:

$$\dot{\theta}_k = \omega_0 - \sum_{j=1}^{N} \sin \theta_{jk} \tag{4}$$

Review

Recall the following which the paper calls Theorem 1:

Theorem 1: The potential $U_1 = \frac{N}{2}|p_\theta|^2$, ($p_\theta$ is the average linear momentum of the swarm), reaches its unique minimum when $p_\theta = 0$ (balancing) and its unique maximum when all phases are identical (synchronization). All other critical points of U_1 are isolated in the shape manifold T^N/S^1 and are *saddle points* of U_1.

Stabilization of Circular Formations

Okay so we can get all of the agents to move in a circle and point in the same direction, so what?

Lets stabilize these agents with the same center and expand the notion of balance, turns out this will also allow us to make our straight line movement more orderly too!

Stabilization of Circular Formations

The paper characterizes this extension, *spacing* control. We need the agents to orbit the same point. Let's define a few things:

$$\begin{aligned} \rho_0 &= |\omega_0|^{-1} \\ c_k &= r_k + i\omega_0^{-1} \\ s_k &= -i\omega_0 c_k = e^{i\theta_k} - i\omega_0 r_k \end{aligned} \tag{5}$$

Where ρ_0 is the radius of the orbit, c_k is the point of the center and s_k can be thought of as a pseudo-center with the sidedness of the center baked in, though no rationale for s_k comes from is given.

Stabilization of Circular Formations

With an all-to-all communication framework we have the Laplacian P and pseudo-center $\mathbf{s}$, we have:

$$P\mathbf{s} = 0 \quad P = I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T \tag{6}$$

when the circle centers coincide. This becomes the root of a Lyapunov analysis that results in the following controller based on stabilizing the system and driving the system to the above condition. The specifics of the proof are outside of the scope of what we want to do, but here are a few highlights:

Stabilization of Circular Formations

The Lyapunov chosen and its derivative are while noting the derivative of s_k:

$$\begin{aligned} S(\mathbf{r},\theta) &= \frac{1}{2}||P\mathbf{s}||^2 \\ \dot{s}_k &= ie^{i\theta_k}(u_k - \omega_0) \\ \dot{S} = \langle P\mathbf{s}, P\dot{\mathbf{s}}\rangle &= \sum_{k=1}^{N} \langle P_k\mathbf{s}, ie^{i\theta_k}\rangle (u_k - \omega_0) \end{aligned} \tag{7}$$

Stabilization of Circular Formations

Choosing the following control law the following holds

$$
\begin{aligned}
u_k &= \omega_0 - \kappa \langle P_k \mathbf{s}, ie^{i\theta_k} \rangle \\
\dot{S} &= -\kappa \sum_{k=1}^{N} \langle P_k \mathbf{s}, ie^{i\theta_k} \rangle^2 \leq 0
\end{aligned}
\tag{8}
$$

This implies that the controller stabilizes the desired function allows our system of agents to orbit a shared center!

Stabilization of Circular Formations

Solving the expression for u_k we find the equation for the control effort for each agent k.

$$
\begin{aligned}
u_k &= \omega_0 - \kappa\langle P_k\mathbf{s}, ie^{i\theta_k}\rangle \\
P_k\mathbf{s} = s_k - \frac{1}{N}\mathbf{1}^{\mathsf{T}}\mathbf{s} &= e^{i\theta_k} - i\omega_0 r_k - (\dot{R} - i\omega_0 R) \\
\langle P_k\mathbf{s}, ie^{i\theta_k}\rangle &= -\langle \omega_0(r_k - R), e^{i\theta_k}\rangle - \langle \dot{R}, ie^{i\theta_k}\rangle \\
&= -\langle \omega_0\tilde{r}_k, e^{i\theta_k}\rangle - \frac{\partial U_1}{\partial \theta_k}
\end{aligned}
\tag{9}
$$

Stabilization of Circular Formations

Key takeaways from this proof:

- You can use a Lyapunov analysis to drive the design of your controller, not just verify its performance.
- The Laplacian of the Multi-Agent system plays a major role in the controller you need to choose.
- The result of picking a control law during this analysis is the sure knowledge that your controller will drive to the location in state space you desire.

Stabilization of Circular Formations

From that analysis we can draw the following conclusion which the paper calls Theorem 2:

Therorem 2: Consider the particle model given with spacing control (derived from the Lyapunov analysis) given by:

$$
\begin{aligned}
u_k &= \kappa \frac{\partial U_1}{\partial \theta_k} + \omega_0 (1 + \kappa \langle \tilde{r}_k, \dot{r}_k \rangle) \\
\tilde{r}_k &= r_k - R
\end{aligned}
\tag{10}
$$

All solutions converge to a relative equilibrium defined by a circular formation of radius $\rho_0 = |\omega_0|^{-1}$ with direction determined by the sign of $\omega \neq 0$.

