Assignment 2

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1 Question 1

Suppose that an athletic field has parallel lines 1 foot apart. Suppose that a $\frac{1}{2}$ foot long rod is randomly dropped on the field. What is probability that the rod will intersect one of the parallel lines? Use a simulation study to approximate this probability. Use $N = 10^i$ realizations for $i = 2, \cdots, 6$ in your study.

1.1 Code

```
#include<iostream>
#include<cmath>
#define half_pi 2.0*atan(1)
#define pi 4.0*atan(1)
using namespace std;

// Use acceptance and rejection method
// consider 2 random numbers u1, u2; u1 determines one end of the rod;
// the rotation of the rod, theta, is determined by the u2,
// due to symmetry, (o,half pi) is considered
// if u1 plus/minus the projection of the rod is >/< 1,
// then the rod crossed the line

int main(){
    double u1, u2 ; //u1,u2~u(0,1)
    double theta; //angle of the rod, theta~u(-pi/2,pi/2)
    int N[5] = {100,1000,10000,100000,1000000};//array to store all iter. numbers
    int counter;
    double prob;
    for(int i=0;i<5;i=i+1){
        counter = 0 ;
        for(int j=0;j<N[i];j=j+1){
            u1 = ((double) rand() / (RAND_MAX));
            u2 = ((double) rand() / (RAND_MAX));
            u1 = u1 * 0.5;
            theta = u2 * pi - half_pi;
            if( u1 + 0.5*sin(theta)> 0.50){
                counter = counter + 1;}
        }
        prob = ((double) counter)/N[i];
        cout<< "Crossing probability for N="<<N[i]<<" is "<< prob << endl;

    }
}
```

1.2 results

```
Crossing probability for N=100 is 0.42
Crossing probability for N=1000 is 0.314
Crossing probability for N=10000 is 0.3102
Crossing probability for N=100000 is 0.3184
Crossing probability for N=1000000 is 0.318615
```

2 Question 2

Give two different algorithms for simulating X having probability mass function $p_j, j = 1, 2, \cdots, 10$ where

$$p_j = \begin{cases} 0.11 & \text{if } j \text{ is odd} \\ 0.09 & \text{if } j \text{ is even} \end{cases} \tag{1}$$

Implement one of your algorithms and check its validity by comparing the p_j you observe with the true values of p_j using $N = 10^i$ realizations for $i = 2, \cdots, 6$.

2.1 Algorithm code a

```
#include<iostream>
#define fgy 0.09/0.11
using namespace std;
// g(x)=1/10, f(x) is p_j, then c = 1.1
int main(){
    double u1,u2; //u~u(0,1)
    double temp;
    int x,y;//y is candidate, x is the acceptance
    int N[5] = {100,1000,10000,100000,1000000};//array to store all iter. numbers
    int counter[10] = {0}; //count the # of diff. ints
    int total_acceptance = 0;
    for(int i=0;i<5;i=i+1){
        for(int j = 0; j < N[i]; j++)
        {
            u1 = ((double)rand()/(RAND_MAX));
            y = (int)( u1*10.0 + 1.0);
            //f(y)=0.11 if y is odd, therefore f(y)/cg(y)=1,and u<1
            if ( y%2 == 1){
                x=y;
                counter[x-1] = counter[x-1] + 1;
                total_acceptance = total_acceptance + 1;
            }else{
                u2 = ((double)rand()/(RAND_MAX));
                if(u2<=fgy){
                    x=y;
                    counter[x-1] = counter[x-1] + 1;
                    total_acceptance = total_acceptance + 1;
                }
            }
        }
        cout<<"N= "<<N[i]<<" , total acceptance is "<<total_acceptance<<endl;
        for (int j=0;j<10;j++){
            temp = (double)counter[j]/total_acceptance;
            cout<<"The probability of "<<(j+1)<<" is "<< temp<<endl;
            counter[j] = 0; //reset
        }
        total_acceptance = 0;//reset
        cout<<"\n"<<endl;
    }
}
```