Stabilization of Circular Formations

Lets take a closer look at this controller.

$$\begin{aligned} u_k &= \kappa \frac{\partial U_1}{\partial \theta_k} + \omega_0 (1 + \kappa \langle \tilde{r}_k, \dot{r}_k \rangle) \\ \tilde{r}_k &= r_k - R \end{aligned} \tag{11}$$

Phase Symmetry Breaking in Circular Formations

Alright, one last component and we will have our first primitive!

We can stabilize around a center and align or balance the phase, lets put the two together. In order to do this we add back in our balancing/synchronizing potential. In general the phase potential need only be smooth and symmetric to rigid rotations. The following theorem informs our design.

Phase Symmetry Breaking in Circular Formations

The authors achieve this in a really nifty and useful way. They just concatenate the previous Lyapunov function! This idea can be powerful, though we will not take the time to walk it through.

$$V(\mathbf{r}, \theta) = \kappa S(\mathbf{r}, \theta) + U(\theta) \tag{12}$$

After the analysis we have proved the following theorem:

Phase Symmetry Breaking in Circular Formations

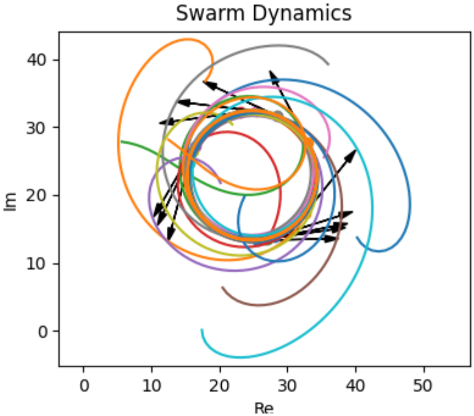
Theorem 3: Consider the particle model given and a smooth phase potential $U(\theta)$ that satisfies $\langle \mathbf{grad}\, U, \mathbf{1} \rangle = 0$. The control law

$$u_k = \omega_0(1 + \kappa\langle \tilde{r}_k, \dot{r}_k \rangle) - \frac{\partial U - \kappa U_1}{\partial \theta_k}, \quad \omega_0 \neq 0 \tag{13}$$

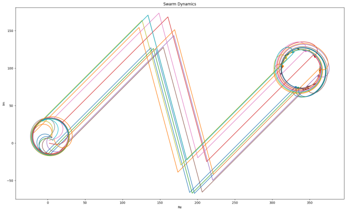
enforces convergence of all solutions to the set of relative equilibria where all particles are on the same circle, with a phase arrangement in the critical set of U. Every (local) minimum of U defines an asymptotically stable set of relative equilibria. Every relative equilibrium wher U does not reach a minimum is unstable.

Results

These results are for the specified phase potential $U = KU_1$.



The Control Primitives



Now that we have our first primitive for circling the center of mass, we can now build the rest of the components to build the capability for the behavior seen above. The piece-wise technique used comes from previous work of the authors, so we will take the stability of the impulses and control switching for granted.

Impulses

The impulses used are instantaneous changes in phase to align the particles in the direction of interest. The direction of interest is calculated then applied to the particle in a single instant.

Circular to Parallel

We need a way to go from travelling in a circle to travelling in a straight line. We will do this using a fixed reference heading. Firstly, we will align all of the agents by applying the following impulse:

$$\Delta\theta_k = \theta_0 - \theta_k \tag{14}$$

Then for agents $k = 1, ..., N - 1$, we use the controller described in (13) (Equation 19 in the paper) with $\omega_0 = \kappa = 0$ and $U = KU_1,\ K < 0$.

Circular to Parallel

For agent $k = N$ we need a different controller. This controller is:

$$u_N = \omega_0(1 + \kappa\langle \tilde{r_N}, \dot{r_N}\rangle) - \frac{\partial U - \kappa U_1}{\partial \theta_N} + d\sin(\theta_0 - \theta_N), \quad d > 0 \tag{15}$$

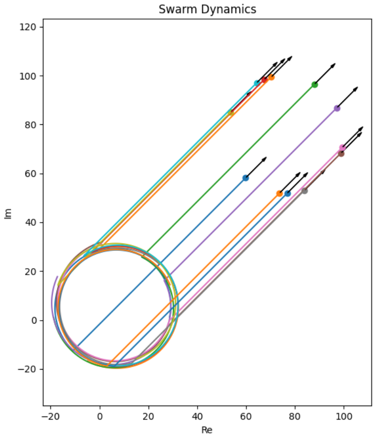
Where θ_0 is the reference heading. They prove this control in the same way as before, taking their existing Lyapunov function V and concatenating it with an additional control and pushing the analysis through.