2.2 result a

```
N= 100 , total acceptance is 91
The probability of 1 is 0.0769231
The probability of 2 is 0.131868
The probability of 3 is 0.10989
The probability of 4 is 0.0989011
The probability of 5 is 0.0659341
The probability of 6 is 0.0769231
The probability of 7 is 0.120879
The probability of 8 is 0.10989
The probability of 9 is 0.0989011
The probability of 10 is 0.10989

N= 1000 , total acceptance is 920
The probability of 1 is 0.102174
The probability of 2 is 0.1
The probability of 3 is 0.113043
The probability of 4 is 0.0913043
The probability of 5 is 0.0956522
The probability of 6 is 0.0880435
The probability of 7 is 0.123913
```

```
The probability of 8 is 0.0847826
The probability of 9 is 0.117391
The probability of 10 is 0.0836957

N= 10000 , total acceptance is 9081
The probability of 1 is 0.109239
The probability of 2 is 0.0906288
The probability of 3 is 0.108028
The probability of 4 is 0.0942627
The probability of 5 is 0.114855
The probability of 6 is 0.0965753
The probability of 7 is 0.103623
The probability of 8 is 0.0882061
The probability of 9 is 0.106486
The probability of 10 is 0.088096

N= 100000 , total acceptance is 90917
The probability of 1 is 0.108583
The probability of 2 is 0.0906651
The probability of 3 is 0.111761
The probability of 4 is 0.0892902
The probability of 5 is 0.10866
The probability of 6 is 0.0902691
The probability of 7 is 0.11164
The probability of 8 is 0.0903681
The probability of 9 is 0.108539
The probability of 10 is 0.0902252

N= 1000000 , total acceptance is 909039
The probability of 1 is 0.10943
The probability of 2 is 0.0898938
The probability of 3 is 0.110047
The probability of 4 is 0.0903229
The probability of 5 is 0.110323
The probability of 6 is 0.0898762
The probability of 7 is 0.110069
The probability of 8 is 0.0902085
The probability of 9 is 0.110167
The probability of 10 is 0.0896628
```

2.3 Algorithm code b

```
#include<iostream>
using namespace std;
//using a naive way
int main(){
    double u1; //u~u(0,1)
    int N[5] = {100,1000,10000,100000,1000000};//array to store all iter. #s
    int counter[10] = {0}; //count the # of diff. ints

    for(int i=0;i<5;i=i+1){
        for(int j = 0; j < N[i]; j++)
        {
            u1 = ((double)rand()/(RAND_MAX));
            if (u1<=0.11){
                counter[0] = counter[0] + 1;
            }else if(u1<=0.2){
                counter[1] = counter[1] + 1;
            }else if(u1<=0.31){
                counter[2] = counter[2] + 1;
            }else if(u1<=0.4){
                counter[3] = counter[3] + 1;
            }else if(u1<=0.51){
                counter[4] = counter[4] + 1;
            }else if(u1<=0.6){
                counter[5] = counter[5] + 1;
            }else if(u1<=0.71){
                counter[6] = counter[6] + 1;
            }else if(u1<=0.8){
                counter[7] = counter[7] + 1;
            }else if(u1<=0.91){
                counter[8] = counter[8] + 1;
            }else {
                counter[9] = counter[9] + 1;
            }
```

```
        }
        cout<<"N="<<N[i]<<endl;
        for (int j=0;j<10;j++){
            cout<<"The␣probability␣of␣"<<(j+1)<<"␣is␣"<< (double)counter[j]/N[i]<<endl;
            counter[j] = 0;   //reset
        }
        cout<<"\n"<<endl;
    }
}
```

2.4 Result b

```
N=100
The probability of 1 is 0.08
The probability of 2 is 0.07
The probability of 3 is 0.1
The probability of 4 is 0.1
The probability of 5 is 0.08
The probability of 6 is 0.1
The probability of 7 is 0.12
The probability of 8 is 0.09
The probability of 9 is 0.13
The probability of 10 is 0.13

N=1000
The probability of 1 is 0.105
The probability of 2 is 0.097
The probability of 3 is 0.111
The probability of 4 is 0.081
The probability of 5 is 0.098
The probability of 6 is 0.091
The probability of 7 is 0.131
The probability of 8 is 0.088
The probability of 9 is 0.116
The probability of 10 is 0.082

N=10000
The probability of 1 is 0.1109
The probability of 2 is 0.091
The probability of 3 is 0.1126
The probability of 4 is 0.0928
The probability of 5 is 0.1092
The probability of 6 is 0.0944
The probability of 7 is 0.104
The probability of 8 is 0.0874
The probability of 9 is 0.1077
The probability of 10 is 0.09

N=100000
The probability of 1 is 0.10926
The probability of 2 is 0.0893
The probability of 3 is 0.11182
The probability of 4 is 0.08942
The probability of 5 is 0.10901
The probability of 6 is 0.09043
The probability of 7 is 0.1105
The probability of 8 is 0.08966
The probability of 9 is 0.10976
The probability of 10 is 0.09084

N=1000000
The probability of 1 is 0.110021
The probability of 2 is 0.089938
The probability of 3 is 0.109603
The probability of 4 is 0.089969
The probability of 5 is 0.1107
The probability of 6 is 0.090014
The probability of 7 is 0.109937
The probability of 8 is 0.089719
The probability of 9 is 0.110188
The probability of 10 is 0.089911
```

3 Question 3

A Weibull random variable X with parameters α and β has distribution

$$F(x) = 1 - e^{-\alpha x^{\beta}}, 0 < x < \infty. \tag{2}$$

Give an algorithm for simulating X. Implement your algorithm for $\alpha = 1$ and $\beta = 2$ and check its validity by plotting a histogram of simulated values and comparing with theoretical density.

3.1 Code part a

```
#include<iostream>
#include<cmath>
#include<fstream>
using namespace std;
//acceptance - rejection, try g(x)=exp(-x);
//therefore, c=2(acceptable)
int main(){
    double u1,u2; //u~u(0,1)
    double temp; // f(y)/c*g(y)
    double x,y;//y is the candidate~exp(1), x is the acceptance
    int N = 1000000;//iter. numbers

    ofstream myfile;
    myfile.open("problem3.csv");
    for(int j = 0; j < N; j++){
        u1 = ((double)rand()/(RAND_MAX));
        u2 = ((double)rand()/(RAND_MAX));
        y = -log(u1);
        temp = y*exp(-y*y+y);
        if(u2<temp){
            x=y;
            myfile<<x<<"\n";
        }
    }
    myfile.close();
    return 0;
}
```

3.2 Code part b

```
load problem3.csv;
figure();
h = histogram(problem3,'Normalization','pdf');%plot histogram, with normalization to pdf
hold on;
X=linspace(0.001,5.0,1000);
f = @(x)(2.*x.*exp(-x.^2));
plot(X,f(X),'r');
legend('Histogram of simulated values','Weibull distribution')
hold off;
```