Circular-to-Parallel

With $\kappa = \omega_0 = 0$, the controller for agent N is drastically simplified to:

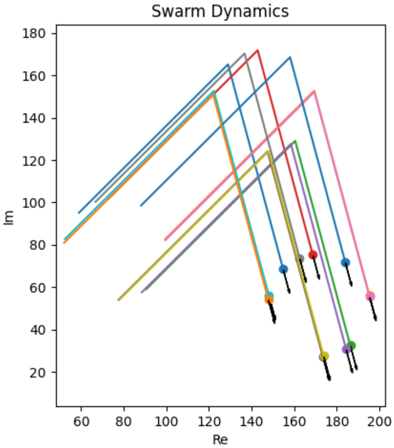
$$-K\frac{\partial U_1}{\partial \theta_N} + dsin(\theta_0 - \theta_N)$$

Essentially the Nth agent will not converge to the average of the group but rather pulls until it is aligned with the reference heading, ensuring the convergence of the group heading to the reference. It is a fully connected graph with a sink.



Parallel-to-Parallel

Starting from parallel motion, this primitive changes the swarm direction to a new reference. It is identical to the previous, and utilizes the same impulse to align the particles initially.



Parallel-to-Circular

In this case we assume $\omega_0 \neq 0$ and $\kappa > 0$. We define R_0 as the initial center of mass when the primitive is instated. We use the impulse:

$$\Delta\theta_k = \arg(i\omega_0 \tilde{r}_k) - \theta_k \tag{16}$$

where $\tilde{r}_k = r_k - R_0$. This makes R_0 a fixed beacon and allows the orbit of the swarm to precisely align with a desired center. The arg measures the angle of the tangent at an agent's location. The impulse aligns the agents exactly to the tangent of the orbit.

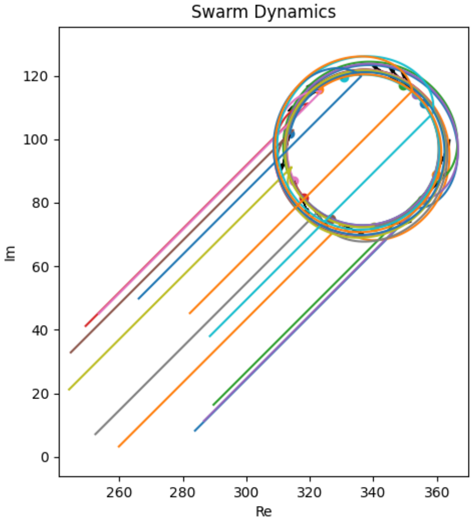
Parallel-to-Circular

The agents are aligned with controller (13) but with a fixed beacon for the calculation of $\tilde{r}_k$ and U_1 removed.

$$\mathrm{u}_k = \omega_0(1 + \kappa\langle r_k - R_0, \dot{r}_k\rangle) - \frac{\partial U}{\partial \theta_k}, \quad \omega_0 \neq 0 \tag{17}$$

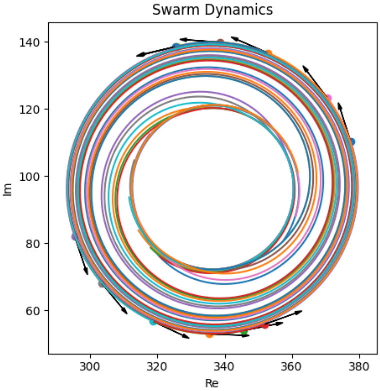
Again the stability of this controller is proved by taking the existing Lyapunov function and concatenating it.

Parallel-to-Circular



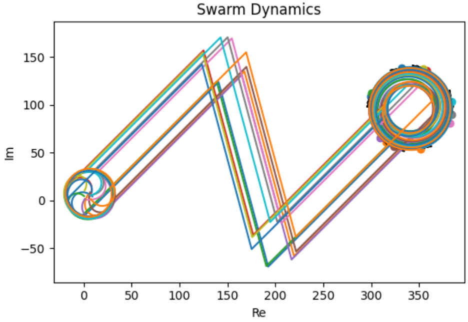
Circular-to-Circular

In this case we assume $\omega_0 \neq 0$ and $\kappa > 0$. There is no need for an impulsive control With the same controller as previous. This is stable for the same reasons.