3.3 Results

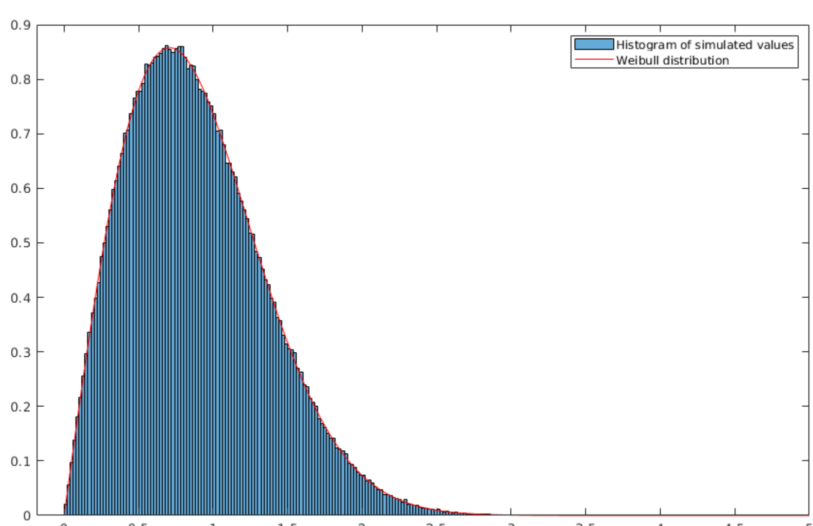

Figure 1: Comparison between normalized histogram and pdf

4 Question 4

Give an algorithm for simulating X having probability density function

$$f(x) = \begin{cases} \frac{x-2}{2} & \text{if } 2 \le x \le 3 \\ \frac{2-\frac{x}{3}}{2} & \text{if } 3 \le x \le 6 \end{cases} \tag{3}$$

Implement your algorithm and check its validity by plotting a histogram of simulated values and comparing with theoretical density.

4.1 Code part a

```
#include<iostream>
#include<cmath>
#include<fstream>
using namespace std;
//acceptance - rejection, try g(x)=1/4,x in (2,6);
//therefore, C=2
int main(){
    double u1,u2; //u~u(0,1)
    double temp;  // f(y)/c*g(y)
    double x,y;   //y is the candidate~exp(1), x is the acceptance
    int N = 1000000;//iter. numbers

    ofstream myfile;
    myfile.open("problem4.csv");
    for(int j = 0; j < N; j++){
        u1 = ((double)rand()/(RAND_MAX));
        u2 = ((double)rand()/(RAND_MAX));
        y = u1*4.0+2.0;//y in (2,6)
        if(y<3){
            temp = (y-2.0);
        }
        else{
            temp = 2.0 - y/3.0;
        }
        if(u2<temp){
            x=y;
            myfile<<x<<"\n";
        }
    }
    myfile.close();
    return 0;
}
```

4.2 Code part b

```
load problem4.csv
figure();
h = histogram(problem4,'Normalization','pdf');%plot histogram, with normalization to pdf
hold on;
X=linspace(2.0,6.0,1000);
syms f(x)
f(x) = piecewise(2<x<3, (x-2)/2, 3<x<6, (2-(x/3))/2);
plot(X,f(X),'r');
legend('Histogram␣of␣simulated␣values','Probability␣density␣function')
```

4.3 Results

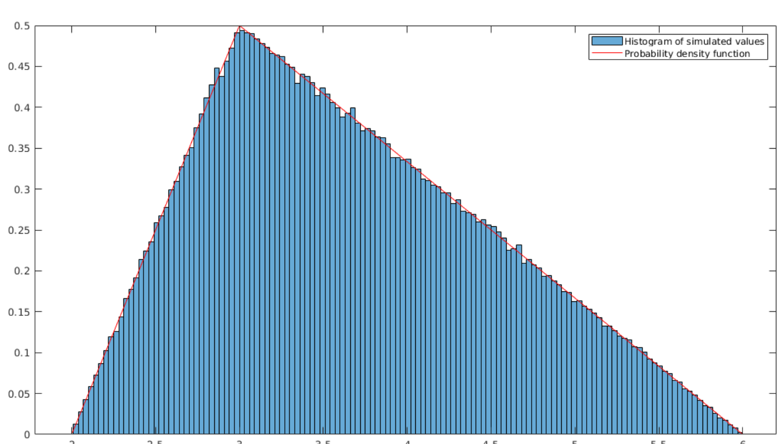

Figure 2: Comparison between normalized histogram and pdf