Combining Primitives

The combination of primitives is as simple as turning off and on controllers as necessary. The code here is designed so that blocks can be added to extend the pathing in whatever way is desired.



Stabilization of Isolated Circular Equilibria

Let's zoom back to the controller described in (13) and pay special attention to the potential U:

$$u_k = \omega_0(1 + \kappa\langle \tilde{r}_k, \dot{r}_k\rangle) - \frac{\partial U - \kappa U_1}{\partial \theta_k}, \quad \omega_0 \neq 0 \tag{18}$$

U allows us to stabilize the agents to a particular orbit. What else can it do?

Stabilization of Isolated Circular Equilibria

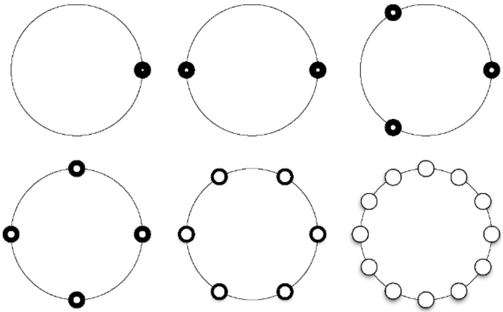
Can we control the spacing of the agents using this? We can! It actually allows us to

$$u_k = \omega_0(1 + \kappa\langle \tilde{r}_k, \dot{r}_k \rangle) - \frac{\partial U - \kappa U_1}{\partial \theta_k}, \quad \omega_0 \neq 0 \tag{19}$$

U allows us to stabilize the agents to a particular orbit. What else can it do?

Stabilization of Isolated Circular Equilibria

With this potential U the paper explores symmetric patterns. These are states of the swarm that push the system to particular stable equilibria. These are (top left) perfect synchrony all the way to perfect balance (bottom right).



Stabilization of Isolated Circular Equilibria

They describe this potential as $U^{M,N}$. Where N is the number of agents and M is a divisor of N. In reality this is the sum of individual potentials:

$$U^{M,N} = \sum_{m=1}^{M} K_m U_m \tag{20}$$

We will vary the K_ms and the U_ms dependent on an agent's number and the result will give us the symmetric and balanced formations we are looking for. Specifically when this potential is at its global minimum.

Stabilization of Isolated Circular Equilibria

The controller that stabilizes to the global minimum, exponentially is:

$$u_k = \omega_0(1 + \kappa\langle \tilde{r}_k, \dot{r}_k \rangle) - \frac{\partial U^{M,N} - \kappa U_1}{\partial \theta_k} \tag{21}$$

The proof of this is involved and involves breaking the eigenvalues of the potential into its Fourier coefficients, solving a nasty integral and drawing conclusions based on the Hessian. The key result is that this shows it *can* stabilize to the global minimum but is not a guarantee. Though by the author's report it does not stabilize generally at other points.

Stabilization of Isolated Circular Equilibria

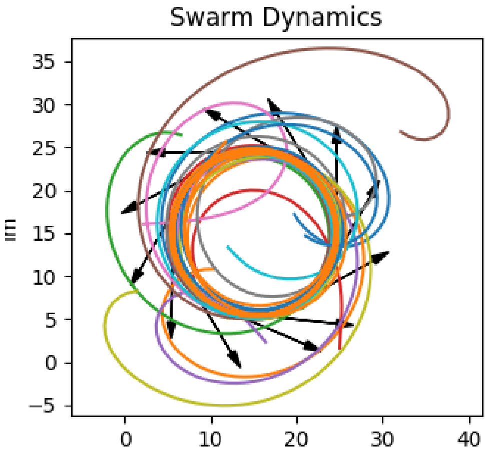
The paper does not give a straight forward set of instructions for creating a $U^{M,N}$ potential. However they do outline the perfect balance state, or splay state potential. THe potential is

$$U^{N,N} = \sum_{m=1}^{\lfloor \frac{N}{2} \rfloor} K_m U_m \tag{22}$$

with $K_m > 0$ for $m = 1, ..., \lfloor (N/2) \rfloor$ and zero otherwise.

Stabilization of Isolated Circular Equilibria

I was able to acheive the following results and interestingly if you floor the sum's range wrong, you get the $N/2, N$ state.



Final Review

This paper demonstrated how a set of primitives can be used to obtain useful formations. This work can be extended to temporal potentials, allowing for temporally evenly space formations. The methods in this paper could possibly be used to extend this to spheres in $\mathbb{R}^3$. In addition it covered how to fuse multiple controllers and extend the aggregate controller driven by Lyapunov analysis.

All code on GitHub:
https://github.com/iandareid/StabilizationOfPlanarCollectiveMotion